

AdS x S _ CFT

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Lectures on AdS/CFT from the Bottom Up

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Abstract

AdS/CFT from the perspective of Effective Field Theory and the Conformal Bootstrap.

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1 Introduction

1.1 Why is Quantum Gravity Different?

Fundamental physics has been largely driven by the reductionist program – look at nature with better and better magnifying glasses and then build theories based on what you see. In a quantum mechanical universe, due to the uncertainty relation, we cannot refine our microscope without using correspondingly larger momenta and energies. Hence the proliferation of larger and larger colliders.

Sometimes when you go to higher energies (shorter distances), physics changes radically at some critical energy scale. Theoretical particle physicists get up in the morning to find and understand the mechanisms behind new, frontier energy scales. For example, in Fermi’s theory of beta decay there was the dimensionful coupling

$$G_F \sim \frac{1}{(293 \text{ GeV})^2} \sim (7 \times 10^{-17} \text{ m})^2 \quad (1.1)$$

and Fermi’s theory was seen to break down at energies somewhere below 293 GeV. Fermi’s theory has been replaced by the electroweak theory, where physics does change drastically before this

energy scale – one starts to produce new particles, the W and Z bosons. Even more dramatically, when physicists first probed the QCD scale, around a GeV (or 2×10^{-14} m), we found a plethora of new strongly interacting particles; the resulting confusion led to the first S-Matrix program, String Theory, and one of the first uses of Effective Field Theory in particle physics. But we’ve since learned to describe both the QCD and the Weak scale, and much else, using local quantum field theories, and they no longer remain a mystery.

Gravity appears to differ qualitatively from these examples. The point is that if you were to try to resolve distances of order the Planck length $\ell_{pl} \sim 10^{-35}$ meters, you would need energies of order the Planck mass, $\hbar/\ell_{pl}$, at which point you would start to make black holes. We are fairly certain of this because the universally attractive nature of gravity permits gedanken experiments in which we could make black holes without passing through a regime of physics we don’t understand. Pumping up the energy further just results in *larger and larger* black holes, and the naive reductionist program comes to an end. So it seems that there’s more to understanding quantum gravity than simply finding a theory of the “stuff” that’s smaller than the Planck length – in fact there is no well-defined notion of smaller than ℓ_{pl} .

In hindsight, we have had many hints for how to proceed, and most of the best hints were decades old even back in 1997, when AdS/CFT was discovered. The classic hint comes from Black Hole thermodynamics, in particular the statement that Black Holes have an entropy

$$S_{BH} = \frac{A}{4\ell_{pl}^2} \tag{1.2}$$

proportional to their surface area, not their volume. Since you can throw anything into a black hole, and entropies must increase, this BH entropy formula should be a fundamental feature of the universe, and not just a property of black holes. The largest amount of information you can store in any region in spacetime will be proportional to its surface area. This is radical, and differs from generic non-gravitational systems, e.g. gases of particles.

So spacetime has to die – at short distances it stops making sense, and it doesn’t seem to store information in its bulk, but only on its boundary. What can we do with this idea? Well, since the 60s it has been known that gravitational energy is not well-defined locally, but it can be made well-defined at, and measured from, infinity. But energy is nothing other than the Hamiltonian. So in gravitational theories, the Hamiltonian should be taken to live at infinity. Perhaps there’s a whole new description of gravitational physics, with not just a Hamiltonian but all sorts of infinity-localized degrees of freedom... a precisely well-defined theory, from which bulk gravity emerges as an approximation?

1.2 AdS/CFT – the Big Picture

AdS/CFT is many things to many people.

For our purposes, AdS/CFT is the observation that any complete theory of quantum gravity in an asymptotically AdS spacetime defines a CFT. For now you can just view AdS as ‘gravity in a box’. By ‘quantum gravity’ we simply mean any theory that is well-approximated by general relativity in AdS coupled to matter, i.e. scalars, fermions, gauge fields, and perhaps more exotic

stuff too, such as membranes and strings. AdS/CFT says that the Hilbert spaces are identical

$$\mathcal{H}_{CFT} = \mathcal{H}_{AdS-QG} \quad (1.3)$$

and all (physical, or ‘global’) symmetries can be matched between the two theories. In particular, the spacetime symmetries or isometries of AdS_{d+1} form the group $SO(2, d)$ and these act on the CFT_d as the conformal group¹, containing the Poincaré group as a subgroup. All states in both theories come in representations of this group. We will explain why quantum field theories in AdS produce CFTs, and we will (hopefully) explain which CFTs have perturbative AdS effective field theory duals.

An incomplete theory of quantum gravity in AdS, such as a gravitational effective field theory (e.g. the standard model of particle physics including general relativity), defines an approximate or effective CFT. Finding a complete theory of quantum gravity, a ‘UV completion’ for any given EFT, amounts to finding an exact CFT that suitably approximates the effective CFT. Among other things, this means that we can make exact statements about quantum gravity by studying CFTs. General theorems about CFTs can be re-interpreted as theorems about all possible theories of quantum gravity in AdS.

When placed in AdS, quantum field theories without gravity define what we’ll refer to as Conformal Theories (CTs). These are ‘non-local’ theories that have CFT-like operators with correlators that are symmetric under the global conformal group and obey the operator product expansion (OPE), but that do not have a local stress-energy tensor $T_{\mu\nu}$. By thinking about simple examples of physics in AdS, we can understand old, familiar results in a new, holographic language.

A famous and revolutionary fact is that (large) black holes in AdS are just CFT states. This means that AdS quantum gravity should be a unitary quantum mechanical theory. Roughly speaking, the temperature and entropy of AdS black holes correspond with the temperature of the CFT and the number of CFT states excited at that temperature, respectively. This suggests that the universality of black hole formation in AdS is dual to the universality of finite temperature physics. Another famous and important thought experiment explains how one obtains QCD-like confinement from strings that dip into AdS.

The purpose of these lectures is to explain these statements in detail. In the next two subsections we will make some philosophical and historical comments; hopefully they will provide some perspective on holography and QFT, but don’t feel discouraged if you do not understand them. They can be skipped on a first pass, especially for more practically or technically minded reader.

1.3 Quantum Field Theory – Two Philosophies

Why is Quantum Field Theory the way it is? Does it follow inevitably from a small set of more fundamental principles? These questions have been answered by two² very different, profoundly

¹In $d = 2$ there are many more symmetries (although no more isometries), and these form the full 2-d conformal group, with Virasoro algebra, as originally shown by Brown and Henneaux.

²There’s a third and oldest viewpoint which is also important, and plays a primary role in most textbooks. It lacks the air of inevitability, although it’s very practical: one just takes a classical theory and ‘canonically quantizes’ it. This was how QED was discovered – we already knew electrodynamics, and then we figured out how to quantize it. This viewpoint is certainly worth understanding; Shankar’s text gives a quick explanation of classical $\rightarrow$ quantum.

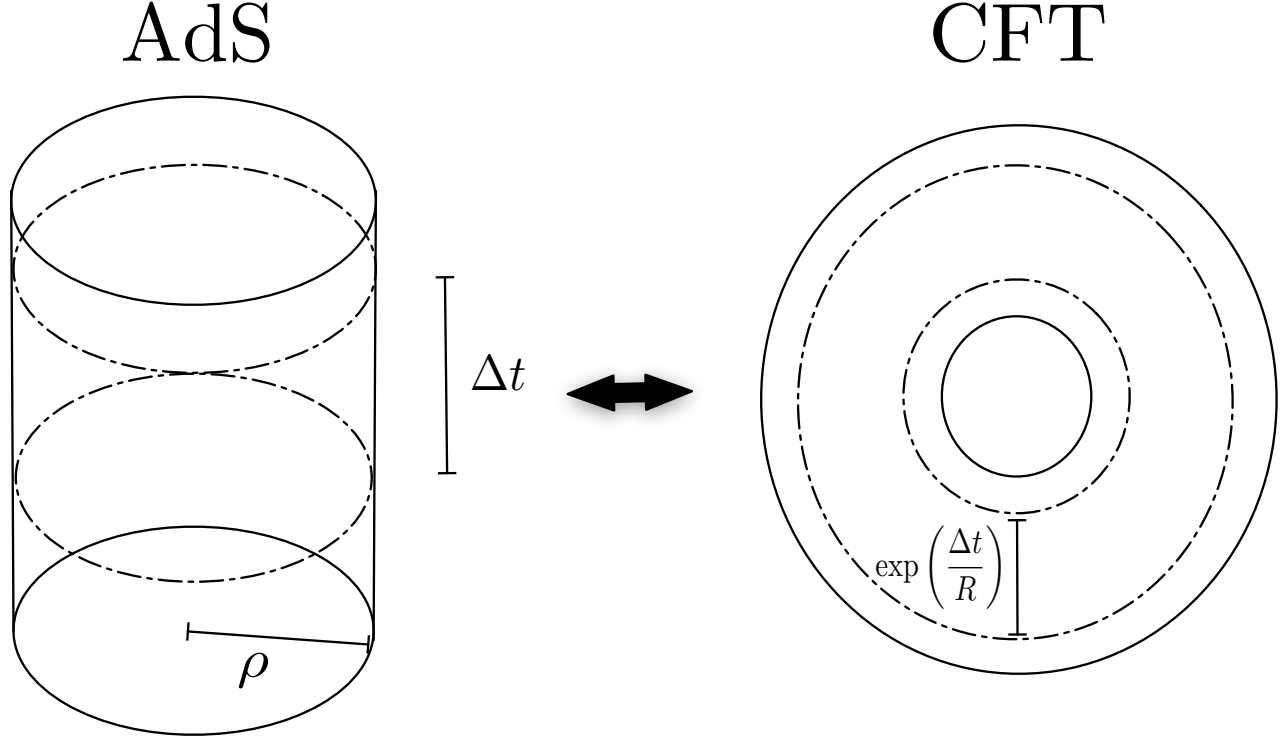


Figure 1: This figure shows how the AdS cylinder in global coordinates corresponds to the CFT in radial quantization. The time translation operator in the bulk of AdS is the Dilatation operator in the CFT, so energies in AdS correspond to dimensions in the CFT. We make this mapping very explicit in section [5.2.2](#).

compatible philosophies, which I will refer to as *Wilsonian* and *Weinbergian*.

The Wilsonian philosophy is based on the idea of zooming out. Two different physical systems that look quite different at short distances may behave the same way at long distances, because most of the short distance details become irrelevant. In particular, we can think of our theories as an expansion in ℓ_{short}/L , where ℓ_{short} is some microscopic distance scale and L is the length scale relevant to our experiment. We study the space of renormalizable quantum field theories because this is roughly equivalent to the space of universality classes of physical systems that one can obtain by ‘zooming out’ to long distances. Here are some famous examples

- The Ising Model is a model of spins on a lattice with nearest-neighbor interactions. We can zoom out by ‘integrating out’ half of the spins on the lattice, leaving a new effective theory for the remainder. However, at long distances the model is described by the QFT with action

$$S = \int d^d x \frac{1}{2} ((\partial\phi)^2 - \lambda\phi^4) \left(\right. \quad (1.4)$$

The details of the lattice structure become ‘irrelevant’ at long distances.

- Metals are composed of some lattice of various nuclei along with relatively free-floating electrons, but they have a universal phase given by a Fermi liquid of their electrons. Note that the Fermi temperature, which sets the lattice spacing for the atoms, is around 10,000 K whereas we are most interested in metals at $\sim 300\text{K}$ and below. At these energies metals are very well described by the effective QFT for the Fermi liquid theory. See [1] for a beautiful discussion of this theory and the Wilsonian philosophy. Research continues to understand the effective QFT that describes so-called strange metals associated with high temperature superconductivity.
- Quantum Hall fluids seem to be describable in terms of a single Chern-Simons gauge field; one can show that this is basically an inevitable consequence of the symmetries of theory (including broken parity), the presence of a conserved current, and the absence of massless particles.
- The Standard Model and Gravity. There are enormous hierarchies in nature, in particular from the Planck scale to the weak scale.
- Within the SM, we also have more limited (and often more useful!) effective descriptions of QED, beta decay, the pions and nucleons, and heavy quarks. Actually, general relativity plus ‘matter’ is another example of an effective description, where the details of the massive matter are unimportant at macroscopic distances (e.g. when we study the motion of the planets, it’s irrelevant what they are made of).

So if there is a large hierarchy between short and long distances, then the long-distance physics will often be described by a relatively simple and universal QFT.

Some consequences of this viewpoint include:

- There may be a true physical cutoff on short distances (large energies and momenta), and it should be taken seriously. The UV theory may not be a QFT. Effective Field Theories with a finite cutoff make sense and may or may not have a short-distance = UV completion.
- UV and IR physics may be extremely different, and in particular a vast number of distinct UV theories may look the same in the IR (for example all metals are described by the same theory at long distances). This means that a knowledge of long-distance physics does not tell us all that much about short-distance physics – TeV scale physics may tell us very little about the universe’s fundamental constituents.
- Symmetries can have crucially important and useful consequences, depending on whether they are preserved, broken, emergent, or anomalous. The spacetime symmetry structure is essential when determining what the theory can be – high-energy physics is largely distinguished from condensed matter because of Poincaré symmetry.
- QFT is a good approach for describing both particle physics and statistical physics systems, because in both cases we are interested in (relatively) long-distance or macroscopic properties. QFT fails to be a good description when there aren’t any interesting degrees of freedom that survive at distances that are long compared to the microscopic scale, e.g. to the lattice spacing.

For a classic review of the Wilsonian picture of QFT see Polchinski [1]. A nice perspective between high-energy and condensed-matter physics is provided by Cardy’s book *Scaling and Renormalization* [2]. Slava Rychkov’s notes give a CFT-oriented discussion of some of these ideas.

A natural question: what if we have a theory that does not change when we zoom out? This would be a scale invariant theory. In the case of high energy physics, where we have Poincaré symmetry, scale invariant theories are basically always conformally invariant QFTs³ which are called Conformal Field Theories (CFTs). It’s easy to think of one example – a theory of free massless particles has this property. One reason why asymptotic freedom in QCD is interesting is that it means that QCD can be viewed as the theory you get by starting with quarks that are arbitrarily close to being free (they have an infinitesimal interaction strength or coupling), and then zooming out. This makes it possible to rigorously define QCD as a mathematical theory.

It seems that all QFTs can be viewed as points along an Renormalization Flow (or RG flow, this is the name we give to the zooming process) from a ‘UV’ CFT to another ‘IR’ CFT. Renormalization flows occur when we deform the UV CFT, breaking its conformal symmetry. QCD was an example of this – we took a free theory of quarks and ‘deformed it’ at high energies by adding a small interaction. This leads to a last implication of the Wilsonian viewpoint as applied to relativistic QFTs:

- Well-defined QFTs can be viewed as either CFTs or as RG flows between CFTs. We can remove the UV cutoff from a QFT (send it to infinite energy or zero length) if it can be interpreted as an RG flow from the vicinity of a CFT fixed point.

So studying the space of CFTs basically amounts to studying the space of all well-defined QFTs. And as we will see, the space of CFTs also includes all well-defined theories of quantum gravity that we currently understand!

The Weinbergian philosophy [3] finds Quantum Field Theory to be the only way to obtain a Lorentz Invariant, Quantum Mechanical (Unitary), and Local (satisfying Cluster Decomposition) theory for the scattering of particles. Formally, a “theory for scattering” is encapsulated by the S-Matrix

$$S_{\alpha\beta} = \langle \alpha_{in} | S | \beta_{out} \rangle \quad (1.5)$$

which gives the amplitude for any “in-state” of asymptotically well-separated particles in the distant past to evolve into any “out-state” of similarly well-separated particles in the future. Some aspects of this viewpoint:

- Particles, the atomic states of the theory, are defined as irreducible representations of the Poincaré group. By definition, an electron is still an electron even if it’s moving, or if I rotate around it! This sets up the Hilbert space of incoming and outgoing multi-particle states as a Fock space of free particles. The fact that energies and momenta of distant particles *add* suggests that we can use harmonic oscillators a_p to describe each momentum p , because the harmonic oscillator has evenly spaced energy levels.

³This statement is still somewhat controversial, and hasn’t been rigorously proven for $d > 2$ dimensions.

- The introduction of creation and annihilation operators for particles is further motivated by the Cluster Decomposition Principle⁴. This principle says that very distant processes don't affect each other; it is the weakest form of locality, and seems necessary to talk sensibly about well-separated particles. Cluster decomposition will be satisfied if and only if the Hamiltonian can be written as

$$H = \sum_{m,n} \int d^3q_i d^3k_i \delta \sum_i \binom{q_i}{i} h_{mn}(q_i, k_i) a^\dagger(q_1) \cdots a^\dagger(q_m) a(k_1) \cdots a(k_n) \quad (1.6)$$

where the function h_{mn} must be a non-singular function of the momenta.

- We want to obtain a Poincaré covariant S-Matrix. The S operator defining the S-Matrix can be written as

$$S = T \exp \left(-i \int_{-\infty}^{\infty} dt V(t) \right) \quad (1.7)$$

Note that this involves some choice of t , which isn't very covariant-looking. However, if the interaction $V(t)$ is constructed from a local Hamiltonian density $\mathcal{H}(x)$ as

$$V(t) = \int d^3\vec{x} \mathcal{H}(t, \vec{x}) \quad (1.8)$$

where the Lorentz-scalar $\mathcal{H}(x)$ satisfies a causality condition

$$[\mathcal{H}(x), \mathcal{H}(y)] = 0 \quad \text{for} \quad (x - y)^2 \text{ spacelike} \quad (1.9)$$

then we will obtain a Lorentz-invariant S-Matrix. How does this come about? The point is that the interactions change the definition of the Poincaré symmetries, so these symmetries do not act on interacting particles the same way they act on free particles. To preserve the full Poincaré algebra with interactions, we need this causality condition.

- Constructing such an $\mathcal{H}(x)$ satisfying the causality condition essentially requires the assembly of local fields $\phi(x)$ with nice Lorentz transformation properties, because the creation and annihilation operators themselves do not have nice Lorentz transformation properties. The $\phi(x)$ are constructed from the creation and annihilation operators for each species of particle, and then $\mathcal{H}(x)$ is taken to be a polynomial in these fields.
- Symmetries constrain the asymptotic states and the S-Matrix. Gauge redundancies must be introduced to describe massless particles with a manifestly local and Poincaré invariant theory.
- One can prove (see chapter 13 of [3]) that only massless particles of spin ≤ 2 can couple in a way that produces long range interactions, and that massless spin 1 particles must couple to conserved currents J_μ , while massless spin 2 particles must couple to $T_{\mu\nu}$. This obviously goes a long way towards explaining the spectrum of particles and forces that we have encountered.

⁴In AdS/CFT we can *prove* that this principle holds in AdS directly from CFT axioms, but in flat spacetime we have to assume it. This very weak form of AdS locality follows from CFT unitarity.

In theories that include gravity, the Weinbergian philosophy accords perfectly with the idea of Holography: that we should view dynamical spacetime as an approximate description of a more fundamental theory in fewer dimensions, which ‘lives at infinity’. Holography was apparently not a motivation for Weinberg himself, and his construction can proceed with or without gravity. But the philosophy makes the most sense when we include gravity, in which case the S-Matrix is the only well-defined observable in flat spacetime.

One of the eventual takeaway points from these lectures is that Weinberg’s derivation of flat space quantum field theory from desired properties of the S-Matrix can be repeated in the case of AdS/CFT, replacing S-Matrix $\rightarrow$ CFT correlators. Specifically, quantum field theory in AdS can be derived in an analogous way from properties of correlation functions

$$\langle \mathcal{O}_1(x_1) \mathcal{O}_2(x_2) \mathcal{O}_3(x_3) \mathcal{O}_4(x_4) \rangle \quad (1.10)$$

of local CFT operators $\mathcal{O}_i$. Specifically:

- Objects in AdS arise as irreducible representations of the conformal group.
- The crossing symmetry of CFT correlators, which gives rise to ‘the Bootstrap Equation’, can be used to prove the Cluster Decomposition Principle in AdS, guaranteeing long-range locality. AdS cluster decomposition is a consequence of unitary quantum mechanics in CFT.
- If the correlators of low-dimension operators in the CFT are approximately Gaussian (determined entirely by 2-pt correlators), then the AdS/CFT spectrum is approximated by a Fock space⁵ of particles in AdS at low energies.
- Crossing symmetric, unitary, interacting (non-Gaussian) CFT correlators built perturbatively on a CFT Fock space will be derivable from an AdS Effective Field Theory description. The cutoff of this effective field theory in AdS can be related to properties of the CFT spectrum. Symmetries play a similarly important role for AdS/CFT.

The key point to notice is that the argument for flat space QFT from the S-Matrix is directly mirrored by the argument for AdS QFT from CFT correlators. The concepts of Poincaré Symmetry, Unitarity, and Cluster Decomposition have been replaced with Conformal Symmetry, Unitarity, and Crossing Symmetry. As we will eventually see, transitioning from flat space to AdS brings additional complications and challenges, but also certain benefits.

The relation between the two approaches is no coincidence – in fact, flat space S-Matrices can be derived from a limit of CFT correlation functions, where the dual AdS length scale $R \rightarrow \infty$. From this point of view, AdS space simply serves as an infrared regulator for flat space.

The converse of both Weinbergian arguments is simpler. It is comparatively straightforward (i.e. it’s the subject of all standard textbooks) to see that local flat space QFT produces a Poincaré covariant S-Matrix. It is also relatively easy to see why AdS QFTs produces good CFT correlation functions. Also, since we obviously have an interest in UV completing quantum gravity, it’s interesting to understand why gravity in AdS *must be* a CFT. So this is the plan of attack for most of these

⁵The Hilbert space formed from any number of non-interacting particles.

lectures – we will start by studying physics in AdS, introducing CFT ideas wherever useful, and show how field theories in AdS naturally produce CFTs⁶. Only at the end will we return to see how AdS EFT actually follows from simple assumptions about the CFT.

1.4 Brief Notes on the History of Holography

Some anachronistic and ideosyncratic comments about the history of Holography:

- In any theory where energy can be measured by a boundary integral, one must have holography, because ‘energy’ is just the Hamiltonian. The fact that this is the case in general relativity has been known for a long time, dating at least to the 1962 work of Arnowitt, Deser, & Misner [4], see for example the classic paper of Regge and Teitelboim [5] for a thorough treatment. Brice DeWitt and others were aware [6] in the 60s that the only exact observables are associated with the boundary, but they didn’t suggest that a different (local!) theory (e.g. a CFT) could live there and compute them.
- Most famously, holography can and usually is motivated via Black Hole thermodynamics, which dates to Bekenstein and Hawking in the 70s. The most important point is that BH’s have an entropy proportional to their area, not their volume. The fact that one can throw anything into a BH, combined with the 2nd law of thermodynamics, means that information in a gravitational universe must follow an area law.
- Work studying the asymptotic symmetries of GR proceeded in the 70s and 80s, leading to the famous Brown and Henneaux result [7] that the asymptotic symmetry group in AdS_3 is the infinite dimensional Virasoro algebra = the conformal algebra in 2 dimensions. Note that this work is quite non-trivial because one has to understand the symmetries of spacetime while allowing spacetime to fluctuate. In a certain sense the full Virasoro symmetry of gravitational AdS_3 is always spontaneously broken to the global $SL(2)$ conformal group.
- The Strominger-Vafa results on BH entropy, which used a CFT and some Brown and Henneaux-esque ideas, were also a precursor to AdS/CFT, and are now understood in these terms.
- ’t Hooft suggested the idea of taking gravity as a boundary theory seriously, and Susskind subsequently named the idea holography. ’t Hooft was unfortunate enough to make a specific and incorrect suggestion for the boundary theory... both suggested the idea that the boundary theory has ‘pixels’. I’m not sure if the intention was to have a local boundary theory.
- Anomaly inflow, and the basic point of Witten that certain anomalies can be viewed as being given by $d + 1$ dimensional integrals, was also suggestive, although this result only relies on the topology and not the specific geometry of the bulk.

⁶Actually, only UV complete theories in AdS produce exact CFTs. Effective field theories in AdS only produce approximate CFTs, which break down when one studies operators of large scaling dimension. If one side of the duality is incomplete, the other must be as well.

- Polyakov had some ideas about strings propagating in an extra dimension being dual to confinement (see his talk ‘String Theory and Quark Confinement’ from 1997), and (anecdotally) these ideas inspired Maldacena’s discovery.
- Amusingly, Dirac wrote a paper in 1936 on singleton representations in AdS_3 (today we would call this a scalar boson in AdS with a particular negative mass squared, so that it corresponds to a free field in the CFT satisfying $\partial^2\phi = 0$); the paper was titled “A Remarkable representation of the 3+2 de Sitter group”.

2 Anti-deSitter Spacetime

So what is Anti-deSitter (AdS) spacetime?

AdS_{d+1} is a maximally symmetric spacetime with negative curvature⁷. It is a solution to Einstein’s equations with a negative cosmological constant. A particularly useful coordinate system for it, often referred to in the literature simply as ‘global coordinates’, is given by

$$ds^2 = \frac{1}{\cos^2\left(\frac{\rho}{R}\right)} \left(dt^2 - d\rho^2 - \sin^2\left(\frac{\rho}{R}\right) d\Omega_{d-1}^2 \right) \quad (2.1)$$

In what follows we will set the AdS length scale $R = 1$. However, it’s important to note that we cannot have an AdS spacetime without choosing some particular distance (and curvature) scale. Lengths and energies in AdS can and usually will be measured in these units.

Here the radial coordinate $\rho \in [0, \frac{\pi}{2})$, while $t \in (-\infty, \infty)$, and the angular coordinates Ω cover a $(d-1)$ -dimensional sphere. For example, if $d = 3$ then we can write

$$d\Omega^2 = d\theta^2 + \sin^2\theta d\phi^2 \quad (2.2)$$

to cover the familiar S^2 . Note that although ρ only runs over a finite range, the spatial distance from any $\rho < \pi/2$ out towards $\pi/2$ diverges, so AdS_{d+1} is not compact. In global coordinates we can picture AdS as the interior of a cylinder, as in figure 2. Note that in these coordinates there is an obvious time translation symmetry, and also an obvious $SO(d)$ symmetry of rotations on the sphere.

By maximally symmetric, we mean that AdS_{d+1} has the maximal number of spacetime symmetries, namely $\frac{1}{2}(d+1)(d+2)$. This is the same number as we have in $d+1$ dimensional flat spacetime, where we have $d+1$ translations, d boosts, and $\frac{1}{2}d(d-1)$ rotations. The easiest way to see the symmetries of AdS is to embed it as the solution of

$$X_A X^A \equiv X_0^2 + X_{d+1}^2 - \sum_{i=1}^d X_i^2 = R^2 \quad (2.3)$$

⁷The use of $d+1$ dimensions is conventional when studying AdS/CFT, because the dual CFT is taken to have d spacetime dimensions.

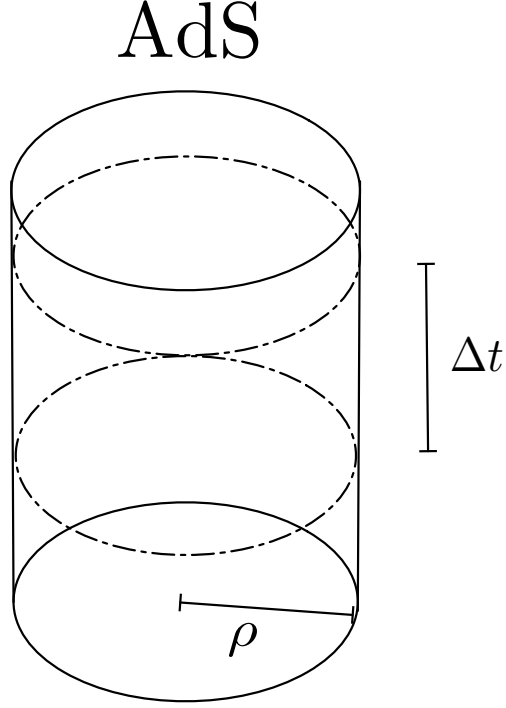


Figure 2: This figure shows AdS in global coordinates. The center is at $\rho = 0$, while spatial infinity is approached as $\rho \rightarrow \pi/2$. The global time coordinate t runs from $-\infty$ to ∞ .

Note again the appearance of the AdS length R . Using the abstruse equation $\cos^2 t + \sin^2 t = 1$ and the related equation $\sec^2 \rho = \tan^2 \rho + 1$ we can map the global coordinates into the X_A via⁸

$$X_0 = R \frac{\cos t}{\cos \rho} \quad (2.4)$$

$$X_{d+1} = R \frac{\sin t}{\cos \rho} \quad (2.5)$$

$$X_i = R \tan \rho \hat{\Omega}_i \quad (2.6)$$

The advantage of the X_A as a presentation of AdS is that all of the symmetries are just the naive rotations and boosts of the X_A . In particular, we have $\frac{1}{2}d(d-1)$ rotations among the X_i with $1 \leq i \leq d$, we have one rotation between the two timelike directions X_0 and X_{d+1} , and then we have $2d$ boosts that mix X_0 and X_{d+1} with the X_i . All of these transformations can be represented by

$$L_B^A = X^A \frac{\partial}{\partial X^B} - X^B \frac{\partial}{\partial X^A} \quad (2.7)$$

⁸To get all of AdS we need to unwrap the X_A to their universal cover, so that t and $t + 2\pi R$ are no longer periodically identified.

which generate the group $SO(2, d)$ of linear transformations of the X_A leaving equation (2.3) invariant. The group $SO(2, d)$ is both the group of isometries of AdS_{d+1} and also the conformal group in d dimensions; so we will often refer to it as the conformal group in what follows. Let's take a look at the 'rotation' in the timeline directions X_0 and X_{d+1} . Note that

$$\frac{\partial}{\partial t} = \frac{\partial X^0}{\partial t} \frac{\partial}{\partial X^0} + \frac{\partial X^{d+1}}{\partial t} \frac{\partial}{\partial X^{d+1}} = L_{d+1}^0 \quad (2.8)$$

so in other words, $L_{0,d+1}$ is the generator of time translations in AdS. So it is the Hamiltonian! The purely space-like generators L_{ij} just generate the rotations of $\hat{\Omega}_d$. These are just the isometries of the sphere S^{d-1} , and they form the group $SO(d)$.

2.1 Euclidean Version and the Poincaré Patch

In many cases it will be useful and important to study the Euclidean version of AdS and the Euclidean conformal group, which is $SO(1, d+1)$. The embedding space becomes

$$X_0^2 - \sum_i^{d+1} X_i^2 = R^2 \quad (2.9)$$

In this case the dt^2 term in equation (2.1) flips sign, and we have $\cos t \rightarrow \cosh t$ and $\sin t \rightarrow \sinh t$ in equation (2.4). This leads to the coordinate identifications in equation (2.10).

There's another coordinate system for AdS, called the Poincaré patch (PP). It's actually used in the literature more often than the global coordinate system. It's the coordinate system to use when studying RS models, holographic QCD, and basically any theory with broken conformal symmetry. The reason for its importance is that it makes the d -dimensional Poincaré subgroup of the conformal group manifest.

The Poincaré Patch is a bit easier to understand in Euclidean signature. The relationship between Euclidean embedding, global, and Poincaré patch coordinates is

$$\begin{aligned} X_0 &= R \frac{\cosh \tau}{\cos \rho} = \frac{1}{2} \left(\frac{z^2 + \vec{r}^2 + R^2}{z} \right) \left(\right. \\ X_{d+1} &= R \frac{\sinh \tau}{\cos \rho} = \frac{1}{2} \left(\frac{z^2 + \vec{r}^2 - R^2}{z} \right) \left(\right. \\ X_i &= R \tan \rho \Omega_i = \frac{R}{z} r_i \end{aligned} \quad (2.10)$$

where $\vec{r}$ is a spatial d -vector, and we have written the global coordinate time as τ . Aspects of this relationship are pictured in figure 3. Note that z runs from 0 to ∞ ; this is necessary so that X_0 has a fixed sign, which is itself required by equation (2.9). In the PP coordinates, dilatations are generated by

$$D = L_{0,d+1} = z \partial_z + r_i \partial_{r_i} \quad (2.11)$$

Euclidean AdS

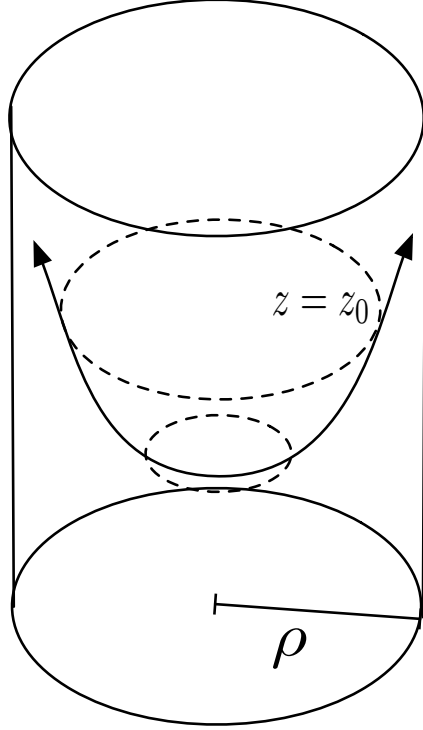


Figure 3: This figure depicts Euclidean AdS. The constant global Euclidean time surfaces are just balls at fixed height on the cylinder. In contrast, the constant z surfaces begin at $\vec{r} = 0$ at some fixed τ with $\rho = 0$, but as $\vec{r} \rightarrow \infty$ we simultaneously have $\rho(r) \rightarrow \pi/2$ with $\tau \rightarrow \infty$, as depicted on the figure. Note that in Euclidean coordinates, both the Poincaré patch and the global coordinates cover all of AdS; in contrast, the PP only covers a portion of AdS in the Lorentzian case.

This operator acts on $\vec{r}$ by stretching the space, as expected. The fact that this is truly $L_{0,d+1}$ can be checked by noting that $(z\partial_z + r_i\partial_{r_i})X_i = 0$, and that

$$z\partial_z + r_i\partial_{r_i} = X^{d+1}\partial_{X^0} - X^0\partial_{X^{d+1}} \quad (2.12)$$

This is easily seen by writing the differential operator $D = z\partial_z + r_i\partial_{r_i}$ as $D(X^A)\partial_{X^A}$.

The relationship in the Lorentzian case is basically given by analytic continuation. The geometry is depicted in figure [4](#). It's most natural to switch the labels of X_d and X_{d+1} to satisfy equation

(2.3). This gives

$$\begin{aligned}
X_0 &= R \frac{\cos \tau}{\cos \rho} = \frac{z}{2} \left(1 + \frac{R^2 + \vec{x}^2 - t^2}{z^2} \right) \left(\right. \\
X_{d+1} &= R \frac{\sin \tau}{\cos \rho} = \frac{R}{z} t \\
X_{i < d} &= R \tan \rho \Omega_i = \frac{R}{z} x_i \\
X_d &= R \tan \rho \Omega_d = \frac{z}{2} \left(1 - \frac{R^2 - \vec{x}^2 + t^2}{z^2} \right) \left(\right.
\end{aligned} \tag{2.13}$$

where $\vec{x}$ is a spatial $d-1$ vector, and we have written the global time as τ to avoid confusion. We can solve for (t, z, x_i) in terms of the global coordinates $(\tau, \rho, \hat{\Omega}_i)$ as

$$\begin{aligned}
t &= R \frac{\sin \tau}{\cos \tau - \Omega_d \sin \rho} \\
z &= R \frac{\cos \rho}{\cos \tau - \Omega_d \sin \rho} \\
\vec{x}_i &= R \frac{\hat{\Omega}_i \sin \rho}{\cos \tau - \Omega_d \sin \rho}
\end{aligned} \tag{2.14}$$

It should be clear from this that even when $t \rightarrow \pm\infty$ we only cover a finite range in τ , because this limit is only achieved by causing the denominators on the RHS to vanish. In these coordinates we obtain the metric

$$ds^2 = \frac{1}{z^2} \left(dt^2 - dz^2 - \sum_{i=1}^{d-1} dx_i^2 \right) \left(\right. \tag{2.15}$$

where the z coordinate only covers the range $0 < z < \infty$. A very commonly used alternative version re-writes $dy = dz/z$ so that $z = e^y$.

We emphasize that in Euclidean signature, the Poincaré patch covers the entire AdS spacetime, just like global coordinates. However, in Lorentzian signature, the Poincaré patch only covers a small sub-region of the full AdS spacetime, bounded by a causal diamond wrapped around the AdS cylinder.

Generally speaking, the global coordinates make an $SO(2) \times SO(d)$ sub-group of the $SO(2, d)$ conformal group obvious, while the Poincaré patch makes the d -dimensional Poincaré group obvious and dilatations (somewhat) manifest.

2.2 Gauss's Law in AdS and Long Range Interactions

Now let's consider Gauss's law in AdS, in order to understand how electromagnetic and gravitational fields behave at long distances.

Lorentzian AdS

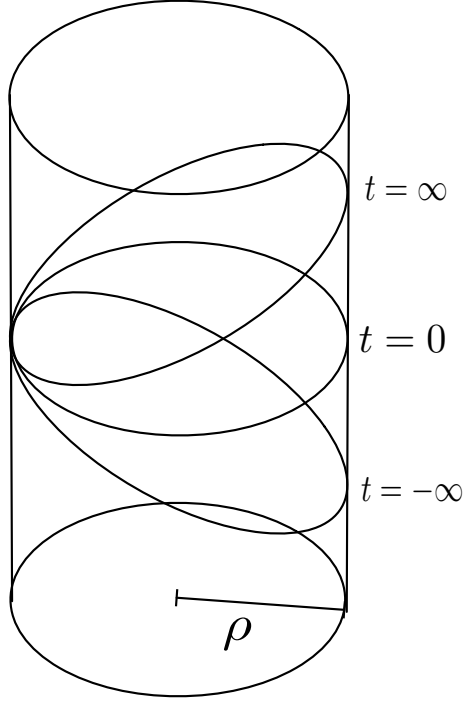


Figure 4: This figure emphasizes the region corresponding to the Poincaré patch, with constant PP times labeled. The PP does not cover all of Lorentzian AdS (in contrast to the Euclidean case).

For this purpose it's natural to transform to a coordinate κ with $d\kappa = \frac{d\rho}{\cos \rho}$, so that $\cosh \kappa = \sec \rho$ and $\sinh \kappa = \tan \rho$. Note that since AdS is maximally symmetric, the point $\kappa = 0$ is equivalent to all other points. In these coordinates the metric takes the form

$$ds^2 = \cosh^2(\kappa) dt^2 - d\kappa^2 - \sinh^2(\kappa) d\Omega_{d-1}^2 \quad (2.16)$$

Now consider a constant time surface, such as the surface $t = 0$, and let's imagine we have a charge sitting at $\kappa = 0$. Gauss's law says that the total amount of flux through any sphere around the charge at $\kappa = 0$ will be a constant. Note that since $g_{\kappa\kappa} = 1$ a point with coordinate κ_* is literally a distance κ_* from the point $\kappa = 0$. The surface area of a sphere of (geometrical) radius κ_* is just $[\sinh(\kappa_*)]^{d-1}$ times the area of an S^{d-1} with radius one. This means that the potential due to a point charge at $\kappa = 0$ will fall off with κ as

$$V(\kappa) = \frac{Q}{[\sinh(\kappa)]^{d-2}} \quad (2.17)$$

In the limit that $\kappa \ll 1$, this is just the usual κ^{2-d} potential appropriate to an (approximately) flat

$d + 1$ dimensional spacetime. But for $\kappa \gg 1$ this becomes

$$V(\kappa) \approx e^{-(d-2)\kappa} \quad (2.18)$$

We can derive these results more systematically by studying the equation of motion for a spherically symmetric, static electric potential.^[9] This follows from studying the action for the electromagnetic field in AdS, namely

$$S = \int \left(d^{d+1} X \sqrt{-g} \frac{1}{4} F_{\mu\nu} F^{\mu\nu} \right) \quad (2.19)$$

The equation of motion we want can be most easily derived by writing $\vec{A} = 0$ and $A_0 = V$, giving

$$\partial_\kappa \left(g^{tt} g^{\kappa\kappa} \sqrt{-g} \partial_\kappa V \right) = \delta(\kappa) \quad (2.20)$$

or

$$\partial_\kappa^2 V + ((d-1) \coth \kappa - \tanh \kappa) \partial_\kappa V = \delta(\kappa) \quad (2.21)$$

which has the solution given in equation (2.17).

These observations have broader implications. For one thing, they imply that IR divergences are regulated by the AdS geometry. But at a more basic level, there are implications for the relationship between angular momentum and radius of orbit. Since a sphere of radius κ will have an area of order $e^{(d-1)\kappa}$, to cover such a sphere with AdS scale resolution requires spherical harmonics up to $\ell \sim e^{(d-1)\kappa}$ as well. This also means that objects in nearly circular orbits with angular momentum ℓ are only a characteristic distance $\kappa \sim R \log \ell$ from the center of AdS.

As a final point, note that the $\cosh^2(\kappa) dt^2$ term in the metric suggests that (at least in the Newtonian limit) an objects energy must increase as it ventures away from $\kappa = 0$. This means that in effect, there is a gravitational force pushing objects in AdS towards the center at $\kappa = 0$. This makes AdS function like a ‘box’. We will see this more directly in the next section, when we study the motion of classical and quantum particles in AdS.

3 A Free Particle in AdS

3.1 Classical Equations of Motion

Now let’s study the simplest possible example of AdS physics, a free scalar particle in AdS₃. The most straightforward way to work out its behavior is to write down a Lagrangian for the particle, and then study the classical equations of motion and their canonical quantization. The action for a free particle of mass m in any spacetime can be written as

$$S = m \int d\tau = m \int dt \sqrt{g_{\mu\nu}(X(t)) \frac{dX^\mu(t)}{dt} \frac{dX^\nu(t)}{dt}} \quad (3.1)$$

⁹It’s worth noting that this differs from the potential for a massless scalar field. In fact, the potential due to a scalar will only agree with this electric potential if the scalar has a slightly negative mass $m^2 = -2(d-2)$ in AdS units, and then it will only agree for $\kappa \gg 1$. Scalars with slightly negative masses can be stable in AdS.

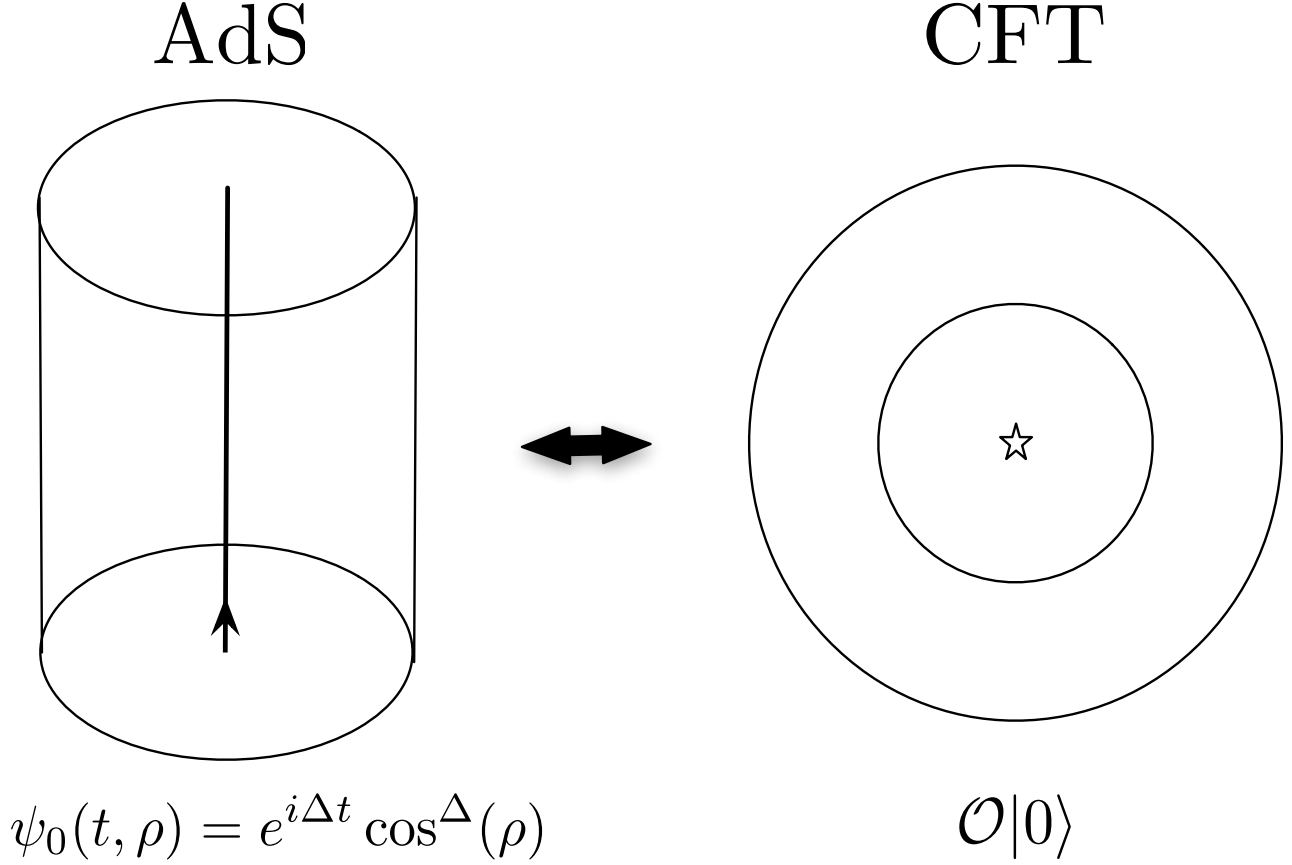


Figure 5: This figure shows an object at rest in AdS and the CFT dual, which is a primary state.

where τ is the proper time, and in the second line $X^\mu(t)$ is the spacetime position of the particle at time t . The fact that this is the correct action for a free particle is equivalent to the statement that free particles move on geodesics. The action computes the geodesic path length in units of $1/m$, because that's the quantum mechanical wavelength of the particle in its own rest frame.

It's worth noting an alternative description (see e.g. [9]). Introducing the lagrange multiplier α , we can write the action as

$$S = \int dt \left(\frac{1}{2\alpha} g_{\mu\nu}(X(t)) \frac{dX^\mu(t)}{dt} \frac{dX^\nu(t)}{dt} + \frac{\alpha}{2} m^2 \right) \quad (3.2)$$

The α equation of motion is trivially solved, and substituting the solution back in gives our first action. But this second version is useful because it eliminates the square root; for example to study massless particles we can just set $m^2 = 0$ from the beginning.

In the simple special case of AdS_3 , we can take $X^\mu(t) = (t, \rho(t), \theta(t))$ in global coordinates, so we obtain the action

$$S = m \int dt \frac{\sqrt{1 - \dot{\rho}^2 - \dot{\theta}^2 \sin^2 \rho(t)}}{\cos \rho(t)} \quad (3.3)$$

If we are interested in the classical physics of our free particle, we could proceed to derive the Euler-Lagrange equations for this action, but this gets messy¹⁰.

Instead, let us take a smarter and more elegant approach. We can view AdS as the surface $X_A X^A = R^2$, as defined in equation (2.3). So why not view a particle in AdS as a particle in $d + 2$ spacetime dimensions constrained to live on this surface? If we write the action for the coordinates $X_A(\tau)$, where τ is an arbitrary worldline parameter, we find¹¹

$$S[X_A, \lambda] = \int d\tau \left[\dot{X}_A \dot{X}^A + \lambda(R^2 - X_A X^A) \right] \quad (3.4)$$

where λ is a Lagrange multiplier that we would also integrate over in the path integral. In fact the Lagrangian formalism was originally invented for exactly this purpose – to describe the motion of objects constrained to live on a surface.

The equations of motion are

$$\ddot{X}_A = -\lambda X_A \quad (3.5)$$

along with the equation of motion from λ , the constraint equation

$$X_A X^A - R^2 = 0 \quad (3.6)$$

The solution to the first set of equations depends on our choice for the variable λ , which is effectively unconstrained. By rescaling the unphysical worldline coordinate τ , we can multiply λ in equation (3.5) by any positive quantity (since the derivative is quadratic). This means that we effectively have three choices: $\lambda = 1, 0$, or -1 .

These three choices for λ correspond to timelike, null, or spacelike trajectories (geodesics) in AdS. Let us focus on the case $\lambda = 1$ for now, which corresponds to the motion of a massive particle in AdS. Then, setting $\tau = t$ from now on, we have the solutions

$$X_A = v_A^c \cos t + v_A^s \sin t \quad (3.7)$$

We see that *particles in AdS always have periodic trajectories*. The other constraint equation becomes

$$R^2 = v_A^c v^{cA} \cos^2 t + v_A^s v^{sA} \sin^2 t + v_A^c v^{sA} \sin(2t) \quad (3.8)$$

This immediately leads to the constraints $v_A^c v^{cA} = 0$ and $v_A^c v^{sA} = v_A^s v^{cA} = R^2$. This leaves us with $2(d + 2) - 3 = 2d + 1$ solutions for the motion of a particle in a $d + 1$ dimensional spacetime. This is one too many solutions; however we can eliminate one by trading the parametric t coordinate for one of the X_A . This gives the correct number of solutions.

The most trivial solution is

$$X_0 = R \cos t \quad \text{and} \quad X_{d+1} = R \sin t \quad (3.9)$$

¹⁰One can still easily check that circular orbits (those with $\dot{\rho} = 0$) exist iff $\dot{\theta} = 1$.

¹¹We should actually be using $\dot{X}^2 \rightarrow \frac{1}{2\alpha} \dot{X}^2 + \frac{\alpha}{2} m^2$ in the action. However, if we make a choice for τ so that $m^2 \alpha^2 = \dot{X}^2$ is a constant, then the effect of including α is simply to rescale λ . We will make such a choice.

with all the $X_i = 0$. This represents a massive particle at rest at $\rho = 0$. A more interesting solution can be easily found by taking

$$\begin{aligned} X_0 &= \frac{R \cos t}{\cos \rho_*} \\ X_{d+1} &= R \sin t \\ X_1 &= R \tan \rho_* \cos t \end{aligned} \tag{3.10}$$

In this case the particle is oscillating back and forth between $\rho = \pm \rho_*$ in the 1 direction. Another canonical example is

$$\begin{aligned} X_0 &= \frac{R \cos t}{\cos \rho_*} \\ X_{d+1} &= \frac{R \sin t}{\cos \rho_*} \\ X_1 &= R \cos t \tan \rho_* \\ X_2 &= R \sin t \tan \rho_* \end{aligned} \tag{3.11}$$

This represents a particle at fixed $\rho = \rho_*$ making a circular orbit in the 1-2 plane. Other more complicated solutions involving various elliptical orbits can be obtained via some simple algebra. The spacelike and null geodesics of AdS can also be obtained.

One might wonder why we have solutions corresponding to distinct periodic orbits, since AdS is a maximally symmetric spacetime – aren't all trajectories equivalent up to symmetries? In fact they are, but by choosing a specific time coordinate t we have chosen some particular center for AdS, and according to this t -evolution objects move in orbits. Any given orbit can be transformed into the trajectory of a particle at rest via a conformal transformation (an AdS isometry).

A very non-trivial feature of these solutions is that they all have the same frequency with respect to the AdS time t . An immediate consequence of this fact is that when we quantize an AdS free particle, we must have energy levels that are integer spaced. In other words, we must find that all energy levels satisfy

$$E = \Delta + m \tag{3.12}$$

for some Δ and for integers $m = 0, 1, 2, \dots$. This follows from both the WKB approximation, and from a consideration of the oscillatory behavior of superpositions of wavefunctions. The only way that every semi-classical linear combination of wavefunctions can have equal periodicity is if all energy levels are quantized in this way. Let us now turn to the quantum theory and confirm this prediction.

3.2 Single Particle Quantum Mechanics

Now we would like to study our single particle in AdS quantum mechanically. Note that there are two physical scales in the problem – the AdS length scale R and the mass of the particle m , from

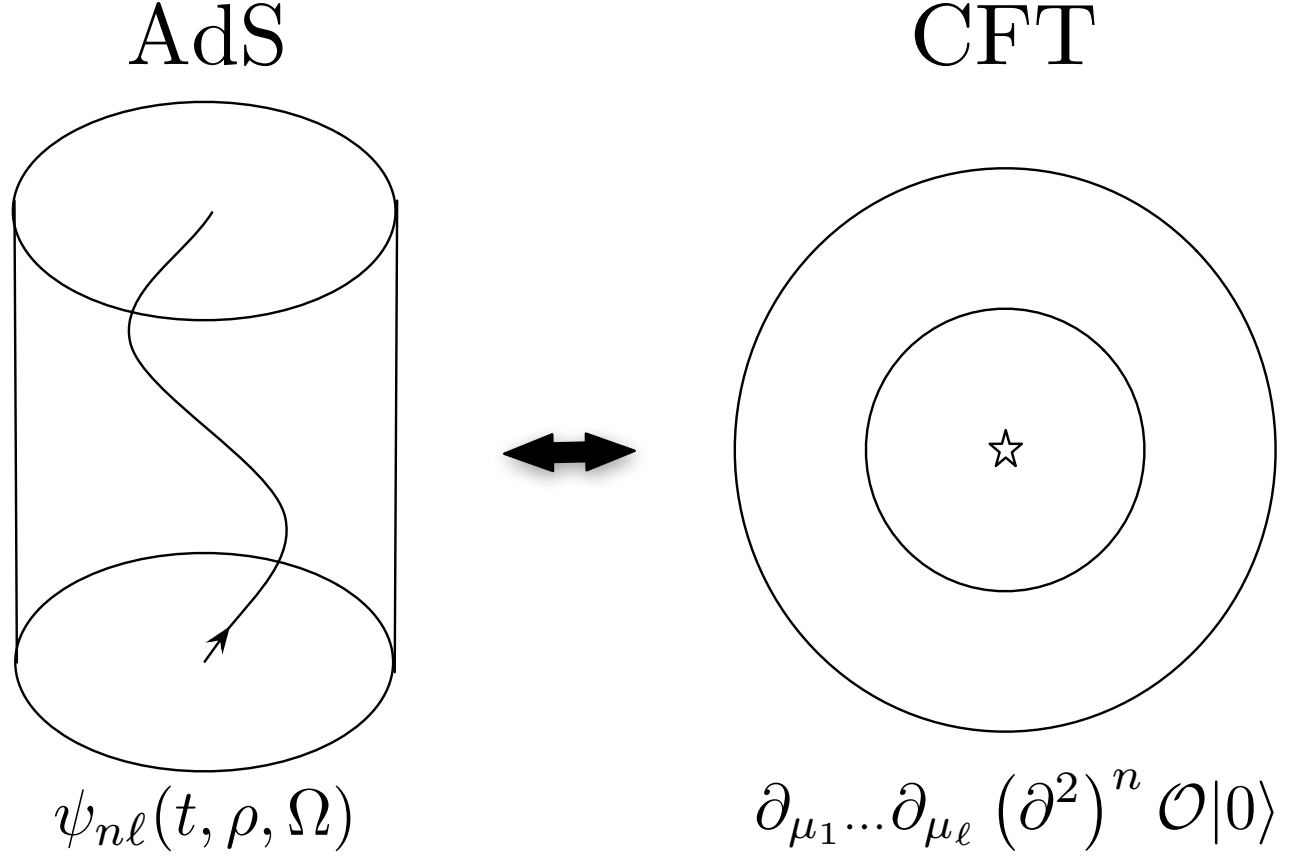


Figure 6: This figure shows an object moving in AdS and the CFT dual, a descendant state.

which we can form the dimensionless number mR on which the wavefunction can depend. This wasn't apparent in the classical analysis because particles move on a timelike geodesics, independent of their masses.

We can straightforwardly derive the AdS Schrodinger equation via canonical quantization of the action for a free particle (there's nothing wrong with describing free relativistic particles using first-quantized quantum mechanics). Let's illustrate this for AdS_2 for simplicity. The action in equation 3.1 in the case of AdS_2 is

$$S = \int \left(dt \frac{m}{\cos \rho} \sqrt{1 - \dot{\rho}^2} \right) \quad (3.13)$$

The canonical momentum conjugate to ρ is

$$P_\rho \equiv \frac{\partial L}{\partial \dot{\rho}} = - \frac{m \dot{\rho}}{\cos \rho \sqrt{1 - \dot{\rho}^2}} \quad (3.14)$$

Using this we can construct the Hamiltonian, which can be written as

$$H = P_\rho \dot{\rho} - L = \sqrt{\left(\frac{m^2}{\cos^2 \rho} + P_\rho^2\right)} \quad (3.15)$$

Quantum mechanically, we must impose the canonical commutation relation $[\rho, P_\rho] = i$, which is conveniently implemented by $P_\rho = -i\partial_\rho$ in the ρ -basis of states. The Schrodinger equation then says that $i\partial_t = H$. Applied to the wavefunction for a particle this would say

$$i\frac{d}{dt}\Psi(t, \rho) = \sqrt{\frac{m^2}{\cos^2 \rho} - \partial_\rho^2} \Psi(t, \rho) \quad (3.16)$$

It's unclear how to solve this equation, but things become easier if we study instead the square of the time evolution operator, $-\partial_t^2 = H^2$, which produces an equation that must also be satisfied when applied to Ψ . So we would want to solve

$$-\partial_t^2 \Psi(t, \rho) = \left(\frac{m^2}{\cos^2 \rho} - \partial_\rho^2\right) \Psi(t, \rho) \quad (3.17)$$

to describe a single quantum mechanical particle in AdS_2 . We have derived a relativistic version of the single-particle Schrodinger equation for a free particle in AdS_2 . We will see later on that this is also compatible with the Klein-Gordon equation in AdS that we obtain from relativistic field theory. In fact, in $d > 2$ there are operator ordering issues that should be resolved to maintain conformal symmetry; the easiest way to do that is simply to use the Klein-Gordon equation $(\nabla^2 + m^2)\psi = 0$.

How do we solve this system? We could just type the differential equation into mathematica, and find a bewildering answer involving hypergeometric functions. But to actually understand the physics, we need to use symmetry. In AdS_2 we actually have 3 symmetry generators, corresponding (in the embedding space) to

$$L_2^0 = \partial_t \quad (3.18)$$

in the language of equation (2.7), and also L_1^0 and L_2^1 . Let's look at one of the latter transformations. In the embedding space with $X_0^2 + X_2^2 - X_1^2 = R^2$ we have

$$X_0 = R \frac{\cos t}{\cos \rho} \quad (3.19)$$

$$X_2 = R \frac{\sin t}{\cos \rho} \quad (3.20)$$

$$X_1 = R \tan \rho \quad (3.21)$$

The operator L_1^0 acts as

$$X_0 \rightarrow X_0 + \epsilon X_1 \quad \text{and} \quad X_1 \rightarrow X_1 + \epsilon X_0 \quad (3.22)$$

We can translate this into a transformation of ρ and t by looking for functions f_t and f_ρ , with $t \rightarrow t + \epsilon f_t$ and $\rho \rightarrow \rho + \epsilon f_\rho$ such that

$$\frac{\cos(t + \epsilon f_t)}{\cos(\rho + \epsilon f_\rho)} \approx \frac{\cos t}{\cos \rho} + \epsilon \tan \rho \quad (3.23)$$

$$\tan(\rho + \epsilon f_\rho) \approx \tan \rho + \epsilon \frac{\cos t}{\cos \rho} \quad (3.24)$$

The solution to these equations is

$$f_t = -\sin t \sin \rho \quad (3.25)$$

$$f_\rho = \cos t \cos \rho \quad (3.26)$$

This tells us that the symmetry generator

$$L_1^0 = -\sin t \sin \rho \partial_t + \cos t \cos \rho \partial_\rho \quad (3.27)$$

Similarly we find that

$$L_2^1 = \cos t \sin \rho \partial_t + \sin t \cos \rho \partial_\rho \quad (3.28)$$

These generators should have the $SO(2, 1)$ commutation relations; in fact we find

$$[L_2^0, L_1^0] = -L_2^1, \quad [L_2^0, L_2^1] = L_1^0, \quad [L_1^0, L_2^1] = L_2^0 \quad (3.29)$$

as the commutation relations of these vector fields. This means that if we define $D = L_2^0$, $P = \frac{1}{2}(L_1^0 + iL_2^1)$, and $K = \frac{1}{2}(L_1^0 - iL_2^1)$ then we find

$$[D, P] = iP, \quad [D, K] = -iK, \quad [K, P] = iD \quad (3.30)$$

This is great – it means that we can choose to label our states as eigenstates of D , so that

$$D|\psi\rangle = \Delta|\psi\rangle \quad (3.31)$$

and with respect to this eigenvalue, P acts as a raising operator while K acts as a lowering operator. In particular, the ground state satisfies

$$K|\psi_0\rangle = 0 \quad (3.32)$$

and all of the other states can be built by applying the operator P to this state. Note that, very explicitly, we have

$$K = \frac{1}{2}e^{-it}(-i \sin \rho \partial_t + \cos \rho \partial_\rho) \quad (3.33)$$

for this operator acting in the (t, ρ) basis of states.

So the ground state of our system must obey the equation

$$-i \sin \rho \partial_t \psi_0(t, \rho) + \cos \rho \partial_\rho \psi_0(t, \rho) = 0 \quad (3.34)$$

Taking $\psi_0 = e^{i\Delta t} \chi(\rho)$ with Δ the eigenvalue of D , we see that

$$\Delta \chi = -\cot \rho (\partial_\rho \chi) \quad (3.35)$$

with the solution

$$\psi(t, \rho) = e^{i\Delta t} \cos^\Delta \rho \quad (3.36)$$

This is the ground state. Now *every other state* in the system can be obtained by acting with the operator

$$P = \frac{1}{2} e^{it} (i \sin \rho \partial_t + \cos \rho \partial_\rho) \quad (3.37)$$

which is the raising operator for our quantum mechanical system. This explains the fact that all AdS orbits have the same period. Note that flipping $t \rightarrow -t$ just exchanges P and K . It turns out that the n th energy level has a wavefunction that can be written explicitly in terms of hypergeometric functions, but we will wait to introduce those until we talk about general AdS $_{d+1}$.

General Dimensions

The fact that all classical solutions oscillate with integer period implies that the energy levels are $E = \Delta + n$ for integers n . Let us understand this fact quantum mechanically. The conformal algebra $SO(d, 2)$ can be re-written in a conventional form as

$$\begin{aligned} [M_{\mu\nu}, P_\rho] &= i(\eta_{\mu\rho} P_\nu - \eta_{\nu\rho} P_\mu), & [M_{\mu\nu}, K_\rho] &= i(\eta_{\mu\rho} K_\nu - \eta_{\nu\rho} K_\mu), & [M_{\mu\nu}, D] &= 0 \\ [P_\mu, K_\nu] &= -2(\eta_{\mu\nu} D + iM_{\mu\nu}), & [D, P_\mu] &= P_\mu, & [D, K_\mu] &= -K_\mu \end{aligned} \quad (3.38)$$

where we have the index $\mu = 1, 2, \dots, d$. Referring to the generators L_{AB} from equation (2.7), the ‘dilatation operator’ $D \equiv -iL_{0,d+1}$, while the ‘momentum generator’ $P_\mu \equiv iL_{d+1,\mu} + L_{0,\mu}$ and ‘special conformal generators’ $K_\mu \equiv iL_{d+1,\mu} - L_{0,\mu}$. The rotation generators $M_{\mu\nu}$ are just the $SO(d)$ generators $-iL_{\mu\nu}$.

Recall that $D = -iL_{0,d+1} = -i\frac{\partial}{\partial t}$. In other words, the dilatation operator D is actually the Hamiltonian for AdS physics, since it is the generator of time translations. For now you can just think of it as the AdS Hamiltonian; soon we’ll learn what role it plays in conformal field theories and thereby understand its name. So what does equation (7.9) mean physically? The first line just indicates that P_μ and K_μ transform as vectors under rotations, while the dilatation operator D is a scalar. Since D commutes with $M_{\mu\nu}$, we can simultaneously diagonalize D and the angular momentum generators. So we can label states by their energy (D eigenvalue) and their angular momentum.

The second line of equation (7.9) gives us even more crucial information. It says that with respect to the Dilatation operator D , P_μ acts as a raising operator while K_μ acts as a lowering

operator. This should remind you of the creation and annihilation operators for the harmonic oscillator. It immediately explains the fact that the energy levels of one-particle states are integer spaced (since energies are just the D eigenvalues). Furthermore, since D is the Hamiltonian, there must be one-particle state with minimum eigenvalue Δ . This state cannot be lowered, so it must be annihilated by all d of the K_μ generators. This gives us an equation

$$K_\mu |\psi_0\rangle = 0 \quad (3.39)$$

where ψ_0 is called (again using CFT terminology) a *primary state*. From the point of view of AdS quantum mechanics, ψ_0 is simply the lowest energy state for our particle, and it has energy Δ . Once we have the primary/ground state ψ_0 , we can construct general states of the form¹²

$$|\psi_{n,\ell}\rangle = (P_\mu^2)^n P_{\mu_1} P_{\mu_2} \cdots P_{\mu_\ell} |\psi_0\rangle \quad (3.40)$$

These states have energy $E_{n,\ell} = \Delta + 2n + \ell$ and angular momentum ℓ . These form a basis for all one-particle states in AdS.

We can compute $\psi_{n,\ell}$ as wavefunctions in AdS by writing out K_μ explicitly as a differential operator. This is a straightforward exercise using the coordinate transformations from equation (2.4), the definitions of L_{AB} from equation (2.7), and the fact that $K_\mu \equiv iL_{d+1,\mu} - L_{0,\mu}$ as we discussed above. For example, in AdS_3

$$K_\pm = ie^{-it \pm i\theta} \left(\sin \rho \partial_t + i \cos \rho \partial_\rho \mp \frac{1}{\sin \rho} \partial_\theta \right) \left(\quad \right) \quad (3.41)$$

$$P_\pm = ie^{it \pm i\theta} \left(\sin \rho \partial_t - i \cos \rho \partial_\rho \pm \frac{1}{\sin \rho} \partial_\theta \right) \left(\quad \right) \quad (3.42)$$

where $K_\pm = K_1 \pm iK_2$ and similarly $P_\pm = P_1 \pm iP_2$. The ground state $\psi_0(t, \rho)$ will be independent of θ so it must satisfy

$$(\sin \rho \partial_t + i \cos \rho \partial_\rho) \psi_0(t, \rho) = 0 \quad (3.43)$$

and this means that it takes the form

$$\psi_0(t, \rho) = e^{i\Delta t} \cos^\Delta(\rho) \quad (3.44)$$

as expected, with $m^2 = \Delta(\Delta - d)$. All of the higher states can be constructed by acting on this state with the differential operators $P_\pm$. The state and these methods also generalize to higher d . In complete generality, the wavefunctions $\psi_{n,\ell}$ in AdS_{d+1} can be written in terms of a hypergeometric function as

$$\psi_{n,\ell J}(t, \rho, \Omega) = \frac{1}{N_{\Delta n \ell}} e^{-iE_{n,\ell} t} Y_{\ell J}(\Omega) \left[\sin^\ell \rho \cos^\Delta \rho {}_2F_1 \left(\left(n, \Delta + \ell + n, \ell + \frac{d}{2}, \sin^2 \rho \right) \right) \right] \left(\quad \right) \quad (3.45)$$

¹²This equation is rather schematic for $\ell > 0$, because we need to arrange the indices to form definite irreducible representations of $SO(d)$, but this is the basic idea.

where $N_{\Delta n \ell}$ is a normalization factor and $E_{n,\ell} = \Delta + 2n + \ell$. The J quantum number just encapsulates other angular momentum quantum numbers, e.g. ‘ m ’ for AdS_4 with $Y_{\ell m}$ spherical harmonics. In the case of Euclidean AdS , we have $t \rightarrow it$.

The easiest way to determine the general solutions in equation (3.45) is to solve a corresponding Schrodinger equation for AdS_{d+1} . We would like to use the symmetries of the problem to our advantage. There are no non-trivial 1st order differential operators that commute with all of the conformal generators $D, K_\mu, P_\mu, M_{\mu\nu}$, so we will not be able to find a fully symmetric 1st order equation. However, just as in the $d = 1$ case we solved explicitly above, we can find a 2nd order differential operator that commutes with all of the conformal generators. This is the *quadratic Casimir* of the conformal group, which takes the form

$$D^2 + \frac{1}{2}(P \cdot K + K \cdot P) + M_{\mu\nu}M^{\mu\nu} = \nabla_{\text{AdS}}^2 \quad (3.46)$$

when the conformal generators are expressed as isometry generators on AdS . The fact that this is the Laplacian is no surprise – the conformal Casimir is unique, and we know that the laplacian is the only isometry-invariant 2nd order differential operator.

Thus we want to solve $(\nabla^2 + m^2)\psi = 0$ as the Schrodinger equation in AdS_{d+1} . We can immediately separate variables into a radial wavefunction and an angular wavefunction. The latter will just take the form of a d -dimensional spherical harmonic $Y_{\ell J}$, which are the eigenfunctions of the Laplacian on the sphere S^{d-1} with eigenvalue $\ell(\ell + d - 2)$. We can then write a Schrodinger equation for the ρ -dependent part of the wavefunction $\psi_\ell(\rho)$, which has energy ω^2 . It takes the form

$$-\psi''(\rho) + \frac{1-d}{\cos \rho \sin \rho} \psi'(\rho) + \left(\frac{\ell(\ell + d - 2)}{\sin^2 \rho} + \frac{m^2}{\cos^2 \rho} \right) \psi(\rho) = \omega^2 \psi(\rho) \quad (3.47)$$

We can simplify this equation by defining

$$\psi(\rho) = \chi(\rho) \sin(\rho) [\cot(\rho)]^{\frac{d}{2}} \quad (3.48)$$

to give the equation for $\chi(\rho)$

$$-\chi''(\rho) + \frac{1}{4} \left(\frac{(2\ell - 3 + d)(2\ell - 1 + d)}{\sin^2 \rho} + \frac{4m^2 + d^2 - 1}{\cos^2 \rho} \right) \chi(\rho) = \omega^2 \chi(\rho) \quad (3.49)$$

We can interpret this equation using intuition from non-relativistic quantum mechanics. The term in parentheses represents an effective potential with an angular momentum barrier and a potential that diverges as $\rho \rightarrow \pi/2$.

It’s not especially obvious how to solve this equation exactly. However, it turns out that if we choose the $\ell = 0$ mode for simplicity, and take $m^2 = \Delta(\Delta - d)$ and $\omega = \Delta$, there is a very simple solution for the ground state

$$\psi(\rho) = \cos^\Delta(\rho) \quad (3.50)$$

which we already found. Quantization and the determination of Δ follows as usual from boundary conditions – the wavefunction must vanish as $\rho \rightarrow \pi/2$ and it must have $\psi'(0) = 0$ to avoid cusps, since ρ is a radial coordinate. Physically, we see that $\psi_0(\rho)$ has its largest support at small ρ and falls to zero as $\rho \rightarrow \pi/2$. For $\Delta \sim 1$ the particle has been confined to a region of order one size R , the AdS length scale. When $\Delta \gg 1$ we have

$$\psi(\rho) \approx e^{-\Delta \frac{\rho^2}{2}} \quad \text{when} \quad \Delta \gg 1 \quad (3.51)$$

so we see that the particle is an approximately Gaussian state so that it is confined to a region of size $R/\sqrt{\Delta}$ due to the AdS curvature.

Actually, there is a slick way of deriving the relation between bulk mass and Δ . We already derived the form of the ground state using the lowering condition $K_\mu \psi_0 = 0$. Once we know its form, we can simply plug it back into the (more complicated) Schrodinger equation, noting that $\omega = \Delta$. A quick computation shows that this equation can then only be satisfied with $m^2 = \Delta(\Delta - d)$.

A Comment on Multiparticle Quantum Mechanics in AdS

Nothing stops us from studying simple examples of multi-particle quantum mechanics in AdS. For example, consider a hydrogen atom. In the usual description in flat space, its wavefunction can be specified in terms of the position of the proton and the position of the electron. Usually it makes more sense to break these up into a center of mass degree of freedom (which is somewhat trivial) and a coordinate representing the relative position between the electron and proton. Then we compute the wavefunctions, energy levels, etc.

We can repeat this same process in AdS, and we could even go on to compute the wavefunctions and binding energies due to the Coulomb attraction. According to what we'll soon learn about AdS/CFT, we could then interpret the results in terms of the scaling dimensions and correlation functions of operators in a putative CFT. We won't take this approach here (although you can read about a very similar approach in [10]), but it's worth emphasizing that even the most elementary results from quantum mechanics can be applied in the AdS/CFT context. Via AdS/CFT we can reinterpret many familiar ideas holographically.

4 Free Fields in AdS

In the previous section we discussed AdS geometry and the physics of a classical or quantum free particle in AdS. Now let us see how to describe any number of quantum free particles in AdS.

The key feature of states constructed from many free particles (Fock space states) is that their quantum numbers are just the sum of the quantum numbers of the individual particles. In particular, their energy (for us the D eigenvalue) is just the sum of the energies of the individual particles. This is why the Hilbert space of free quantum field theory is built using harmonic oscillator creation and annihilation operators – the harmonic oscillator has evenly spaced energy levels, so any two states can be combined by just adding their energies.

For this reason, to describe any number of free particles in AdS we simply introduce the creation and annihilation operators $a_{n\ell J}$ and $a_{n\ell J}^\dagger$ with the usual commutator

$$[a_{n\ell J}, a_{n\ell J}^\dagger] = 1 \quad (4.1)$$

A basis for many particle states can be constructed as usual; a generic k -particle state is

$$a_{n_1\ell_1 J_1}^\dagger a_{n_2\ell_2 J_2}^\dagger \cdots a_{n_k\ell_k J_k}^\dagger |0\rangle \quad (4.2)$$

where $|0\rangle$ is the vacuum, which has no particles. In terms of these operators, the AdS Hamiltonian D is simply

$$D = \sum_{n,\ell J} (\Delta + 2n + \ell) a_{n\ell J}^\dagger a_{n\ell J} \quad (4.3)$$

because it sums the energy of all the particle, and $a_{n\ell J}^\dagger a_{n\ell J}$ is the number operator that counts the number of particles with quantum numbers n, ℓ, J . The other conformal generators can also be expressed in terms of the creation and annihilation operators. So we have an infinite collection of harmonic oscillators in AdS.

It's straightforward to see that this is just what we get from quantizing a free scalar field in AdS, with action

$$S = \int_{AdS} d^{d+1}x \sqrt{-g} \left(\frac{1}{2} (\nabla_A \phi)^2 - \frac{1}{2} m^2 \phi^2 \right) \quad (4.4)$$

Before we explain why, let's detour to review everything you ever wanted to know about free fields but were afraid to ask.

4.1 Classical and Quantum Fields Anywhere – Canonical Quantization

Here we will review, at a rather formal level, how one goes about solving for the time evolution of a classical field. Then we will explain how this relates to quantization. These ideas are worth understanding for reasons that go beyond AdS – for example, they are an elementary ingredient in Hawking's derivation of black hole evaporation.

Let us study a scalar field theory in a completely general spacetime, with free action

$$S = \int d^d x dt \sqrt{-g} \left(\frac{1}{2} g^{\mu\nu} \nabla_\mu \phi \nabla_\nu \phi - \frac{m^2}{2} \phi^2 \right) \quad (4.5)$$

This action implies the usual equations of motion

$$g^{\mu\nu} \nabla_\mu \nabla_\nu \phi + m^2 \phi = 0 \quad (4.6)$$

Let us imagine that we find the general solutions to this equation, functions $f_n(t, x)$ labeled by an index n that may be discrete or continuous. Here 'solving the equation' is a less precise way of saying

that we have diagonalized the differential operator $g^{\mu\nu}\nabla_\mu\nabla_\nu + m^2$, obtaining its eigenfunctions and eigenvalues. For example in flat spacetime we usually take¹³ $n = \vec{k}$ and $f_k(t, x) = e^{i\omega t}e^{i\vec{k}\cdot\vec{x}}$ with $\omega^2 = \vec{k}^2 + m^2$. In AdS we find solutions labeled by n, ℓ, J .

Even if we are only solving the theory at a classical level, we would still like to be able to write a general solution as (we will get to quantization below)

$$\phi(t, x) = \sum_n \left(c_n f_n(t, x) + c_n^\dagger f_n^\dagger(t, x) \right) \quad (4.7)$$

given some initial data consisting of $\phi_0(0, x)$ and $\dot{\phi}_0(0, x)$ on the $t = 0$ surface. The initial data just corresponds to the initial position and velocity of the ϕ field. We need to know both initial conditions to solve for $\phi(t, x)$ because its equation of motion is second order.

We know that $\phi(t, x)$ can be expressed as a sum over the modes $f_n(t, x)$. But given some initial condition, how do we solve for the c_n coefficients? We need an *inner product defined on spacelike surfaces at fixed time*. A natural place to start is with the equations of motion, since these are the equations we have diagonalized. One often obtains inner products by noting that some integral $\int f dg = - \int dfg$, and if the operator “ d ” can be diagonalized, then this integral must vanish unless f and g have the same eigenvalue under the action of d .

Therefore for *any* two functions ψ_1 and ψ_2 on spacetime (vanishing sufficiently fast at spatial infinity) we can define the integral

$$\iint_\Omega d^d x dt \sqrt{-g} \left[\psi_1^\dagger (g^{\mu\nu} \nabla_\mu \nabla_\nu - m^2) \psi_2 - \left((g^{\mu\nu} \nabla_\mu \nabla_\nu - m^2) \psi_1^\dagger \right) \psi_2 \right] \quad (4.8)$$

over some region Ω . Let’s imagine that Ω spans all of space, but ends at some finite times t_i and t_f on Cauchy surfaces Σ_i and Σ_f . We can re-write this integral as a boundary integral

$$\iint_{\Sigma_f - \Sigma_i} d^d x \sqrt{-g} g^{00} \left[\psi_1^\dagger \nabla_t \psi_2 - (\nabla_t \psi_1^\dagger) \psi_2 \right] \quad (4.9)$$

where we assumed that the t direction is orthogonal to Σ , for simplicity, and we took t pointing forward in time for both surfaces. If both ψ_1 and ψ_2 obey the equations of motion, then our original integral must vanish, so we see that

$$\langle \psi_1, \psi_2 \rangle \equiv i \int_{\Sigma_f} d^d x \sqrt{-g} g^{00} \left[\psi_1^\dagger \nabla_t \psi_2 - \psi_2 \nabla_t \psi_1^\dagger \right] \left(= i \int_{\Sigma_i} d^d x \sqrt{-g} g^{00} \left[\psi_1^\dagger \nabla_t \psi_2 - \psi_2 \nabla_t \psi_1^\dagger \right] \right) \quad (4.10)$$

defines a *conserved* inner product. In particular, if we normalize $\langle f_n, f_m \rangle = \delta_{mn}$ and $\langle f_n^\dagger, f_m \rangle = 0$ at some time then this normalization will be time independent. This gives the usual normalization

¹³Actually, the t dependence should technically be forced to be either $\cos(\omega t)$ or $\sin(\omega t)$ so that the solutions are real, assuming we are studying a real scalar field. Then we would express the full solution as a sum over sines and cosines, instead of complex exponentials with complex coefficients. The real case is simpler, but the complex case is more useful and standard. It’s also noteworthy that if the t direction points along a Killing vector, then we have a time translation symmetry, and so the time dependence will always just be $e^{\pm i\omega t}$.

$\frac{1}{\sqrt{2k_0}}e^{ik \cdot x}$ for modes in flat spacetime (noting that in that case, since k is continuous, we get a delta function in place of δ_{mn}).

If we write $\phi(t, x)$ as in equation (4.7) then the c_n will be time independent, as desired. More importantly, it means that we can compute the c_n from

$$c_n = \langle \phi(0, x), f_n \rangle = i \int_{\mathbb{R}^d} d^d x \sqrt{-g} g^{00} \left[\phi_0^\dagger(x) \dot{f}_n(x) - f_n(x) \dot{\phi}_0^\dagger(x) \right] \quad (4.11)$$

using our inner product along with both the $\phi(0, x)$ and $\dot{\phi}(0, x)$ initial conditions. So we see that our inner product automatically knows how initial conditions work in classical theories. At the classical level, if we the kinetic term had been a higher order differential operator, then we would end up with an inner product involving higher time derivatives, in keeping with expectations for the initial value problem.

Now consider the canonical momentum

$$\pi(x) = \frac{\delta L}{\delta \dot{\phi}} = \sqrt{-g(x)} g^{00}(x) \dot{\phi}(x) \quad (4.12)$$

In either the classical or the quantum theory we need to satisfy the canonical commutation relations (or Poisson brackets)

$$[\pi(x), \phi(y)] = -i\delta^d(x - y) \quad (4.13)$$

and as usual, when dealing with the quantum theory we can write

$$\phi(t, x) = \sum_n \left(\frac{1}{N_n} [f_n(t, x) a_n + f_n^\dagger(t, x) a_n^\dagger] \right) \quad (4.14)$$

in terms of annihilation and creation operators a_n and $a_n^\dagger$, and a normalization factor N_n that we have added in case we need it. When we quantize, the coefficients c_n of the n th classical harmonic mode has been replaced with creation and annihilation operators for quantum versions of that mode.

Now we must demand that the field and its canonical momentum satisfy the canonical commutation relations. A simple computation shows that these will be satisfied if

$$\sqrt{-g(x)} g^{00}(x) \sum_n \left(\frac{1}{N_n^2} [\dot{f}_n(0, x) f_n^\dagger(0, y) - \dot{f}_n^\dagger(0, x) f_n(0, y)] \right) = -i\delta^d(x - y) \quad (4.15)$$

However, this can be re-interpreted as an expression for the coefficients c_n that would be obtained if the function $\delta^d(x - y)$ (viewed as a function of y , with x serving as a fixed paramter) is expanded in the $f_n(0, y)$ basis. This means we can use the inner product to write

$$\begin{aligned} \sqrt{-g(x)} g^{00}(x) \frac{1}{N_n^2} \dot{f}_n(0, x) &= \int d^d y \sqrt{-g(y)} g^{00}(y) \delta^d(x - y) \dot{f}_n(0, y) \\ &= \sqrt{-g(x)} g^{00}(x) \dot{f}_n(0, x) \end{aligned} \quad (4.16)$$

in this very special case. This determines that all the normalization constants $N_n = 1$ if we work in terms of modes f_n normalized with our inner product as $\langle f_n, f_m \rangle = \delta_{mn}$. So we have obtained a canonically quantized scalar field $\phi(t, x)$ – it obeys the correct EoM, and it has the correct commutation relations.

Note that the choice of time coordinate t played an essential role in our logic. Had we chosen a different notion of time, we would have had different mode functions, a different inner product, and a different quantization. Hawking originally derived black hole radiation by comparing the different natural quantizations that exist before and after black hole formation.

4.2 Canonical Quantization of a Scalar Field in AdS

Let us return and consider the AdS_3 example for concreteness. Then the action is

$$S = \int dt d\rho d\theta \frac{\sin \rho}{\cos \rho} \frac{1}{2} \left(\dot{\phi}^2 - (\partial_\rho \phi)^2 - \frac{1}{\sin^2 \rho} (\partial_\theta \phi)^2 - \frac{m^2}{\cos^2 \rho} \phi^2 \right) \quad (4.17)$$

The canonical momentum conjugate to ϕ is

$$P_\phi = \frac{\delta L}{\delta \dot{\phi}} = \frac{\sin \rho}{\cos^2 \rho} \dot{\phi} \quad (4.18)$$

So we will want to impose the canonical commutations relations

$$[\phi(t, X), P_\phi(t, Y)] = i(2\pi)^2 \delta^2(X - Y) \quad (4.19)$$

We will do this by first solving the equations of motion for ϕ , and then writing ϕ in terms of a sum over harmonic oscillator modes. As we discussed in generality above, the canonical commutation relation then gives a normalization condition for the modes. Note that we are working in global coordinates; we would obtain different modes, with a different interpretation, if we worked in the Poincaré patch using its notion of time.

The equations of motion are

$$\ddot{\phi} - \frac{\cos^2 \rho}{\sin \rho} \partial_\rho (\sin \rho \cos^{-2} \rho \partial_\rho \phi) \left(\frac{1}{\sin^2 \rho} \partial_\theta^2 \phi + \frac{m^2}{\cos^2 \rho} \phi \right) = 0 \quad (4.20)$$

Taking $\phi = e^{i\omega t + i\ell\theta} \psi(\rho)$, we have

$$-\psi''(\rho) - \frac{1}{\cos \rho \sin \rho} \psi'(\rho) + \left(\frac{\ell^2}{\sin^2 \rho} + \frac{m^2}{\cos^2 \rho} \right) \psi(\rho) = \omega^2 \psi(\rho) \quad (4.21)$$

which is just the equation for a one-particle wavefunction in $d = 2$ that we obtained previously. The solutions were given in equation [\(3.45\)](#).

The quantized field which obeys canonical commutation relations is

$$\phi(t, \rho, \Omega) = \sum_{n, \ell} \left(\psi_{n\ell J}(t, \rho, \Omega) a_{n\ell} + \psi_{n\ell J}^*(t, \rho, \Omega) a_{n\ell}^\dagger \right), \quad (4.22)$$

with the wavefunctions $\psi_{n\ell}$ that were defined in equation (3.45). To find the normalizations of the $\psi_{n\ell}$ we impose

$$\langle \psi_{n\ell}, \psi_{n'\ell'} \rangle = \delta_{nn'} \delta_{\ell\ell'} = \int_{\text{AdS}} d^d x \sqrt{-g} g^{00} (\psi_{n\ell} \partial_t \psi_{n'\ell'} - \partial_t \psi_{n\ell} \psi_{n'\ell'}) \quad (4.23)$$

which translates in AdS_3 to

$$1 = \frac{1}{(N_{\Delta n\ell})^2} \int d\rho d\theta \sin \rho \, 2(\Delta + 2n + \ell) \psi_{n\ell}^\dagger(\rho, \theta) \psi_{n\ell}(\rho, \theta) \quad (4.24)$$

It is easy to see that modes with different ℓ are orthogonal, as the θ integral then vanishes. The orthogonality of modes with different n can be verified directly. One finds that the normalizations of the $\psi_{n\ell J}$ are

$$N_{\Delta n\ell} = (-1)^n \sqrt{\frac{n! \Gamma^2(\ell + \frac{d}{2}) \Gamma(\Delta + n - \frac{d-2}{2})}{\Gamma(n + \ell + \frac{d}{2}) \Gamma(\Delta + n + \ell)}}. \quad (4.25)$$

by a direct computation. This can be checked by hand for small n and ℓ .

A crucial condition obeyed by $\phi(x)$ is that, as in flat spacetime, $[\phi(x), \phi(y)] = 0$ whenever x and y are spacelike separated in AdS . This means that if we construct local interactions $\mathcal{V}[\phi(x)]$ then we will also have $[\mathcal{V}(x), \mathcal{V}(y)] = 0$ outside the lightcone. Weinberg [3] showed (see section 3.5 of his book) that this condition was necessary for QFT to produce a sensible Poincaré invariant S-Matrix. In section 2.4 of [10] this reasoning was generalized to AdS and the conformal algebra.

We saw in section 3.2 that a single free particle in AdS transforms under an irreducible representation of the conformal group. The particle's ground state is a primary state of the conformal algebra, because it is annihilated by all the 'lowering operators', the special conformal generators K_μ . All the other states can be built with P_μ , the momentum generator or 'raising operator' with respect to D . Soon we will discuss analagous statements for the quantum field $\phi(x)$.

5 Approaching the Boundary of AdS

Now we would like to consider the asymptotic structure of AdS , or the 'boundary at infinity'. First we will discuss what the boundary is and how it inherits structure from the bulk (the inside) of the spacetime. Then we will discuss how the boundary inherits a full fledged theory, with its own correlation functions, from the bulk. This is actually more familiar than one might think – the usual method of obtaining a flat space S-Matrix from a bulk QFT can be interpreted in the same spirit.

5.1 Asymptotia and Penrose Diagrams

The purpose of a Penrose diagram (or 'causal diagram' or 'conformal diagram') is to encapsulate the causal structure of the entire infinite spacetime in a compact picture. To make a Penrose diagram, we (vastly) distort the geometry in order to map the entire spacetime into a finite region, while

maintaining the notions of timelike, spacelike, and null separations between points. Penrose diagrams are also useful for understanding the structure of (or ‘boundary at’) infinity because they turn infinity into a finite codimension one region with a causal structure that is inherited from the bulk.

How can we distort the geometry without affecting causality? An easy way is to multiply the entire metric by a Weyl factor $f(x)$, sending $g_{\mu\nu}(x) \rightarrow f(x)g_{\mu\nu}(x)$.

5.1.1 The Penrose Diagram for Flat Spacetime

Let’s review how this works in flat spacetime. If we write the flat space metric as

$$ds^2 = dt^2 - dr^2 - r^2 d\Omega^2 \quad (5.1)$$

then we can introduce coordinates $2 \tan U = t + r$ and $2 \tan V = t - r$ so that U and V run from $-\pi/2$ to $\pi/2$. In these coordinates the metric takes the form

$$ds^2 = \frac{4dUdV}{\cos^2 U \cos^2 V} + (\tan U - \tan V)^2 d\Omega^2 \quad (5.2)$$

We can perform a Weyl transformation and multiply by $\cos^2 U \cos^2 V$. Now using the new coordinates $T = U + V$ and $R = U - V$ and a trig identity, we find

$$ds^2 = dT^2 - dR^2 + \sin^2(R) d\Omega^2 \quad (5.3)$$

with T and R only running over a finite diamond. This is the derivation of the usual diamond shaped Penrose diagram for flat spacetime.

The boundary (or ‘conformal infinity’) is a null diamond. There are special points where time-like and space-like geodesics terminate, and then the null surfaces where all light rays approach infinity. Notice that the boundary is not a nice spacetime; for example, it does not inherit a Lorentzian metric with timelike directions. This explains why it is non-trivial to find a holographic description of flat spacetime – the holographic theory cannot be a conventional QFT, since it would have to live on a bizarre (null) spacetime.

5.1.2 The Penrose Diagram and the Boundary of Global AdS

When we study AdS/CFT we will constantly refer to the boundary of AdS. The factor of $1/\cos^2(\rho)$ multiplies the entire metric in equation (2.1), so we can determine the AdS Penrose diagram by simply dropping it (or more precisely, by multiplying the AdS metric by the Weyl factor $\cos^2(\rho)$). This gives a new metric

$$ds^2 = dt^2 - d\rho^2 - \sin^2(\rho) d\Omega^2 \quad (5.4)$$

Since $0 \leq \rho < \pi/2$, we can now draw a spatially finite Penrose diagram. Time still runs from $-\infty$ to ∞ , and we cannot alter this fact while maintaining the feature that light rays move at 45 degree angles in the t - ρ plane.

In terms of the global coordinates that cover the entirety of the AdS manifold, the boundary of AdS is the cylinder $R \times S^{d-1}$ that we obtain by taking $\rho \rightarrow \frac{\pi}{2}$, its limiting value at spatial infinity.

We can simply use the coordinates t and Ω to parameterize this boundary cylinder. Note that unlike in the flat case, the boundary of AdS is a nice spacetime with a metric that inherits the signature of the AdS metric.

5.1.3 What About the Poincaré Patch?

Recall that in the Poincaré Patch coordinates the AdS metric is

$$ds^2 = \frac{1}{z^2} \left(dt^2 - dz^2 - \sum_{i=1}^{d-1} x_i^2 \right) \quad (5.5)$$

where the z coordinate only covers the range $0 < z < \infty$.

Clearly if we multiply by the Weyl factor z^2 we simply have the metric for a flat $d+1$ dimensional spacetime, so the Penrose diagram must look the same as that for half of Minkowski spacetime. In other words, it will look like a half-diamond.

What does this mean for the boundary of the Poincaré patch? Well, the surface $z = 0$ must be part of the boundary, since the Penrose diagram abruptly ends there. From the identification in equations (2.13) we see that $z \rightarrow 0$ corresponds with $\rho \rightarrow \pi/2$, as expected. We also see that $z \rightarrow \infty$ corresponds to $\rho \rightarrow \pi/2$, but only at the unique point on the boundary cylinder where $\Omega_i = 0$ and $\tau = 0$ (in global coordinates). If we take some combinations of $\vec{x}$ and t to infinity we can reach the end of the Poincaré patch, but not the boundary of the spacetime, because we are free to extend the spacetime past this ‘infinity’, which can actually be reached in a finite proper time.

But the real punchline is that by taking $z \rightarrow 0$, we find a different (but only partial) boundary for the AdS Poincaré patch. This boundary is parameterized by $(t, \vec{x})$, and it just looks like it inherits the geometry of a d -dimensional Minkowski spacetime.

5.1.4 Cosmology and DeSitter Space

Let us make a couple of comments about cosmology, which naturally transpires (during the early epoch of inflation) in quasi-deSitter spacetime. dS_{d+1} is the hyperboloid

$$X_0^2 - \sum_{i=1}^{d+1} X_i^2 = -R^2 \quad (5.6)$$

Setting $R = 1$ (in other words, the Hubble constant is 1) and applying $\sinh^2 t - \cosh^2 t = -1$ we can parameterize dS as

$$ds^2 = dt^2 - \cosh^2(t) d\Omega^2 \quad (5.7)$$

Now using $\tan(\eta/2) = \tanh(t/2)$ we can re-write this metric as

$$ds^2 = \frac{1}{\cos^2 \eta} (d\eta^2 - d\Omega^2) \quad (5.8)$$

Note the similarity to AdS in global coordinates. Now we have a metric that has an obvious Penrose diagram. Note that spatially *there is no boundary*, and the spacetime only ends in time, at $\eta = \pm\pi/2$. This is the origin of the usual statement that deSitter space only has a boundary in the infinite past and infinite future. The absence of any notion of time on the boundary is one (mild) reason to worry about attempts at holography in deSitter spacetime... the deeper problems are that dS is unstable to bubble nucleation, that classically there will be a spacelike singularity in the past (due to the singularity theorems), and that dS behaves like a thermal system, so it is ‘asymptotically hot’ – fields at infinity never cease their fluctuations.

5.2 General Approaches to the Boundary

Penrose diagrams describe the causal structure of the bulk and boundary. Now let us see how in the case of AdS, we can also ascribe a metric to the boundary. We will see that the boundary metric is not uniquely determined, but can be multiplied by an arbitrary Weyl factor that depends on how we approach infinity.

5.2.1 Lessons from the Poincaré vs Global Approach

It seems we have two different ways to approach the boundary – in global coordinates, we naturally obtain a cylinder parameterized by $(\tau, \hat{\Omega})$ when we take $\rho \rightarrow \pi/2$ at the same rate for all $(\tau, \hat{\Omega})$. In contrast, in the Poincaré patch coordinates we find a Minkowski space parameterized by $(t, \vec{x})$ when we take $z \rightarrow 0$ at the same rate for all $(t, \vec{x})$. Actually, there are infinitely more possibilities.

The idea is that in the global coordinate metric of equation (2.1), we can approach different points on the boundary, parameterized by $(t, \hat{\Omega}_i)$, at different rates. This effectively allows us to perform further Weyl transformations as we take $\rho \rightarrow \pi/2$ in order to alter the geometry of the boundary. Specifically, we can choose some function $f(t, \hat{\Omega}_i)$ in the neighborhood $\rho = \pi/2 - \epsilon f(t, \hat{\Omega}_i)$ with small ϵ , and then take $\epsilon \rightarrow 0$. This results in an effective metric on the boundary set by

$$\begin{aligned}
ds^2 &= \frac{1}{\cos^2(\rho(\epsilon, f))} (dt^2 - \sin^2(\rho) d\Omega_i^2) \left(\right. \\
&\approx \frac{1}{\epsilon^2 [f(t, \Omega)]^2} (dt^2 - (1 - \epsilon)^2 d\Omega_i^2) \left(\right. \\
&\rightarrow \left[f(t, \hat{\Omega}_i) \right]^{-2} (dt^2 - d\hat{\Omega}_i^2) \left(\right.
\end{aligned} \tag{5.9}$$

which is related to the metric on the cylinder by a Weyl transformation. So by taking the limit as we approach the boundary in different ways, we can obtain a CFT on any conformally flat manifold, a.k.a. any manifold related to flat spacetime by a Weyl transformation. Note that a cylinder is conformally flat since we can take $f(t, \hat{\Omega}_i) = e^{-t}$ and switch to the coordinate $r = e^t$ to find a flat space metric (in Euclidean signature).

5.2.2 The Euclidean Version

We parameterized Euclidean AdS_{d+1} using global or Poincaré coordinates as

$$\begin{aligned} X_0 &= R \frac{\cosh \tau}{\cos \rho} = \frac{1}{2} \left(\frac{z^2 + \vec{r}^2 + R^2}{z} \right) \left(\right. \\ X_{d+1} &= R \frac{\sinh \tau}{\cos \rho} = \frac{1}{2} \left(\frac{z^2 + \vec{r}^2 - R^2}{z} \right) \left(\right. \\ X_i &= R \tan \rho \Omega_i = \frac{R}{z} r_i \end{aligned} \quad (5.10)$$

where $\vec{r} = (\vec{x}, t)$. Note that this means that $\vec{r}_i = \hat{\Omega}_i e^\tau \sin \rho$ and $z = e^\tau \cos \rho$. A nice thing about the Euclidean version is that as $z \rightarrow 0$, we have $\rho \rightarrow \pi/2$ with a coordinate mapping

$$\frac{e^{\tau/2} \hat{\Omega}_i}{\sqrt{\frac{\pi}{2} - \rho}} = \frac{r_i}{\sqrt{z}} \quad (5.11)$$

This naturally produces a flat spacetime metric $ds^2 = d\vec{r}^2$ for the boundary.

We can also see this directly in global coordinates without using the Poincaré patch description – as we approach the boundary of Euclidean AdS we naturally obtain a flat space metric in polar coordinates (which we will soon associate with radial quantization of CFT). In particular, with the metric

$$ds^2 = \frac{1}{\cos^2 \rho} (d\tau^2 + \sin^2 \rho d\Omega^2) \left(\right. \quad (5.12)$$

we can simply approach the boundary as

$$\rho = \frac{\pi}{2} - \epsilon e^{-\tau} \quad (5.13)$$

with $\epsilon \rightarrow 0$. Then we find the effective boundary metric

$$ds_{\partial \text{AdS}}^2 \rightarrow e^{2\tau} (d\tau^2 + d\Omega^2) \rightarrow dr^2 + r^2 d\Omega^2 \quad (5.14)$$

with $r = e^\tau$. We see that $r = 0$, the origin of the Euclidean plane, maps to $t = -\infty$, the infinite past in Euclidean AdS, while $r = \infty$ corresponds to the infinite future.

5.2.3 Getting a DeSitter Boundary Geometry

Finally, note that the d -dimensional deSitter metric from equation (5.8) is also conformally flat, so by approaching the boundary of AdS_{d+1} in an appropriate manner, we obtain a deSitter boundary metric. If we choose

$$\rho = \frac{\pi}{2} - \epsilon \cos(t) \quad (5.15)$$

Then we find

$$\frac{1}{\cos^2 \rho(\epsilon; t)}(dt^2 - \sin^2 \rho d\Omega^2) \rightarrow \frac{1}{\cos^2 t}(dt^2 - \sin^2 \rho d\Omega^2) \quad (5.16)$$

This is precisely the metric for global deSitter space, where in equation (5.8) we called t the variable η . In particular, now we only have $-\pi/2 < t < \pi/2$. This scaling makes it possible to obtain a CFT living in a deSitter spacetime background! We could use this to study a holographic CFT present during inflation. It is equally easy to find an f that makes the boundary look like an FRW cosmology.

5.2.4 Lorentzian Poincaré Patch Version

Finally, let us see how we could obtain the Lorentzian Poincaré patch from a boundary limit of the global coordinates via an appropriate choice of $f(t, \Omega)$. The boundary of the Poincaré patch is obtained by taking $z \rightarrow 0$ with fixed $t, \vec{x}$, so let us translate this into the global coordinates $\rho, \tau, \hat{\Omega}$. Using the relations of equation (2.13), we see that if we take the coordinate ρ towards $\pi/2$ at a rate dictated by

$$\rho(z; \tau, \hat{\Omega}_i) = \frac{\pi}{2} - \frac{z}{R^2}(\cos \theta - \cos \tau) \quad \text{as } z \rightarrow 0 \quad (5.17)$$

where θ is the complement of the angle Ω_d . This means that in the limit as we approach the boundary we obtain the metric

$$ds^2 = \frac{1}{(\cos \theta - \cos \tau)^2}(d\tau^2 - d\theta^2 - \sin^2 \theta d\Omega_{d-1}^2) \quad (5.18)$$

This is familiar from our discussion of the Penrose diagram for flat spacetime – if we first define $2U = -\theta + \tau$ and $2V = \theta + \tau$ then we obtain a new metric

$$ds^2 = \frac{1}{\sin^2 U \sin^2 V} (dU dV - \sin^2(U + V) d\Omega_{d-1}^2) \quad (5.19)$$

Now we can define $\cot u = U$ and $\cot v = V$ so that the metric becomes simply

$$ds^2 = dudv - \frac{(u + v)^2}{4} d\Omega_{d-1}^2 \quad (5.20)$$

Finally, we can write this in terms of $2t = u - v$ and $2r = u + v$ and see that it is in fact the flat spacetime metric.

5.2.5 Embedding Space Coordinates and the Projective Null Cone

It is natural to translate all of these statements into the more abstract embedding space coordinates, where the full conformal symmetry is manifest. For convenience, we recall that the identification of

coordinates in Euclidean signature is

$$\begin{aligned} X_0 &= R \frac{\cosh \tau}{\cos \rho} = \frac{1}{2} \left(\frac{z^2 + \bar{r}^2 + R^2}{z} \right) \left(\right. \\ X_{d+1} &= R \frac{\sinh \tau}{\cos \rho} = \frac{1}{2} \left(\frac{z^2 + \bar{r}^2 - R^2}{z} \right) \left(\right. \\ X_i &= R \tan \rho \Omega_i = \frac{R}{z} r_i \end{aligned} \quad (5.21)$$

The limit $\rho = \frac{\pi}{2} - \epsilon f(\tau, \Omega)$ with $\epsilon \rightarrow 0$ sends all of the X_A to infinity. We can obtain a finite limit by fixing the *null projective cone* with coordinates

$$P_A \equiv \epsilon X_A \quad (5.22)$$

in the limit $\epsilon \rightarrow 0$. Since $X_A X^A$ is fixed, the null projective coordinates satisfy

$$P_A P^A = 0 \quad \text{and} \quad P_A \cong \lambda P_A \quad (5.23)$$

where the second statement means that the P_A and λP_A are identified – the overall positive rescaling is a redundancy of description, or a ‘gauge symmetry’. The conformal group $SO(1, d+1)$ acts on the P_A in the obvious way inherited from the X_A .

Note that the redundant rescalings by λ have the effect of sending $\epsilon \rightarrow \epsilon/\lambda$. So $P_A \rightarrow \lambda P_A$ rescales the boundary metric by an overall factor, enacting a global dilatation. This means that the λ -redundancy can be used to determine the overall scaling of a function $f(P_A)$. If f has overall scaling dimension Δ , then $f(\lambda P_A) = \lambda^{-\Delta} f(P_A)$. If f does not transform homogeneously in this way, then it is not a conformally invariant function. To be concrete, in a conformal theory

$$\langle \mathcal{O}(P_1) \mathcal{O}(P_2) \rangle = \frac{1}{(P_1 \cdot P_2)^\Delta} \quad (5.24)$$

scales by $\lambda^{-2\Delta}$, as it should. In this example we also see another very useful feature of the embedding coordinates, namely that the only invariant that can be formed is $P_1 \cdot P_2$.

In the Euclidean case where the conformal group is $SO(1, d+1)$, the P_A can be identified with flat space, or any conformally flat manifold. For example, using our choice

$$\rho = \frac{\pi}{2} - \epsilon e^{-\tau} \quad (5.25)$$

with $\epsilon \rightarrow 0$ means that

$$P_A = (P_+, P_-, P_\mu) = (e^{2\tau}, 1, \Omega_i e^\tau) \quad (5.26)$$

where $P_\pm = P_0 \pm P_{d+1}$ and we used the identifications of the X_A above. If we take $r_i = e^\tau \Omega_i$, where r_i is a coordinate in Euclidean flat spacetime, then we obtain flat space as a *section* of the null projective cone. This terminology just means that we are studying the intersection of the hyperplane

$P_- = 1$ with the null projective cone, fixing the projective λ -redundancy and obtaining d -spacetime coordinates.¹⁴ In general we explicitly obtain

$$\begin{aligned} P_0 &= \frac{\cosh \tau}{f(\tau, \Omega)} \\ P_{d+1} &= \frac{\sinh \tau}{f(\tau, \Omega)} \\ P_i &= \frac{\Omega_i}{f(\tau, \Omega)} \end{aligned} \tag{5.27}$$

in the Euclidean case, with the equation $\cos \rho = f$ determining the section of the cone, ie the surface that intersects the cone to fix the geometry.

5.3 Implementing Holography

The basic idea of holography is that physics in $d + 1$ dimensions can be reproduced by a ‘hologram at infinity’, in d or fewer dimensions. It’s always a good idea to take ideas very seriously, make them as concrete and specific as possible, and then see where you are led. Now that we have constructed a quantum field $\phi(x)$ depending on the AdS coordinate x , we can do just that – we can study $\phi(x)$ in the limit that x approaches infinity and see what we find. Why is this a reasonable guess? If there exists a holographic dual to effective field theory in AdS, then it must reproduce at least some of the familiar observables in AdS. Correlation functions of operators $\phi(x)$ as $x \rightarrow \infty$ would be a reasonable starting point, since they can be easily computed in the AdS QFT, while they are also sensible data ‘at infinity’. The S-Matrix is a flat space analog of these observables.

Let us begin with an even broader approach by studying ϕ as an quantum operator with $x \rightarrow \infty$. For this purpose we will use the Euclidean version of AdS. We saw in section 5.1 that the boundary of AdS (aka ‘infinity’) is just the limit $\rho \rightarrow \pi/2$ in global coordinates. Being a bit more careful, we saw in section 5.2.2 that we need to take this limit as

$$\rho(\epsilon) = \frac{\pi}{2} - \epsilon e^{-\tau} \tag{5.28}$$

with $\epsilon \rightarrow 0$ in a coordinate independent way in order to recover a Euclidean flat spacetime metric on the boundary.

Before we consider $\phi(t, \rho, \Omega)$ in the limit $\rho \rightarrow \pi/2$, let us think about how we expect ϕ to behave near infinity. As usual in QFT, we have been assuming that $\phi \rightarrow 0$ as $\rho \rightarrow \pi/2$ – were this not the case, then ϕ would not be normalizable. Thus naively we expect $\phi = 0$ ‘at’ $\rho = \pi/2$. However, in the presence of sources with finite energy and charge, we expect ϕ to have a universal asymptotic

¹⁴Sometimes it is claimed that knowledge of the CFT on one spacetime allows us to ‘lift’ the correlators to the projective null cone, and then to change to a different section, transforming the CFT to some other manifold related to the original one by a Weyl transformation, $g_{\mu\nu} \rightarrow e^{2\Omega(x)} g_{\mu\nu}$. In fact, this procedure is ambiguous for CFTs in $d > 2$ dimensions. Knowledge of the CFT correlators on e.g. flat spacetime does not uniquely determine the correlators on other conformally flat manifolds. General Weyl transformations are not part of the global conformal group, so conformal symmetry is insufficient for determining how the correlators transform.

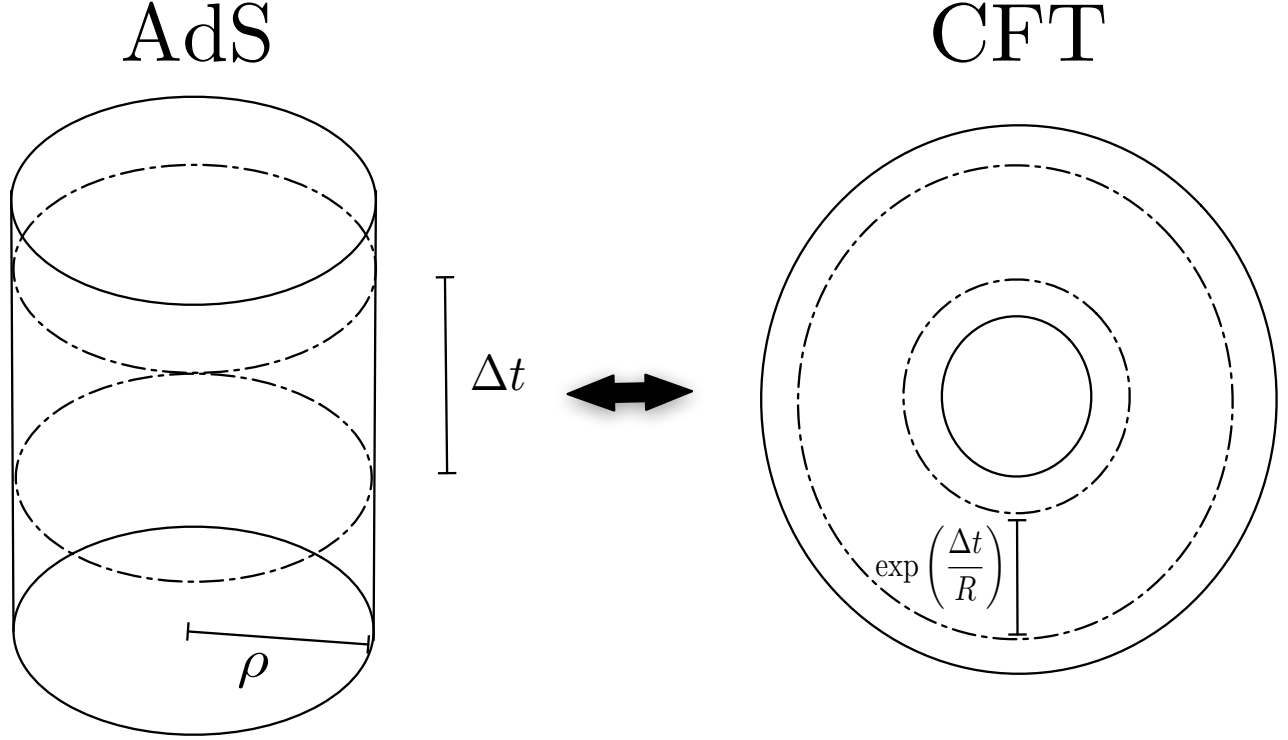


Figure 7: This figure shows how the AdS cylinder in global coordinates corresponds to the CFT in radial quantization. The time translation operator in the bulk of AdS is the Dilatation operator in the CFT, so energies in AdS correspond to dimensions in the CFT. We made this mapping very explicit in section [5.2.2](#).

behavior *near* infinity. For example, in flat spacetime gauge fields fall off as $1/r$ in $3 + 1$ dimensions, whereas massive fields fall off exponentially. So we can define a non-vanishing field ‘at infinity’ by multiplying by a compensating factor, and only then taking the asymptotic limit.

As we can see from the explicit wavefunctions in equation [\(3.45\)](#), as $\rho \rightarrow \pi/2$ we have $\psi_{n\ell J} \rightarrow 0$ as $\cos^\Delta \rho$. One can interpret this as the statement that finite energy particles in AdS have a vanishing probability of making it to a specific point on the boundary, due to the AdS curvature. In other words, the response of the field $\phi(x)$ to finite energy sources/objects/ ϕ -particles in the bulk leads to a $\cos^\Delta \rho$ profile for ϕ near infinity.

Thus we can define a new quantum operator $\mathcal{O}(t, \Omega)$, ‘dual to ϕ ’, or more accurately, representing ϕ ‘at infinity’, via^{[15](#)}

$$\mathcal{O}(t, \Omega) \equiv \lim_{\epsilon \rightarrow 0} \frac{\phi(t, \rho(\epsilon), \Omega)}{\epsilon^\Delta} \quad (5.29)$$

¹⁵Note that we are *not* rescaling ϕ with $\cos \rho$, but only with ϵ . This is necessary to preserve the conformal (AdS isometry) transformation properties of ϕ , so that they are inherited by the CFT operator $\mathcal{O}$.

which is finite on the boundary of AdS. Computing this limit for the wavefunction $\psi_{n\ell J}$ we note that

$$\sin(\rho) \rightarrow 1 \quad (5.30)$$

$$\cos(\rho) \rightarrow \epsilon e^{-\tau\Delta} \quad (5.31)$$

which means that since

$$\psi_{n\ell J} \propto e^{-E_{n,\ell}\tau} Y_{\ell J}(\Omega) \left[\sin^\ell \rho \cos^\Delta \rho {}_2F_1 \left(\left(n, \Delta + \ell + n, \ell + \frac{d}{2}, \sin^2 \rho \right) \right) \right] \left(\quad (5.32)$$

we have

$$\psi_{n\ell J} \rightarrow \frac{1}{N_{\Delta n\ell}} e^{-(2\Delta+2n+\ell)\tau} Y_{\ell J}(\Omega) {}_2F_1 \left(-n, \Delta + \ell + n, \ell + \frac{d}{2}, 1 \right) \left(\quad (5.33)$$

$$\psi_{n\ell J}^\dagger \rightarrow \frac{1}{N_{\Delta n\ell}} e^{(2n+\ell)\tau} Y_{\ell J}^\dagger(\Omega) {}_2F_1 \left(-n, \Delta + \ell + n, \ell + \frac{d}{2}, 1 \right) \left(\quad (5.34)$$

Note that the factors of $\cos^\Delta(\rho)$ are responsible for a shift in the t -dependence. Also, a useful fact about hypergeometric functions is that

$${}_2F_1 \left(-n, \Delta + \ell + n, \ell + \frac{d}{2}, 1 \right) \left(= \frac{\Gamma(\ell + \frac{d}{2})\Gamma(\frac{d}{2} - \Delta)}{\Gamma(n + \ell + \frac{d}{2})\Gamma(\frac{d}{2} - n - \Delta)} \quad (5.35)$$

This means that the operator $\mathcal{O}$ inherits a normalization

$$\frac{1}{N_{\mathcal{O}n\ell}} \propto \frac{(-1)^n \Gamma(\ell + \frac{d}{2})\Gamma(\frac{d}{2} - \Delta)}{\Gamma(n + \ell + \frac{d}{2})\Gamma(\frac{d}{2} - n - \Delta)} \frac{n! \Gamma^2(\ell + \frac{d}{2})\Gamma(\Delta + n - \frac{d-2}{2})}{\Gamma(n + \ell + \frac{d}{2})\Gamma(\Delta + n + \ell)} \Big)^{-1/2} \quad (5.36)$$

$$= \sqrt{\frac{\Gamma(\Delta + n + \ell)\Gamma(\Delta + n - \frac{d-2}{2})\Gamma(\frac{d}{2})}{n!\Gamma(\Delta)\Gamma(\Delta - \frac{d-2}{2})\Gamma(\frac{d}{2} + n + \ell)}} \left(\quad (5.37)$$

where in the second line we multiplied by a constant factor of $\sqrt{\Gamma(\Delta + \frac{d-2}{2})/\Gamma(\Delta)}$ so that $\mathcal{O}$ will have a 2-pt correlator with a simple normalization. Now we can write the quantum operator

$$\mathcal{O}(t, \Omega) = \sum_{n,\ell} \left(\frac{1}{N_{\mathcal{O}n\ell}} \left(e^{-(2\Delta+2n+\ell)\tau} Y_{\ell J}(\Omega) a_{n\ell} + e^{(2n+\ell)\tau} Y_{\ell J}^\dagger(\Omega) a_{n\ell}^\dagger \right) \right) \quad (5.38)$$

living on the boundary of AdS, in the coordinates determined by our specific choice of approach to the boundary. Let us now see why this operator has the correlators of a local CFT operator in flat spacetime.

5.4 Correlators

Now let us look at the correlation function of our new operator $\mathcal{O}$. Then in the next section we will study its conformal transformation properties, the operator state correspondence, and the OPE.

5.4.1 Computation from AdS

In the limit of large distances (which is all we need because we are taking an asymptotic limit in AdS), the 2-pt correlator of ϕ is

$$\langle \phi(X)\phi(Y) \rangle \approx \frac{\Gamma(\Delta)}{\Gamma(\Delta + 1 - \frac{d}{2})} e^{-\Delta\sigma(X,Y)} \quad (5.39)$$

where $\sigma(X, Y)$ is the geodesic distance between X and Y in AdS. One can compute this 2-point correlation function in many ways, e.g. by directly summing over modes created by the oscillators $a_{n\ell J}$. We will give a version of that calculation below. It is also guaranteed to take this form at large distances (up to a normalization) due to the AdS isometries and basic facts about the behavior of solutions to the Klein-Gordon equation.

Geodesic distances in the ρ direction are just given by an integral $\int^\rho \sec(\rho) d\rho$. With $\rho = \frac{\pi}{2} - \epsilon e^{-\tau}$, in the $\epsilon \rightarrow 0$ limit this is just $\log(\pi/2 - \rho(\epsilon)) \approx \tau - \log \epsilon$. Without loss of generality we can take X and Y to be on opposite sides of the Euclidean AdS cylinder at some fixed time τ . This gives a 2-pt correlator for $\mathcal{O}$

$$\langle \mathcal{O}(x_1)\mathcal{O}(x_2) \rangle = \frac{\Gamma(\Delta)}{\Gamma(\Delta + 1 - \frac{d}{2})} \frac{\exp[-\Delta(2\tau - 2\log \epsilon)]}{\epsilon^{2\Delta}} = \frac{\Gamma(\Delta)}{\Gamma(\Delta + 1 - \frac{d}{2})} \frac{1}{\epsilon^{2\Delta\tau}} \quad (5.40)$$

Restated in terms of the boundary points $x_1 = \hat{\Omega}e^\tau$ and $x_2 = -\hat{\Omega}e^\tau$ and dropping the normalization factor for simplicity, this is

$$\langle \mathcal{O}(x_1)\mathcal{O}(x_2) \rangle \propto \frac{1}{(x_1 - x_2)^{2\Delta}} = \frac{1}{(2P_1 \cdot P_2)^\Delta} \quad (5.41)$$

where we have also stated the result in terms of the projective null cone coordinates. Although we computed this result directly, the result actually follows entirely from conformal symmetry. The only conformal invariant that can be made from P_1 and P_2 is $P_1 \cdot P_2$, and then the power of Δ follows from the D transformation properties that $\mathcal{O}$ inherits from ϕ .

Note that since we were studying a free theory in AdS, all correlators of the ϕ field can be reduced to Wick contractions leading to products of 2-pt (Gaussian) correlators. So in fact we already know all correlation functions of $\mathcal{O}(x)$ as well, for the same reason.

5.4.2 Computation from the CFT Operator

As we will soon see, $\mathcal{O}$ inherits conformal transformation properties from the AdS isometry transformations of ϕ . In particular, it certainly inherits translations, so WLOG we can compute

$$\langle \mathcal{O}(\vec{r})\mathcal{O}(0) \rangle \quad (5.42)$$

with, as usual $\vec{r} = e^t \hat{\Omega}$. However, note that

$$\begin{aligned} \mathcal{O}(0)|0\rangle &= \sum_{n,\ell} \left(\frac{1}{N_{\mathcal{O}n\ell}} e^{-(2\Delta+2n+\ell)t} Y_{\ell J}(\Omega) a_{n\ell} + e^{(2n+\ell)t} Y_{\ell J}^\dagger(\Omega) a_{n\ell}^\dagger \right) |0\rangle \\ &= a_{0,0}^\dagger |0\rangle \end{aligned} \quad (5.43)$$

This follows because all the annihilation operators destroy the vacuum, while the other creation operators are multiplied by positive powers of $\vec{r} = 0$. Now evaluating the correlator is extremely easy, because the only term that survives is proportional to $a_{0,0}a_{0,0}^\dagger$, and we simply find

$$\langle \mathcal{O}(\vec{r}) \mathcal{O}(0) \rangle = \frac{1}{r^{2\Delta}} \quad (5.44)$$

which is exactly what we expected. Notice that it was crucial that we scaled ρ appropriately to obtain $\mathcal{O}$ from ϕ – if we had simply taken $\rho \rightarrow \pi/2$ uniformly then $\mathcal{O}$ would have been proportional to an additional $e^{-t\Delta}$; this would have given the correct correlators on a cylinder, but not in flat spacetime!

6 Generalized Free Theories and ‘AdS/CT’

In this section we will show how our Generalized Free Theory, defined by the holographic limit of the AdS bulk field ϕ , satisfies and in many cases physically illustrates the general abstract properties expected for a local operator in a conformal theory. We refrain from using the words “CFT” because the conformal theory we are discussing lacks a stress-energy tensor.

6.1 The Operator/State Correspondence

First of all, what are these so-called operators?

Clearly we have the object $\mathcal{O}(\vec{r})$ that we defined previously in terms of the AdS field ϕ . To study the question in general, let us specialize for convenience to the case of AdS_3 . The reason is that we can significantly simplify the notation in that case by writing

$$\vec{r} = (x, y) \quad (6.1)$$

in terms of

$$z = x + iy \quad \text{and} \quad \bar{z} = x - iy \quad (6.2)$$

Since we are studying the boundary of Euclidean AdS_3 as the 2-dimensional Euclidean plane, the angular coordinate $\Omega \rightarrow \theta$ just parameterizes a single circle. In terms of z and $\bar{z}$ we have

$$r^2 = z\bar{z} \quad \text{and} \quad e^{2i\theta} = \frac{z}{\bar{z}} \quad (6.3)$$

and the $Y_\ell = e^{i\ell\theta}$. Therefore we can write

$$\mathcal{O}(z, \bar{z}) = \sum_{n,\ell} \left(\frac{1}{N_{n,\ell}^{\mathcal{O}}} \left[(z\bar{z})^{-(\Delta+n+\frac{|\ell|}{2})} \left(\frac{z}{\bar{z}} \right)^{\frac{\ell}{2}} a_{n\ell} + (z\bar{z})^{n+\frac{|\ell|}{2}} \left(\frac{\bar{z}}{z} \right)^{\frac{\ell}{2}} a_{n\ell}^\dagger \right] \right) \quad (6.4)$$

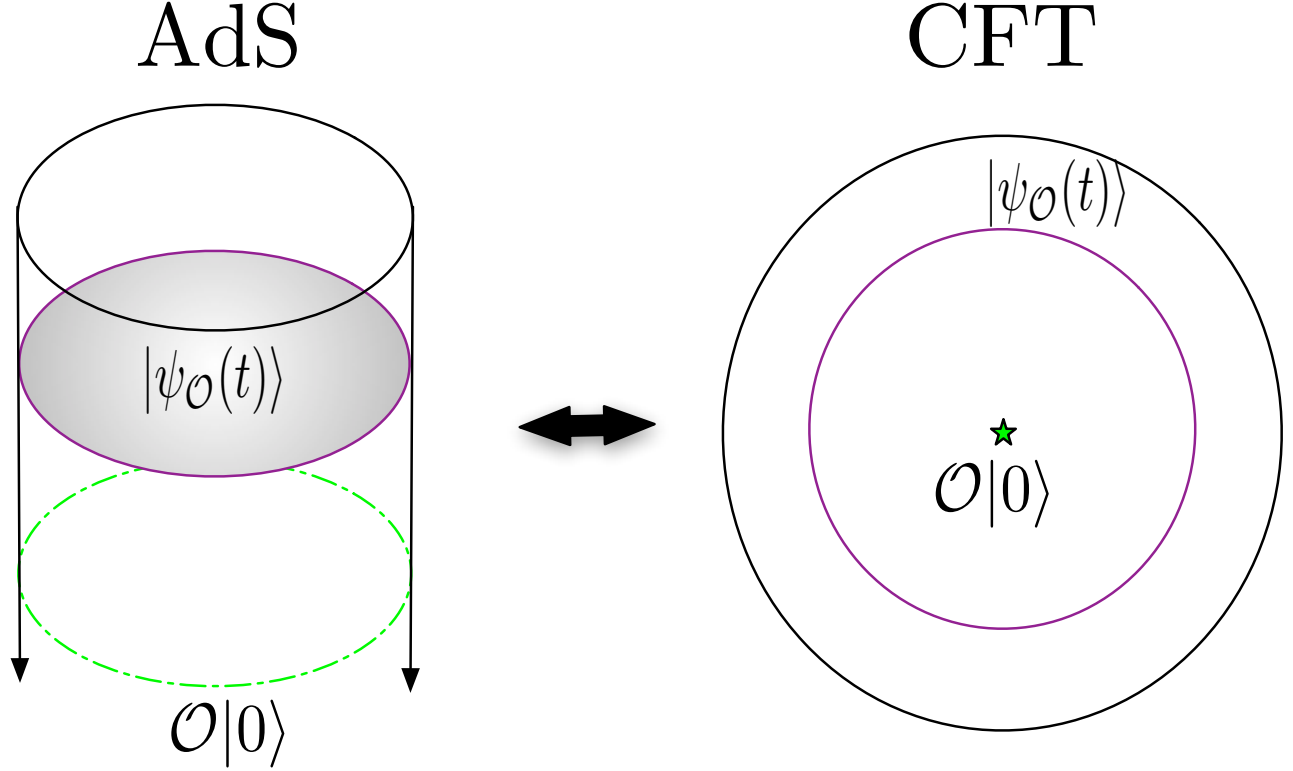


Figure 8: This figure portrays radial quantization in the CFT, while also providing the corresponding picture in AdS. The insertion of an operator at the origin in the CFT defines a specific AdS/CFT state; interpreted in AdS this sets up an initial state in the infinite past that evolves to $\psi_{\mathcal{O}}(t)$ at a finite time. Note that $e^t \sim |\vec{r}|$, so the global coordinate time is the logarithm of the radius of a circle about the origin in the CFT.

When we work in terms of z and $\bar{z}$, it's simpler to define the operator in terms of integers h and $\bar{h}$ with

$$\mathcal{O}(z, \bar{z}) = \sum_{h, \bar{h}=0}^{\infty} \frac{1}{N_{h, \bar{h}}^{\mathcal{O}}} \left[(z\bar{z})^{-\Delta} z^{-h} \bar{z}^{-\bar{h}} a_{h, \bar{h}} + z^h \bar{z}^{\bar{h}} a_{h, \bar{h}}^{\dagger} \right] \quad (6.5)$$

where $2n + |\ell| = h + \bar{h}$ and $\ell = h - \bar{h}$, so that $n = \min(h, \bar{h})$.

Above we saw that when we act $\mathcal{O}(0)$ on the vacuum, we obtain the state

$$\mathcal{O}(0)|0\rangle = a_{0,0}^{\dagger}|0\rangle \quad (6.6)$$

Now let us consider what we get from P_{μ} acting on $\mathcal{O}$. We can simply write the momentum operator in terms of a basis as $P_z = \partial_z$ and $P_{\bar{z}} = \partial_{\bar{z}}$ (we will derive the fact that P_{μ} acts on $\mathcal{O}$ in this way in the next section). Thus we find

$$(P_z \mathcal{O})(0) = \frac{1}{N_{h=1, \bar{h}=0}^{\mathcal{O}}} a_{h=1, \bar{h}=0}^{\dagger} |0\rangle \quad (6.7)$$

Similarly, we can act with any number of P_z and $P_{\bar{z}}$ on $\mathcal{O}$. So we have a whole host of operators of the form

$$\left((\partial_z)^h (\partial_{\bar{z}})^{\bar{h}} \mathcal{O}\right) \Big|_{z, \bar{z}=0} \quad (6.8)$$

These create all of the states $a_{h, \bar{h}}^\dagger$! Crucially, these are just *the one-particle states in AdS_3 labeled by n and ℓ* . So we see that all one-particle states can be created by some particular CT operator! This is the beginning of the *operator/state correspondence*; it's not an abstract idea but a very concrete concept that can be visualized directly in AdS.

What about multiparticle states? We would like to define an operator like

$$\mathcal{O}^2(0) \quad (6.9)$$

and we would expect it to create 2 AdS particles, both in their ground state. But if we evaluate this directly, there appears to be a singularity. For example, if we look at

$$\begin{aligned} \mathcal{O}(z, \bar{z}) \mathcal{O}(w, \bar{w}) &= \left(\sum_{h, \bar{h}=0}^{\infty} \left(\frac{1}{N_{h, \bar{h}}^{\mathcal{O}}} \left[(z\bar{z})^{-\Delta} z^{-h} \bar{z}^{-\bar{h}} a_{h, \bar{h}} + z^h \bar{z}^{\bar{h}} a_{h, \bar{h}}^\dagger \right] \right) \right) \left(\right. \\ &\quad \times \left. \left(\sum_{h, \bar{h}=0}^{\infty} \left(\frac{1}{N_{h, \bar{h}}^{\mathcal{O}}} \left[(w\bar{w})^{-\Delta} w^{-h} \bar{w}^{-\bar{h}} a_{h, \bar{h}} + w^h \bar{w}^{\bar{h}} a_{h, \bar{h}}^\dagger \right] \right) \right) \right) \left(\right. \end{aligned} \quad (6.10)$$

then we have singularities from the limit of $\omega \rightarrow 0$ in the second operator, as well as singularities that arise when the annihilation operators in $\mathcal{O}(z, \bar{z})$ hit the creation operators in $\mathcal{O}(w, \bar{w})$. Thus we can define $\mathcal{O}^2(0)$ as the normal ordered object (annihilation operators artificially moved to the right, without using the commutation relations) with these singularities removed, so that we get

$$(\mathcal{O}^2) \binom{0}{0} \equiv \left(\frac{1}{N_{0,0}^{\mathcal{O}}} \right)^2 a_{0,0}^\dagger a_{0,0}^\dagger \quad (6.11)$$

Similarly, we can define

$$(\mathcal{O}^3) \binom{0}{0} \equiv \left(\frac{1}{N_{0,0}^{\mathcal{O}}} \right)^3 a_{0,0}^\dagger a_{0,0}^\dagger a_{0,0}^\dagger \quad (6.12)$$

and also the operator

$$(\mathcal{O} P_z \mathcal{O}) \binom{0}{0} \equiv \left(\frac{1}{N_{0,0}^{\mathcal{O}} N_{1,0}^{\mathcal{O}}} \right) a_{0,0}^\dagger a_{1,0}^\dagger \quad (6.13)$$

We can proceed to define operators with an arbitrarily long string of P_z , $P_{\bar{z}}$, and $\mathcal{O}$ s. Again, the crucial point is that by acting with general linear combinations of such operators on the vacuum $|0\rangle$,

we can obtain every single state in the theory – in particular, *there is a one-to-one correspondence between AdS/CFT states in the Hilbert space and local operators acting at the origin, $z = 0$. This is the operator/state correspondence for the CT defined by a free field in AdS.* From the CT point of view, this system is called a Generalized Free Theory, a Gaussian CFT, a CFT at infinite N , or a mean field theory. I will usually stick with the first term.

In closing, we should note that throughout this section we have used *radial quantization* without drawing much attention to it. This is the natural quantization scheme in mapping from AdS in global coordinates to a CFT on a plane.

6.2 Conformal Transformations of Operators

We obtained the Conformal Theory operator $\mathcal{O}(\vec{r})$ from a limit of $\phi(t, \rho, \Omega)$, a free field in AdS with an action symmetric under the AdS isometries aka conformal transformations. This means that $\mathcal{O}$ naturally inherits conformal transformation properties from AdS – so what are they? The intelligent way to approach this question would be to use the embedding space coordinates. However, we can also be somewhat less clever and see how it works directly in the global coordinate system.

We already computed the dilatation operator D as the AdS Hamiltonian, it is

$$D = \sum_{n, \ell, J} (\Delta + 2n + \ell) a_{n\ell J}^\dagger a_{n\ell J} \quad (6.14)$$

and by definition, since in AdS there was no explicit time dependence, we have that

$$[D, \phi(t, \rho, \Omega)] = \partial_t \phi \quad (6.15)$$

in Euclidean signature. Let's compute its action on $\mathcal{O}$ in two different ways. Note that if we had defined $\mathcal{O}$ by simply taking $\rho \rightarrow \pi/2$ in a t -independent way, then we would find that $\mathcal{O}$ obeys the same relation as ϕ . That would be the correct result for a CFT living on the cylinder $R \times S^{d-1}$. However, in order to obtain a CT in flat Euclidean space, we took $\rho = \pi/2 - \epsilon e^{-t}$, producing an extra explicit t -dependence.

In particular, an infinitesimal shift $t \rightarrow t + a$ leads to

$$\begin{aligned} [D, \mathcal{O}(t, \Omega)] &= \lim_{\epsilon \rightarrow 0} \left(\frac{d}{dt} \frac{\phi(t, \rho(\epsilon; t), \Omega)}{\epsilon^\Delta} - \frac{\partial \rho}{\partial t} \frac{\partial}{\partial \rho} \frac{\phi(t, \rho(\epsilon; t), \Omega)}{\epsilon^\Delta} \right) \left(\right. \\ &\approx \frac{d}{dt} \mathcal{O}(t, \Omega) - \lim_{\epsilon \rightarrow 0} \frac{\partial \rho}{\partial t} \frac{\partial}{\partial \rho} \frac{\phi(t, \rho, \Omega)}{\epsilon^\Delta} \\ &\approx \left(\Delta + \frac{\partial}{\partial t} \right) \mathcal{O}(t, \Omega) \end{aligned} \quad (6.16)$$

with $\rho(\epsilon; t) = \pi/2 - \epsilon e^{-t}$ as usual. The crucial step comes in the first line, where we recognize that time derivatives acting on $\phi(t, \rho(\epsilon; t), \Omega)$ will also act on $\rho(\epsilon; t)$, and so we have to subtract the second term to compensate. A similar procedure will be necessary for P_μ and K_μ . The final result can be written in $\vec{r} = e^t \hat{\Omega}$ coordinates as

$$[D, \mathcal{O}(\vec{r})] = \left(\Delta + r \frac{\partial}{\partial r} \right) \mathcal{O}(\vec{r}) \quad (6.17)$$

This is the correct conformal transformation rule for a CFT operator of dimension Δ living on the plane. The lesson is that radial time derivatives on $\mathcal{O}$ act both explicitly (via partial derivative) and with a shift of Δ in order to cancel the t dependence that arises through $\rho = \pi/2 - \epsilon f(t, \Omega)$. We can also compute this a different way, using our explicit formula for $\mathcal{O}$ and D in terms of oscillators in $\text{AdS}_3/\text{CFT}_2$. We find

$$\begin{aligned}
& \left[\sum_{h, \bar{h}} (\Delta + h + \bar{h}) a_{h, \bar{h}}^\dagger a_{h, \bar{h}}, \left(\sum_{h, \bar{h}=0}^{\infty} \frac{1}{N_{h, \bar{h}}^{\mathcal{O}}} \left[(z\bar{z})^{-\Delta} z^{-h} \bar{z}^{-\bar{h}} a_{h, \bar{h}} + z^h \bar{z}^{\bar{h}} a_{h, \bar{h}}^\dagger \right] \right) \right] \left(\right. \\
&= \sum_{h, \bar{h}=0}^{\infty} \frac{1}{N_{h, \bar{h}}^{\mathcal{O}}} \left[\left((\Delta + h + \bar{h}) (z\bar{z})^{-\Delta} z^{-h} \bar{z}^{-\bar{h}} a_{h, \bar{h}} + (\Delta + h + \bar{h}) z^h \bar{z}^{\bar{h}} a_{h, \bar{h}}^\dagger \right) \right] \\
&= (\Delta + z\partial_z + \bar{z}\partial_{\bar{z}}) \mathcal{O}(z, \bar{z})
\end{aligned} \tag{6.18}$$

as promised, so the two methods agree.

In AdS_3 there is only one angular momentum generator, proportional to ∂_θ . This does not change when we take it to the boundary, and so we just find

$$L = -i\partial_\theta = \frac{1}{2} (z\partial_z - \bar{z}\partial_{\bar{z}}) \tag{6.19}$$

in the 2-dimensional plane.

In AdS_3 we looked at the momentum operator $P_\pm$. Converting to the Euclidean case, it is

$$P_\pm = e^{-t \pm i\theta} \left(\sin \rho \partial_t + \cos \rho \partial_\rho \pm \frac{i}{\sin \rho} \partial_\theta \right) \left(\right. \tag{6.20}$$

We cannot immediately drop $\cos \rho \partial_\rho$ because it scales as a constant as $\rho \rightarrow \pi/2$. This term just produces $-\Delta$ when acting on ϕ . This cancels with a shift of Δ from converting ∂_t from an action on ϕ to an action on $\mathcal{O}$, equivalent to that which we observed when we examined the dilatation operator. So as we approach the boundary, we simply find that it acts on $\mathcal{O}$ as

$$\begin{aligned}
P_\pm &\rightarrow e^{t \pm i\theta} (-\partial_t \pm i\partial_\theta) \\
&= \frac{1}{\sqrt{z\bar{z}}} \left(\frac{z}{\bar{z}} \right)^{\pm \frac{1}{2}} \left[\frac{1}{2} (z\partial_z + \bar{z}\partial_{\bar{z}}) \mp \frac{1}{2} (z\partial_z - \bar{z}\partial_{\bar{z}}) \right] \\
&= \partial_{\bar{z}/z}
\end{aligned} \tag{6.21}$$

This just translates into the statement that $P_- = P_z$ and $P_+ = P_{\bar{z}}$. So the P_μ isometry of AdS just turns into the translation operator, as we would expect.

Finally, we can consider the special conformal generators, which take the form

$$K_\pm = e^{t \pm i\theta} \left(\sin \rho \partial_t - \cos \rho \partial_\rho \mp \frac{i}{\sin \rho} \partial_\theta \right) \left(\right. \tag{6.22}$$

in AdS_3 . Here the contribution of $\cos \rho \partial_\rho$ does not cancel, instead it contributes additively with the shift in ∂_t . Otherwise this operator behaves similarly to the translation operator, and we find

$$\begin{aligned} K_\pm &\rightarrow e^{t \pm i\theta} (\partial_t + 2\Delta \mp i\partial_\theta) \\ &= \sqrt{z\bar{z}} \left(\frac{z}{\bar{z}} \right)^{\pm \frac{1}{2}} \left[\frac{1}{2} (z\partial_z + \bar{z}\partial_{\bar{z}}) + 2\Delta \pm \frac{1}{2} (z\partial_z - \bar{z}\partial_{\bar{z}}) \right] \end{aligned} \quad (6.23)$$

This leads to

$$K_+ = z^2 \partial_z + 2z\Delta \quad (6.24)$$

which is the correct result, up to an overall normalization. Note that the holomorphic and anti-holomorphic parts of the algebra can be decoupled from each other. We have derive the action of all of the conformal generators on our CT operator $\mathcal{O}(z, \bar{z})$.

To summarize, we have found that the 2d global conformal algebra $SO(1, 3)$ acts as

$$[D, \mathcal{O}(x)] = \left(\Delta + \frac{1}{2} z \partial_z + \frac{1}{2} \bar{z} \partial_{\bar{z}} \right) \mathcal{O}(z, \bar{z}) \quad (6.25)$$

$$[P_\mu, \mathcal{O}(x)] = \partial_\mu \mathcal{O}(z, \bar{z}) \quad (6.26)$$

$$[K_z, \mathcal{O}(x)] = (2z\Delta + z^2 \partial_z) \mathcal{O}(z, \bar{z}) \quad (6.27)$$

$$[K_{\bar{z}}, \mathcal{O}(x)] = (2\bar{z}\Delta + \bar{z}^2 \partial_{\bar{z}}) \mathcal{O}(z, \bar{z}) \quad (6.28)$$

$$[L, \mathcal{O}(x)] = \frac{1}{2} (z\partial_z - \bar{z}\partial_{\bar{z}}) \mathcal{O}(z, \bar{z}) \quad (6.29)$$

We derived all of these relations from AdS. When we study general CFTs in the next section, these relations will take on a fundamental character – in fact, one way of *defining* a CT is as a theory of abstract operators $\mathcal{O}(z, \bar{z})$ transforming in exactly this way. But we have understood these transformation properties as an inevitable consequence of AdS physics.

We can also write these symmetry generators in terms of $a_{h, \bar{h}}$, just as we did for D . There is an algorithm to do this – we just need to find the conformal generators in terms of the AdS field ϕ , and then use the expansion of that field in creation and annihilation operators. We can also try to get the answer by guessing. For example, to obtain P I find we need to take

$$\begin{aligned} P_z &\equiv \sum_{h, \bar{h}} \left(\sqrt{\min(h, \bar{h})(\Delta + \min(h, \bar{h}) - 1)} a_{h, \bar{h}}^\dagger a_{h-1, \bar{h}} \right. \\ P_{\bar{z}} &\equiv \sum_{h, \bar{h}} \left(\sqrt{\min(h, \bar{h})(\Delta + \min(h, \bar{h}) - 1)} a_{h, \bar{h}}^\dagger a_{h, \bar{h}-1} \right. \end{aligned} \quad (6.30)$$

Note that (crucially!) these operators also satisfy the commutation relations $[D, P_\mu] = P_\mu$ and $[P_z, P_{\bar{z}}] = 0$. It's fairly trivial to find an operator P_z that acts on $\mathcal{O}(z, \bar{z})$ as ∂_z , the property that actually makes this a Conformal Theory is that the operators D, P_μ, K_μ , and $M_{\mu\nu}$ that act appropriately on $\mathcal{O}(x)$ also satisfy the commutation relations of the conformal algebra.

Finally, note that in Lorentzian AdS we had the relation $P_\mu^\dagger = K_\mu$. This has a simple meaning in AdS – it just says that since complex conjugation sends $it \rightarrow -it$, conjugation exchanges the past and future. This is what we are used to in Quantum Mechanics; for example when considering scattering we have

$$(|\text{in}\rangle)^\dagger = \langle \text{out}| \quad (6.31)$$

as usual. However, when we go from AdS $\rightarrow$ CFT, we went to Euclidean space and took $r^2 = z\bar{z} = e^{2t}$. Thus in the CFT, $t \rightarrow -t$ means

$$r \rightarrow \frac{1}{r} \quad (6.32)$$

or in other words, *complex conjugation corresponds to performing a spacetime inversion about the origin*. The states built at the origin transform like

$$\mathcal{O}(0)|0\rangle \rightarrow \lim_{r \rightarrow \infty} r^{2\Delta} \langle 0|\mathcal{O}(r) \quad (6.33)$$

under this transformation. More generally, for a primary operator $\mathcal{O}$ we have

$$[\mathcal{O}(\vec{r})]^\dagger = r^{-2\Delta} \mathcal{O}^* \left(\frac{\vec{r}}{r^2} \right) \quad (6.34)$$

where the $*$ means that we take $a \rightarrow a^\dagger$. The reason for the factor of r can be seen explicitly in equation (5.38). There we see that the $a_{n\ell J}$ annihilation operators are accompanied by this factor, whereas the creation operators are not, so we need to multiply by $r^{-2\Delta}$ to insure that the operator is Hermitian, ie that $[\mathcal{O}(r)]^\dagger = \mathcal{O}(r)$.

6.3 Primary and Descendant Operators

In our earlier discussions we noted that states in a theory with conformal symmetry can be classified as *primary* or *descendant*. Primary states have the property that

$$K_\mu |\psi_0\rangle = 0 \quad (6.35)$$

so they are annihilated by all the special conformal (or lowering) operators. One example of a primary states is $a_{0,0}^\dagger |0\rangle$; this corresponds to a free particle in its quantum mechanical ground state in AdS₃. States formed by acting on this state with P_μ are descendants.

But there are many many more primaries. One example is

$$\mathcal{O}^k(0)|0\rangle = \left(a_{0,0}^\dagger\right)^k |0\rangle \quad (6.36)$$

but this is just the beginning. This can be interpreted as a condensate of k particles in AdS₃, all in their ground state.

Let us pause to recall why this is a primary operator. In AdS, note that the CoM wavefunction is

$$\psi_k(t_{CoM}, \rho_{CoM}) = \langle \phi^k(t_{CoM}, \rho_{CoM}) \mathcal{O}^k(0) \rangle \quad (6.37)$$

$$= e^{-k\Delta t_{CoM}} \cos^{k\Delta} \rho_{CoM} \quad (6.38)$$

So we see it has the correct form for a primary wavefunction with dimension $= k\Delta$. In particular, it will be annihilated by all the SCT K_μ in AdS, which act as differential operators in the variables t_{CoM} and ρ_{CoM} . In the CFT, we can test if this state is primary by computing

$$K_\mu \mathcal{O}^k(0) |0\rangle = 0 \quad (6.39)$$

We can calculate this by commuting K_μ through so that it hits the vacuum, where we obtain 0. So in other words, this state will be a primary if

$$[K_\mu, \mathcal{O}^k(0)] \neq 0 \quad (6.40)$$

This is the condition for an *operator to be a primary*. Note that the operator must be evaluated at the origin. A primary operator is just an operator that creates a primary state when it acts at the origin. The converse is also true, namely if $\mathcal{O}(0)|0\rangle$ is a primary state, then we must have $[K_\mu, \mathcal{O}(0)] = 0$, because this commutator must have vanishing correlators with other local operators.

A more interesting and non-trivial example is the state

$$[\partial_z \mathcal{O} \partial_{\bar{z}} \mathcal{O} - \mathcal{O} \partial_z \partial_{\bar{z}} \mathcal{O}] |0\rangle \quad (6.41)$$

where the operators $\mathcal{O}$ are normal ordered. One can check that this is always a primary state by using the conformal commutation relations. Note that

$$\partial_z \mathcal{O} = [P_z, \mathcal{O}] \quad (6.42)$$

and so this state takes the form

$$([P_z, \mathcal{O}][P_{\bar{z}}, \mathcal{O}] - \mathcal{O}[P_z, [P_{\bar{z}}, \mathcal{O}]] |0\rangle \quad (6.43)$$

To check that it is a primary we need to show

$$K_{z/\bar{z}} ([P_z, \mathcal{O}][P_{\bar{z}}, \mathcal{O}] - \mathcal{O}[P_z, [P_{\bar{z}}, \mathcal{O}]] |0\rangle = 0 \quad (6.44)$$

and this will be true iff

$$[K_z, ([P_z, \mathcal{O}][P_{\bar{z}}, \mathcal{O}] - \mathcal{O}[P_z, [P_{\bar{z}}, \mathcal{O}]] (0)) = 0 \quad (6.45)$$

and similarly for $z \rightarrow \bar{z}$, since the conformal generators act on commutators via the Lie algebra commutator. Note that the operator we wish to show is primary is evaluated at the origin. We could also compute this directly in AdS using the known wavefunctions.

Carrying out the computations using $[A, [B, C]] + [B, [C, A]] + [C, [A, B]] = 0$, we find that

$$[K_\mu, [P_z, \mathcal{O}][P_{\bar{z}}, \mathcal{O}]] = 2\Delta \mathcal{O} (\delta_{z\mu} [P_{\bar{z}}, \mathcal{O}] + \delta_{\bar{z}\mu} [P_z, \mathcal{O}]) \quad (6.46)$$

while a similar computation gives

$$[K_\mu, \mathcal{O}[P_z, [P_{\bar{z}}, \mathcal{O}]] = 2\Delta \mathcal{O}(\delta_{z\mu}[P_{\bar{z}}, \mathcal{O}] + \delta_{\bar{z}\mu}[P_z, \mathcal{O}]) \quad (6.47)$$

so the linear combination we wrote down is in fact a primary operator. Note that it was crucial for these computations that we are dealing with a generalized free (or quadratic) theory. This assumption was used whenever we assumed that

$$\lim_{x \rightarrow 0} [A, \mathcal{O}(x)\mathcal{O}(0)] = [A, \mathcal{O}(0)]\mathcal{O}(0) + \mathcal{O}(0)[A, \mathcal{O}(0)] \quad (6.48)$$

an identity we used implicitly above. We also implicitly normal ordered the operators to subtract off the identity contribution.

This state corresponds to 2 particles in AdS_3 in an excited state with $\ell = 0$ and with their center of mass degree of freedom in its ground state. However, this state has a non-trivial s -wave momentum, so neither particle is expected to be in its ground state. Because the center of mass degree of freedom in AdS_3 is at rest, this state is primary – from the AdS point of view, that’s the physical definition of primary-ness. Acting on this state with P_μ will generate descendant 2-particle states.

We already established a one-to-one operator/state correspondence. So we see that not only states, but also *operators can be classified as either primaries or descendants*. Usually this classification is stated in terms of the conformal transformation properties of operators, but the operator/state correspondence and AdS provide an alternate physical picture.

Primary operators transform according to the rules similar to those we derived above for $\mathcal{O}(z, \bar{z})$; we will give the general story below. Descendants do not transform as simply because they take the form

$$\mathcal{O}_{desc} \sim [P_{\mu_1}, [P_{\mu_2}, [\dots, [P_{\mu_x}, \mathcal{O}_{prim}(x)] \dots]] \quad (6.49)$$

and so the conformal generators act on them in a relatively complicated way. In general CFTs primary operators are often defined as the operators with the ‘best’ or ‘simplest’ transformation properties.

In a unitary quantum mechanical theory, all states must have positive normalization. However, since different states are connected by raising and lowering using P and K , the norm of one state can be computed in terms of the norm of another, and this puts restrictions on the theory. Unitarity bounds follow from evaluating the matrix element (for a scalar state/operator)

$$\langle \Delta | K_\alpha K^\alpha P_\mu P^\mu | \Delta \rangle \quad (6.50)$$

and demanding that it is positive definite, since it must be the norm of a physical state. Note that this follows because $P_\mu^\dagger = K_\mu$; this relation must hold so that e.g.

$$(P_\mu |\psi\rangle)^\dagger = \langle \psi | K_\mu \quad (6.51)$$

and raising and lowering work correctly. Let us derive the unitarity relation. First we consider

$$\langle \Delta | K_\alpha P^\mu | \Delta \rangle = \langle \Delta | P^\mu K_\alpha + 2\eta_{\alpha\mu} D + 2M_{\alpha\mu} | \Delta \rangle \quad (6.52)$$

$$= 2\Delta \langle \Delta | \Delta \rangle \quad (6.53)$$

and from this we learn something – namely that $\Delta \geq 0$ is required in order to have a positive (or zero) norm descendant. Note that in fact $\Delta = 0$ is allowed – this is the case of the identity operator 1, which clearly does not have any descendants at all, since $[P_\mu, 1] = \partial_\mu 1 = 0$. The identity transforms in the trivial representation of the conformal group.

We will now see that if $\Delta \neq 0$ then we must have a stricter bound. Let us go on and evaluate the norm of a 2nd level descendant. This is

$$\langle \Delta | K_\alpha K^\alpha P_\mu P^\mu | \Delta \rangle = \langle \Delta | K_\alpha P_\mu K^\alpha P^\mu + 2K^\alpha (\eta_{\mu\alpha} D + M_{\mu\alpha}) P^\mu | \Delta \rangle \quad (6.54)$$

$$= \langle \Delta | 2K^\alpha P^\mu (\eta_{\mu\alpha} D + M_{\mu\alpha}) + 2K \cdot P (\Delta + 1 + (1-d)) | \Delta \rangle \quad (6.55)$$

$$= 2(2\Delta + 2 - d) \langle \Delta | K \cdot P | \Delta \rangle \quad (6.56)$$

by assumption the matrix element in the last line must be positive, and we assume that $\langle \Delta | K \cdot P | \Delta \rangle$ is positive since if it was not then the theory already would not have been unitary at the first level. This means that we must have

$$\Delta \geq \frac{d-2}{2} \quad (6.57)$$

as promised in order to have a unitary representation. For operators/states with spin ℓ a similar analysis shows that

$$\Delta \geq d - 2 + \ell \quad (6.58)$$

for unitarity.

6.4 The Operator Product Expansion

Finally, we are ready to discuss one of the other key attributes of a conformal theory – the Operator Product Expansion (OPE), as applied to generalized free theories. The idea behind the OPE in AdS/CFT is pictured in figure 9.

Let us examine the general operator quantity

$$\begin{aligned} \mathcal{O}(z, \bar{z}) \mathcal{O}(w, \bar{w}) &= \left(\sum_{h, \bar{h}=0}^{\infty} \left(\frac{1}{N_{h, \bar{h}}^{\mathcal{O}}} \left[(z\bar{z})^{-\Delta} z^{-h} \bar{z}^{-\bar{h}} a_{h, \bar{h}} + z^h \bar{z}^{\bar{h}} a_{h, \bar{h}}^\dagger \right] \right) \right) \left(\sum_{h, \bar{h}=0}^{\infty} \left(\frac{1}{N_{h, \bar{h}}^{\mathcal{O}}} \left[(w\bar{w})^{-\Delta} w^{-h} \bar{w}^{-\bar{h}} a_{h, \bar{h}} + w^h \bar{w}^{\bar{h}} a_{h, \bar{h}}^\dagger \right] \right) \right) \left(\right) \end{aligned} \quad (6.59)$$

We cannot say much about this quantity that is useful, in general. However, if it is the case that *between these two operators the CFT is in its vacuum state*, then we can translate $w \rightarrow 0$ for simplicity and note that this operator product creates the linear combination of states

$$\mathcal{O}(z, \bar{z}) \mathcal{O}(0) |0\rangle = \left(\sum_{h, \bar{h}=0}^{\infty} \left(\frac{1}{N_{h, \bar{h}}^{\mathcal{O}}} \left[(z\bar{z})^{-\Delta} z^{-h} \bar{z}^{-\bar{h}} a_{h, \bar{h}} + z^h \bar{z}^{\bar{h}} a_{h, \bar{h}}^\dagger \right] \right) \right) a_{0,0}^\dagger |0\rangle \quad (6.60)$$

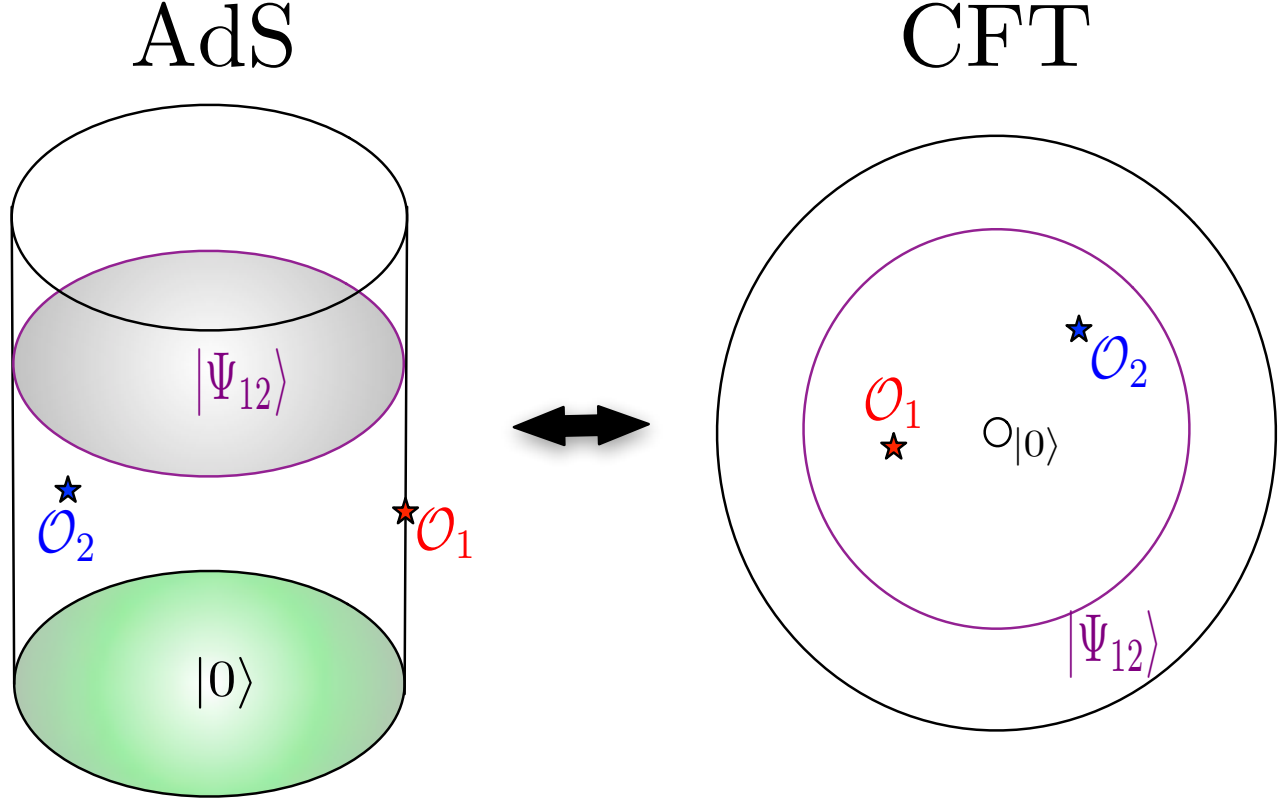


Figure 9: This figure shows how the OPE can be derived in the CFT, and how the derivation is mirrored in AdS. The final point left off the diagram is that the state ψ_{12} can be created by a local operator acting at the origin in the CFT, corresponding to an operator in the infinite past in global coordinate AdS. This follows from the operator/state correspondence of radial quantization.

That's quite a bit simpler!

The AdS picture is the following: assume that the theory is in its vacuum state in the infinite past. Then if we insert two operators before some time $t = \log(z\bar{z})$ we obtain a state $|\psi_{12}(t)\rangle$. But then we can evolve that state all the back to $t = -\infty$, replacing $|\psi_{12}(t)\rangle$ with $|\psi_{12}(-\infty)\rangle$. All matrix elements of the operator product with local operators that appear at times $t > \log(z\bar{z})$ can be computed as matrix elements with this new initial state $|\psi_{12}(-\infty)\rangle$.

Now we can ask what linear combination of local operators, acting at the origin, creates the state $|\psi_{12}(-\infty)\rangle$? We already saw that every CFT state is associated with an operator, so can write the answer to this question in terms of a sum over primary and descendant operators. Since all descendants are just ∂_z or $\partial_{\bar{z}}$ acting on a primary, we must have that

$$\mathcal{O}(z, \bar{z})\mathcal{O}(0) = \sum_{\text{primary } n, \ell} \left(C_{n, \ell}(z, \bar{z}, \partial_z, \partial_{\bar{z}}) \mathcal{O}_{n, \ell} \right) (0) \quad (6.61)$$

as an operator statement, for some coefficient function $C_{n, \ell}$. The $\mathcal{O}_{n, \ell}$ are just the primary operators

made from two of the fundamental $\mathcal{O}$ operators. This is the OPE in a generalized free theory. The coefficients $C_{n,\ell}$ can be computed once and for all, as they are determined by conformal symmetry, up to a constant (which can also be determined once and for all in generalized free theories).

Let us go back and look at the generalized free theory result more explicitly. We can write

$$\begin{aligned}
\mathcal{O}(z, \bar{z})\mathcal{O}(0)|0\rangle &= \left(\sum_{h,\bar{h}=0}^{\infty} \left(\frac{1}{N_{h,\bar{h}}^{\mathcal{O}}} \left[(z\bar{z})^{-\Delta} z^{-h} \bar{z}^{-\bar{h}} a_{h,\bar{h}} + z^h \bar{z}^{\bar{h}} a_{h,\bar{h}}^{\dagger} \right] \right) \right) a_{0,0}^{\dagger} |0\rangle \\
&= \left[\frac{1}{(z\bar{z})^{\Delta}} + \left(\sum_{h,\bar{h}=0}^{\infty} \left(\frac{1}{N_{h,\bar{h}}^{\mathcal{O}}} z^h \bar{z}^{\bar{h}} a_{h,\bar{h}}^{\dagger} a_{0,0}^{\dagger} \right) \right) \right] |0\rangle \\
&= \left[\left(\frac{1}{(z\bar{z})^{\Delta}} + \frac{1}{N_{0,0}^{\mathcal{O}}} \mathcal{O}^2(0) + \frac{\bar{z}}{N_{0,1}^{\mathcal{O}}} \mathcal{O} \partial_{\bar{z}} \mathcal{O}(0) + \frac{z}{N_{1,0}^{\mathcal{O}}} \mathcal{O} \partial_z \mathcal{O}(0) + \dots \right) \right] |0\rangle
\end{aligned} \tag{6.62}$$

so we can explicitly see various operators appearing. Note also the appearance of the identity operator $\equiv 1$, from the Wick contraction.

In fact, the OPE has a much more precise structure determined by conformal symmetry. The point is that once we write down the OPE as a sum over all of the various operators that can appear on the RHS, including both primaries and descendants, we can then act with conformal symmetry generators on both sides to find relations among the coefficients. In fact, the only free coefficients are those in front of the primary operators! We will return to explain this fact in detail when we discuss general CFTs.

6.5 Conformal Symmetry of Correlators in GFT

Recall that in d Euclidean dimensions, a free scalar boson has the 2-pt function

$$\langle \phi(x) \phi(0) \rangle = \frac{1}{x^{d-2}} \tag{6.63}$$

All of the other correlators are just obtained from this one by Wick contraction. Another way to discover generalized free theories is to define (really just make up!) a theory that generalizes this in a simple and conformally invariant way. It will be a theory whose entire Hilbert space is generated by the action of a local operator $\mathcal{O}(x)$ that has purely ‘Gaussian’ correlators – in other words, all correlation functions of $\mathcal{O}(x)$ will be determined by its 2-pt function via Wick contraction. To be specific, the 2-pt function has the form

$$\langle \mathcal{O}(x) \mathcal{O}(0) \rangle = \frac{1}{x^{2\Delta}} \tag{6.64}$$

where $\Delta \geq \frac{d}{2} - 1$ is a number that we are free to specify. The 3-pt correlator of $\mathcal{O}$ vanishes, as do all correlators with an odd number of $\mathcal{O}$ s. The 4-pt correlator is

$$\langle \mathcal{O}(x_1) \mathcal{O}(x_2) \mathcal{O}(x_3) \mathcal{O}(x_4) \rangle = \frac{1}{x_{12}^{2\Delta}} \frac{1}{x_{34}^{2\Delta}} + \frac{1}{x_{13}^{2\Delta}} \frac{1}{x_{24}^{2\Delta}} + \frac{1}{x_{14}^{2\Delta}} \frac{1}{x_{23}^{2\Delta}} \tag{6.65}$$

Working out any n -pt correlator is just an exercise in combinatorics. These correlators show why this is a ‘generalized free theory’ – in a free theory we would have $\Delta = \frac{d}{2} - 1$, but here we are letting the exponent be a general parameter.

So how do we see the conformal symmetry of these correlators? Obviously these correlators are invariant under translations and rotations. To study the other transformations, we can use the conformal algebra directly, recalling that $[D, \mathcal{O}] = (\Delta + r\partial_r)\mathcal{O}$ so that we have

$$0 = \langle D\mathcal{O}(r_1)\mathcal{O}(r_2) \rangle \quad (6.66)$$

$$= \langle [D, \mathcal{O}(r_1)]\mathcal{O}(r_2) \rangle + \langle \mathcal{O}(r_1)[D, \mathcal{O}(r_2)] \rangle \quad (6.67)$$

This means that the 2-pt correlator must obey

$$(r_1 \cdot \partial_{r_1} + r_2 \cdot \partial_{r_2}) \langle \mathcal{O}(\vec{r}_1)\mathcal{O}(\vec{r}_2) \rangle = -2\Delta \langle \mathcal{O}(\vec{r}_1)\mathcal{O}(\vec{r}_2) \rangle \quad (6.68)$$

which is exactly what we find. Special conformal symmetries can be checked in the same way. In the next section we will use the projective null cone coordinates to make conformal invariance manifest.

6.6 Connection to Infinite N

Many physicists would refer to a Generalized Free Theory as ‘a CFT at infinite N ’. There are a whole host of examples. For instance, we might have

- A large N gauge theory with gauge field A_{ab}^μ , as well as matter in the adjoint such as ψ_{ab} and Φ_{ab} fields, or vector-like matter ψ_a , perhaps with flavor quantum numbers. The classic AdS/CFT example is the maximally supersymmetric $\mathcal{N} = 4$ SYM Theory.
- A vector-like theory such as the $O(N)$ model, with scalar fields ϕ_a and an $O(N)$ symmetry.
- Any other theory with a large N Lie Group symmetry forcing the observables to take a symmetric form, such as $\phi_a\phi^a$ or $\text{Tr}[(F_{\mu\nu}^{ab})^k]$.

Let us understand why large N relates to AdS/CFT. Consider a free theory of an $SO(N)$ matrix valued field ϕ_{ab} where these indices a, b run from $a = 1, 2, \dots, N$. The action will simply be

$$\begin{aligned} S &= \int d^d x \frac{1}{2} \sum_{a,b} \left(\partial_\mu \phi_{ab} \partial^\mu \phi_{ab} \right) \\ &= \int d^d x \frac{1}{2} \text{Tr} [\partial_\mu \phi \partial^\mu \phi] \end{aligned} \quad (6.69)$$

Thus the ϕ_{ab} fields have extremely simple correlation functions, namely

$$\langle \phi_{ab}(x) \phi_{cd}(y) \rangle = \frac{\delta_{ac}\delta_{bd} + \delta_{ad}\delta_{bc}}{(x-y)^{d-2}} \quad (6.70)$$

or it can be written as

$$\langle \phi_A(x) \phi_B(y) \rangle = \frac{\delta_{AB}}{(x-y)^{d-2}} \quad (6.71)$$

where A indexes the generators T_{ab}^A of $SO(N)$, since ϕ is in the adjoint representation. None of these details matter much for our purposes here. All of the n -pt correlators of ϕ are just built on these via Wick contraction, since this is just a free theory with N^2 free fields.

The interesting point is that now we can consider more general operators made out of ϕ . For example, consider

$$\mathcal{O}_2(x) = \frac{1}{N\sqrt{2}} \text{Tr} [\phi^2(x)] = \frac{1}{N\sqrt{2}} \sum_{ab} \phi_{ab}(x) \phi^{ba}(x) \quad (6.72)$$

What is the 2-pt correlator of $\mathcal{O}_2$? It is easy to compute

$$\langle \mathcal{O}(x) \mathcal{O}(y) \rangle = \frac{1}{(x-y)^{2(d-2)}} \quad (6.73)$$

This is the form of a generalized free field with dimension $\Delta = (d-2)$. If we were to consider $\mathcal{O}_k = \text{Tr} [\phi^k]$ then we could get any $\Delta = k(d-2)/2$.

But now let's consider a 4-pt correlator

$$\begin{aligned} \langle \mathcal{O}_2(x_1) \mathcal{O}_2(x_2) \mathcal{O}_2(x_3) \mathcal{O}_2(x_4) \rangle = &= \frac{1}{x_{12}^{2\Delta}} \frac{1}{x_{34}^{2\Delta}} + \frac{1}{x_{13}^{2\Delta}} \frac{1}{x_{24}^{2\Delta}} + \frac{1}{x_{14}^{2\Delta}} \frac{1}{x_{23}^{2\Delta}} \\ &+ \frac{1}{N} (\dots) \end{aligned} \quad (6.74)$$

In the limit that $N \rightarrow \infty$, the 4-pt correlator of $\mathcal{O}_2$, and in fact of all properly normalized $\mathcal{O}_k$, will be given entirely by evaluating pairs of 2-pt functions. Thus in the large N limit, this theory reproduces what we wanted – a generalized free theory with many choices for Δ , the dimension of the operator. It's that last point that differentiates perturbation theory in $1/N$ from the more familiar examples of perturbation theory in a small coupling. For further reading, Witten's 'Baryons and the Large N Expansion' [11] is a great intro to the $1/N$ expansion.

An interesting point to note here is that the quantum mechanical expansion of ϕ_{ab} is in terms of free field oscillator modes, which are not much like the AdS oscillator modes of a generalized free field. So the relationship between an operator like $\text{tr}[\phi^k]$ and $\mathcal{O}$ isn't straightforward quantum mechanically.

7 General CFT Axioms from an AdS Viewpoint

Now we will discuss general CFTs from a bottom-up, or 'axiomatic' point of view. The reader should compare and contrast the properties of general CFTs with those of the generalized free theories analyzed in the previous section. Here is a list of the main ideas:

- Conformal transformations consist of the Poincaré group of transformations, plus scale transformations and special conformal transformations. The extra symmetries can be most quickly derived by demanding Poincaré invariance plus an additional symmetry under inversions about a point, where $x^i \rightarrow x^i/x^2$. They are best understood as coordinate transformations that do not leave the metric invariant, but rescale it by an overall spacetime-dependent factor, ie the symmetries are conformal killing vectors.

- CFTs can be quantized in the standard way, along flat spacelike surfaces that evolve with time. However, an alternative *radial quantization* in Euclidean space is often more useful, and will play a large role in what follows; in this quantization we start at a point and evolve outwards on expanding spheres. This idea is pictured in figure 7.
- Radial quantization associates the entire Hilbert space of CFT states with an arbitrarily small region about a point. Associating states on arbitrarily small circles with operators at the center of these circles leads to the *operator-state correspondence*. This can be made very explicit when there is a path integral description of the CFT, but the existence of a path integral is not necessary.
- Using conformal symmetry we can classify states according to irreducible representations of the conformal group, $SO(1, d + 1)$ in Euclidean space, and therefore describe every state as a linear combination of primaries and their descendants; the latter are obtained from primaries by acting with the momentum generators.
- The conformal symmetry allows us to move these operators around, so that from $\mathcal{O}(0)$ at the origin we obtain $\mathcal{O}(x)$ at any point x . The correlators of any 2 or 3 CFT operators are entirely fixed by symmetry, up to a finite set of constants.
- As usual in a quantum mechanical theory, we can multiply any two operators to obtain some other (new) operator. So what about operators at different points in space? The operator-state correspondence applied to a product $\mathcal{O}_1(x_1)\mathcal{O}_2(x_2)$ leads us to the *Operator Product Expansion* (OPE), which has a finite radius of convergence in any CFT. This can be made very explicit when there is a path integral description of the theory (see e.g. Weinberg Volume 2), but the existence of a path integral is not necessary. Operator products and OPEs in general CFTs are more subtle than in the generalized free theories we studied in the previous section.
- Local conserved currents are special and extremely important. Spin-1 currents J_μ satisfying $\partial^\mu J_\mu = 0$ generate global symmetries, while the spin-2 conserved current $T_{\mu\nu}$ is even more special. Conventionally a theory is only defined to be a CFT if a conserved $T_{\mu\nu}$ exists; if there is no conserved spin 2 current the lore holds that the theory is ‘non-local’¹⁶ as we will discuss.

7.1 What is a C(F)T?

Although later on we will take a more abstract point of view, let’s review the standard approach to CFTs. Conventionally, conformal isometries are defined as a change of coordinates or diffeomorphism $x_\mu \rightarrow x'_\mu(x)$ such that the metric

$$dx^2 \rightarrow dx'^2 = \Omega^2(x)dx^2 \quad (7.1)$$

where $\Omega(x)$ is an arbitrary function of the coordinates. This specification preserves angles, but not distances. This means that in the Lorentzian case, it always preserves the causal structure of the spacetime.

¹⁶in quotes because this phrase is vague... different physicists use it to mean very different things.

Thinking infinitesimally, we have $x'_\mu = x_\mu + v_\mu(x)$, where we treat v_μ as small. The new metric is

$$dx'_\alpha dx'^\alpha = dx^\mu dx^\nu (\eta_{\mu\nu} + \partial_\mu v_\nu + \partial_\nu v_\mu + \dots) \quad (7.2)$$

and so we must have $\partial_\mu v_\nu + \partial_\nu v_\mu = \omega(x)\eta_{\mu\nu}$, where $\Omega = 1 + \omega/2$. This immediately gives the equation

$$\partial_\mu v_\nu + \partial_\nu v_\mu - \frac{2}{d}\partial_\alpha v^\alpha \eta_{\mu\nu} = 0 \quad (7.3)$$

A well-known fact is that in 2-dimensions there are infinitely many solutions, whereas in $d > 2$ dimensions there are only a finite number of solutions, the transformations that form the conformal group $SO(2, d)$ (in Lorentzian signature).

Note that there is a significant difference between a Weyl transformation $g_{\mu\nu}(x) \rightarrow W(x)g_{\mu\nu}(x)$, where we are changing the physical metric, and a conformal isometry, which is a change of coordinates (diffeomorphism) $x_\mu \rightarrow x'_\mu(x)$ such that in the new coordinates the metric takes the form of the old metric multiplied by some function $\Omega^2(x)$.

To make this all clearer, let's consider why a free massless scalar field theory is invariant under a conformal isometry. We have $\phi(x) \rightarrow \phi(x + v) \approx \phi(x) + v_\mu(x)\partial^\mu\phi(x)$. Then the free Lagrangian transforms as

$$\begin{aligned} \delta L &= \partial_\mu\phi(x)\partial^\mu(v_\alpha(x)\partial^\alpha\phi(x)) \\ &= \partial_\mu v_\alpha \left(\partial^\mu\phi\partial^\alpha\phi - \frac{1}{2}\eta_{\mu\alpha}(\partial\phi)^2 \right) \left(= \partial_\mu v_\alpha T^{\mu\alpha} \right) \end{aligned} \quad (7.4)$$

When we take $v_\mu = \lambda x_\mu$, corresponding to a dilatation, or the more complicated $v_\mu = b_\mu x^2 - 2x_\mu b \cdot x$ for a special conformal transformation, we find that $\delta L \propto T^\mu_\mu$, the trace of the energy momentum tensor, as expected.

You might notice that $T_{\mu\nu}$ for the scalar is only traceless in $d = 2$ spacetime dimensions; this is because we have automatically (and in general incorrectly) assigned ϕ a scaling dimension of 0 under dilatations. This can be fixed by adding a local improvement term to $T_{\mu\nu}$, namely some multiple of the conserved total derivative $(\partial_\mu\partial_\nu - \eta_{\mu\nu}\partial^2)\phi^2$. The point is that the scale and special conformal transformations can involve a combination of $T_{\mu\nu}$ and other local conserved currents; see [12] for more details. For example, for the case of dilatations as long as we can define

$$S_\mu = x^\nu T_{\mu\nu} - V_\mu \quad (7.5)$$

with $T^\mu_\mu = \partial^\mu V_\mu$ then the current S_μ is a scale current, which generates dilatations as we desire. In the case of a free theory we have $V_\mu \propto \partial_\mu\phi^2$.

This can be understood more directly. When we computed $T_{\mu\nu}$ above we assumed that $\phi(x) \rightarrow \phi(x + v) \approx \phi(x) + v_\mu(x)\partial^\mu\phi(x)$, but this is not correct for e.g. dilatations for any choice of v_μ . We know that ϕ will be a primary operator of dimension $\Delta = \frac{d}{2} - 1$, and so it should transform as $\delta\phi(x) \rightarrow (\Delta + x \cdot \partial_x)\phi(x)$. But taking $v_\mu = \epsilon x_\mu$ does not lead to this transformation rule. To obtain a scale current we therefore want to take $\delta\phi \rightarrow \epsilon(x)(\Delta + x \cdot \partial_x)\phi$. This leads to a scale current

$$S_\mu = x^\nu \left(\partial^\mu\phi\partial^\alpha\phi - \frac{1}{2}\eta_{\mu\alpha}(\partial\phi)^2 \right) \left(- \Delta\phi\partial_\mu\phi \right) \quad (7.6)$$

which accords with the form we found above by the inclusion of an explicit improvement term into $T_{\mu\nu}$. The effect of the improvement term is simply to shift the scaling dimension of ϕ without altering the conservation or symmetry (as a matrix) of the stress-energy tensor.

It's worth noting that in the theories we are used to studying, e.g. $\lambda\phi^4$ theory, gauge theories, etc., the fact that conformal symmetry is broken quantum mechanically means that $T_\mu^\mu \propto \beta(\lambda)$, where β is the usual beta function for the running coupling. This means that these classically conformal theories have an anomaly.

The preceding remarks were an example of how CFTs are usually introduced. We will take a rather different and more abstract point of view that follows more naturally from thinking about AdS theories. Conformal Theories (CTs) are defined as quantum mechanical theories where

- All states fall into representations of the d -dimensional conformal algebra:

$$\begin{aligned} [M_{\mu\nu}, P_\rho] &= i(\eta_{\mu\rho}P_\nu - \eta_{\nu\rho}P_\mu), & [M_{\mu\nu}, K_\rho] &= i(\eta_{\mu\rho}K_\nu - \eta_{\nu\rho}K_\mu), & [M_{\mu\nu}, D] &= 0 \\ [P_\mu, K_\nu] &= -2(\eta_{\mu\nu}D + iM_{\mu\nu}), & [D, P_\mu] &= P_\mu, & [D, K_\mu] &= -K_\mu \end{aligned} \quad (7.7)$$

- There are local operators $\mathcal{O}_i(x)$ called primaries that transform covariantly. We already saw what this means for scalar operators in Generalized Free Theories in the previous section. As we will see again in the next section, there is a one-to-one correspondence between all the operators at any point x and all of the states in the Hilbert space.
- We can multiply operators $\mathcal{O}_1(x)\mathcal{O}_2(y)$. If the CFT is in its vacuum state throughout the space intervening between these operators, then the product can be re-written as a convergent sum over all the operators in the theory at any point between x and y . This is called the Operator Product Expansion (OPE), to be derived in general below. The coefficients of all descendant are determined by conformal symmetry and the coefficients of the primaries.

From these properties many others follow, for example that the correlators must take a very constrained form, and for the theory to be unitary all operators have to have scaling dimensions (eigenvalues of the dilatation operator D) greater than the unitarity bound, as derived in the previous section.

For a theory to be a Conformal *Field* Theory (CFT):

- The theory must contain in the spectrum of operators a spin 2 tensor $T_{\mu\nu}$ with dimension $\Delta = d$, the spacetime dimension, or equivalently, this stress tensor must be conserved. This stress tensor certainly was *not* present in the Generalized Free Theory we studied in the last section, except when $\Delta = \frac{d}{2} - 1$, in which case the GFT reduces to an ordinary free scalar theory.

We will discuss below why $T_{\mu\nu}$ is so uniquely important, and how its special properties relate to the universality of gravity. For now it should at least be apparent from our discussion above that $T_{\mu\nu}$ is related to implementing the conformal symmetries in a way that is manifestly local in spacetime ('local' because the symmetries are expressed in terms of $T_{\mu\nu}(x)$, which is localized *at the point* x).

7.2 Radial Quantization and Local Operators – A General Story

Let us start with some general comments on the Hilbert space construction in QFT.¹⁷ This procedure is linked to the choice of foliation of spacetime with fixed time surfaces. Time evolution connects states on one surface to states on the other surfaces.

Each leaf of the foliation becomes endowed with its own Hilbert space. We create in states ψ_{in} by inserting operators or directly specifying a wavefunction in the past of a given surface. Analogously we deal with out states by inserting operators or directly specifying the wavefunction in the future. The overlap of an in and out states living on the same surface $\langle \psi_{out} | \psi_{in} \rangle$ is equal to the correlation function of operators which create these in and out states (or to the S-matrix element if the in and out states consist of well-separated particles).

In standard textbook discussions of QFT, states are defined on flat spacelike surfaces in Minkowski spacetime. The time evolution operator is ‘ P_0 ’, the momentum generator in the time direction. Since the momenta all commute, states can be classified according to their total momentum and energy. To summarize

- States are defined on flat spacelike surfaces in Minkowski spacetime
- Since the momenta all commute

$$[P_\mu, P_\nu] = 0 \quad (7.8)$$

states can be classified according to their total momentum and energy.

- The time evolution operator is ‘ P_0 ’, the momentum generator in the time direction. Via Lorentz transformations we can consider other compatible (boosted) notions of P_0 .

In CFTs we additionally have the symmetries D = dilatations and K_μ = special conformal transformations. Could still use standard picture, but more natural to think about states in a new way – Radial Quantization. CFT algebra:

$$\begin{aligned} [M_{\mu\nu}, P_\rho] &= i(\eta_{\mu\rho}P_\nu - \eta_{\nu\rho}P_\mu), & [M_{\mu\nu}, K_\rho] &= i(\eta_{\mu\rho}K_\nu - \eta_{\nu\rho}K_\mu), & [M_{\mu\nu}, D] &= 0 \\ [P_\mu, K_\nu] &= -2(\eta_{\mu\nu}D + iM_{\mu\nu}), & [D, P_\mu] &= P_\mu, & [D, K_\mu] &= -K_\mu \end{aligned} \quad (7.9)$$

Some implications

- Simultaneously diagonalize D and $M_{\mu\nu}$ = angular momentum. Label states by Δ or $\tau = \Delta - \ell$ and angular momentum ℓ , so have

$$D|\tau, \ell J\rangle = (\tau + \ell)|\tau, \ell J\rangle \quad (7.10)$$

- P_μ is raising, K_μ lowering operator wrt D .
- Define primary as state killed by K_μ . Get discrete spectrum.

¹⁷One can find a very similar discussion in Slava’s notes.

- Radial quantization with Hamiltonian is D , unitary evolution via $e^{iD\tau}$ with $\tau = \log r$.

The vacuum $|0\rangle$ is annihilated by all the conformal generators, since we assume conformal symmetry is unbroken. Associated to each *primary state* is a *primary operator* $\mathcal{O}(0)$ with twist τ and angular momentum ℓ , so that acting on the vacuum

$$\mathcal{O}(0)|0\rangle = |\tau, \ell\rangle \quad (7.11)$$

This is the local operator/state correspondence (isomorphism or identification).

Note that radial quantization and the operator/state correspondence can be treated in a natural way in the path integral formalism. In that case we define states by wave functionals on Cauchy surfaces, and the path integral allows us to evolve from one Cauchy surface to another. Radial quantization is then immediate. The operator/state correspondence follows if we evolve back in time towards a point. Then the wave functional at the origin is equivalent to some insertion of a functional of the fields directly into the path integral, and this insertion is itself the desired operator.

Let us now see how the conformal transformation properties of $\mathcal{O}(0)$ completely determine the transformation properties of $\mathcal{O}(x)$, and therefore define the operator everywhere. This is interesting because it means that once we know about the primary state $\mathcal{O}(0)|0\rangle$ we can derive everything we would like to know about the corresponding local operator at any point in spacetime. The conformal generators act on $\mathcal{O}(0)$ according to

$$[K_\mu, \mathcal{O}(0)] = 0 \quad (7.12)$$

$$[D, \mathcal{O}(0)] = -i\Delta\mathcal{O}(0) \quad (7.13)$$

$$[M_{\mu\nu}, \mathcal{O}(0)] = -i\Sigma_{\mu\nu}^\ell\mathcal{O}(0) \quad (7.14)$$

Along with the assumption that

$$[P_\mu, \mathcal{O}(x)] = -i\partial_\mu\mathcal{O}(x) \quad (7.15)$$

this entirely determines the action of the conformal algebra on the local operator $\mathcal{O}(x)$, assuming the commutation relations of the conformal group. At a general point in flat Euclidean space, we have the commutators

$$[D, \mathcal{O}(x)] = -i(\Delta + x^\mu\partial_\mu)\mathcal{O}(x) \quad (7.16)$$

$$[P_\mu, \mathcal{O}(x)] = -i\partial_\mu\mathcal{O}(x) \quad (7.17)$$

$$[K_\mu, \mathcal{O}(x)] = -i(2x_\mu\Delta + 2x^\alpha\Sigma_{\alpha\mu} + 2x_\mu x^\alpha\partial_\alpha - x^2\partial_\mu)\mathcal{O}(x) \quad (7.18)$$

$$[M_{\mu\nu}, \mathcal{O}(x)] = -i(\Sigma_{\mu\nu} + x_\mu\partial_\nu - x_\nu\partial_\mu)\mathcal{O}(x) \quad (7.19)$$

where Δ is the scaling dimension of $\mathcal{O}$ and $\Sigma_{\mu\nu}$ is a finite dimensional spin matrix encoding the angular momentum representation of $\mathcal{O}$.

For fun let us derive the first infinitesimal approximation of the first equation above. We know from equation (7.15) that we can write

$$\begin{aligned} \mathcal{O}(\epsilon) &\approx \mathcal{O}(0) + \epsilon^\mu\partial_\mu\mathcal{O}(0) + \dots \\ &= \mathcal{O}(0) + i\epsilon^\mu[P_\mu, \mathcal{O}(0)] + \dots \end{aligned} \quad (7.20)$$

Now if we act the dilation operator on $\mathcal{O}(\epsilon)$ we find

$$\begin{aligned}
[D, \mathcal{O}(\epsilon)] &\approx [D, \mathcal{O}(0)] + i\epsilon^\mu [D, [P_\mu, \mathcal{O}(0)]] \\
&= -i\Delta \mathcal{O}(0) + i\epsilon^\mu ([D, P_\mu], \mathcal{O}(0)) - [P_\mu, [\mathcal{O}, D]] \\
&= -i\Delta \mathcal{O}(0) + i\epsilon^\mu ([-iP_\mu, \mathcal{O}(0)] - i\Delta [P_\mu, \mathcal{O}(0)]) \\
&= -i\Delta \mathcal{O}(0) + i\epsilon^\mu (-\partial_\mu \mathcal{O}(0) - \Delta \partial_\mu \mathcal{O}(0)) \\
&\approx -i(\Delta + \epsilon^\mu \partial_\mu) \mathcal{O}(\epsilon)
\end{aligned} \tag{7.21}$$

so we have derived the first of the equations (7.16) when $x = \epsilon$ is near to 0.

The AdS dual of radial quantization is pictured in figure 7 – it’s just ordinary quantization in AdS global coordinates. Specifically, ‘ordinary quantization’ in global coordinates means that we use the global time t to define our clocks, and so the dilatation operator D is the Hamiltonian (the generator of time translations). The Cauchy surfaces in AdS on which states are defined are just the constant t surfaces, parameterized by the coordinates ρ and Ω .

7.3 General CFT Correlators and the Projective Null Cone

We already saw some non-trivial examples of the way that conformal symmetry acts on and constrains CFT correlators. In general, all two-point correlators are completely determined up to a normalization, while 3-pt correlators are determined up to a finite number of coefficients (in the case of three scalar operators, there is only one overall coefficient). Higher point correlators can only depend on the conformal cross ratios.

We can study the conformal transformation properties in an elegant fashion by using the projective null cone coordinates P_A , which satisfy $P_A P^A = 0$ and are identified under $P_A \sim \lambda P_A$. Since we have been thinking about CFTs from the AdS point of view, it is natural to understand CFT correlators as expressed in terms of the P_A with reference to AdS.

Recall that we defined a CFT operator $\mathcal{O}$ in terms of an AdS field $\phi(X_A)$ as

$$\mathcal{O}(t, \Omega) = \lim_{\epsilon \rightarrow 0} \frac{\phi(t, \rho(\epsilon; t, \Omega), \Omega)}{\epsilon^\Delta} \tag{7.22}$$

where for e.g. scalars $m^2 = \Delta(\Delta - d)$ relates the AdS mass and CFT dimension, and $\rho(\epsilon; t, \Omega) = \frac{\pi}{2} - \epsilon f(t, \Omega)$ controls the way we approach the boundary of AdS. Furthermore, using our coordinate identifications we can write

$$\begin{aligned}
X_0 &= R \frac{\cosh t}{\cos \rho} \rightarrow \frac{\cosh t}{\epsilon f(t, \Omega)} = \frac{1}{\epsilon} P_0 \\
X_{d+1} &= R \frac{\sinh t}{\cos \rho} \rightarrow \frac{\sinh t}{\epsilon f(t, \Omega)} = \frac{1}{\epsilon} P_{d+1} \\
X_i &= R \tan \rho \Omega_i \rightarrow \frac{\Omega_i}{\epsilon f(t, \Omega)} = \frac{1}{\epsilon} P_i
\end{aligned} \tag{7.23}$$

The choice of function $f(t, \Omega)$ here represents a choice of a section of the null cone. Note that with our favorite $f = e^{-t}$ we obtain

$$(P_0, P_{d+1}, P_i) = \left(\frac{e^{2t} + 1}{2}, \frac{e^{2t} - 1}{2}, e^t \Omega_i \right) \left(\quad \right) \quad (7.24)$$

as usual for Euclidean flat space.

Now in general, instead of taking $\rho = \frac{\pi}{2} - \epsilon f(t, \Omega)$ we could have equivalently taken $\rho(\epsilon; t, \Omega) \rightarrow \frac{\pi}{2} - \frac{1}{\lambda} \epsilon f(t, \Omega)$ for a constant $\lambda > 0$. This has precisely the effect of sending $P_A \rightarrow \lambda P_A$ in our definition. But we also know from the definition of $\mathcal{O}$ that this would transform

$$\mathcal{O}(\lambda P_A) = \lambda^{-\Delta} \mathcal{O}(P_A) \quad (7.25)$$

for a conformal primary operator obtained from an AdS theory. Note that we could use a different λ for each operator in a correlator – we do not have to rescale them all in the same way, since we can use a different ϵ for each operator.

Furthermore, one can reverse the logic, and show that this transformation rule must obtain for any CFT primary expressed in terms of the null projective cone, regardless of whether it comes from AdS. It accords with all the results we have derived before by other means.

This puts powerful constraints on CFT correlation functions when we combine it with the fact that only $P_i \cdot P_j$ are conformally covariant objects. In particular, it enforces that

$$\langle \mathcal{O}_1(P_1) \mathcal{O}_2(P_2) \rangle = 0 \quad \text{if} \quad \Delta_1 \neq \Delta_2 \quad (7.26)$$

and that this correlator takes the conventional form otherwise. Furthermore, in the case of 3-pt correlators of scalars it implies

$$\langle \mathcal{O}_1(P_1) \mathcal{O}_2(P_2) \mathcal{O}_3(P_3) \rangle = \frac{C_{123}}{(P_1 \cdot P_2)^{\alpha_{12,3}} (P_2 \cdot P_3)^{\alpha_{23,1}} (P_1 \cdot P_3)^{\alpha_{13,2}}} \quad (7.27)$$

where we must have

$$\alpha_{ij,k} = \frac{\Delta_i + \Delta_j - \Delta_k}{2} \quad (7.28)$$

to achieve the correct scalings under $P_i \rightarrow \lambda_i P_i$ for each $i = 1, 2, 3$.

In general, it's much easier to work using the projective null cone coordinates. Conformally invariant correlators can only be a function of the conformal cross ratios

$$u_{ijkl} \equiv \frac{(P_i \cdot P_j)(P_k \cdot P_l)}{(P_i \cdot P_k)(P_j \cdot P_l)} \quad (7.29)$$

that are invariant under scalings of the individual P_i . In particular, when we study the 4-pt correlator, we only have two cross ratios, conventionally called u and v

$$u = \frac{(P_1 \cdot P_2)(P_3 \cdot P_4)}{(P_1 \cdot P_3)(P_2 \cdot P_4)} \quad \text{and} \quad v = \frac{(P_1 \cdot P_4)(P_2 \cdot P_3)}{(P_1 \cdot P_3)(P_2 \cdot P_4)} \quad (7.30)$$

Note that in terms of these variables, our generalized free theory correlator can be written as

$$\langle \mathcal{O}(x_1)\mathcal{O}(x_2)\mathcal{O}(x_3)\mathcal{O}(x_4) \rangle_{GFT} = \frac{1}{(P_1 \cdot P_3)^\Delta (P_2 \cdot P_4)^\Delta} (u^{-\Delta} + 1 + v^{-\Delta}) \left(\quad \right) \quad (7.31)$$

The factor out front just encodes the necessary kinematical data associated with the scaling dimension of $\mathcal{O}$, while the terms in parentheses contain all of the dynamical information about the theory. In a general theory this correlator would take the form

$$\langle \mathcal{O}(x_1)\mathcal{O}(x_2)\mathcal{O}(x_3)\mathcal{O}(x_4) \rangle = \frac{1}{(P_1 \cdot P_3)^\Delta (P_2 \cdot P_4)^\Delta} \mathcal{A}(u, v) \quad (7.32)$$

where the function $\mathcal{A}(u, v)$ cannot be determined from symmetry alone.

7.4 Deriving the OPE from the CFT, and from an AdS QFT

A key property of all local quantum field theories is the existence of an Operator Product Expansion. In CFTs this expansion converges in a finite region, and so it can be used to make various exact statements. Let us first understand^[18] what the OPE is and what it has to do with locality. Then we will show that any field theory in global AdS gives rise to CFT correlators that satisfy an OPE. The idea of the derivation is pictured in figure 9.

The OPE says that for any local operators ϕ_1 and ϕ_2 we have

$$\phi_1(x)\phi_2(0) = \sum_{\mathcal{O}} \left(\mathcal{C}(x, \partial) \mathcal{O}(0) \right) \quad (7.33)$$

This relation is derived via path integral methods in Weinberg V2 [13], although it can be easily derived without any explicit action for the CFT. Let us review it, and then we will show that any local QFT in AdS will give rise to CFT correlators that satisfy an OPE. As we will see, the derivation of the OPE in AdS is very similar to the derivation in the CFT.

To derive the OPE in the CFT, we consider two operators $\mathcal{O}_1(x_1)$ and $\mathcal{O}_2(x_2)$. Now draw a circle around x_1 and x_2 , centered at some point x – the circle should have a radius r larger than $|x - x_1|$ and $|x - x_2|$ so that it contains both points, but does not contain an insertion of any other operators. Choose to radially quantize the CFT about x . If we imagine the radial evolution from x outwards, we start with the vacuum and then we eventually hit the operators $\mathcal{O}_1$ at x_1 and $\mathcal{O}_2(x_2)$. Thus on the circle of radius r , we have some state

$$|\psi_{12}(r)\rangle = \mathcal{O}_1(x_1)\mathcal{O}_2(x_2)|0\rangle \quad (7.34)$$

because the CFT was in its vacuum at x . This state will be some linear combination of all of the states in the Hilbert space. By the operator state correspondence, there exists some local operator $\mathcal{O}_{\psi_{12}}(x)$ such that

$$\mathcal{O}_{\psi_{12}}(y)|0\rangle = |\psi_{12}(r)\rangle = \mathcal{O}_1(x_1)\mathcal{O}_2(x_2)|0\rangle \quad (7.35)$$

¹⁸In Weinberg V.2 the OPE is derived from the path integral, and in Slava's notes its derived from radial quantization.

The operator $\mathcal{O}_{\psi_{12}}$ will certainly not have a definite dimension or spin – it’s an extremely abstract object. However, we can express $\mathcal{O}_{\psi_{12}}$ as a sum over all of the primary operators of the theory and their descendants, and categorized according to their dimension and spin. This gives the OPE

$$\mathcal{O}_1(x_1)\mathcal{O}_2(x_2) = \sum_{\Delta,\ell} \left(f_{\Delta,\ell} C_{\Delta,\ell}(x_1 - y, x_2 - y, \partial_y) \mathcal{O}_{\Delta,\ell}(y) \right) \quad (7.36)$$

Note that we are free to use translation symmetry to take $x_1 = y = 0$ for convenience, in which case the function on the RHS just becomes $f_{\Delta,\ell}(x_1, \partial)$. In fact, this function is entirely determined by conformal symmetry, so the only non-trivial information in the OPE is the value of the OPE coefficients $c_{\Delta,\ell}$, which are labeled entirely by the dimension and spin of the primary operators in the theory. However, there are still an infinite number of these OPE coefficients. If this is your first sight of the OPE, then it probably seems very abstract, and in fact there’s much more to say about and do with it, but before any of that let’s discuss how it’s derived from QFT in AdS.

In fact, the AdS derivation mostly copies from the CFT version, as pictured in figure [9](#). The key point is that the Hilbert space of the theory lives on complete spacelike Cauchy surfaces in AdS. As $t \rightarrow -\infty$ in global AdS coordinates, the theory is in the vacuum. If we insert some operators $\mathcal{O}_1(x_1)$ and $\mathcal{O}_2(x_2)$ at some finite times, then we can always study a time t in the future of x_1 and x_2 . On this time slice we will find a wavefunction $\psi_{12}(t)$, where the radius of the corresponding CFT circle is just $r = e^t$ in some units. But this state $\psi_{12}(t)$ could just as well have been created by setting it up at some point arbitrarily far into the past. In the limit that we setup ψ_{12} in the infinite past, we can just create it by acting with a local operator at the origin in the CFT. This local operator can be written as a sum over primary operators of definite dimension (AdS energy) and spin, and this expansion reproduces the OPE for the CFT.

Now let us look at the OPE in a more refined way, in order to relate the OPE coefficients of descendants to the coefficients of their associated primary operator. For example, let us consider some particular term associated with a primary operator $\mathcal{O}$

$$\mathcal{O}_1(x)\mathcal{O}_2(0)|0\rangle = \frac{c}{x^p} [\mathcal{O}_{\Delta,\ell}(0) + \dots] |0\rangle + \text{other primaries} \quad (7.37)$$

One might immediately guess that the power p is fixed the dimensions Δ_1 , Δ_2 , and Δ . We can verify this by acting on both sides with the dilatation operator (and ignoring other contributions), giving

$$\begin{aligned} D\mathcal{O}_1(x)\mathcal{O}_2(0)|0\rangle &= (\Delta_1 + x \cdot \partial_x) \mathcal{O}_1(x)\mathcal{O}_2(0)|0\rangle + \Delta_2 \mathcal{O}_1(x)\mathcal{O}_2(0)|0\rangle \\ &= (\Delta_1 + \Delta_2 - p) \frac{c}{x^p} [\mathcal{O}_{\Delta,\ell}(0) + \dots] |0\rangle \end{aligned} \quad (7.38)$$

However, we can also compute this directly as an action on $\mathcal{O}$, giving

$$D \frac{c}{x^p} [\mathcal{O}_{\Delta,\ell}(0) + \dots] |0\rangle = \frac{c}{x^p} [\Delta \mathcal{O}_{\Delta,\ell}(0) + \dots] |0\rangle \quad (7.39)$$

so we find that $p = \Delta_1 + \Delta_2 - \Delta$ for consistency. More interesting relations can be obtained by considering not just the primary $\mathcal{O}_{\Delta,\ell}$ but also the descendants that make up the ellipsis. The next term will be

$$\mathcal{O}_1(x)\mathcal{O}_2(0) = \frac{c}{x^p} [\mathcal{O}_{\Delta,\ell}(0) + \kappa x^\mu \partial_\mu \mathcal{O}_{\Delta,\ell}(0) + \dots] |0\rangle \quad (7.40)$$

and we can determine the coefficient c using conformal symmetry. Let us act on this equation with K_μ . Note that this SCT annihilates $\mathcal{O}_2(0)$ because its primary, so we find

$$\begin{aligned} K_\mu \mathcal{O}_1(x) \mathcal{O}_2(0) |0\rangle &= (2x_\mu \Delta_1 + 2x^\alpha \Sigma_{\alpha\mu} + 2x_\mu x^\alpha \partial_\alpha - x^2 \partial_\mu) \mathcal{O}_1(x) \mathcal{O}_2(0) |0\rangle \\ &= (2x_\mu \Delta_1 + 2x^\alpha \Sigma_{\alpha\mu} + 2x_\mu x^\alpha \partial_\alpha - x^2 \partial_\mu) \frac{c}{x^p} [\mathcal{O}_{\Delta,\ell}(0) + \kappa x^\mu \partial_\mu \mathcal{O}_{\Delta,\ell}(0) + \dots] |0\rangle \end{aligned} \quad (7.41)$$

This must match with the action of K_μ directly on the RHS, but this is

$$K_\mu \frac{c}{x^p} (\mathcal{O}_{\Delta,\ell}(0) + \kappa x^\alpha \partial_\alpha \mathcal{O}_{\Delta,\ell}(0) + \dots) |0\rangle = \frac{c}{x^p} (\kappa x^\alpha [K_\mu, [P_\alpha, \mathcal{O}_{\Delta,\ell}]](0) + \dots) |0\rangle \quad (7.42)$$

where we note that K_μ annihilates the primary operator $\mathcal{O}(0)$, and we have written the last part very explicitly in terms of the conformal generators to emphasize that it can be evaluated just using the algebra.

To match the two sides, note that in equation (7.41) the first two terms in K_μ actually cancel against the action of the derivative operators on x^{-p} , so that all terms proportional to $\mathcal{O}_{\Delta,\ell}(0)$ drop out. Then for $\ell = 0$ we are left with the first term as

$$2\kappa \Delta x^\mu \mathcal{O}_{\Delta,0} = (2\Delta_1 - p) x^\mu \mathcal{O}_{\Delta,0} \quad (7.43)$$

so we find that

$$\kappa = \frac{1}{2} + \frac{\Delta_1 - \Delta_2}{2\Delta} \quad (7.44)$$

for the first coefficient. In principle we can compute all subsequent coefficients this way, although it would be rather laborious.

Finally, now that we have demonstrated that the appearance of descendant operators in the OPE is entirely determined by the appearance of primaries, let us consider how to compute the primary OPE coefficients. Actually, this is easy, and also provides a way to compute $C(x, \partial)$. The method is to take the OPE expansion and compute its correlator with a single operator $\mathcal{O}_{\Delta,\ell}$, giving

$$\langle \mathcal{O}_{\Delta,\ell}(z) \mathcal{O}_1(x) \mathcal{O}_2(y) \rangle = \left\langle \mathcal{O}_{\Delta,\ell}(z) \sum_{\text{primary } \mathcal{O}} \left(\lambda_{\Delta,\ell}^{12} C(x-y, \partial_y) \mathcal{O}(y) \right) \right\rangle \quad (7.45)$$

where in our current notation $\lambda_{\Delta,\ell}^{12}$ is the OPE coefficient. Now note that CFT 2-pt functions always vanish unless the operators have the same dimension, charge, and spin, so on the RHS we have projected out the 2-pt function of a single primary operator $\mathcal{O}_{\Delta,\ell}$. On the LHS we just have a CFT 3-pt function, and these are fixed up to a finite number of constants (in the case of a scalar-scalar-spin ℓ correlator there is a unique answer kinematically, so we only have one overall constant). Considering the scalar case for simplicity, we find that

$$c_{\Delta,\ell=0}^{12} \frac{1}{(x-y)^{\alpha_{12\Delta}} (z-y)^{\alpha_{2\Delta 1}} (x-z)^{\alpha_{1\Delta 2}}} = \lambda_{\Delta,0}^{12} C(x-y, \partial_y) \frac{1}{(z-y)^{2\Delta}} \quad (7.46)$$

So we see that there is a very simple relationship between the coefficient $c_{\Delta,0}^{12}$ of the 3-pt function and the OPE coefficient $\lambda_{\Delta,0}^{12}$. Furthermore, we can also determine the function $C(x-y, \partial_y)$ by matching the kinematics on both sides of this equation.

7.5 The Conformal Partial Wave Expansion and the Bootstrap

We derived the OPE in the case of generalized free theory, and then we explained that in fact the OPE exists with a finite radius of convergence in any CFT. Crucially, we found that all coefficients for descendant operators in the OPE are determined kinematically in terms of the coefficients of primary operators. We also saw how one can derive the OPE directly by thinking about the AdS field theory description.

Let us now see what happens when we apply the OPE to a 4-pt CFT correlator. This is straightforward; we see that

$$\begin{aligned}
\langle \phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4) \rangle &= \sum_{\tau,\ell} \left(\lambda_{\tau,\ell} C_{\tau,\ell}(x_{12}, \partial_1) \langle \mathcal{O}_{\tau,\ell}(x_2)\phi(x_3)\phi(x_4) \rangle \right) \\
&= \sum_{\tau,\ell} \lambda_{\tau,\ell} C_{\tau,\ell}(x_{12}, \partial_2) \sum_{\tau',\ell'} \left(\lambda_{\tau',\ell'} C_{\tau',\ell'}(x_{34}, \partial_4) \langle \mathcal{O}_{\tau,\ell}(x_2)\mathcal{O}_{\tau',\ell'}(x_4) \rangle \right) \\
&= \sum_{\tau,\ell} \left(\lambda_{\tau,\ell}^2 [C_{\tau,\ell}(x_{12}, \partial_2) C_{\tau',\ell'}(x_{34}, \partial_4) \langle \mathcal{O}_{\tau,\ell}(x_2)\mathcal{O}_{\tau,\ell}(x_4) \rangle] \right) \\
&= \frac{1}{(x_1 - x_3)^{2\Delta}(x_2 - x_4)^{2\Delta}} \sum_{\tau,\ell} \lambda_{\tau,\ell}^2 g_{\tau,\ell}(u, v)
\end{aligned} \tag{7.47}$$

In the first line we applied the OPE in 12, while in the second we applied it in 34. In the third line we simplified by noting that

$$\langle \mathcal{O}_{\tau,\ell}(x_2)\mathcal{O}_{\tau',\ell'}(x_4) \rangle \propto \delta_{\tau\tau'}\delta_{\ell\ell'} \tag{7.48}$$

so this term is only present when $\tau = \tau'$ and $\ell = \ell'$. In the last line we noted that

$$\frac{1}{(x_1 - x_3)^{2\Delta}(x_2 - x_4)^{2\Delta}} g_{\tau,\ell}(u, v) \equiv (C_{\tau,\ell}(x_{12}, \partial_2) C_{\tau,\ell}(x_{34}, \partial_4) \langle \mathcal{O}_{\tau,\ell}(x_2)\mathcal{O}_{\tau,\ell}(x_4) \rangle) \tag{7.49}$$

is entirely determined by conformal symmetry – in other words, this *conformal partial wave* is a purely kinematic quantity. The conformal block coeffs are squares of OPE coeffs:

$$\lambda_{\tau,\ell}^2 > 0 \tag{7.50}$$

This is how *Unitarity* shows up.

Another equivalent way to think about it is via the insertion of complete set of states in the middle. Organize according to irreps of conformal symmetry. This is the same thing as using the OPE, as is obvious from the radial quantization derivation. Contribution of a single irrep is a *conformal partial wave* or *conformal block*:

$$\frac{g_{\tau,\ell}(u, v)}{(x_{13}^2 x_{24}^2)^{\Delta_\phi}} = \langle \phi(x_1)\phi(x_2) \sum_{\tau,\ell \text{ irrep}} |\alpha\rangle\langle\alpha| \rangle \left(\phi(x_3)\phi(x_4) \right) \tag{7.51}$$

This is an even easier way of explaining these conformal partial waves – it is just what we get when we insert **1** into the correlator as a sum over a particular set of states.

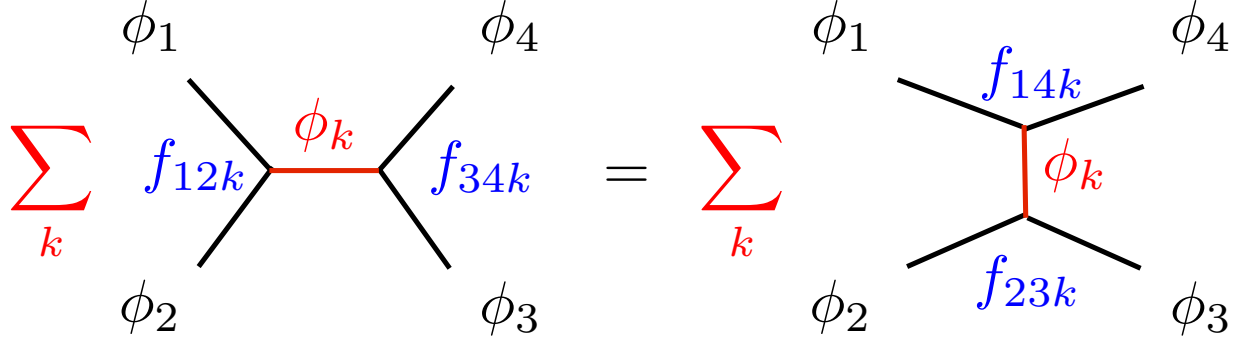


Figure 10: This figure illustrates the bootstrap equation relating the conformal partial wave expansion in two different channels. We see the explicit appearance of various OPE coefficients in the different channels.

The conformal partial wave expansion is in all respects analogous to the partial wave expansion of scattering amplitudes. The analogy connects angular momentum to angular momentum, while the twist τ or dimension $\Delta = \tau + \ell$ plays the role of the center of mass energy.

The conformal blocks are known (complicated) functions in many, but not all cases. For example, for CFTs in $d = 2$ dimensions the conformal partial waves are

$$g_{\tau,\ell}(z, \bar{z}) = k_{\tau+2\ell}(z)k_{\tau}(\bar{z}) + k_{\tau+2\ell}(\bar{z})k_{\tau}(z) \quad (7.52)$$

where $u = z\bar{z}$ and $v = (1-z)(1-\bar{z})$ and

$$k_{2\beta}(x) = x^{\beta} {}_2F_1(\beta, \beta, 2\beta; x) \quad (7.53)$$

where this is the hypergeometric function. The reason for the notation of z and $\bar{z}$ is that if we take the correlator

$$\langle \phi(0)\phi(z)\phi(1)\phi(\infty) \rangle \quad (7.54)$$

then we see that

$$u = \frac{x_{12}^2 x_{34}^2}{x_{13}^2 x_{24}^2} = \frac{(z\bar{z})\infty^2}{1^2 \infty^2} \rightarrow z\bar{z} \quad (7.55)$$

and we obtain a similarly simple result with v . So z and $\bar{z}$ are literally complex coordinates in a two dimensional plane.

Recall that in the previous section we noted various identities between CFT correlators when we exchanged $2 \leftrightarrow 3$ and $2 \leftrightarrow 4$. The bootstrap equation uses this *crossing symmetry* to equate the conformal partial wave expansions in different channels:

$$u^{-\Delta_{\phi}} + u^{-\Delta_{\phi}} \sum_{\tau,\ell} P_{\tau,\ell} g_{\tau,\ell}(u, v) = v^{-\Delta_{\phi}} + v^{-\Delta_{\phi}} \sum_{\tau,\ell} P_{\tau,\ell} g_{\tau,\ell}(v, u) \quad (7.56)$$

This is pictured in figure [10](#). Let us now see how the conformal partial wave decomposition works in our favorite example, that of a Generalized Free Theory. In the next section we will study what happens when we add AdS interactions.

The GFT the 4-pt correlator can be written as

$$\langle \mathcal{O}(x_1)\mathcal{O}(x_2)\mathcal{O}(x_3)\mathcal{O}(x_4) \rangle = \frac{1}{(x_{13}^2 x_{24}^2)^\Delta} \left(\frac{1}{(z\bar{z})^\Delta} + 1 + \frac{1}{(1-z)^\Delta (1-\bar{z})^\Delta} \right) \left(\quad \right) \quad (7.57)$$

We want to equate this with the conformal block decomposition, so in other words we are looking for

$$\frac{1}{(z\bar{z})^\Delta} \sum_{\tau,\ell} P_{\tau,\ell} g_{\tau,\ell}(z, \bar{z}) = 1 + \frac{1}{(1-z)^\Delta (1-\bar{z})^\Delta} \quad (7.58)$$

where we have already canceled off the contribution of the identity operator on both sides. Now let us expand the RHS at small z and $\bar{z}$; we find

$$1 + \frac{1}{(1-z)^\Delta (1-\bar{z})^\Delta} \approx 2 + \Delta(z + \bar{z}) + \frac{\Delta(\Delta+1)}{2}(z^2 + \bar{z}^2) + \Delta^2(z\bar{z}) + \dots \quad (7.59)$$

Notice that all of the powers that appear are integral. However, note that the terms on the LHS with the conformal partial waves behave as

$$\frac{1}{(z\bar{z})^\Delta} g_{\tau,\ell}(z, \bar{z}) = (z\bar{z})^{\frac{\tau}{2}-\Delta} (z^\ell + \bar{z}^\ell + \dots) \quad (7.60)$$

So matching both sides around $z, \bar{z} \sim 0$ immediately implies that we must always have $\tau = 2\Delta + 2n$ for a generalized free theory. We have just re-discovered the fact that the only operators that can appear in the OPE of $\mathcal{O}(x)\mathcal{O}(0)$ are

$$\mathcal{O}(x)\mathcal{O}(0) \sim 1 + a\mathcal{O}^2(0) + b\mathcal{O}\partial^2\mathcal{O} + \dots \quad (7.61)$$

so that they involve $\mathcal{O}\partial^{2n}\partial_{\mu_1}\dots\partial_{\mu_\ell}\mathcal{O}$ and have dimension $2\Delta + m$. By matching both sides of equation [\(7.60\)](#) we can compute the conformal partial wave coefficients (and therefore the OPE coefficients) for a generalized free theory. When we add AdS interactions in the next section we will see that the various operators get anomalous (shifted) dimensions proportional to the coupling constants. Note that shifting τ produces $(z\bar{z})^\gamma \approx 1 + \gamma \log(z\bar{z})$, so anomalous dimension appear as logarithms in CFT correlators.

8 Adding AdS Interactions

Let us compare the conformal partial wave expansion with various perturbative methods. The reader is likely already familiar with Feynman diagram perturbation theory in QFT, and perhaps also with ‘Old Fashioned Perturbation Theory’. Let us compare these descriptions of correlators to the Conformal Partial Wave (aka Conformal Block) decomposition and the bootstrap equation.

Feynman Diagram Perturbation Theory aka Witten Diagrams in an AdS context, provide a perturbative expansion that keeps the spacetime symmetries (conformal symmetry) manifest at every step of the calculation. However, Feynman diagrams obscure which physical states contribute to a given process.

Specifically, both single and multi-particle states are exchanged even in tree-level diagrams. This feature of perturbation theory is not very prominent in flat spacetime, where partial waves are continuous functions of the scattering energy, but it is manifest in AdS/CFT.

Old Fashioned Perturbation Theory is another perturbative method that's closely connected to conventional quantum mechanical perturbation theory. It requires an explicit choice of time slicing, with an accompanying specification of the Hilbert space of states on those constant time surfaces. As a consequence, OFPT breaks spacetime symmetries that explicitly involve time. However, OFPT has the advantage of being formulated directly in terms of physical states.

The Conformal Partial Wave Expansion aka the conformal block decomposition of a CFT correlation function – this is a non-perturbative expansion that keeps all spacetime symmetries manifest, and that only involves physical states. When combined with crossing symmetry, the conformal block decomposition leads to the bootstrap equation, which provides powerful constraints on correlators, but unlike the previous methods, it is not a systematic technique for solving the theory order-by-order in an expansion parameter.

However, given a correlator computed in perturbation theory, one can always decompose it into conformal partial waves. This leads to a clear specification of which states (operators) appear in various OPEs. Note that the identity operator, which is dual to ‘free propagation’ in AdS, has a very non-trivial conformal partial wave decomposition in other channels. This means that it's not always easy to isolate ‘free propagation’ from ‘interactions’ in the conformal block decomposition.

Now let us see how an interacting QFT in AdS spacetime give rise to an interacting C(F)T. Then we will see how abstract CFT data like OPE coefficients and anomalous dimensions can be computed in AdS/CFT, and then we will translate these ideas into the much more mundane-sounding physics of wavefunction overlaps and the binding energies between particles.

8.1 Anomalous Dimensions from Basic Quantum Mechanics

In the following sections we will discuss AdS Feynman diagrams and perform some computations in $\lambda\phi^4$ theory. For now we will discuss a simpler, old-fashioned perturbation theory approach [10] that basically amounts to an application of undergraduate quantum mechanics. This approach is actually easier and more efficient than Feynman diagrams in many cases.

Consider a situation where we have a free Hamiltonian $D^{(0)}$ that we have diagonalized, and we add a small interaction Hamiltonian D^I . Everyone knows that to first order in perturbation theory, the energy of a $D^{(0)}$ eigenstate $|\psi\rangle$ simply changes by

$$\gamma_\psi = \langle\psi|D^I|\psi\rangle \quad (8.1)$$

We can apply this formula directly in the case of $\lambda\phi^4$ theory in AdS in order to compute the energy shifts of 2-particle states in AdS. Recall that 2-particle states in AdS are created by the CFT operators $[\mathcal{OO}]_{n\ell}$ where $\mathcal{O} = \lim_{\rho \rightarrow \pi/2} \phi$ as usual. So computing the energy shifts of these states is equivalent to computing the ‘anomalous dimensions’ of these operators.

Let us consider $\lambda\phi^4$ theory in AdS_3 for simplicity. The (Lorentzian) action is

$$S[\phi] = \int_{\text{AdS}} d^3x \sqrt{-g} \left(\frac{1}{2} (\nabla\phi)^2 - \frac{m^2}{2} \phi^2 - \frac{\lambda}{4!} \phi^4 \right) \quad (8.2)$$

We derive the Hamiltonian as usual, by defining canonical momenta and computing $D = P\dot{q} - L$. This gives the Hamiltonian (Dilatation operator) for the free theory plus an interaction part, which is

$$D^I(t) = \frac{\lambda}{4!} \int_{\text{AdS}} d^2x \sqrt{-g} \phi^4(t, x) \quad (8.3)$$

This interaction vanishes except for the scalar or $\ell = 0$ partial wave, so $[\mathcal{OO}]_{n\ell}$ do not receive anomalous dimensions from $\lambda\phi^4$ for $\ell > 0$. This is directly analogous to the fact that $\lambda\phi^4$ interaction only contributes to the $\ell = 0$ partial wave when we study scattering in flat spacetime. Specializing to the case of $\ell = 0$ we want to compute the matrix element

$$\begin{aligned} \gamma(n) &= \langle n, 0 | D^I | n, 0 \rangle \\ &= \frac{\lambda}{4!} \int_{\text{AdS}} d^2x \sqrt{-g} \langle n, 0 | \phi^4(x) | n, 0 \rangle \end{aligned} \quad (8.4)$$

where

$$|n, \ell\rangle = [\mathcal{OO}]_{n\ell} |0\rangle \quad (8.5)$$

is the primary state corresponding to two particles in AdS. Now we can evaluate this matrix element directly. In order for it to be non-vanishing, two of the $\phi(x)$ operators must destroy the initial state particles while two destroy the final state particles, so we find

$$\gamma(n) = \frac{\lambda}{4} \int_0^{2\pi} d\theta \int_0^{\frac{\pi}{2}} d\rho \frac{\sin \rho}{\cos^3 \rho} \langle n, 0 | \phi^2(\rho, \theta) | 0 \rangle \langle 0 | \phi^2(\rho, \theta) | n, 0 \rangle \quad (8.6)$$

However, note that these states are scalar primaries, and $\phi^2(x)$ is a scalar operator, so by symmetry we must have

$$\langle 0 | \phi^2(\rho, \theta) | n, 0 \rangle = \frac{1}{\sqrt{2\pi}} (e^{it} \cos \rho)^{2\Delta+2n} \quad (8.7)$$

since these are primary wavefunctions that are annihilated by K_μ . Note that this matrix element would vanish for $\ell > 0$ – this is how one can see that $\lambda\phi^4$ does not produce anomalous dimensions at first order for $\ell > 0$ operators. The correct normalization can be computed [10], although the calculation is a bit non-trivial. I have simply written the correct result for AdS_3 . Performing the integrals gives

$$\gamma(n) = \frac{\lambda}{8\pi(2\Delta + 2n - 1)} \quad (8.8)$$

so we have computed the anomalous dimension of all $[\mathcal{OO}]_{n\ell}$ in $\lambda\phi^4$ theory in AdS. As we have emphasized before, anomalous dimensions are just AdS energy shifts due to interactions.

8.2 Perturbative Interactions and $\lambda\phi^4$ Theory

Interacting QFTs in AdS can be studied in exactly the same way that we study such theories in flat spacetime. In particular, we can expand perturbatively in the coupling constants and compute using AdS Feynman diagrams. To derive the Feynman rules we can use either an operator or a path integral formalism, as usual; the result are rules for propagators, which are just the 2-pt functions of AdS fields, and vertices, which follow as direct generalizations of those of flat spacetime. A significant technical difference is that in AdS spacetime we lack a set of commuting translation operators, and so we cannot use the Fourier transform to momentum space to simplify computations. Conceptually this may almost be an advantage, but computationally it means that AdS Feynman diagrams are much more cumbersome to calculate.

Let us consider a standard example, that of $\lambda\phi^4$ theory in AdS. We have an AdS path integral

$$Z = \int \mathcal{D}\phi e^{-S[\phi]} \quad (8.9)$$

where the Euclidean action is

$$S[\phi] = \int_{AdS} d^{d+1}x \sqrt{-g} \left(\frac{1}{2} (\nabla\phi)^2 + \frac{m^2}{2} \phi^2 + \frac{\lambda}{4!} \phi^4 \right) \quad (8.10)$$

The bulk-to-bulk propagators have a simple expression in terms of the geodesic distance $\sigma(X, Y)$ between X and Y :

$$G_\Delta(X, Y) = \mathcal{C}_\Delta z^{\Delta/2} {}_2F_1(\Delta, h, \Delta + 1 - h, z), \quad (8.11)$$

where $z = e^{-2\sigma}$.^[19] The factor $\mathcal{C}_\Delta = \Gamma(\Delta)/(2\pi^h \Gamma(\Delta + 1 - h))$ with $h = d/2$. This propagator can be obtained as usual as the solution to the Klein Gordon equation in Euclidean AdS with a delta function source, namely

$$(\nabla_X^2 + m^2) G_\Delta(X, Y) = (2\pi)^{d+1} \delta_{AdS}^{d+1}(X - Y) \quad (8.14)$$

It is also given as a sum over modes if we compute

$$\langle \phi(X) \phi(Y) \rangle = G_\Delta(X, Y) \quad (8.15)$$

using the expansion of $\phi(X)$ in terms of creation and annihilation operators with corresponding wavefunctions.

¹⁹This normalization of the bulk-to-bulk propagator differs from that in e.g. [10] by a factor of $\mathcal{C}_\Delta$. Other useful representations of the bulk-to-bulk propagator are

$$G_\Delta(X, Y) = \mathcal{C}_\Delta \frac{y^{\Delta/2}}{2^\Delta} {}_2F_1\left(\frac{\Delta}{2}, \frac{\Delta}{2} + \frac{1}{2}, \Delta + 1 - h, y\right) \quad (8.12)$$

$$= \frac{\mathcal{C}_\Delta}{u^\Delta} {}_2F_1\left(\Delta, \Delta - h + \frac{1}{2}, 2\Delta - 2h + 1, -\frac{4}{u}\right), \quad (8.13)$$

where $u = (X - Y)^2$ and $y^{-\frac{1}{2}} = \cosh \frac{\sigma}{R} = 1 + \frac{u}{2R^2}$.

The Feynman rules for the theory are identical to the position space Feynman rules for $\lambda\phi^4$ theory in flat spacetime, except that we replace flat space propagators with $G_\Delta(X, Y)$, and we integrate over AdS. For example, the 4-pt correlator of ϕ to first order in λ is

$$\langle \phi(X_1)\phi(X_2)\phi(X_3)\phi(X_4) \rangle = \lambda \iint_{\text{AdS}} d^{d+1}Y \sqrt{-g} G_\Delta(X_1, Y) G_\Delta(X_2, Y) G_\Delta(X_3, Y) G_\Delta(X_4, Y) \quad (8.16)$$

If we had a $g\phi^3$ theory instead we would have

$$\begin{aligned} \langle \phi(X_1)\phi(X_2)\phi(X_3)\phi(X_4) \rangle &= g^2 \iint_{\text{AdS}} d^{d+1}Y_1 \sqrt{-g(Y_1)} G_\Delta(X_1, Y_1) G_\Delta(X_2, Y_1) \\ &\times \int d^{d+1}Y_2 \sqrt{-g(Y_2)} G_\Delta(Y_1, Y_2) G_\Delta(X_3, Y_2) G_\Delta(X_4, Y_2) \end{aligned} \quad (8.17)$$

Loops can be calculated in the same way, with an AdS position-space integral for every vertex, and a propagator for each line.

Now that we know how to compute correlators of fields in AdS, we can apply our usual dictionary to compute the correlation functions of CFT operators. We need only apply our formula

$$\mathcal{O}(t, \Omega) = \lim_{\epsilon \rightarrow 0} \frac{\phi\left(t, \frac{\pi}{2} - \epsilon f(t, \Omega), \Omega\right)}{\epsilon^\Delta} \quad (8.18)$$

for each of the AdS fields in an AdS correlator. Applying this formula to any of the examples above, we immediately notice the appearance of the function

$$G_{B\partial}(t, \Omega; Y) = \lim_{\epsilon \rightarrow 0} \frac{1}{\epsilon^\Delta} G_\Delta\left(t, \frac{\pi}{2} - \epsilon f(t, \Omega), \Omega; Y\right) \quad (8.19)$$

which we refer to as the *bulk-boundary propagator*. For example, to compute the CFT 4-pt correlator at tree level in $\lambda\phi^4$ theory in AdS, we use the formula

$$\langle \mathcal{O}(\vec{r}_1)\mathcal{O}(\vec{r}_2)\mathcal{O}(\vec{r}_3)\mathcal{O}(\vec{r}_4) \rangle = \lambda \iint_{\text{AdS}} d^{d+1}Y \sqrt{-g} G_{B\partial}(\vec{r}_1, Y) G_{B\partial}(\vec{r}_2, Y) G_{B\partial}(\vec{r}_3, Y) G_{B\partial}(\vec{r}_4, Y) \quad (8.20)$$

This is an example of how to use AdS/CFT to compute non-trivial CFT correlation functions.

Fortunately, the bulk-boundary propagator $G_{B\partial}$ is much simpler than the bulk-bulk propagator G_Δ . In terms of the projective null cone coordinates P , we simply have

$$G(P, X) = \lim_{\epsilon \rightarrow 0} \frac{\mathcal{C}_\Delta e^{-\sigma(X, Y(\epsilon))} {}_2F_1(\Delta, h, \Delta + 1 - h, e^{-2\sigma(X, Y(\epsilon))})}{\epsilon^\Delta}, \quad (8.21)$$

where in this limit $Y \rightarrow \frac{1}{\epsilon}P$ becomes the boundary point. In this limit the geodesic distance goes to infinity, and we simply obtain

$$G_{B\partial}(P, X) = \frac{\mathcal{C}_\Delta}{(2P \cdot X)^\Delta} = \frac{1}{\Gamma(\Delta - h + 1)} \int_0^\infty \frac{dt}{t} t^\Delta e^{-2tP \cdot X} \quad (8.22)$$

where as usual

$$\mathcal{C}_\Delta = \frac{\Gamma(\Delta)}{2\pi^h \Gamma(\Delta - h + 1)} \quad (8.23)$$

The last formula in terms of a t integral is useful for computations.

This formula for $G_{B\partial}$ was essentially forced on us by conformal symmetry – the bulk boundary propagator could only depend on the conformal invariant $P \cdot X$, and the scaling relation $\mathcal{O}(\lambda P) = \lambda^{-\Delta} \mathcal{O}(P)$ forces the power-law relation that we derived. In this normalization the 2-pt function is

$$\langle \mathcal{O}(P_1) \mathcal{O}(P_2) \rangle = \frac{\mathcal{C}_\Delta}{(2P_1 \cdot P_2)^\Delta} \quad (8.24)$$

for an operator of dimension Δ . This could be derived by taking the point $X \rightarrow P_2$ on the boundary of AdS.

Let us now try to compute the 4-pt correlator to first order in $\lambda\phi^4$ theory. This is

$$\langle \mathcal{O}(P_1) \mathcal{O}(P_2) \mathcal{O}(P_3) \mathcal{O}(P_4) \rangle = \frac{\lambda}{[2\pi^h \Gamma(\Delta - h + 1)]^4} \int_{AdS} d^{d+1}Y \prod_{i=1}^4 \left(\int_0^\infty \frac{dt_i}{t_i} t_i^\Delta e^{-2t_i P_i \cdot Y} \right) \quad (8.25)$$

Now we can exchange the orders of integration to write

$$\langle \mathcal{O}(P_1) \mathcal{O}(P_2) \mathcal{O}(P_3) \mathcal{O}(P_4) \rangle = \frac{\lambda}{[2\pi^h \Gamma(\Delta - h + 1)]^4} \prod_{i=1}^4 \left(\int_0^\infty \frac{dt_i}{t_i} t_i^\Delta \int_{AdS} d^{d+1}Y e^{-2(\sum_i t_i P_i) \cdot Y} \right) \quad (8.26)$$

We can do the integral over AdS most easily using the Euclidean Poincaré patch coordinates. We can use conformal symmetry to rotate so that the only non-vanishing component of Q is Q_0 , and we have $Y_0 = \frac{1}{2} \left(\frac{z^2 + \vec{r}^2 + R^2}{z} \right)$, (so that in this frame $2Q \cdot Y = |Q|(1 + z^2 + r^2)/z$, leading to

$$\begin{aligned} \int_{AdS} d^{d+1}Y e^{2Q \cdot Y} &= \int_0^\infty \frac{dz}{z} z^{-d} \int \left(d^d \vec{r} e^{-(1+z^2+r^2)|Q_0|/z} \right) \\ &= \pi^h \int_0^\infty \frac{dz}{z} (z|Q|)^{-h} e^{-(1+z^2)|Q_0|/z} \\ &= \pi^h \int_0^\infty \frac{dz}{z} z^{-h} e^{-z+Q^2/z} \end{aligned} \quad (8.27)$$

where we note that $Q_0^2 = Q^2$ and we have now restored manifest conformal symmetry. This means that

$$\langle \mathcal{O}(P_1) \mathcal{O}(P_2) \mathcal{O}(P_3) \mathcal{O}(P_4) \rangle = \frac{\lambda}{[2\pi^h \Gamma(\Delta - h + 1)]^4} \prod_{i=1}^4 \left(\int_0^\infty \frac{dt_i}{t_i} t_i^\Delta \int_0^\infty \frac{dz}{z} z^{-h} e^{-z + \frac{1}{z} \sum_{i,j=1}^4 t_i t_j P_i \cdot P_j} \right) \quad (8.28)$$

Now we can rescale $t_i \rightarrow t_i \sqrt{z}$ and factor out the z dependence completely. Then we find that

$$\langle \mathcal{O}(P_1) \mathcal{O}(P_2) \mathcal{O}(P_3) \mathcal{O}(P_4) \rangle \propto \int_0^\infty \prod_{i=1}^4 \frac{dt_i}{t_i} t_i^\Delta e^{-\sum_{i,j=1}^4 t_i t_j P_i \cdot P_j} \quad (8.29)$$

This last integral can be written using the Symanzik ‘star formula’. This formula arises by noting

$$e^{-x} = \int_{i\infty}^{\infty} d\delta \Gamma(\delta) x^{-\delta} \quad (8.30)$$

Inserting this identity for each of the $e^{-t_i t_j P_i \cdot P_j}$ term and performing the t_i integrals leads to

$$\int_0^\infty \prod_{i=1}^n \left(\frac{dt_i}{t_i} t_i^\Delta e^{-\sum_{i,j=1}^n t_i t_j P_i \cdot P_j} \right) = \frac{1}{2} \int_{i\infty}^{\infty} \prod_{i<j}^n \frac{d\delta_{ij}}{2\pi i} \Gamma(\delta_{ij}) (2P_i \cdot P_j)^{-\delta_{ij}} \quad (8.31)$$

where the δ_{ij} are constrained by

$$\sum_{j \neq i}^n \delta_{ij} = \Delta_i \quad (8.32)$$

although in this case all $\Delta_i = \Delta$, the dimension of $\mathcal{O}$. Note that if we imagine that $\delta_{ij} = p_i \cdot p_j$, then these constraints are just what we would find from momentum conservation plus the on-shell condition $p_i^2 = \Delta_i$. One should think of the δ_{ij} as analogous to the Mandelstam invariants s_{ij} for a scattering amplitude.

This last form for the correlator has transformed it into ‘Mellin space’, where the δ_{ij} variables parameterize the Mellin space. The Mellin amplitude is defined through

$$\langle \mathcal{O}(P_1) \mathcal{O}(P_2) \mathcal{O}(P_3) \mathcal{O}(P_4) \rangle = \int \left(\prod_{i<j}^4 \frac{d\delta_{ij}}{2\pi i} \Gamma(\delta_{ij}) (2P_i \cdot P_j)^{-\delta_{ij}} M(\delta_{ij}) \right) \quad (8.33)$$

where $M(\delta_{ij})$ is the Mellin Amplitude. Here we have found that a $\lambda \phi^4$ interaction in AdS space leads to a Mellin amplitude

$$M(\delta_{ij}) = \lambda \quad (8.34)$$

so it is simply a Mellin space-independent constant. This should remind you of the simplicity of momentum space scattering amplitudes.

We can also re-write the Mellin form by eliminating the extra δ_{ij} variables, giving the non-trivial part of the correlator as

$$\mathcal{A}(u, v) = \int_{i\infty}^{\infty} \frac{d\delta_{12} d\delta_{14}}{(2\pi i)^2} M(\delta_{12}, \delta_{14}) \Gamma^2(\delta_{12}) \Gamma^2(\delta_{14}) \Gamma^2(\Delta - \delta_{12} - \delta_{14}) u^{\Delta - \delta_{12}} v^{-\delta_{14}} \quad (8.35)$$

We see that the Mellin amplitude is just the transform in the conformally invariant cross-ratios. For general reference, the Mellin transform of a function $f(x)$ is

$$M(s) = \int_0^\infty x^{s-1} f(x) dx \quad (8.36)$$

$$f(x) = \int_{i\infty}^{\infty} \frac{ds}{2\pi i} x^{-s} M(s) \quad (8.37)$$

with the inverse also listed. Note that this is basically just a Fourier transform in the logarithm of the usual variables. So for the 4-pt correlator the Mellin amplitude is just a Mellin transform in the cross-ratios u and v .

What about the position-space correlator? In general, it is a special function referred to in the literature as a ‘D-function’, which are usually written as $D_{\Delta_1\Delta_2\Delta_3\Delta_4}$ as they depend on the external dimensions. The dependence on the spacetime dimension d only comes about through an overall normalization factor. The choice

$$D_{1111}(z, \bar{z}) = \frac{2\text{Li}_2\left(\frac{1}{\bar{z}}\right) \left(\frac{2\text{Li}_2\left(\frac{1}{z}\right) \left(\log(z\bar{z}) \log\left(\frac{\bar{z}(z-1)}{z(\bar{z}-1)}\right) \right)}{z \bar{z}} \right)}{(8.38)$$

can be expressed in terms of logarithms and dilogarithms. The other D-functions with integer Δ_i can be computed in terms of this one by taking derivatives, but in general for non-integral Δ_i there is no such simple expression. Here $z\bar{z} = u$ and $(1-z)(1-\bar{z}) = v$ as usual.

8.3 Conformal Partial Waves and $\lambda\phi^4$

Let us consider $\lambda\phi^4$ theory in AdS, and see how the interaction effects the CFT. For concreteness, let us imagine that we are working in AdS_4 , so $d = 3$, and we have fixed the squared mass of the ϕ field to be $m^2 = \Delta(\Delta - 3) = -2/R^2$, so that $\Delta = 1$. Then the correlator of the operator $\mathcal{O}$ obtained from taking ϕ to the boundary is

$$\begin{aligned} \langle \mathcal{O}(x_1)\mathcal{O}(x_2)\mathcal{O}(x_3)\mathcal{O}(x_4) \rangle &= \frac{1}{x_{13}^2 x_{24}^2} \left(1 + \frac{1}{z\bar{z}} + \frac{1}{(1-z)(1-\bar{z})} \right) \left(\right. \\ &\quad \left. - \lambda \frac{\mathcal{N}}{x_{13}^2 x_{24}^2} \left(\frac{2\text{Li}_2\left(\frac{1}{\bar{z}}\right) \left(\frac{2\text{Li}_2\left(\frac{1}{z}\right) \left(\log(z\bar{z}) \log\left(\frac{\bar{z}(z-1)}{z(\bar{z}-1)}\right) \right)}{z \bar{z}} \right) \right) \right) \right) \end{aligned} \quad (8.39)$$

where $\mathcal{N}$ is a normalization constant that just depends on our choice $\Delta = 1$ and $d = 3$. We chose these parameters so that we could utilize the simplest non-trivial D -function.

In section [7.5](#) we discussed the conformal partial wave expansion of correlators, and the specific partial waves that appear in the case of a generalized free theory. Now we would like to see how the conformal partial wave expansion changes due to the $\lambda\phi^4$ interaction. We are interested in this because it tells us about the change in the OPE coefficients that are proportional to 3-pt functions such as

$$\langle \mathcal{O}(x_1)\mathcal{O}(x_3)[\mathcal{OO}]_{n,\ell}(y) \rangle \quad (8.40)$$

where $[\mathcal{OO}]_{n,\ell}$ is the primary operator that would have dimension $2\Delta + 2n + \ell$ and spin ℓ in the generalized free theory. It can be viewed schematically as

$$[\mathcal{OO}]_{n,\ell} \sim \mathcal{O} \left(\overset{\leftrightarrow}{\partial} \right)^{2n} \overset{\leftrightarrow}{\partial}_{\mu_1} \cdots \overset{\leftrightarrow}{\partial}_{\mu_\ell} \mathcal{O} \quad (8.41)$$

although there are a lot of combinatorial coefficients and traces to remove to get the precise primary operator.

Furthermore, the correlator also tells us about the *anomalous dimensions* $\gamma(n, \ell)$ of these operators – this is the shift in their dimension due to the interaction. Recall that these operators simply correspond to 2-particle states in AdS, and that the dilatation operator is the AdS Hamiltonian, so an anomalous dimension in this context is simply the interaction energy (attractive or repulsive) between the two particles. As we will see, this can be computed directly using old-fashioned quantum mechanical perturbation theory.

Let us expand the correlator in $z, \bar{z}$ near 0, since this is the OPE limit. Ignoring the normalization factor and just keeping track of the parametric dependence on λ , we find

$$\mathcal{A}(z, \bar{z}) = \frac{1}{z\bar{z}} + 2 + z + \bar{z} - \frac{\lambda}{2}(4 + z + \bar{z} - (2 + z + \bar{z}) \log(z\bar{z})) + \dots \quad (8.42)$$

where the ellipsis denotes higher order terms in $z, \bar{z}$. Recall that conformal partial waves behave as

$$\frac{1}{(z\bar{z})^{\Delta_\phi}} g_{\tau, \ell}(z, \bar{z}) = (z\bar{z})^{\frac{\tau}{2} - \Delta_\phi} (z^\ell + \bar{z}^\ell + \dots) \quad (8.43)$$

The terms that are independent of λ are just what we have in a generalized free theory. But let us examine the order λ terms more closely. The correlator contains

$$\mathcal{A}(z, \bar{z}) \supset (2 - 2\lambda + \lambda \log(z\bar{z})) \quad (8.44)$$

Recall that the OPE in a generalized free theory begins (at small $z, \bar{z}$) with

$$\mathcal{O}(z, \bar{z})\mathcal{O}(0) = \left[\frac{1}{(z\bar{z})^\Delta} + \frac{1}{N_{0,0}^{\mathcal{O}}} \mathcal{O}^2(0) + \frac{\bar{z}}{N_{0,1}^{\mathcal{O}}} \mathcal{O} \partial_{\bar{z}} \mathcal{O}(0) + \frac{z}{N_{1,0}^{\mathcal{O}}} \mathcal{O} \partial_z \mathcal{O}(0) + \dots \right] \left(\quad (8.45)$$

The terms we have found in the correlator $\mathcal{A}$ are of order $z^0 \bar{z}^0$, so we can identify them as corresponding to the operator $\mathcal{O}^2(0)$. In other words, this is the operator $[\mathcal{O}\mathcal{O}]_{n,\ell}$ with $n = \ell = 0$. We see that in the generalized free theory the conformal block coefficient was simply 2, but we have found that this has shifted from $2 \rightarrow 2 + 2\lambda$. This also means that the OPE coefficient has shifted – since the conformal block coefficient is the square of the OPE coefficient, we have

$$c_{0,0} = \sqrt{2} - \frac{\sqrt{2}}{2} \lambda \quad (8.46)$$

so that $(c_{0,0})^2$ gives $2 + 2\lambda$, the conformal block coefficient (although recall that the precise numerical coefficient isn't meaningful since we didn't keep track of the normalization $\mathcal{N}$).

Now let us consider the $\lambda \log(z\bar{z})$ term in $\mathcal{A}$. Note that

$$\lambda \log(z\bar{z}) \approx (z\bar{z})^\lambda - 1 \quad (8.47)$$

in perturbation theory in λ . In other words, *we should interpret this logarithm as a shift in the scaling dimension of $\mathcal{O}^2(0)$* . From this logarithm we can compute the anomalous dimension $\gamma(n, \ell)$ for $n = \ell = 0$, namely

$$\gamma(0, 0) = (z\partial_z + \bar{z}\partial_{\bar{z}}) [\lambda \log(z\bar{z})] = 2\lambda \quad (8.48)$$

Note that the anomalous dimension is positive for positive λ , so the dimension of $\mathcal{O}^2$ has increased. This is due to the fact that the corresponding 2-particle state in AdS is experiencing a repulsive force.

Let us now revisit the utility of the Mellin amplitude representation of the correlator. We noted that for the 4-pt correlator we can write the contour integral

$$\mathcal{A}(u, v) = \iint_{i\infty}^{\infty} \frac{d\delta_{12} d\delta_{14}}{(2\pi i)^2} M(\delta_{12}, \delta_{14}) \Gamma^2(\delta_{12}) \Gamma^2(\delta_{14}) \Gamma^2(\Delta - \delta_{12} - \delta_{14}) u^{\Delta - \delta_{12}} v^{-\delta_{14}} \quad (8.49)$$

We have just seen that perturbative AdS interactions shift the OPE coefficients and operator/state dimensions in the CFT, and that these shifts can be extracted by looking at the coefficients of power-law monomials in $\mathcal{A}(u, v)$. Furthermore, we have seen that anomalous dimensions are associated with logarithms.

Note that a single pole in an integral of the form

$$\iint_{i\infty}^{\infty} \frac{d\delta}{2\pi i} \frac{1}{\delta + \Delta} u^{-\delta} = u^{\Delta} \quad (8.50)$$

while a double pole gives

$$\iint_{i\infty}^{\infty} \frac{d\delta}{2\pi i} \frac{1}{(\delta + \Delta)^2} u^{-\delta} = \frac{d}{d\Delta} u^{\Delta} = u^{\Delta} \log u \quad (8.51)$$

So the OPE coefficients and anomalous dimensions in a CFT are associated with the *residues of poles* in the Mellin integrand. Much of the power of Mellin space is that it allows us to use complex analysis for the study of AdS/CFT correlators.

8.4 Generalities About AdS Interactions and CFT

We will discuss expectations for the behavior of AdS interactions, a formal proof that (appropriate) perturbative interactions in AdS do not destroy conformal symmetry, and the meaning of EFT in AdS for the CFT.

8.4.1 Physical Discussion of AdS/CFT with Interactions

In perturbation theory the free propagator plays a crucial role. Let us examine various physical limits of the bulk-to-bulk propagator $G_{\Delta}(X, Y)$. In the short distance limit, it must agree with the flat spacetime propagator in $d + 1$ dimensions. This is the limit where $z = e^{-2\sigma(X, Y)} \approx e^{-2r/R_{AdS}}$ where $r \ll R_{AdS}$, $1/m$ is the distance between X and Y . Note that

$${}_2F_1(\Delta, h, \Delta + 1 - h, e^{-2r/R_{AdS}}) \propto \frac{1}{\Gamma(\Delta)} \left(\frac{R_{AdS}}{r} \right)^{d-1} \quad (8.52)$$

where I have dropped some factors of 2 and π , and approximated the hypergeometric function near 1. Thus we find that in the very short distance limit

$$G_{\text{short}}(r) \propto \left(\frac{R_{AdS}}{r} \right)^{d-1} \quad (8.53)$$

which is what we would expect for a field in $d + 1$ dimensional flat spacetime at very short distances. At very long distances $z \ll 1$, and we find that

$$G_{\text{long}}(r) \propto e^{-\Delta \frac{r}{R_{\text{AdS}}}} \quad (8.54)$$

which falls off very quickly at long-distances in AdS, due to the mass and AdS curvature. There is also a more delicate intermediate regime, where $\Delta \gg \frac{R_{\text{AdS}}}{r} \gg 1$ in which the usual Yukawa potential is reproduced. This can be obtained from a saddle-point approximation to the hypergeometric function at large Δ .

Let us consider whether the interactions disturb the limit where we take AdS QFT operators towards the boundary. There is an analogous question in flat spacetime. When we construct the flat spacetime S-Matrix, a key physical point is that we can study initial and final states composed of asymptotically well-separated particles. This is actually rather subtle, and in fact there are well-known IR divergences in four and fewer dimensions that must be treated with great care.

In AdS/CFT the question is whether when we take bulk operators $\phi(t, \rho, \Omega)$ to the boundary in an interacting theory, the interactions effectively shut off. The answer is yes – in fact, interactions shut off in AdS much more robustly than in flat spacetime, due to the AdS curvature. Recall that in section [2.2](#) we showed that even massless fields in AdS decrease exponentially fast at large distances from their sources. This means that well-separated sources in AdS decouple very quickly at large distances, i.e. as we approach the boundary of AdS.

In particular, we would like to see that as we take our AdS fields to the boundary, their interactions are dominated by points deep within the bulk (i.e. not arbitrarily close to the boundary). Note that when we approach the boundary we take $\phi(t, \rho(\epsilon; t, \Omega), \Omega)$ with $\epsilon \rightarrow 0$ but with fixed t, Ω . This means that when we have a correlator

$$\langle \phi_1(t_1, \rho(\epsilon; t_1, \Omega_1), \Omega) \phi_2(t_2, \rho(\epsilon; t_2, \Omega_2), \Omega) \cdots \phi_n(t_n, \rho(\epsilon; t_n, \Omega_n), \Omega) \rangle \quad (8.55)$$

as we take $\epsilon \rightarrow 0$, the geodesic distance $\sigma(X_i, X_j)$ between X_i and X_j will always go to infinity. In other words, any two distinct points on the boundary of AdS are infinitely far from each other. This by itself does not prevent us from having interactions near the boundary, since we are rescaling the ϕ_i by $\epsilon^{-\Delta_i}$ to obtain CFT operators.

However, consider some interaction vertex (from a Feynman diagram) localized at Y in AdS. If Y follows X_i as $X_i \rightarrow P_i$ on ∂AdS , then the vertex at Y will be further and further from all of the other vertices, and so bulk-to-bulk propagators connecting it to the rest of the AdS Feynman diagram will be exponentially suppressed. This is the reason that non-trivial interactions do not follow our bulk fields out to the boundary. Note that if all interaction vertices move together out towards the boundary then there is no suppression, but this uniform displacement does not lead to non-trivial interactions at infinity. This is analogous to the fact that in flat spacetime, the integration over the center of mass coordinate just leads to an overall momentum conserving delta function enforcing translation invariance.

8.4.2 A Weinbergian Proof of Conformal Symmetry

We saw by explicit computations that free fields in AdS give rise to Generalized Free Theories, which are conformal theories. Now we are studying AdS field theories with interactions, so how can we be

sure that the boundary theory is a conformal theory?

This is not so obvious. In a free theory we have the conformal symmetry generators $D^{(0)}$, $P_\mu^{(0)}$, $K_\mu^{(0)}$, and $M_{\mu\nu}^{(0)}$. But when we add interactions we are changing the dilatation operators = bulk Hamiltonian by

$$D = D^{(0)} + D^I \quad (8.56)$$

where D^I represents the effects of interactions. The free generators may have satisfied the conformal algebra, but now the presence of D^I will (naively) ruin it. In order to restore conformal symmetry we need to deform the other generators.

If we are dealing with a local relativistic QFT in AdS spacetime, then the preservation of the conformal algebra including interactions is actually pretty easy to see. The point is that the AdS isometries²⁰ are explicitly a symmetry of the action, and so we can use Noether's theorem to compute the symmetry generators as classical or quantum operators. Then these operators also act naturally on the fields when we take them to the boundary of AdS, $\phi \rightarrow \mathcal{O}$, and the discussion of section 8.4.1 shows that in fact the interactions decouple so that the CFT operators $\mathcal{O}$ transform just as we found in the free AdS theory.

Let us make this argument in a more general and formal way. We expect the interaction to satisfy

$$[D^I, M_{\mu\nu}] = 0 \quad (8.57)$$

so that we do not need to deform the AdS rotations, and we can continue to use angular momentum quantum numbers. This should not be surprising, since angular momentum is a discrete quantum number, so it would be difficult to imagine deforming it continuously. However, the generators $P_\mu^{(0)}$ and $K_\mu^{(0)}$ do not commute with $D^{(0)}$ and so we should expect that they will need to be deformed by P_μ^I and K_μ^I in the presence of the interaction D^I . Specifically, if we consider K_μ we must have

$$[D^{(0)} + D^I, K_\mu^{(0)} + K_\mu^I] \neq -K_\mu^{(0)} - K_\mu^I \quad (8.58)$$

so we need to define K_μ^I and ensure that this commutation relation holds. We already know that the zeroth order commutation relations work, so we need

$$[D^{(0)}, K_\mu^I] \neq [D^I, K_\mu^{(0)}] \neq [D^I, K_\mu^I] \neq -K_\mu^I \quad (8.59)$$

Let us assume that

$$D^I(t) = \iint_{AdS} d^d x \sqrt{-g} \mathcal{V}(t, x) \quad (8.60)$$

where we have included the t dependence to emphasize that the integral is a spatial integral at some fixed time t . In the interaction picture, the time evolution of $\mathcal{V}$ is given by

$$\partial_t \mathcal{V}(t, x) = -i[D^{(0)}, \mathcal{V}(t, x)] \quad (8.61)$$

²⁰Note that in the case of AdS₃ in the presence of gravity, we have more symmetries than just the AdS₃ isometries, but this more subtle point can only be seen by studying the asymptotic symmetry group of the spacetime.

So now let us look at the commutator of $K_\mu^{(0)}$ with D^I . We worked these generators out explicitly for AdS_3 , so specializing to that case note that

$$[K_\pm^{(0)}, D^I] = -i \int d^2x \frac{\sin \rho}{\cos^3 \rho} e^{-it \pm i\theta} \left(\sin \rho \partial_t + i \cos \rho \partial_\rho \mp \frac{1}{\sin \rho} \partial_\theta \right) \mathcal{V}(t, \rho, \theta) \quad (8.62)$$

We can replace the first term with the commutator, while integrating the last two terms by parts gives

$$[K_\pm^{(0)}, D^I] = - \iint d^2x \sqrt{-g} \sin \rho e^{-it \pm i\theta} ([D^{(0)}, \mathcal{V}(t, x)] + \mathcal{V}(t, x)) \quad (8.63)$$

This means that if we define

$$K_\pm^I = \iint d^2x \sqrt{-g} \sin \rho e^{-it \pm i\theta} \mathcal{V}(t, x) \quad (8.64)$$

then we will satisfy the desired commutation relations if the term $[D^I, K_\mu^I]$ vanishes. This will be possible if

$$[\mathcal{V}(t, x), \mathcal{V}(t, y)] = 0 \quad (8.65)$$

This is the condition we wanted to derive – we can preserve the conformal symmetry if *the interaction density commutes with itself outside the lightcone*. Weinberg derived precisely this condition in order to ensure the Poincaré invariance of the flat spacetime S-Matrix; here we see that it is necessary to preserve the conformal invariance of a theory living in AdS spacetime. The same condition is sufficient to preserve the interacting commutators of the P_μ generators.

8.4.3 Effective Conformal Theory and the Breakdown of AdS EFT

Thus far we have been talking about quantum field theory in AdS as though it is immutable and UV complete. However, in most cases²¹ it is more sensible to view QFT as an effective field theory that breaks down at an energy scale Λ , or (more physically) at a short-distance scale $1/\Lambda$. So it becomes natural to ask how the cutoff scale Λ in AdS translates to the CFT. Roughly speaking, one gets the answer immediately by noting that an AdS energy scale Λ is an eigenvalue of the AdS Hamiltonian D , so it must translate into a *dimension* ΛR_{AdS} in the CFT. An AdS field theory with a UV cutoff Λ corresponds to a CFT where we only keep states and operators with dimension $\Delta < \Delta_\Lambda$ with $\Delta_\Lambda = \Lambda R_{\text{AdS}}$.

As a more detailed comment – to discuss EFT in an invariant way, we would like to talk about integrating out states with large invariant mass. This means that if we have some AdS/CFT state $|\psi\rangle$ and some AdS operator that creates it, then we have a wavefunction

$$\nabla_{\text{AdS}}^2 \langle 0 | \Phi_{\text{AdS}}(X) | \psi \rangle \propto M^2 \langle 0 | \Phi_{\text{AdS}}(X) | \psi \rangle \quad (8.66)$$

²¹In the case of AdS/CFT this is always true, because once we involve gravity the QFT description can only be an approximation.

But the ∇_{AdS}^2 operator is just a combination of conformal generators, in fact it is the conformal Casimir operator

$$\nabla_{AdS}^2 = D^2 + \frac{1}{2}(K \cdot P + P \cdot K) + M_{\mu\nu}M^{\mu\nu} \quad (8.67)$$

and so states of definite invariant mass have definite conformal Casimir, where the eigenvalues of this operator are $M^2 = \Delta(\Delta - d) + \ell(\ell + d - 2)$ for general spins. So integrating out the high mass states means integrating out operators/states with large conformal Casimir. Since single particles typically have small spin, large Casimir naturally corresponds to large dimension Δ .

We know from experience that EFTs neglecting higher dimension operators are a good systematic approximation for lower-energy physics. This translates into the statement that corrections to correlators involving operators of dimension Δ from neglected higher dimension effects should be suppressed by powers of Δ/Δ_Λ . In fact, some recent results show that the approximation can be even better than this. For example, consider the conformal partial wave expansion of a correlator

$$\langle \phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4) \rangle = \frac{1}{(x_{13})^{2\Delta}(x_{24})^{2\Delta}} \sum_{\Delta, \ell}^{\Delta_\Lambda} \lambda_{\Delta, \ell}^2 g_{\Delta, \ell}(u, v) \quad (8.68)$$

where instead of summing over all Δ, ℓ , we have restricted the sum to $\Delta < \Delta_\Lambda$. One can prove [14] that in the Euclidean region this results in an *exponentially* good approximation to the CFT correlator. This is better than one might have expected from effective field theory in AdS, where corrections are generally power-laws. The reason it is better is that we are assuming that the OPE coefficients and operator dimensions are exactly correct for $\Delta < \Delta_\Lambda$, and that all corrections come from dropping the conformal partial waves with larger dimension. In an EFT description we do not simply leave out the states with large energy – we also miss power-law corrections to the low-energy states. This is (most-likely) why dropping high dimension partial waves gives a better approximation than one might guess from EFT.

One can see AdS EFT directly by studying theories with massive particles, moving the cutoff, and integrating these heavy states out. Then the CFT correlators, OPE coefficients, and anomalous dimensions can be matched between the high-energy and low-energy description. This analysis was carried out explicitly in [10].

8.5 Gravitational Interactions in AdS₃

We studied a $\lambda\phi^4$ interaction in the previous sections. Unfortunately, studying even that simple interaction required an involved calculation in AdS resulting in a not-very-illuminating special function. To get more intuition, let us study an important example that's also soluble – gravitational interactions between a heavy source and a light probe in AdS₃.

We will be studying a 4-pt correlator of two CFT₂ operators $\mathcal{O}_H$ and $\mathcal{O}_L$, which we write as

$$\langle \mathcal{O}_H(\infty)\mathcal{O}_L(1)\mathcal{O}_L(z, \bar{z})\mathcal{O}_H(0) \rangle \quad (8.69)$$

Conceptually, we will be computing the effects pictured in figure [11]. But instead of using Feynman diagrams, we will simplify the calculation in a series of steps.

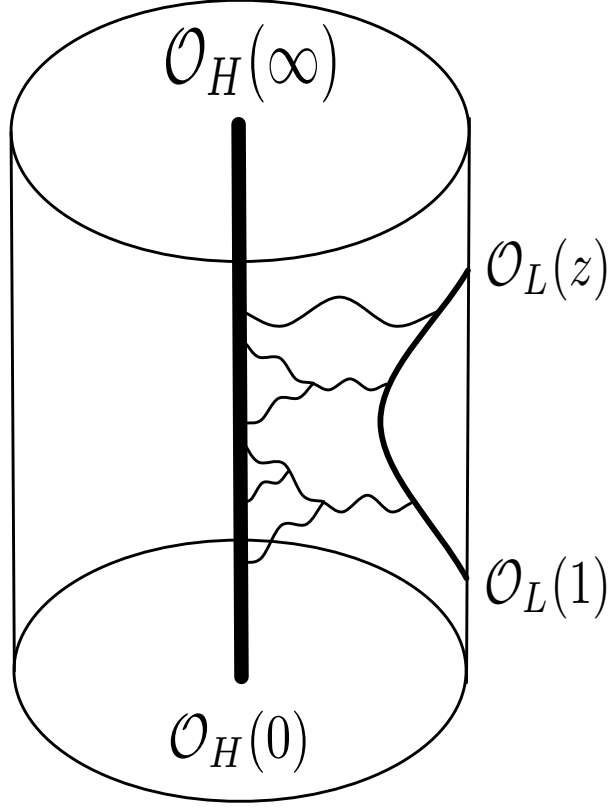


Figure 11: This figure provides a suggestive depiction of how graviton exchanges in AdS build up the classical field experienced by a light probe.

First of all, note that $|\mathcal{O}_H(0)\rangle$ and $\langle\mathcal{O}_H(\infty)|$ correspond to primary states, which means that they represent an object at rest in the center of global AdS. So we can interpret our calculation as the evaluation of the $\mathcal{O}_L(1)\mathcal{O}_L(z)$ 2-pt correlator in a background determined by $\mathcal{O}_H$. We are allowing $\mathcal{O}_H$ and $\mathcal{O}_L$ to interact gravitationally, but that just means that $\mathcal{O}_H$ will create a gravitational field, and then the AdS dual of $\mathcal{O}_L$ will move in that non-trivial field. In other words, we can compute the correlator by solving for the gravitational field created by $\mathcal{O}_H$, and then solving for the 2-pt function of $\mathcal{O}_L$ in that perturbed geometry.

There are many strategies that we can use for the second step, where we compute $\mathcal{O}_L(1)\mathcal{O}_L(z)$ in the gravitational background. One method would be to quantize a field $\phi_L(X)$ in the appropriate geometry, compute its 2-pt correlator, and then extrapolate to the boundary. But since we are treating ϕ_L as a free field, its 2-pt correlator is simply the solution to a differential equation

$$\nabla^2 K(X, Y) = \delta^{d+1}(X - Y) \quad (8.70)$$

So a natural method is to just solve this equation in the appropriate geometry, taking care of the

boundary conditions. Then the correlator will just be (we re-labeled $\mathcal{O}_L \rightarrow \mathcal{O}$)

$$\langle \mathcal{O}_H(\infty) \mathcal{O}(1) \mathcal{O}(z) \mathcal{O}_H(0) \rangle = \lim_{\epsilon \rightarrow 0} \frac{K(z, \rho_X(\epsilon); 1, \rho_Y(\epsilon))}{\epsilon^{2\Delta}} \quad (8.71)$$

Another method follows from the fact that we are neither creating nor destroying ϕ particles, so we can use a first quantized action for ϕ . As we discussed for the pure AdS case, the first quantized action is just $m \int d\tau$, the mass times the proper time in the particle's frame. In the semi-classical limit that $\Delta_L \sim (m R_{AdS}) \gg 1$ the correlator will be dominated by a saddle point, giving

$$\langle \mathcal{O}_H(\infty) \mathcal{O}(1) \mathcal{O}(z, \bar{z}) \mathcal{O}_H(0) \rangle = e^{-mS[z,1]} \quad (8.72)$$

where $S[z, 1]$ is the length of a geodesic stretching through the bulk from the boundary points z and 1 . Thus the computation of the correlator can be reduced to that of a geodesic in the background created by $\mathcal{O}_H$.

Let's proceed with the calculation; we will first find the background, and then we will just compute $K(X, Y)$ by using the method of images. The state $|\mathcal{O}_H(0)\rangle$ has an AdS wavefunction centered on the origin. It will give rise to a spherically symmetric vacuum solution of Einstein's equations, the AdS-Schwarzschild solution. In AdS₃ this is simply

$$ds^2 = -(r^2 + 1 - \mu) \left(dt^2 + \frac{dr^2}{r^2 + 1 - \mu} + r^2 d\phi^2 \right) \quad (8.73)$$

where the case $\mu = 0$ corresponds to the AdS vacuum. We can write

$$\mu = 8G_N M_H = 12 \frac{\Delta_H}{c} = \frac{24h_H}{c} \quad (8.74)$$

where c is the central charge of the dual 2d CFT and Δ_H is the dimension of $\mathcal{O}_H$. We will discuss central charges like c and their relationship with gravity in later sections; for now it's enough to know that c sets the strength of AdS₃ gravity via $c = \frac{3}{2G_N}$.

Note that for $\mu > 1$ the geometry changes dramatically, as there is an event horizon at $r^2 = \mu - 1 \equiv r_+^2$. This geometry represents a BTZ black hole. In the case $\mu < 1$ there is a deficit angle in ϕ ; this can be seen by studying the spatial geometry in the vicinity of $r = 0$, where there is a conical singularity.

Now we need to compute the correlators in this background. It turns out that locally the BTZ metric is just AdS₃; it only differs from BTZ by global identifications. This means that it can be computed via the method of images. Furthermore, the metric and the relevant results are analytic functions r_+ or equivalently, μ . For $\mu < 1$ we can re-write the metric as

$$ds^2 = -(1 - \mu) (r^2 + 1) \left(dt^2 + \frac{dr^2}{r^2 + 1} + (1 - \mu) r^2 d\phi^2 \right) \quad (8.75)$$

which makes it manifest that the deficit angle is $\sqrt{1 - \mu}$. Thus we can convert from AdS₃ to the deficit angle background by using the *method of images*. This works just as in electromagnetism –

we take the free field propagator and add shifted copies of it to itself in order to guarantee that we have smooth boundary conditions.

In fact, we can write the entire BTZ geometry in terms of the hyperboloid coordinates $X_A X^A = 1$, by making the identification

$$X_3 \pm X_2 \sim e^{2\pi r_+} (X_3 \pm X_2) \quad (8.76)$$

where we identify our friendly X_A with the usual coordinates via

$$\begin{aligned} X_3 \pm X_2 &= \frac{r}{r_+} e^{\pm r_+ \phi} \\ X_1 \pm X_0 &= \frac{\sqrt{r^2 - r_+^2}}{r_+} e^{\pm r_+ t} \end{aligned} \quad (8.77)$$

The limit of pure AdS_3 is just $r_+ = i$. Deficit angle metrics have $0 < -ir_+ < 1$ while real r_+ corresponds to BTZ. So the propagators are obtained by adding images in order to ensure continuity across these identifications. The result for the bulk-to-boundary correlator (from hep-th/0212277) is

$$K(X, b) = \sum_{n=-\infty}^{\infty} \frac{1}{\left(-\frac{\sqrt{r^2 - r_+^2}}{r_+} \cosh(r_+ \delta t) + \frac{r}{r_+} \cosh(r_+ (\delta \phi + 2\pi n)) \right)^\Delta} \quad (8.78)$$

where X is in the bulk and b is a boundary point, and the sum on n fixes the boundary conditions.

We want to compute the CFT 2-pt correlator, which comes from the $r \rightarrow \infty$ limit of this bulk correlator, after a rescaling by r^{2h} . In fact, that's not quite what we want, because we would like to study the correlator *on the plane*, not on the cylinder, and so we also need to multiply by an overall factor of $e^{-\Delta\tau}$. The result is

$$\langle \mathcal{O}(\tau, \phi) \mathcal{O}(0, 0) \rangle_{BTZ} = \sum_{n=-\infty}^{\infty} \left(\frac{e^{\Delta\tau}}{(\cosh(r_+ (\phi + 2\pi n)) - \cos(r_+ \tau))^\Delta} \right) \quad (8.79)$$

where we switched to $\tau = it$, the Euclidean time. We also assumed that the operator is a scalar so that $h = \bar{h} = \Delta/2$. Now we see that the sum over images simply guarantees that the correlator will be periodic in ϕ . You should also note that the Hawking temperature is

$$T_H = \frac{r_+}{2\pi} = \frac{\sqrt{\frac{24h_H}{c} - 1}}{2\pi} \quad (8.80)$$

as can be seen via the periodicity in Euclidean time.

We would like to expand this correlator in the $\mathcal{O}(z)\mathcal{O}(0)$ OPE channel in order to understand which operators are being exchanged between $\mathcal{O}$ and the heavy operator that has created the black hole. For this purpose it's useful to write

$$\langle \mathcal{O}(\tau, \phi) \mathcal{O}(0, 0) \rangle_{BTZ} = \sum_{n=-\infty}^{\infty} \left(\frac{e^{\Delta\tau}}{(2 \sin(\pi T_H (\tau + i\phi + 2\pi in)) \sin(\pi T_H (\tau - i\phi - 2\pi in)))^\Delta} \right) \quad (8.81)$$

Now we can write

$$1 - z = e^{-\tau + i\phi}, \quad 1 - \bar{z} = e^{-\tau - i\phi} \quad (8.82)$$

to translate this into $z, \bar{z}$ coordinates, where we used $1 - z$ so that $z \rightarrow 0$ corresponds with the OPE limit. Note that now each term has an independent z and $\bar{z}$ dependence, so to understand the OPE it suffices to look at

$$\frac{e^{-h \log(1-z)}}{[\sin(\pi T_H (\log(1-z) + 2\pi i n))]^{2h}} \quad (8.83)$$

where for convenience we write $\Delta = 2h$. The cases where $n = 0$ and $n \neq 0$ are very different. The series expansions are

$$\frac{1}{(\pi T_H z)^{2h}} \left(1 + \frac{h}{12} (4\pi^2 T_H^2 + 1) \left(z^2 + \dots \right) \right) \quad (8.84)$$

for the $n = 0$ case versus

$$\frac{1}{(\sin^2(2\pi^2 n i T_H))^h} \left(1 + h z - 4\pi h z T_H \cot(2\pi^2 n i T_H) + \dots \right) \quad (8.85)$$

for the $n \neq 0$ case. Note that final term cancels between n and $-n$ when we sum, so that we would be left with $1 + h z + \dots$ as an overall coefficient.

We see that there is a clear physical distinction between the two cases. When $n = 0$, we have a singularity $\frac{1}{z^\Delta}$, signalling the presence of the **1** operator in the $\mathcal{O}(z)\mathcal{O}(0)$ OPE. The first correction, at order z , vanishes, because identity does not have any global conformal descendants, and the probe operator $\mathcal{O}$ and the black hole do not exchange any fields of dimension 1. However, there is a contribution at order z^2 , which corresponds to the exchange of *one graviton*, ie the stress tensor quasi-primary state $T(0)|0\rangle$, between the black hole and the light probe. As expected, the graviton has dimension 2 in a CFT₂. Also note that

$$\frac{h}{12} (4\pi^2 T_H^2 + 1) \left(\frac{2h h_H}{c} \right) \quad (8.86)$$

which is what we would expect from the various OPEs – the graviton couples to the product of h and h_H , divided by $c/2$. Further multigraviton states can be seen if we expand the $n = 0$ case to higher orders.

In contrast, the $n \neq 0$ terms begin with a constant – they are not singular in the OPE limit. This means that they involve the exchange of operators of dimension 2Δ . These are precisely the ‘double-trace’ primary operators $\mathcal{O}^2$ that we discovered when we studied generalized free theories. These operators have descendants, and so we have corrections at every integer order z^k . In particular, notice that ratio of coefficients between the order z term and the order 1 term is h , which is exactly what we should expect from the ratio of first descendant and primary for an operator with holomorphic dimension $2h$, ie for $\mathcal{O}^2$.

Furthermore, if we looked at higher orders we could also identify other operators corresponding to 2-particle states in AdS_3 , such as $\mathcal{O}\partial^2\mathcal{O}$ etc. Matching to the (global) conformal block expansion in this channel would allow us to identify the OPE coefficients of general $\mathcal{O}\partial^k\mathcal{O}$ primary operators with the operator $\mathcal{O}_H$ that creates and destroys the black hole. We would also find the contributions from operators like $\mathcal{O}\mathcal{O}T$, $\mathcal{O}\mathcal{O}TT$, $\mathcal{O}\partial^r\mathcal{O}T\partial^sT$, etc involving two $\mathcal{O}$ particles accompanied by any number of gravitons.

9 On the Existence of Local Currents J_μ and $T_{\mu\nu}$

When we first learn QFT (or perhaps just classical mechanics), one of the earliest lessons is the beautiful and important Noether's theorem, which connects symmetries to conservation laws. When we have a Lagrangian description, Noether's theorem allows us to connect three different ideas:

1. the existence of a symmetry of the theory
2. the existence of a local current $J_\mu(x)$ (so called because it depends on a point in spacetime, and because we usually implicitly assume that it is insensitive to very distant physics) that is conserved, so that $\partial^\mu J_\mu(x) = 0$
3. the existence of a charge (no spacetime dependence) Q that measures the conserved quantity associated with the symmetry; if J_μ and Q both exist then we can usually write $Q = \int d^3x J^0(x)$

Global symmetry currents J_μ and the stress-energy tensor $T_{\mu\nu}$ form an extremely important class of operators in CFTs. As we mentioned above, conventionally CFTs are *defined* to have a conserved $T_{\mu\nu}$, although one can study 'CTs' or 'almost-CFTs' that have all the other features of a CFT. So let us discuss these operators in more detail in order to understand their physical properties and to see why they play such an important role. In particular, we will discuss

1. The difference between a simple conserved charge Q versus a local symmetry current. It is possible to have J_μ without Q or a charge Q without a corresponding J_μ .
2. How local currents such as $J_\mu(x)$ and $T_{\mu\nu}(x)$ generate symmetry transformations that produce new states, thereby telling us something about the Hilbert space of the theory.
3. The fact that when we *gauge* local symmetries, we identify all these different states, thereby 'mod-ing out' by the symmetry. Gauge symmetries are really redundancies of the description.

Although we will discuss physics quantum mechanically in this section, all of our statements also pertain to the classical phase space of classical theories, and in fact might be more natural and elementary in that context.

9.1 An $SU(2)$ Example

To understand these issues very concretely, let's study a very simple free theory of a pair of complex bosons, (Φ_1, Φ_2) with an $SU(2)$ symmetry. The action is just

$$S = \int \left(d^d x \sum_{i=1}^2 \left(\partial_\mu \Phi_i^\dagger \partial^\mu \Phi_i - m^2 \Phi_i^\dagger \Phi_i \right) \right) \quad (9.1)$$

where $i = 1, 2$. The local conserved current associated to this $SU(2)$ global symmetry is the operator

$$J_a^\mu(x) = i\sigma_a^{ij} \left(\Phi_i^\dagger(x) \partial^\mu \Phi_j(x) - \partial^\mu \Phi_i^\dagger(x) \Phi_j(x) \right) \quad (9.2)$$

where σ_a^{ij} are the $SU(2)$ generators (Pauli matrices). Let's check that it is conserved

$$\begin{aligned} \partial_\mu J_a^\mu(x) &= i\sigma_a^{ij} \left(\partial_\mu \Phi_i^\dagger(x) \partial^\mu \Phi_j(x) + \Phi_i^\dagger(x) \partial_\mu \partial^\mu \Phi_j(x) - \partial_\mu \partial^\mu \Phi_i^\dagger(x) \Phi_j(x) - \partial^\mu \Phi_i^\dagger(x) \partial_\mu \Phi_j(x) \right) \\ &= i\sigma_a^{ij} m^2 (\Phi_i^\dagger(x) \Phi_j(x) - \Phi_j^\dagger(x) \Phi_i(x)) = 0 \end{aligned} \quad (9.3)$$

Note that the momentum conjugate to Φ_i is

$$\Pi_i(x) \equiv \frac{\delta \mathcal{L}}{\delta \dot{\Phi}_i} = \dot{\Phi}_i^\dagger \quad (9.4)$$

so that the time component of the current can also be written as

$$J_a^0(x) = i\sigma_a^{ij} \left(\Phi_i^\dagger(x) \Pi_j^\dagger(x) - \Pi_i(x) \Phi_j(x) \right) \quad (9.5)$$

This means that the commutators of $J_a^0(x)$ can be computed easily using the canonical commutation relations, giving

$$[J_a^0(t, x), \Phi^i(t, y)] = i\delta^{d-1}(\vec{x} - \vec{y}) \sigma_a^{ij} \Phi_j(t, x) \quad (9.6)$$

so as we would expect, the current locally rotates the Φ_i field in $SU(2)$ space. We can measure a total charge using the integrated quantity

$$Q_a = \int \left(d^{d-1} x J_a^0(x) \right) \quad (9.7)$$

and note that it is time-independent using the conservation of J_a^μ and the vanishing of fields at infinity.

Although Q_a can be used to measure the total charge (in this case, an $SU(2)$ representation) associated with a given state, it always adds up the contribution of the entire universe. This severely limits the utility of Q_a by itself.

However, the local current J_a^μ is much more powerful, because we can use it to turn one state into another. The point is that we can smear J^0 over a small region surrounding some isolated

sub-system, so that it only transforms that sub-system, but leaves the rest of the universe alone. When you pick up an object and spin it around, you are (in some sense) taking advantage of the existence of local translation and rotation currents. We will come back to this point directly below when we discuss $T_{\mu\nu}$, but for now let us stick with our $SU(2)$ current.

For example, say we have a state $|\Psi_K\rangle$ composed of K well-separated Φ_1 particles, located around $x_1, x_2, \dots, x_K$. Then we can immediately construct an $SU(2)$ matrix operator R_a via

$$R_{ij}(t) = \exp \left[i \frac{\pi}{2} \int_{B_i \supset x_i} d^{d-1} y J_1^0(t, y) \sigma^1 \right] \left(\quad \right) \quad (9.8)$$

where the region B_i is some compact ball containing only the i th particle, and σ_1 is the 1th or x th Pauli matrix (chosen as a random example). Note that $[R_{ij}(t), R_{kl}(t)] = 0$ since they are compactly supported away from each other. Acting with R_{ij} on the state $|\psi_K\rangle$ will take the i th particle from $(1, 0)$ to $(0, 1)$, but leave the others alone. In other words, out of the state $|\Psi_K\rangle$ with all the bosons in the $(1, 0)$ configuration, we can generate all $SU(2)$ configurations for the isolated bosons! This obviously would not be possible with only the global charge Q_a . Note that if we had only studied an abelian current then all we could do is multiply the state with a phase, which is rather boring – this is why we chose an $SU(2)$ current as our example.

An important feature of R_{ij} is that it takes eigenstates of the Hamiltonian to other eigenstates. In other words, it guarantees that if a particle in the $(1, 0)$ state is an eigenstate, then $(0, 1)$ is an eigenstate as well. The conservation of J_a^μ is crucial for this, because it means that the time derivative of the R_i is

$$\begin{aligned} \partial_t R_{ij} &= R_{ij} \frac{i\pi}{2} \int_{B_i \supset x_i} d^{d-1} y \partial_t J_1^0(y) \sigma^1 \\ &= R_{ij} \frac{i\pi}{2} \int_{B_i \supset x_i} d^{d-1} y \partial_j J_1^j(y) \sigma^1 \\ &= R_{ij} \frac{i\pi}{2} \int_{\partial B_i} d^{d-2} s \hat{n}_j J_1^j(s) \sigma^1 \end{aligned} \quad (9.9)$$

and this last quantity vanishes when it acts on a state where the particles are well-separated, because the current will vanish on the surface of the ball B_i . This means that operators like R_{ij} are the exponential of local charges that are conserved in states where the particles are well-separated.

9.2 A Very Special Current – $T_{\mu\nu}$

The conserved energy-momentum tensor $T_{\mu\nu}$ plays a special role because it generates the spacetime symmetry transformations, including the conformal transformations in CFTs.

To understand more concretely what we can do with $T_{\mu\nu}(x)$, let's revisit the ultra-familiar symmetry of rotations in the case of a free QFT. If we take a free ϕ field in $2 + 1$ dimensions, we have the action

$$S = \int \left(dt d\mathbf{r} dr d\theta \frac{1}{2} ((\partial\phi)^2 - m^2 \phi^2) \right) \left(\quad \right) \quad (9.10)$$

In these specially chosen coordinates, rotations are just $\phi \rightarrow \phi + \epsilon \partial_\theta \phi$, and we find that the 0 component of the current is $T_\theta^0 = r \dot{\phi} \partial_\theta \phi$. Note that since the canonical conjugate to ϕ is $\pi(x) = \dot{\phi}(x)$, we can also write this as

$$T_\theta^0(t, x) = r \partial_\theta \phi(t, x) \pi(t, x) \quad (9.11)$$

so that it's clear what the commutation relations of $T_\theta^0(x)$ are with $\phi(y)$, namely

$$[T_\theta^0(t, x), \phi(t, y)] = i \delta^2(x - y) [r \partial_\theta \phi(t, x)] \quad (9.12)$$

so clearly T_θ^0 generates rotations, as desired. The global charge is

$$Q = \int \left(dr d\theta r \dot{\phi} \partial_\theta \phi \right) \quad (9.13)$$

However, this charge simply rotates the entire universe. So it isn't very useful physically – we cannot observe that we have rotated the entire universe.

Fortunately, we can make a local version using T_θ^μ . Using a radius dependent function $\eta(r)$, we can now consider the rotation operator

$$R[\eta] = \int \left(dr d\theta \eta(r) (r \dot{\phi} \partial_\theta \phi) \right) \quad (9.14)$$

As an example, we might have a gas of particles well-inside some sphere of radius R , along with some far away gas. So if we choose $\eta(r) = \Theta(R - r)$ then we have a rotation that turns off at R – we are just rotating the region inside the sphere while holding the outside universe constant.

We would now like to check that by performing this transformation we have not changed the energy of our state. Since $\partial_\mu J^\mu = 0$, we see that

$$\begin{aligned} \partial_t R[\eta] = i [H, Q[\eta]] &= \int \left(dr d\theta \eta(r) \partial_i J^i(r, \theta) \right) \\ &= \int \left(dr d\theta [\partial_r \eta(r)] J^r(r, \theta) \right) \\ &= \int \left(d\theta J^r(R, \theta) \right) \end{aligned} \quad (9.15)$$

where the last line follows because the r -derivative of $\eta(r)$ is simply a delta function. But this result makes a lot of sense – we see that $R[\eta]$ is time independent, and therefore commutes with the Hamiltonian, as long as we are working in a state where J^r doesn't see any particles at the surface of the sphere. So we can rotate everything inside a ball separately from the rest of the universe and obtain an equivalent but new state as long as there aren't any particles on the surface of the ball.

So this is just a spacetime version of the $SU(2)$ operator we defined in equation (9.8). Instead of performing an internal $SU(2)$ rotation on everything within a ball, it performs a spacetime rotation on the contents of the ball, creating a physically distinct state in the Hilbert space.

9.3 Separating Q and J_μ – Spontaneous Breaking and Gauging

We mentioned above that there are situations where a current J_μ exists but Q does not. This can happen when the integral defining Q is not well-defined, because it doesn't converge in infinite volume. A very physically important example of this phenomenon is when the symmetry is *spontaneously broken*. In that case the current J^μ still exists and is conserved, so $\partial_\mu J^\mu = 0$, however J_μ does not annihilate the vacuum:

$$J_\mu(x)|0\rangle = |\pi_{J_\mu(x)}\rangle \neq 0 \quad (9.16)$$

This makes it obvious that if we try to integrate over all of (an infinite) spacetime, the result will be ill-defined, assuming we have not also broken translational symmetry. This state $|\pi_{J_\mu(x)}\rangle$ is a one-goldstone boson state. A more physical state is

$$\frac{p_\mu}{F_\pi \sqrt{2p_0}} |\pi_p\rangle = \int \left(d^{d-1}x e^{ip \cdot x} J_\mu(x) |0\rangle \right) \quad (9.17)$$

and up to a normalization, this is just a state consisting of a single goldstone boson with momentum $\vec{p}$. The non-existence of Q corresponds with the fact that a goldstone boson state with $\vec{p} = 0$ is not normalizable. Note that current conservation implies that

$$\frac{-ip^\mu p_\mu}{F_\pi} |\pi_p\rangle = \int \left(d^{d-1}x e^{ip \cdot x} \partial^\mu J_\mu(x) |0\rangle \right) = 0 \quad (9.18)$$

so that we must have

$$p^2 = 0 \quad (9.19)$$

for a one-goldstone boson state. In other words, we learn that goldstone bosons must be massless. This basically shows that $J_\mu(x)$ acts on the vacuum to create a one-particle state, so goldstone bosons really are single particles. One can read more about all of this in Weinberg V2 [13].

There are also situations where Q exists but J_μ does not. The example that is most relevant for these notes is the ‘CT’ dual²² of non-gravitational $d+1$ dimensional field theories in AdS. In these d -dimensional boundary theories we have global symmetry charges associated with the full conformal group, enlarging the Poincaré generators, but these do not arise from the integral of a local current (these local currents would be made out of $T_{\mu\nu}$ if it existed, but it does not).

An alternate way to have a charge Q without J_μ is to ‘gauge’ the symmetry corresponding to J_μ . This means that we *identify* all of the states related by the action of J_μ , turning the *symmetry into a redundancy* and eliminating many degrees of freedom. To be precise, in a gauge theory the various states related by the action of operators such as the R_i of equation (9.8) would all be viewed as the same state. As an example where degrees of freedom really are eliminated from a low-energy theory, consider a complex field

$$\phi(x) = (v + h(x))e^{i\theta(x)} \quad (9.20)$$

²²A theory with conformal symmetry but not a ‘Field Theory’.

If $v \neq 0$ then $\theta(x)$ is a true goldstone boson degree of freedom. However, if we gauge the symmetry under which $\theta(x) \rightarrow \theta(x) + \epsilon$, then we eliminate this degree of freedom, as it is ‘eaten’ by the abelian gauge field $A_\mu(x)$.

In the case of gravity, when we gauge spacetime symmetries we remove local spacetime observables from the exact quantum mechanical theory; the only well-defined observables are ‘holographic’ – they are associated with infinity. Let us be a bit more precise about this. The energy-momentum tensor $T_{\mu\nu}$ is really the current for the d different charges P_μ , which correspond to translations (not rotations, or dilatations and special conformal transformations – these are associated with currents like $x^\nu T_{\mu\nu}$). When we place a theory with energy-momentum tensor $T_{\mu\nu}$ in a dynamical spacetime, we are therefore gauging, or redundantly identifying, the spacetime translations. Local translations occur when one object in spacetime moves with respect to another object. In general relativity, distances between objects are gauged, although obviously just as with spin one gauge theories, the gravitational field also changes under these gauge transformations (diffeomorphisms).

These ideas see a concrete realization when one studies membranes (or actually just particles) in spacetime. Particles, strings, and membranes can all be thought of as objects that spontaneously break translation symmetry! If we have a $d - 1$ dimensional membrane in d -dimensional spacetime, then we can parameterize it using a goldstone boson field

$$\pi_d(t, x_1, \dots, x_{d-1}) \tag{9.21}$$

that tells us that the spacetime position of the membrane is $(t, x_1, \dots, x_{d-1}, \pi_d)$. There is a universal low-energy action for π_μ called the DBI action

$$T \int \left(d^{d-1}x \sqrt{1 - \partial_\mu \pi_d \partial^\mu \pi_d} \right) \tag{9.22}$$

which just measures the world-volume of the membrane. The parameter T is the tension of the membrane. But the interesting point for us is that when we turn on gravity, thereby ‘gauging’ the translation symmetry, the goldstone boson π_d gets ‘eaten’ by the gravitational field, just like a standard goldstone boson in a spin one gauge theory. How do we see this? Actually, it’s trivial – we can always just choose a spacetime coordinate system built on top of the membrane so that the membrane is at $\pi_d = 0$ by definition. With this gauge choice, obviously the membrane fluctuations have been absorbed into fluctuations of the spacetime metric!

These statements are also directly related to the well-known Weinberg-Witten theorem [15], which says that theories with massless spin-1 particles cannot have an associated conserved current, and that theories with massless spin 2 particles cannot have a conserved stress-energy tensor transforming appropriately under spacetime symmetries. The point with this theorem is that gauging the symmetry in order to include a spin one or spin two massless particle necessarily alters the Poincaré transformation properties of the associated current.

The statement that gauge theories have fewer degrees of freedom may seem counter-intuitive, because the gauge theories we usually study have more degrees of freedom – either additional gauge bosons or gravitons (viewing gravity as a gauging of the spacetime symmetries). Essentially, this is the case because we need to introduce a fluctuating gauge field A_μ or gravitational field $g_{\mu\nu}$ in order to identify states related by symmetry without sacrificing locality.

Finally, we should mention that in the case of theories such as QED and non-abelian gauge theories, the existence of Q without J_μ is fairly trivial (sometimes physicists simply neglect its existence entirely). The reason is that this global charge is conserved, so once the universe exists in an eigenstate of Q the eigenvalue can never change. Thus we can view universes with different values of Q as different super-selection sectors, or as different theories entirely – they can never ‘talk to each other’. Also, in theories such as Yang-Mills or QCD where charge is confined, a non-zero charged state in an infinite spacetime would have infinite energy, so if we are only interested in finite energy states, then we must always have $Q = 0$, rendering Q completely trivial.

10 Universal Long-Range Forces in AdS and CFT Currents

The central point of this section is that *spin one gauge fields A_a in AdS are dual to spin one conserved currents J_μ in the CFT, while the spin two graviton in AdS is dual to the spin two energy momentum tensor $T_{\mu\nu}$ of the CFT*. So we will begin by explaining how this correspondence works. Then our major theme will be that the universality of the interactions of gauge bosons and gravitons in AdS translates into the universality of the correlators of J_μ and $T_{\mu\nu}$ in the CFT, due to Ward identities.

Some of the ideas from the previous section can be stated more formally in terms of Ward identities for conserved currents in CFTs. We will derive these CFT identities and then we will explain how they relate to the universality of long-range forces in AdS.

10.1 CFT Currents and Higher Spin Fields in AdS

Before we discuss symmetries and dynamics, let us note a few kinematic features of conserved currents in CFTs and massless particles in AdS. The basic point will be to note how the counting of degrees of freedom match between AdS and the CFT. This is essentially a quick version of the analysis we went through in the first several sections of these notes for a scalar particle, where we showed that one-particle scalar states in AdS correspond to the states created by $\mathcal{O}(0)$ acting on the AdS/CFT vacuum. Here the point is that $J_\mu(0)|0\rangle$ corresponds to a vector particle in AdS in its ground state, while $T_{\mu\nu}(0)|0\rangle$ corresponds to a spin two particle in AdS in the ground state. Conservation of the currents corresponds to massless particles in AdS, which have fewer degrees of freedom due to a gauge redundancy. If the AdS theory is free (corresponding to coupling $g = 0$ or $G_N = 0$) then we just have Gaussian correlators for J_μ or $T_{\mu\nu}$. Let us give a quick explanation of some of these statements.

Conserved currents satisfy

$$\partial^\nu J_{\nu\mu_2\cdots\mu_\ell} = 0 \quad \text{or} \quad [P^\nu, J_{\nu\mu_2\cdots\mu_\ell}] = 0 \quad (10.1)$$

We will assume that these are symmetric traceless tensors. At the level of the CFT states we have

$$P^\nu J_{\nu\mu_2\cdots\mu_\ell}|0\rangle = 0 \quad (10.2)$$

In other words, this descendant state and all others that normally would have followed from further applications of P^μ vanish. This means that a large number of states that would have been present for a current that wasn’t conserved have been eliminated from the Hilbert space of the theory.

For a spin 1 current J_μ , we have $d - 1$ independent primary DoFs, where we are comparing to the case of a scalar operator with 1 DoF. For a traceless spin 2 current $T_{\mu\nu}$, we naively have $d(d + 1)/2 - 1$ degrees of freedom, but another d are eliminated by the conservation condition, leaving us with only $d(d - 1)/2 - 1$ DoFs. Similar counts can be applied to higher spin currents.

A simple calculation determines that $\Delta = d - 2 + \ell$ for conserved currents. This is the CFT scaling dimension of a massless spin ℓ field in AdS. Our prior derivation of the unitarity bound proves that if $\Delta = d - 2 + \ell$ then the current must be conserved. For the converse, note that since J is primary we have

$$\begin{aligned} 0 &= [K_\alpha, [P_\beta, J_{\mu_1 \dots \mu_\ell}]] + [J_{\mu_1 \dots \mu_\ell}, [K_\alpha, P_\beta]] \\ &= [K_\alpha, [P_\beta, J_{\mu_1 \dots \mu_\ell}]] + 2\Delta_J J_{\mu_1 \dots \mu_\ell} \eta_{\alpha\beta} + 2 \sum_i \left(\eta_{\alpha\mu_i} J_{\mu_1 \dots \beta \dots \mu_\ell} - \eta_{\beta\mu_i} J_{\mu_1 \dots \alpha \dots \mu_\ell} \right) \end{aligned} \quad (10.3)$$

Now if we take the trace via $\eta^{\beta\mu_1}$ then the first commutator term vanishes by conservation, giving

$$0 = \Delta_J J_{\alpha\mu_2 \dots \mu_\ell} - (d + \ell - 2) J_{\alpha\mu_2 \dots \mu_\ell} \quad (10.4)$$

where the -2 comes from one missing term when $i = 1$ and a single non-vanishing contribution from the first term in the summand, also when $i = 1$. So we have shown that $\Delta_J = d - 2 + \ell$.

The point of these observations is that they agree with the DoF counting for massless fields with spin – specifically, a massless gauge boson A_A in AdS_{d+1} contains the same degrees of freedom as a conserved current J_μ in a CFT_d , and a massless graviton h_{AB} in AdS_{d+1} has the same degrees of freedom as the stress-energy tensor $T_{\mu\nu}$ in a CFT_d . In the case of gravitons and gauge bosons, the extra degrees of freedom between the massive and massless case are eliminated by the gauge redundancy of either a gauge theory with Lie Group symmetry or the diffeomorphism redundancy of General Relativity. So to summarize, *conserved spin ℓ CFT currents are dual to massless AdS fields of spin ℓ whose description necessarily involves extra redundancy in order to eliminate unphysical degrees of freedom.*

Now let us see more explicitly how the mapping works for a simple example. Chapter 7 of Raman’s notes [8] also give a nice discussion. First consider a free massive spin one boson with action

$$S = - \int_{\text{AdS}} d^{d+1} X \sqrt{-g} \left(\frac{1}{4} F_{AB} F^{AB} + \frac{1}{2} m^2 A_B A^B \right) \quad (10.5)$$

where $F_{AB} = \nabla_A A_B - \nabla_B A_A$ as usual, and indices are raised and lower with the AdS metric. This theory does not have a gauge symmetry. Alternatively, one can view this as a gauge theory that has been higgsed and written in unitarity gauge.

However, as in any theory where the kinetic term comes from F^2 , the component A_0 does not have a kinetic term (there is no term of the form $\partial_t A_0$ anywhere in the action). This means that A_0 is non-dynamical, so we can use its equations of motion to eliminate it, after which point we will have $d + 1 - 1 = d$ separate degrees of freedom. Concretely, the A_B field can create d distinct states corresponding to a single particle at rest; these are the d states created by a general current $J_\mu(0)$ in the CFT when it acts on the CFT vacuum.

The equation of motion for A_B is

$$\nabla^M F_{MN} - m^2 A_N = 0 \quad (10.6)$$

as usual. In the Poincaré patch coordinates for e.g. AdS_5 from the equations of motion we obtain

$$z^3 \partial_z \left(\frac{A_z}{z^3} \right) = \partial_\mu A^\mu \quad (10.7)$$

so that if we wish we can express the A_z component in terms of the others, instead of eliminating the A_0 component as is conventional. Explicitly, we have

$$A_z(z, x) = z^3 \int_0^1 dw \frac{\partial_\mu A^\mu(x, w)}{w^3} \quad (10.8)$$

The utility of this method is that we can write

$$\lim_{z \rightarrow 0} \frac{A_\mu(z, x)}{z^{\Delta-1}} = J_\mu(x) \quad (10.9)$$

and so these components become the (not conserved) CFT current J_μ of dimension Δ . Note that to get a finite result we scale A_μ by its twist $\tau = \Delta - \ell$, not its dimension, as we approach the boundary. One can obtain similar results for massive tensor fields. But we also see what happened to A_z . We have

$$\lim_{z \rightarrow 0} \frac{A_z(z, x_\mu)}{z^\Delta} = \lim_{z \rightarrow 0} \frac{\partial^\mu A_\mu(z, x_\mu)}{z^{\Delta-1}} = \partial^\nu J_\nu(x_\mu) \quad (10.10)$$

A_z is a redundant degree of freedom which becomes the (also redundant) descendant operator $\partial^\nu J_\nu$ in the CFT.

Now let us consider the case of a massless gauge boson in AdS. In this case the action has the gauge redundancy under

$$A_M(X) \rightarrow A_M(X) + \nabla_M \Lambda(X) \quad (10.11)$$

for any scalar function $\Lambda(X)$ on AdS. However, the field strength F_{MN} is gauge invariant in the abelian case, and it transforms as an ordinary adjoint field in the non-abelian case. Thus we can use its components $F_{\mu z}$ to directly define

$$J_\mu(x) = \lim_{z \rightarrow 0} \frac{F_{\mu z}(z, x)}{z^{d-3}} \quad (10.12)$$

Note that here we have that $\tau = \Delta - \ell = d - 2$, and we removed an additional power of z from the denominator in this definition because of the derivatives in $F_{\mu z}$.

Another natural way to obtain this result is to fix the gauge $A_z(z, x) = 0$. There is a residual gauge symmetry – we can shift A_μ by $\partial_\mu \Lambda(x)$ independent of z – and so we can also take $\partial_\mu A^\mu = 0$ in the directions orthogonal to z . This immediately implies that if we take

$$\begin{aligned} J_\mu(x) &= \lim_{z \rightarrow 0} \frac{F_{\mu z}(z, x)}{z^{d-3}} \\ &= \frac{\partial_z A_\mu(z, x)}{z^{d-3}} \\ &= \frac{A_\mu(z, x)}{z^{d-2}} \end{aligned} \tag{10.13}$$

and furthermore that we will have $\partial_\mu J^\mu(x) = 0$ from our residual gauge condition, so $J_\mu(x)$ will be a conserved current, as expected.

A similar analysis applies to massive and massless linearized gravitational fields h_{MN} in AdS. See Raman’s notes [\[8\]](#) for the details.

10.2 CFT Ward Identities and Universal OPE Coefficients

Ward identities are a formal consequence of symmetries as applied to correlation functions in QFTs. They play an especially important role in CFTs because they give us a lot of information about the interactions of currents J_μ and especially the energy momentum tensor $T_{\mu\nu}$ with other operators in the theory. In particular, Ward identities for the CFT currents

$$\langle J_\mu(y) \mathcal{O}_1(x_1) \mathcal{O}_2(x_2) \rangle \quad \text{and} \quad \langle T_{\mu\nu}(y) \mathcal{O}_1(x_1) \mathcal{O}_2(x_2) \rangle \tag{10.14}$$

directly relate to the universality of gauge and gravitational forces in AdS.

The Ward identity states that

$$\frac{\partial}{\partial y^\mu} \langle J^\mu(y) \mathcal{O}_1(x_1) \mathcal{O}_2(x_2) \cdots \mathcal{O}_n(x_n) \rangle = -i \sum_{m=1}^n \delta^d(y - x_m) \langle \mathcal{O}_1(x_1) \cdots [G_m \mathcal{O}_m(x_m)] \cdots \mathcal{O}_n(x_n) \rangle \tag{10.15}$$

where G_m enacts (represents in the group theory sense) the symmetry transformation corresponding to J_μ on a given operator $\mathcal{O}_m$. If $\mathcal{O}_m$ does not transform under this symmetry, then it does not contribute to the right hand side. Let us give the path integral derivation of this Ward identity for the conserved current $J_\mu(x)$.

To derive the identity, consider the path integral computation of a correlator

$$\langle \mathcal{O}_1(x_1) \cdots \mathcal{O}_n(x_n) \rangle = \int \left([\mathcal{D}\phi] \mathcal{O}_1(x_1) \cdots \mathcal{O}_n(x_n) e^{-S[\phi]} \right) \tag{10.16}$$

where the operators $\mathcal{O}_i$ are some general operators in the theory with fields represented by ϕ . Now if we perform an infinitesimal symmetry transformation under which

$$\delta_\epsilon \mathcal{O}_i = -i\epsilon(x) G_i \mathcal{O}_i \tag{10.17}$$

then the action and the operators appearing within the path integral will both transform. By assumption, the action would be invariant if $\epsilon(x)$ were a constant, and the measure of integration is invariant, so we must have

$$\langle \mathcal{O}_1(x_1) \cdots \mathcal{O}_n(x_n) \rangle = \int \left([\mathcal{D}\phi] (1 + \delta_\epsilon) (\mathcal{O}_1(x_1) \cdots \mathcal{O}_n(x_n)) e^{-S[\phi] - \int d^d x \partial_\mu J^\mu(x) \epsilon(x)} \right) \quad (10.18)$$

Expanding to first order in ϵ tells us that

$$\langle \delta_\epsilon (\mathcal{O}_1(x_1) \cdots \mathcal{O}_n(x_n)) \rangle = \int \left(d^d x \epsilon(x) \partial_\mu \langle J^\mu(x) \mathcal{O}_1(x_1) \cdots \mathcal{O}_n(x_n) \rangle \right) \quad (10.19)$$

In order for this identity to hold for any infinitesimal function $\epsilon(x)$, we must have the Ward identity of equation (10.15).

If it exists, the conserved charge associated with J_μ is

$$Q = \int \left(d^{d-1} x J^0(t, x) \right) \quad (10.20)$$

This charge will be time-independent because $\partial_\mu J^\mu = 0$. We can also use the Ward identity to give a formal proof that

$$[Q, \mathcal{O}_m] = -i G_m \mathcal{O}_m \quad (10.21)$$

for Lorentzian correlators (where it is more obvious what ‘time’ means). For this purpose let $\mathcal{O}_m(t_m, x_m)$ have t_m distinct from the other t_i in any n -pt correlator. Now let us take the Ward identity and integrate over the spatial y and between times $t_y \pm \epsilon$ for an infinitesimal ϵ . Let us choose $m = 1$ for convenience. This gives

$$\langle [Q(t_1 + \epsilon) \mathcal{O}_1(x_1)] \cdots \mathcal{O}_n(x_n) \rangle - \langle [Q(t_1 - \epsilon) \mathcal{O}_1(x_1)] \cdots \mathcal{O}_n(x_n) \rangle = -i \langle G_1 \mathcal{O}_1(x_1) \cdots \mathcal{O}_n(x_n) \rangle \quad (10.22)$$

Since these are Lorentzian correlators, analytically continued from Euclidean spacetime, they will be time ordered. Since the other operators appearing in the correlator were completely arbitrary, this identity can be interpreted as equation (10.21).

Our proof of the Ward identity relied on a path integral for the CFT. One can also derive the Ward identity more directly from the CFT axioms [16, 17] by expanding the OPE of $J_\mu(x) \mathcal{O}_i(0)$ and constraining the possible terms that appear using conformal invariance.

Now let us consider the Ward identity for the correlator between an abelian spin one current and any number of operators. If we integrate the identity over all y , we find that the LHS vanishes, and so we must have that

$$\sum_i q_i = 0 \quad (10.23)$$

for the correlator to be non-vanishing, where $G_i = q_i$ for an abelian current. This also follows from the fact that the vacuum is uncharged, so $Q|0\rangle = 0$. If we apply the Ward identity to a non-vanishing 3-pt correlator with the current, we see

$$\partial_\mu \langle J^\mu(y) \mathcal{O}(x_1) \mathcal{O}^\dagger(x_2) \rangle = -iq \left[\delta^d(y - x_1) \langle \mathcal{O}(x_1) \mathcal{O}^\dagger(x_2) \rangle - \delta^d(y - x_2) \langle \mathcal{O}(x_1) \mathcal{O}^\dagger(x_2) \rangle \right] \quad (10.24)$$

Recall that 3-pt functions such as $\langle J\mathcal{O}\mathcal{O}^\dagger \rangle$ are completely determined by conformal invariance up to some overall constant. Once the operator $\mathcal{O}$ is normalized, the Ward identity completely fixes this constant to be the charge q of $\mathcal{O}$. In other words, the Ward identity tells us that *conserved currents must have a 3-pt coupling proportional to charge*. Of course this 3-pt coupling also sets the OPE coefficient for the appearance of J_μ in the OPE of $\mathcal{O}(x)\mathcal{O}^\dagger(0)$. The Ward identity is the CFT version of the statement that massless gauge bosons must couple to a conserved charge.

Now let us consider the analogous statements for the energy momentum tensor $T_{\mu\nu}$. In this case we have the ‘obvious’ Ward identities associated with the conservation condition $\partial_\mu T^{\mu\nu} = 0$ which give information about translations. However, there are also various non-obvious identities associated with rotations, dilatations, and special conformal transformations. The first also follows from conservation of $T^{\mu\nu}$, but the latter are associated with the tracelessness condition $T^\mu_\mu = 0$. This is the conservation condition for the scale current

$$S_\mu(x) = x^\nu T_{\mu\nu} \quad (10.25)$$

We already know that D is the charge associated with the scale current, and so it acts as $[D, \mathcal{O}(x)] = -i(\Delta + x \cdot \partial_x)\mathcal{O}(x)$. Thus we see that

$$\begin{aligned} \partial_\mu \langle S^\mu(y)\mathcal{O}(x_1)\mathcal{O}^\dagger(x_2) \rangle &= \langle T^\mu_\mu(y)\mathcal{O}(x_1)\mathcal{O}^\dagger(x_2) \rangle + \langle y_\nu \partial_\mu T^{\mu\nu}(y)\mathcal{O}(x_1)\mathcal{O}^\dagger(x_2) \rangle \\ &= -i\Delta [\delta^d(y-x_1)\langle \mathcal{O}(x_1)\mathcal{O}^\dagger(x_2) \rangle + \delta^d(y-x_2)\langle \mathcal{O}(x_1)\mathcal{O}^\dagger(x_2) \rangle] \left(\right. \\ &\quad \left. -i [\delta^d(y-x_1)x_1 \cdot \partial_{x_1}\langle \mathcal{O}(x_1)\mathcal{O}^\dagger(x_2) \rangle + \delta^d(y-x_2)x_2 \cdot \partial_{x_2}\langle \mathcal{O}(x_1)\mathcal{O}^\dagger(x_2) \rangle] \right) \end{aligned} \quad (10.26)$$

We see that the final term on the first line is equal to the terms on the last line – this identity is just the Ward identity for translations. This means that

$$\langle T^\mu_\mu(y)\mathcal{O}(x_1)\mathcal{O}^\dagger(x_2) \rangle = -i\Delta [\delta^d(y-x_1)\langle \mathcal{O}(x_1)\mathcal{O}^\dagger(x_2) \rangle + \delta^d(y-x_2)\langle \mathcal{O}(x_1)\mathcal{O}^\dagger(x_2) \rangle] \left(\right. \quad (10.27)$$

This identity immediately tells us that the 3-pt correlation function $\langle T_{\mu\nu}(y)\mathcal{O}(x_1)\mathcal{O}^\dagger(x_2) \rangle$ has a *universal coefficient proportional to Δ* . Note that this 3-pt function exists for *all primary operators in the theory*, since $T_{\mu\nu}$ generates the conformal transformations on all operators/states. This contrasts with the abelian current J_μ we considered above, which only acts on the sector of charged states. The Ward identity we have derived is the CFT version of the statement that gravity couples universally to energy-momentum, whose role is played here by the dimension-charge Δ .

As a final comment, let us consider the 3-pt correlator

$$\langle T_{\mu_1\nu_1}(x_1)T_{\mu_2\nu_2}(x_2)T_{\mu_3\nu_3}(x_3) \rangle \quad (10.28)$$

In general d this correlator can involve various tensor structures, and it needs to be normalized. It is natural to normalize the 2-pt function of $T_{\mu\nu}$ via

$$\langle T_{\alpha\beta}(x)T_{\mu\nu}(0) \rangle = \frac{C_T}{x^{2d}} I_{\alpha\beta,\mu\nu}(x) \quad (10.29)$$

with

$$I_{\alpha\beta,\mu\nu}(x) \equiv \frac{1}{2} (I_{\alpha\mu}I_{\beta\nu} + I_{\alpha\nu}I_{\beta\mu}) - \frac{1}{d}\eta_{\alpha\beta}\eta_{\mu\nu} \quad \text{with} \quad I_{ab} \equiv \eta_{ab} - 2\frac{x_a x_b}{x^2} \quad (10.30)$$

A natural normalization of $T_{\mu\nu}$ occurs because various integrals of $T_{\mu\nu}$ give the conformal generators; e.g. P_μ which must act on the conformal primaries in a specific way. This fixes C_T , a *central charge of the CFT*. Roughly speaking, one can think of C_T as counting the number of degrees of freedom of the theory. This follows intuitively because the direct product of two CFTs with central charges C_1 and C_2 yields a new CFT with central charge $C_1 + C_2$, since the energy-momentum tensors in the two CFTs are decoupled.

Once the energy-momentum tensor is normalized the coefficient of each tensor structure in its 3-pt correlator gives dynamical information about the theory. Since all $\langle T_{\mu\nu} \mathcal{O} \mathcal{O}^\dagger \rangle$ correlators must be proportional to $\Delta_{\mathcal{O}}$, by looking at the universal energy-momentum 3-pt correlator we can fix the proportional constant once and for all for a given theory. For more details of this see [17]. This proportionality constant is the inverse square root of a central charge of the CFT. This central charge is proportional to a power of $M_{pl} R_{AdS}$ in the AdS theory.

In general d there are several central charges, and they are more conventionally discussed as coefficients of the expansion of the ‘trace anomaly’, which is $\langle T^\mu_\mu \rangle$ evaluated when the CFT is placed in a general curved spacetime. For example, in $d = 4$ we have

$$\langle T^\mu_\mu \rangle = \frac{c}{16\pi^2} W_{abcd} W^{abcd} - \frac{a}{16\pi^2} (R_{abcd} R^{abcd} - 4R_{ab} R^{ab} + R^2) \left(\frac{a'}{16\pi^2} \nabla^2 R \right. \quad (10.31)$$

where the terms on the right hand side are various curvature tensors for the 4-d spacetime in which the CFT is living. Note that $W_{abcd} W^{abcd} = R_{abcd} R^{abcd} - 2R_{ab} R^{ab} + R^2/4$ is the square of the Weyl tensor, and the terms proportional to a are the 4-d Euler density. Also, the quantity a' is scheme dependent, but both a and c are physically meaningful, and a is known to decrease under RG flows. These coefficients can also be computed directly from the 3-pt correlator of $T_{\mu\nu}$ in flat spacetime, see [17]. In AdS_5 theories where gravity has only an Einstein-Hilbert action term, it turns out that $a = c$, and so both are proportional to $(M_5 R_{AdS})^3$, where M_5 is the 5-d Planck scale.

10.3 Ward Identities from AdS and Universality of Long-Range Forces

In the previous section we saw how CFT Ward identities imply that conserved spin one currents must couple to conserved charges, while the stress-energy tensor must have couplings proportional to the dimension Δ . In both cases by a ‘coupling’ we meant the magnitude of the universal 3-pt correlators

$$\langle J_\mu(y) \mathcal{O}_1(x_1) \mathcal{O}_2(x_2) \rangle \quad \text{and} \quad \langle T_{\mu\nu}(y) \mathcal{O}_1(x_1) \mathcal{O}_2(x_2) \rangle \quad (10.32)$$

which also set the OPE coefficients for the appearance of J_μ and $T_{\mu\nu}$ in the OPE of $\mathcal{O}_1(x) \mathcal{O}_2(0)$. Note that these operators $\mathcal{O}_1$ and $\mathcal{O}_2$ were *any* operators in the theory, not just the operators dual to single-particle states in AdS. However, in the case where we have some AdS field theory involving a gauge boson A_μ or a graviton $h_{\mu\nu}$ coupling to some other field ϕ , we will have some linearized couplings in AdS of the schematic form

$$q \int d^{d+1} X \sqrt{-g} (\phi^\dagger \partial_\mu \phi - \partial_\mu \phi^\dagger \phi) A^\mu \quad \text{or} \quad \sqrt{G_N} \int d^{d+1} X \sqrt{-g} \partial_\mu \phi \partial_\nu \phi h^{\mu\nu} \quad (10.33)$$

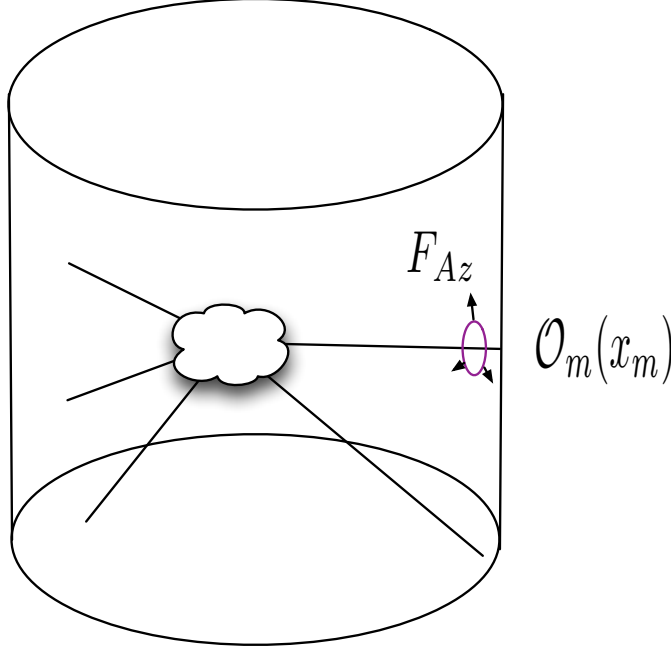


Figure 12: This figure illustrates the derivation of the Ward identity for a CFT current from Gauss's Law in AdS. Taking the surface integral towards the boundary of AdS gives the integral form of the Ward Identity given in equation (10.34).

If we compute the AdS correlator $\langle \phi(X_1)\phi(X_2)A_\mu(X_3) \rangle$ or $\langle \phi(X_1)\phi(X_2)h_{\mu\nu}(X_3) \rangle$ and then send $X_i \rightarrow P_i$ on the boundary of AdS as usual, then we will recover the 3-pt CFT correlators of a current or the stress-energy tensor from equation (10.32). The normalization of these 3-pt correlators will then either be the charge or the energy in Planck units, for A_μ and $T_{\mu\nu}$, respectively.

Let us give a quick derivation/interpretation of the CFT Ward identity for a gauge field A_M in AdS in order to better understand the physics. The Ward identity can be re-expressed as the statement that

$$\iint_{B_m} d^d y \frac{\partial}{\partial y^\mu} \langle J^\mu(y) \mathcal{O}_1(x_1) \mathcal{O}_2(x_2) \cdots \mathcal{O}_n(x_n) \rangle = -i \langle \mathcal{O}_1(x_1) \cdots [G_m \mathcal{O}_m(x_m)] \cdots \mathcal{O}_n(x_n) \rangle \quad (10.34)$$

where B_m is a d -dimensional ball containing the point x_m and none of the others; we can make the ball B_m as small as we like. Now the left hand side can be re-expressed as a surface integral

$$\iint_{\partial B_m} d^{d-1} s_\mu \langle J^\mu(y) \mathcal{O}_1(x_1) \mathcal{O}_2(x_2) \cdots \mathcal{O}_n(x_n) \rangle = -i \langle \mathcal{O}_1(x_1) \cdots [G_m \mathcal{O}_m(x_m)] \cdots \mathcal{O}_n(x_n) \rangle \quad (10.35)$$

where the normal to the surface of B_m is contracted with J_μ . Now let us use the definition of the current from the bulk gauge field, equation (10.13), to write this as

$$\lim_{z \rightarrow 0} z^{3-d} \iint_{\partial B_m} d^{d-1} s_\mu \langle F_{\mu z}(y, z) \mathcal{O}_1(x_1) \cdots \mathcal{O}_n(x_n) \rangle = -i \langle \mathcal{O}_1(x_1) \cdots [G_m \mathcal{O}_m(x_m)] \cdots \mathcal{O}_n(x_n) \rangle \quad (10.36)$$

Finally, let us interpret this equation by viewing all of the operators as $z \rightarrow 0$ limits of bulk fields $\phi_i(x, z)$. In that case, the surface integral, pictured in figure [12](#), is just

$$\int_{\partial B_m} d^{d-1} s_A E^A(s, z) = q_m \quad (10.37)$$

where the integral is being performed at fixed small z in AdS. This is the integral of the AdS ‘electric field’ $E^A = F^{Az}$ around the state created by the m th operator. The other particles/blobs/states are irrelevant because they are infinitely far away in the limit that $z \rightarrow 0$. So the Ward identity in the CFT just corresponds to Gauss’s law in AdS!

Now we will review an old argument of Weinberg that proves that in flat spacetime, massless spin one particles must couple to conserved charges while massless spin two particles must couple to energy-momentum. The argument also shows that higher-spin massless particles cannot couple in a way that would lead to long-range forces. This classic theorem goes a long way towards explaining why the spectrum of elementary particles we observe ends at the spin two graviton. The argument itself is independent of AdS/CFT, but it is probably the best general argument one can give in order to explain why massless spin one and spin two particles couple in a universal way at long distances.

We begin by considering some scattering amplitude $\mathcal{M}_n(p_i)$ for n particles in flat spacetime. Given this n -particle process, what is the amplitude for emitting an additional very soft photon with momentum q , so that $q \cdot p_i \ll p_j \cdot p_k$? This question is of great importance because the exchange of particles with small energy and momentum is what leads to long-range forces.

If the soft photon is emitted from any external line in the scattering amplitude then we find a contribution

$$\mathcal{M}_n(p_i) \sum_{m=1}^n \left(\frac{e_m p_m^\mu}{p_m \cdot q - i\epsilon} \epsilon_\mu \right) \quad (10.38)$$

where ϵ_μ is the polarization vector for the soft photon, and e_m is the coupling constant for the m th particle. One might also wonder if the photon can couple to vectors made from the spin of the m th particle, but this will also not give a pole as $q \rightarrow 0$ since multipole moments cannot give long-distance effects. There will be other contributions from emission of the photon from internal parts of the diagram, but these will not have a pole as $q \rightarrow 0$, so they will be subdominant in the soft limit. The soft emission amplitude is dominated by contributions from long time periods of order $1/q$ associated with the nearly free propagation of the initial and final state particles.

The pole we have found literally corresponds to a sum over contributions of the form

$$\begin{aligned} e^{ie_m \int dt \hat{n}_m^\mu A_\mu(x(t))} - 1 &\approx ie_m \int dt \hat{n}_m^\mu A_\mu(x(t)) + \dots \\ &= ie_m \epsilon \cdot \hat{n} \int dt \hat{e}^{iq \cdot x(t)} \end{aligned} \quad (10.39)$$

which is just the amplitude for a charged particle to source an electromagnetic field, with $x(t) = t\hat{n}$, because the action for such charged particle is just the integral of $A_\mu \dot{x}^\mu(t)$ along the world-line of

the particle. This gives the contribution

$$\sum_{m=1}^n \left(\epsilon_m \cdot \hat{n}_m \int_0^\infty dt e^{it\hat{n}_m \cdot q} \right) \quad (10.40)$$

where the integral over time proceeds from the scattering process out to infinity, and $\hat{n}_m$ is a unit vector pointing in the direction of p_m . Performing the integral gives the soft emission amplitude.

Consider the Lorentz transformation properties of this scattering amplitude. In particular, note that the polarization tensor ϵ_μ for a massless particle does not transform as a vector, instead it transforms as

$$\epsilon_\mu \rightarrow \Lambda_\mu^\nu \epsilon_\nu + \alpha(q, \Lambda) q_\mu \quad (10.41)$$

where α is some function. So in other words, the polarization tensor shifts by q_μ under general Lorentz transformations. This means that to recover a Lorentz invariant S-Matrix we must have

$$\mathcal{M}_n(p_i) \sum_{m=1}^n \left(\frac{e_m p_m^\mu}{p_m \cdot q - i\epsilon} q_\mu \right) = \mathcal{M}_n(p_i) \sum_{m=1}^n e_m = 0 \quad (10.42)$$

The sum of the charges of the scattering particles must vanish. In other words, *a massless spin one particle must couple to a conserved charge*.

Consider the Lorentz transformation properties of this object. In particular, note that the polarization tensor ϵ_μ for a massless particle does not transform as a vector, instead it transforms as

$$\epsilon_\mu \rightarrow \Lambda_\mu^\nu \epsilon_\nu + \alpha(q, \Lambda) q_\mu \quad (10.43)$$

where α is some function. So in other words, the polarization tensor shifts by q_μ under general Lorentz transformations. This means that to recover a Lorentz invariant S-Matrix we must have

$$\mathcal{M}_n(p_i) \sum_{m=1}^n \frac{e_m p_m^\mu}{p_m \cdot q - i\epsilon} q_\mu = \mathcal{M}_n(p_i) \sum_{m=1}^n e_m = 0 \quad (10.44)$$

The sum of the charges of the scattering particles must vanish. In other words, *a massless spin one particle must couple to a conserved charge*.

In the case of a massless spin two particle, the soft emission amplitude must be

$$\mathcal{M}_n(p_i) \sum_{m=1}^n \left(\frac{G_m p_m^\mu p_m^\nu}{p_m \cdot q - i\epsilon} \epsilon_{\mu\nu} \right) \quad (10.45)$$

where $\epsilon_{\mu\nu}$ is the polarization tensor of the graviton, and G_m are its coupling constants. The polarization tensor transforms as

$$\epsilon_{\mu\nu} \rightarrow \Lambda_\mu^\alpha \Lambda_\nu^\beta \epsilon_{\alpha\beta} + \alpha_\mu q_\nu + \alpha_\nu q_\mu \quad (10.46)$$

and so in order to have a Lorentz invariant S-Matrix, we must have

$$\sum_{m=1}^n G_m p_m^\mu q_\mu = 0 \quad (10.47)$$

But the only way this can be zero for all q_μ without putting a restriction on the allowed momenta p_μ for scattering is if $G_m = G$ is a universal constant, so that the equation is satisfied due to momentum conservation. Thus *a massless spin two particle must couple universally to energy-momentum*. Repeating this argument for higher spin massless particles shows that they cannot couple in a way that will give rise to long-range forces, which necessarily involve propagators with poles in the soft $q \rightarrow 0$ limit.

As promised, we have shown that massless particles must couple to conserved charges or momenta, meaning that they couple to both fundamental and composite particles/objects/states with universal 3-pt vertices. Interpreted in AdS, these universal 3-pt vertices are just Feynman diagrams that give rise to universal correlators for J_μ and $T_{\mu\nu}$ with any operators $\mathcal{O}_1$ and $\mathcal{O}_2$.

11 AdS Black Holes and Thermality

Let us begin with a thought experiment. Consider a CFT living on the Lorentzian cylinder $R \times S^{d-1}$, and let us slowly heat it up. Since the dilatation operator serves as the Hamiltonian, as we increase the temperature the CFT will be in a state characterized by larger and larger operator/state dimensions.

Since the AdS Hilbert space is identical to that of the CFT, we can interpret our hot CFT as a thermal state in AdS. But what will this state consist of? At low temperatures we will just have a thermal gas made up of the light particles in AdS. Due to the AdS geometry, these particles will mostly move around near the center of AdS, with only occasional excursions further away. This means that as we increase the temperature, we will be cramming more and more energy into a region of roughly fixed size. In the presence of dynamical gravity, this cannot go on forever – eventually, at some critical temperature T_c , the hot gas will collapse to form a black hole in AdS. *Our thought experiment shows that black holes in AdS must correspond to a hot CFT!*

Now let us consider the thermodynamics more carefully. We will use the microcanonical ensemble, fixing the total energy of the system, as we will see that the microcanonical and canonical ensembles look somewhat different in equilibrium, due to the existence of a phase with negative specific heat.

Let us nevertheless use the notation T to denote the average energy of a graviton in the gas of gravitons. In that case the occupation number of the n, ℓ mode will be

$$\frac{1}{1 - e^{-\beta(\Delta+2n+\ell)}} \approx \frac{R_{AdS}T}{\Delta + 2n + \ell} \quad (11.1)$$

when this ratio is large. For gravitons $\Delta = d$, and there will be of order T^d such modes with $\Delta + 2n + \ell < T$, so we expect that

$$E_{tot} \sim T^{d+1} R_{AdS}^d \quad (11.2)$$

for a gas of gravitons with $T \gg 1/R_{AdS}$. This is what we would expect for a box of size R_{AdS} in $d+1$ dimensions. As we increase T , E_{tot} will eventually grow so large that its corresponding Schwarzschild radius will be of order R_{AdS} , at which point the gas will collapse to form a large black hole. In $d+1$ dimensional flat spacetime the Schwarzschild radius would be

$$R_S = (G_{d+1} E_{tot})^{\frac{1}{d-2}} \quad (11.3)$$

where $G_{d+1} = (M_{d+1})^{1-d}$ is the Newton constant and M_{d+1} is the $d+1$ dimensional Planck mass. If we are studying a homogeneous gas, collapse will first occur on the largest scales, because gravitational effects are cumulative with energy. This suggests that a black hole only forms once $R_S \sim R_{AdS}$, when the energy E_{tot} is within its own Schwarzschild radius, so that

$$T_{collapse} \sim (G_{d+1} R_{AdS}^2)^{-\frac{1}{d+1}} \quad (11.4)$$

For example, in the case of AdS_4 this would be a temperature $\sqrt{M_4/R_{AdS}}$, at an intermediate energy between the AdS scale and the Planck scale.

However, this analysis in the microcanonical ensemble has been rather misleading. First of all, in most examples considered in the literature there are other regimes aside from the gas of gravitons and the large AdS black hole – there is also a Hagedorn stringy regime, and a regime where small black holes (much smaller in size than R_{AdS}) can be produced. Furthermore, the parametric relationships change due to the presence of extra dimensions of R_{AdS} size. Finally, this analysis was misleading because in actuality, the large AdS black holes have such a large entropy that in the canonical ensemble they start to dominate already at $T \sim 1/R_{AdS}$. In other words, the huge number of different AdS black hole states is able to make up for their $e^{-\beta E}$ Boltzmann factor.

11.1 The Hagedorn Temperature and Negative Specific Heat

Let us briefly discuss the Hagedorn phenomenon, since it is interesting and simple. The canonical ensemble partition for any physical system is

$$Z[\beta] = \sum_{\psi} e^{-\beta E_{\psi}} \quad (11.5)$$

where as usual $\beta = 1/T$, and the sum is over all quantum mechanical states ψ in the theory. The sum can be re-organized as

$$Z[\beta] = \sum_E e^{S(E) - \beta E} \quad (11.6)$$

where $e^{S(E)}$ is the number of states with energy E – this defines the entropy as a function of E . Note that if $S(E)$ grows faster than $\beta_* E$ for some β_* in the limit that $E \rightarrow \infty$, then this sum diverges and partition function is ill defined!

To understand this phenomenon better, note that the expectation value of the (total) energy is

$$\langle E \rangle = -\frac{\partial}{\partial \beta} \log Z[\beta] \quad (11.7)$$

Let us consider what happens if we have the borderline case $S(E) \rightarrow \beta_* E$ in the limit that $E \rightarrow \infty$. Then we find

$$Z[\beta] \approx \sum_E e^{(\beta_* - \beta)E} = \frac{1}{\beta_* - \beta} \quad (11.8)$$

so we find that the expectation value of the energy is

$$\langle E \rangle = \frac{1}{\beta_* - \beta} \quad (11.9)$$

Thus we see that as $\beta \rightarrow \beta_*$, the expectation value of the energy diverges. Reversing the logic, it takes an infinite amount of energy to achieve the temperature $T_* = 1/\beta_*$. *This T_* is the Hagedorn temperature, and it is the maximum possible temperature for the system.* Hagedorn behavior is relevant because it is the thermodynamic behavior of a gas of strings. Thus if a stringy regime exists in AdS, then there will be a range of temperatures showing a Hagedorn behavior.

There is an even more extreme behavior, where $S(E)$ grows more rapidly than $\beta_* E$ for any fixed β_* . For example, in $d + 1$ dimensional flat spacetime, black holes have an entropy

$$S(R_S) \sim \frac{R_S^{d-1}}{G_{d+1}} \quad (11.10)$$

which translates into

$$S(E) \sim (G_{d+1})^{\frac{1}{2-d}} E^{\frac{d-1}{d-2}} \quad (11.11)$$

This means that $S(E)$ grows with a power of E greater than one. This implies the famous statement that flat space black holes have a specific heat

$$\frac{\partial}{\partial T} \langle E \rangle < 0 \quad (11.12)$$

Increasing the temperature actually decreases the black hole mass, because the black hole entropy dominates the free energy. This is relevant to AdS because this is the behavior of ‘small’ AdS black holes, with $R_S \ll R_{AdS}$. So these states can also contribute to the partition function in AdS/CFT.

11.2 Thermodynamic Details from AdS Geometry

To make all of this more precise, let us study the Euclidean CFT on $S^1 \times S^{d-1}$, following [21]. Now we will be using the canonical ensemble, so we will be studying states that minimize the free energy, which includes the effects of entropy. Compactifying the Euclidean time direction with period β corresponds to studying the partition function of the theory on S^{d-1} at a temperature $T = 1/\beta$, as usual. Note that the S^{d-1} also has some radius β' , and since we are studying a CFT, the physics only depends on the ratio of β/β' . For convenience we can fix $\beta' = 1$, although one can also find a scaling limit [21] where $\beta' \rightarrow \infty$, so that we get a Euclidean CFT on $S^1 \times R^{d-1}$.

We can write Euclidean AdS in the coordinates

$$ds_{AdS}^2 = \left(\frac{r^2}{R_{AdS}^2} + 1 \right) \left(dt^2 + \frac{dr^2}{\frac{r^2}{R_{AdS}^2} + 1} + r^2 d\Omega^2 \right) \quad (11.13)$$

This coordinate system follows immediately from our usual global coordinates by taking $r = R_{AdS} \tan \rho$. We have introduced this particular coordinate system so that we can naturally write the AdS-Schwarzschild solution

$$ds_{Sch}^2 = \left(\frac{r^2}{R_{AdS}^2} + 1 - \frac{\omega_d M}{r^{d-2}} \right) \left(dt^2 + \frac{dr^2}{\frac{r^2}{R_{AdS}^2} + 1 - \frac{\omega_d M}{r^{d-2}}} + r^2 d\Omega^2 \right) \quad (11.14)$$

where M is the physical mass of a black hole, and the constant

$$\omega_d = \frac{16\pi G_{d+1}}{(d-1)\text{Vol}(S^{d-1})} \quad (11.15)$$

is written in terms of the $d+1$ dimensional Newton constant G_{d+1} . There is a horizon at the largest root r_+ of

$$\frac{r^2}{R_{AdS}^2} + 1 - \frac{\omega_d M}{r^{d-2}} = 0 \quad (11.16)$$

The physical space ends at r_+ , since after this point the metric is no longer Euclidean. The metric for the full $S^1 \times S^{d-1}$ will only be smooth if we must avoid conical singularities in t . In the vicinity of r_+ , we can write $r = r_+ + \delta r$ and the metric looks like

$$ds^2 \approx \delta r \left(\frac{dr_+^2 + (d-2)R_{AdS}^2}{R_{AdS}^2 r_+} \right) dt^2 + \frac{1}{\delta r} \left(\frac{R_{AdS}^2 r_+}{dr_+^2 + (d-2)R_{AdS}^2} \right) \left(d(\delta r)^2 + r_+^2 d\Omega^2 \right) \quad (11.17)$$

We can define a new coordinate $x = 2\sqrt{\delta r}$, in which case we see that we have the metric for a cone in x and t . To avoid a conical singularity at $x = 0$, we must take the periodicity in $t \rightarrow t + \beta$ with

$$\beta = \frac{4\pi R_{AdS}^2 r_+}{dr_+^2 + (d-2)R_{AdS}^2} \quad (11.18)$$

This result means that *the temperature $1/\beta$ is fixed in terms of the black hole mass M* . Furthermore, β has a maximum as a function of r_+ , so we see that for $\beta > \beta_{min}$, which means

$$T < T_{min} = \frac{1}{R_{AdS}} \frac{\sqrt{d(d-2)}}{2\pi} \quad (11.19)$$

there is no thermodynamically stable black hole. In other words, when $T < T_{min}$ the thermodynamically stable state of the spacetime will be a gas of particles (and perhaps strings and small black holes) in otherwise empty AdS, while for $T > T_{min}$ we can also have the AdS-Schwarzschild spacetime describing a large black hole.

The phase transition between these two states is called the Hawking-Page phase transition. We can compute the transition temperature by studying the difference in Free Energies between AdS and the AdS-Schwarzschild solution. It is computed in [21], and the result is

$$F = \frac{\text{Vol}(S^{d-1})(R_{AdS}^2 r_+^2 - r_+^{d+1})}{4G_{d+1}(dr_+^2 + (d-2)R_{AdS}^2)} \quad (11.20)$$

We see that in the large r_+ limit the free energy of AdS-Schwarzschild dominates. The two free energies are equal when $r_+ = R_{AdS}$, so that we have a phase transition temperature of

$$T_c = \frac{1}{R_{AdS}} \frac{d-1}{2\pi} \quad (11.21)$$

In other words, heating up the AdS/CFT system above this temperature (in the canonical ensemble) will result in an equilibrium state consisting of a large AdS black hole.

12 Partition Functions, Sources, and Holographic RG

12.1 Various Dictionaries

In these notes we have defined CFT operators directly in terms of limits of bulk fields, via

$$\mathcal{O}(t, \Omega) = \lim_{\epsilon \rightarrow 0} \frac{\phi(t, \rho(\epsilon; t, \Omega), \Omega)}{\epsilon^\Delta} \quad (12.1)$$

where $\rho(\epsilon; t, \Omega) = \frac{\pi}{2} - \epsilon f(t, \Omega)$ and the function $f(t, \Omega)$ determines the geometry of the boundary on which the CFT lives. This allowed us to compute correlators for CFT operators $\mathcal{O}$ in terms of limits of AdS correlators using AdS Feynman diagrams.

In fact, there are two AdS/CFT ‘dictionaries’, and our choice has been the less common one in the literature. Our choice has the great advantage that it makes it obvious how to identify the Hilbert spaces of the AdS and CFT theories, and it also seems more immediate and intuitive to me. The moreconventional dictionary has the advantage that it is easier to use when considering finite deformations of the CFT.

The conventional dictionary involves computing the AdS path integral, where the bulk fields $\phi(t, \rho, \Omega) \rightarrow \phi_0(t, \Omega)$ for some classical field configuration $\phi_0(t, \Omega)$ on the boundary of AdS. This gives a partition functional

$$Z_{AdS}[\phi_0] = \int_{\phi_0} D\phi e^{-S_{AdS}[\phi]} \quad (12.2)$$

Alternatively, we could imagine computing a CFT partition function

$$Z_{CFT}[\phi_0] = \int_{\phi_0} D\chi e^{-S_{CFT}[\chi] + \int d^d x \phi_0(x) \mathcal{O}(x)} \quad (12.3)$$

where $\mathcal{O}(x)$ is a CFT operator and $\phi_0(x)$ is a classical source for this operator. The claim of the usual AdS/CFT dictionary is that

$$Z_{AdS}[\phi_0] = Z_{CFT}[\phi_0] \quad (12.4)$$

In particular, this means that we can compute CFT correlators by differentiating Z_{AdS} with respect to ϕ_0 and then setting it to zero, giving

$$\langle \mathcal{O}(x_1) \cdots \mathcal{O}(x_n) \rangle = \frac{\delta}{\delta \phi_0(x_1)} \cdots \frac{\delta}{\delta \phi_0(x_n)} Z_{AdS}[\phi_0]_{\phi_0=0} \quad (12.5)$$

This form of the dictionary is particularly useful because it tells us how to compute the partition function of the CFT for finite values of the sources ϕ_0 , which we can then interpret as coupling constants, since in effect, $\phi_0 \mathcal{O}$ has been added to the CFT action.

Let us see why these two dictionaries are equivalent, following [22]. To make the argument, let us use the Poincaré patch for convenience. We will introduce two reference surfaces, one at some $z = \epsilon$ and another at $z = \ell$, with $\epsilon \ll \ell \ll 1$. The bulk path integral can be broken up as

$$\begin{aligned} Z_{AdS}[\phi_0] &= \int \left(\mathcal{D}\phi_{z<\ell} e^{-S_{z<\ell}[\phi]} \int \mathcal{D}\tilde{\phi}_{z=\ell} \int \left(\mathcal{D}\phi_{z>\ell} e^{-S_{z>\ell}[\phi]} \right. \right. \\ &= \int \left(\mathcal{D}\tilde{\phi} \Psi_{UV}[\phi_0, \tilde{\phi}; \epsilon, \ell] \Psi_{IR}[\tilde{\phi}, \ell] \right) \end{aligned} \quad (12.6)$$

where $\tilde{\phi}(x)$ is the field $\phi(\ell, x)$ at the surface $z = \ell$ (so it only depends on x , not z), and we have written the path integrals over the intervals $[\epsilon, \ell]$ and $[\ell, \infty]$ in terms of a ‘UV’ and ‘IR’ wavefunction. Only the UV wavefunction knows about ϕ_0 . Our usual dictionary is defined by

$$\langle \mathcal{O}(x_1) \cdots \mathcal{O}(x_n) \rangle = \lim_{\ell \rightarrow 0} \ell^{-n\Delta} \int \left(\mathcal{D}\tilde{\phi} \Psi_{IR} \tilde{\phi}(x_1) \cdots \tilde{\phi}(x_n) \Psi_{UV} \right)_{\phi_0=0, \epsilon=0} \quad (12.7)$$

The other dictionary based on partition functions will be equivalent if

$$\lim_{\ell \rightarrow 0} \int \mathcal{D}\tilde{\phi} \Psi_{IR} \left[\left(\frac{\delta}{\delta \phi_0(x_1)} \cdots \frac{\delta}{\delta \phi_0(x_n)} \Psi_{UV} \right) \right]_{\phi_0=0} \propto \lim_{\ell \rightarrow 0} \ell^{-n\Delta} \int \left(\mathcal{D}\tilde{\phi} \Psi_{IR} \tilde{\phi}(x_1) \cdots \tilde{\phi}(x_n) \Psi_{UV} \right)_{\phi_0=0} \quad (12.8)$$

The ‘UV’ region plays a simple role – it is only present so that we can differentiate Ψ_{UV} to obtain the field $\tilde{\phi}$ at the intermediate surface at $z = \ell$.

We have previously discussed the fact that AdS interactions become irrelevant as we approach the boundary, so we can assume that Ψ_{UV} is computed purely from the free theory. In this case we can just calculate it directly. It is

$$\Psi_{UV}[\phi_0, \tilde{\phi}] = \int_{\phi(\epsilon)=\phi_0 \epsilon^{d-\Delta}}^{\phi(\ell)=\tilde{\phi}} \mathcal{D}\phi \exp \left[-\frac{1}{2} \int_{\epsilon}^{\ell} d^d x dz \sqrt{-g} ((\nabla \phi)^2 - m^2 \phi^2) \right] \quad (12.9)$$

This is just a Gaussian integral, so we can evaluate it by solving the equations of motion for ϕ with the correct boundary conditions and then inputting the solution back into the action. The equations of motion

$$\nabla^2 \phi - m^2 \phi = 0 \quad (12.10)$$

will have two solutions, which we can refer to as $G_k^\epsilon(z, x)$ and $G_k^\ell(z, x)$. One can compute these G_k in terms of bessel functions in the case of pure AdS. We choose a linear combination of solutions so that $G_k^\epsilon(\epsilon, x) = 1$ and $G_k^\epsilon(\ell, x) = 0$, and vice versa for $G_k^\ell(z, x)$. The label k is a momentum label for the x_μ coordinates, although in principle it could be any label associated with a complete basis for solutions. Now the solution for $\phi(z, k)$ (we have gone to momentum space for convenience) with the desired boundary conditions is simply

$$\phi(z, k) = \phi_0(k) \epsilon^{d-\Delta} G_k^\epsilon(z) + \tilde{\phi}(k) G_k^\ell(z) \quad (12.11)$$

When we evaluate the action with this solution it vanishes up to a total derivative term, and so the result only involves boundary terms of the form $\phi \nabla_z \phi$ evaluated at ϵ and ℓ . Terms quadratic in ϕ_0 or $\tilde{\phi}$ will not contribute when we differentiate Ψ_{UV} and then set $\phi_0 \rightarrow 0$. Thus the relevant terms will be

$$\Psi_{UV}[\phi_0, \tilde{\phi}] \propto \exp \left[-\frac{1}{2} \int \left(d^d k \phi_0(-k) \tilde{\phi}(k) (\epsilon^{-\Delta} \nabla_z G_k^\ell(\epsilon) - \ell^{-d} \nabla_z G_k^\epsilon(\ell)) \right) \right] \quad (12.12)$$

where we note that the z derivatives of the G solutions can be non-vanishing on both ends of the interval. Evaluating the derivatives for the actual solutions gives

$$\Psi_{UV} = \exp \left[-(2\Delta - d) \ell^{-\Delta} \int \left(d^d x \phi_0(x) \tilde{\phi}(x) + \dots \right) \right] \quad (12.13)$$

where the ellipsis denotes other terms that either vanish more quickly as $\ell \rightarrow 0$ or that are quadratic in ϕ_0 or $\tilde{\phi}$. So we see that

$$\left(\frac{1}{\Psi_{UV}} \frac{\delta \Psi_{UV}}{\delta \phi_0} \right)_{\phi_0=0} = \ell^{-\Delta} (2\Delta - d) \tilde{\phi}(x) \quad (12.14)$$

as desired, and the equivalence between the two dictionaries has been proven.

12.2 Fefferman-Graham and the Holographic RG Formalism

12.3 Breaking of Conformal Symmetry

In AdS/CFT, the conformal symmetry of the CFT is realized geometrically as the group of AdS isometries. If we want to use holography to study theories that are not conformally invariant, then we must study quantum field theories in spacetimes with less symmetry than AdS. In order to do minimal violence to the conformal symmetry, we can study Poincaré invariant theories that are

approximately conformal in some regime, for example at very high or very low energies. However, we should note that recently AdS/CFT type ideas have been used to attempt to study toy non-relativistic systems in order to make contact with condensed matter physics.

Since we wish to preserve Poincaré symmetry, it is most natural to view AdS in the coordinates

$$ds^2 = \frac{1}{z^2} (dz^2 + \eta^{\mu\nu} dx_\mu dx_\nu) \quad (12.15)$$

Now we would like to break conformal symmetry. The laziest option is to simply declare, by fiat, that there exist membranes, conventionally called either the ‘UV Brane’ and the ‘IR Brane’, that end spacetime at some finite positions $z_{UV} \ll z_{IR}$. In this case the AdS isometries will only be broken by the presence of these membrane, in the same way that a brick wall breaks translation invariance.

Note that the positions z_{UV} and z_{IR} transform under dilatations

$$D = z\partial_z + x^\mu\partial_\mu \quad (12.16)$$

as lengths, so they specify a short and long distance scale, respectively. These obviously correspond to energy scales $\mu_{UV} = 1/z_{IR}$ and $\mu_{IR} = 1/z_{UV}$.

Let us first consider the case when $z_{IR} = \infty$, so that the IR brane is absent and we only have a UV brane. If we like *we can add whatever d dimensional quantum field theory we like by including an action localized to the UV brane.* We can (roughly) interpret the UV Brane as setting a UV cutoff μ_{UV} for the CFT, and the fields localized to the UV brane represent some other theory that couples to the CFT. In particular, when we quantize gravity in this space, the boundary conditions (or UV brane action) for the graviton will be important. If we do not impose Dirichlet boundary conditions then the graviton will have a normalizable zero mode, and so *our theory will include d -dimensional gravity.*

Now let us consider an IR brane at z_{IR} in the absence of a UV Brane. In this case there can be no dynamical gravity, or any other degrees of freedom aside from the CFT. However, what we do have is some kind of conformal symmetry breaking associated with the energy scale μ_{IR} . Since this IR brane comes out of nowhere, we do not have any particular insight into the nature of the conformal symmetry breaking in the CFT. However, what we can say is that the IR brane will fluctuate in position (do to gravitational effects if nothing else), and in the absence of other fields these fluctuations will be a massless goldstone boson mode. This mode is often called the radion or dilaton; it is the goldstone boson of broken conformal symmetry.

We can also put d -dimensional quantum fields directly on the IR brane. These can be very roughly interpreted as composite or emergent degrees of freedom that arise when conformal symmetry is broken. Again, this is rather ad hoc from the CFT point of view.

With both an IR and a UV brane, we have both a UV cutoff for the CFT and conformal symmetry breaking in the IR. In this case the dynamics become a bit more interesting, because we can stabilize the position z_{IR} by giving fields in the ‘slice’ of AdS appropriate boundary conditions. The idea is that the configuration of the AdS fields will depend on z_{IR} , and therefore the total bulk energy will depend on z_{IR} . Minimizing this energy determines the equilibrium value of z_{IR} . The original implementation of these ideas was called the Goldberger-Wise mechanism. It was interesting

because this setup, when applied to weak-scale physics, is called a Randall-Sundrum model, and it has the potential to explain the hierarchy problem.

One can consider more involved and also much more physically plausible models of conformal symmetry breaking where the AdS metric itself is modified to a metric of the form

$$ds^2 = \frac{1}{A(z)^2} (dz^2 + \eta^{\mu\nu} dx_\mu dx_\nu) \quad (12.17)$$

where $A(z)$ is some function that asymptotes to $A(z) \sim z$ for $z \sim 0$. Such a model represents a theory that is approximately conformal in the UV, where $z \rightarrow 0$, but that breaks conformal symmetry in a more complex and interesting way. Such models have been studied in great detail in the literature, both phenomenologically and as precise string theory backgrounds.

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An Overview of AdS/CFT

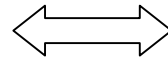
David Tong



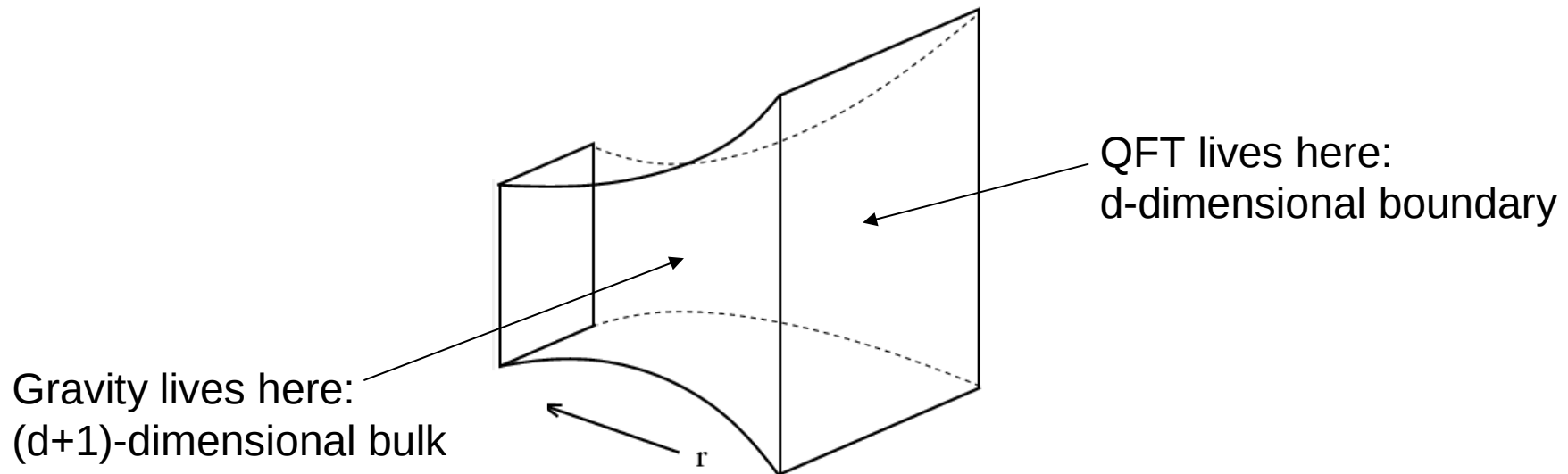
Imperial, January 2011

The AdS/CFT Correspondence

Strongly interacting QFT
in d -dimensions

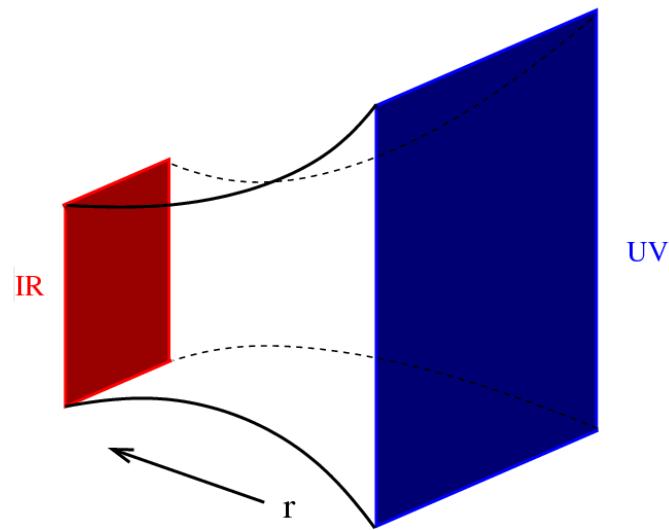


General relativity in (at least)
 $(d+1)$ -dimensions



$$[G,R]=0$$

- The extra direction, r , should be thought of as *energy scale*.
- Objects occurring on different scales live in different r -slices of bulk
- AdS/CFT is the geometrization of Wilsonian RG flow.



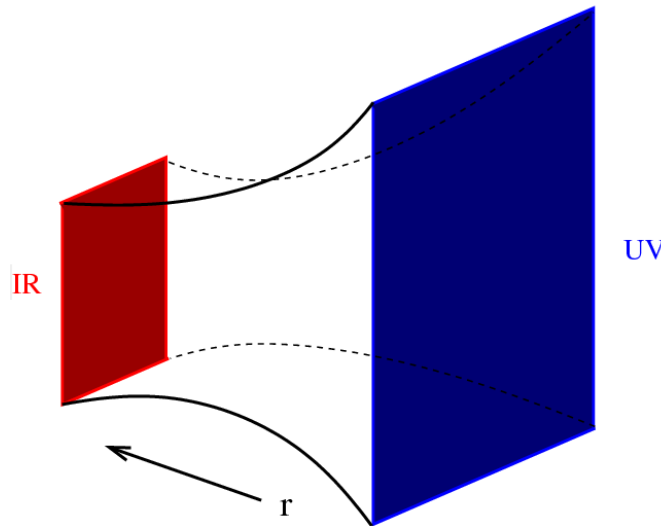
What is AdS?

- AdS = Anti-de Sitter Space

$$ds^2 = \frac{L^2}{r^2} (dr^2 + \eta_{\mu\nu} dx^\mu dx^\nu)$$

slice of Minkowski space

horizon at
 $r \rightarrow \infty$

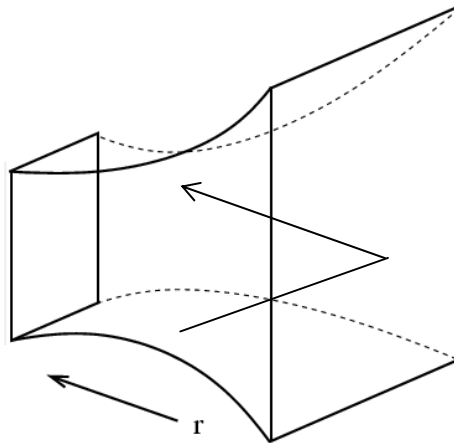


boundary at $r = 0$

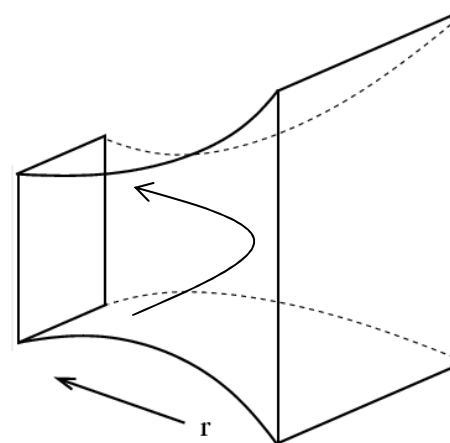
Properties of AdS

$$ds^2 = \frac{L^2}{r^2} (dr^2 + \eta_{\mu\nu} dx^\mu dx^\nu)$$

- Solution of Einstein's equations with *negative* cosmological constant.
- Isometry group is translations + $SO(d,2)$
- Geodesics



lightlike geodesics hit the boundary



timelike geodesics don't



The Basics of AdS/CFT

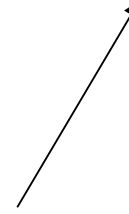


Generating Function

$$Z_{\text{QFT}}[\phi_0] = \int \mathcal{D}A \exp \left(\frac{i}{\hbar} S_{\text{QFT}}[A] + \phi_0 \mathcal{O}(A) \right)$$



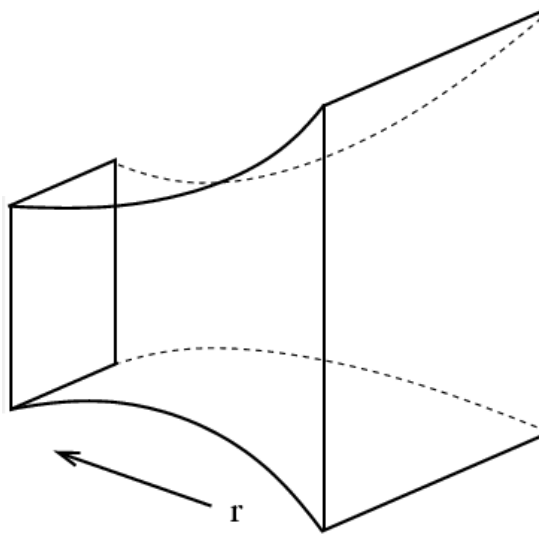
source



operator

Idea: Make the Sources Come Alive

$$\phi(\vec{x}, r) \rightarrow \phi_0(\vec{x})$$



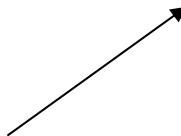
How to Calculate: GKPW Formula

(Gubser, Klebanov, Polyakov; Witten)

$$Z_{\text{QFT}}[\phi_0] = Z_{\text{Quantum Grav}}[\phi \rightarrow \phi_0]$$

$$\approx e^{iS_{\text{gravity}}(\phi)} \Big|_{\phi \rightarrow \phi_0}$$

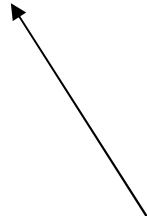
when classical
gravity is valid



on-shell action
for AdS bulk



boundary conditions



AdS/CFT Will Not Solve Your Favourite Theory

- Tricky Part: Find the map

$$S_{\text{QFT}} \rightarrow S_{\text{gravity}}$$

- We have several classes of well explored examples, but no proof.
 - e.g. Maximally supersymmetry Yang-Mills = IIB string theory on $AdS_5 \times S^5$
- Simpler method: take your favourite gravity theory to *define* the QFT on the boundary
- Classical gravity means large N
 - Matrix large N, not vector large N



Building the Dictionary



The Dictionary

Bulk

Fields, $\phi(\vec{x}, r)$

Mass, m

Spin, Charge

Boundary

Operators, $\mathcal{O}(\vec{x})$

Dimension, $\Delta(\Delta - d) = m^2 L^2$

Spin, Charge

The Dictionary

Bulk

Fields, $\phi(\vec{x}, r)$

Mass, m

Spin, Charge

Boundary

Operators, $\mathcal{O}(\vec{x})$

Dimension, $\Delta(\Delta - d) = m^2 L^2$

Spin, Charge

Some special fields

Metric, $g_{\mu\nu}$

Gauge Fields, A_μ

Stress Tensor, $T_{\mu\nu}$

Conserved Current, J_μ

A Scalar Field

- The Lagrangian for a scalar field in AdS is

$$S_{\text{scalar}} = -\frac{1}{2} \int d^{d+1}x \sqrt{-g} \left(g^{AB} \partial_A \phi \partial_B \phi + m^2 \phi^2 \right)$$

- Special property of AdS: stability requires

$$m^2 \geq -\frac{d^2}{4L^2}$$

(Breitenlohner-Freedman bound)

- Relevant boundary operator: $m^2 < 0$
- Marginal boundary operator: $m^2 = 0$
- Irrelevant boundary operator: $m^2 > 0$

Solution near the Boundary

- Solve the equation of motion near $r=0$

$$\phi(\vec{x}, r) \rightarrow \left(\frac{r}{L}\right)^{\Delta_-} [\phi_0(\vec{x}) + \dots] + \left(\frac{r}{L}\right)^{\Delta_+} [\phi_1(\vec{x}) + \dots]$$

$$\Delta_{\pm} = \frac{d}{2} \pm \sqrt{\left(\frac{d}{2}\right)^2 + m^2 L^2}$$

- $\phi_0(\vec{x})$ is source. (I lied before!)
- What is interpretation of $\phi_1(\vec{x})$

Response

- Can use GKPW formula to show that

$$\phi_1(\vec{x}) = \langle \mathcal{O}(\vec{x}) \rangle$$

- i.e. response in presence of source $\phi_0(\vec{x})$

Every Single “Holographic” Calculation Ever...

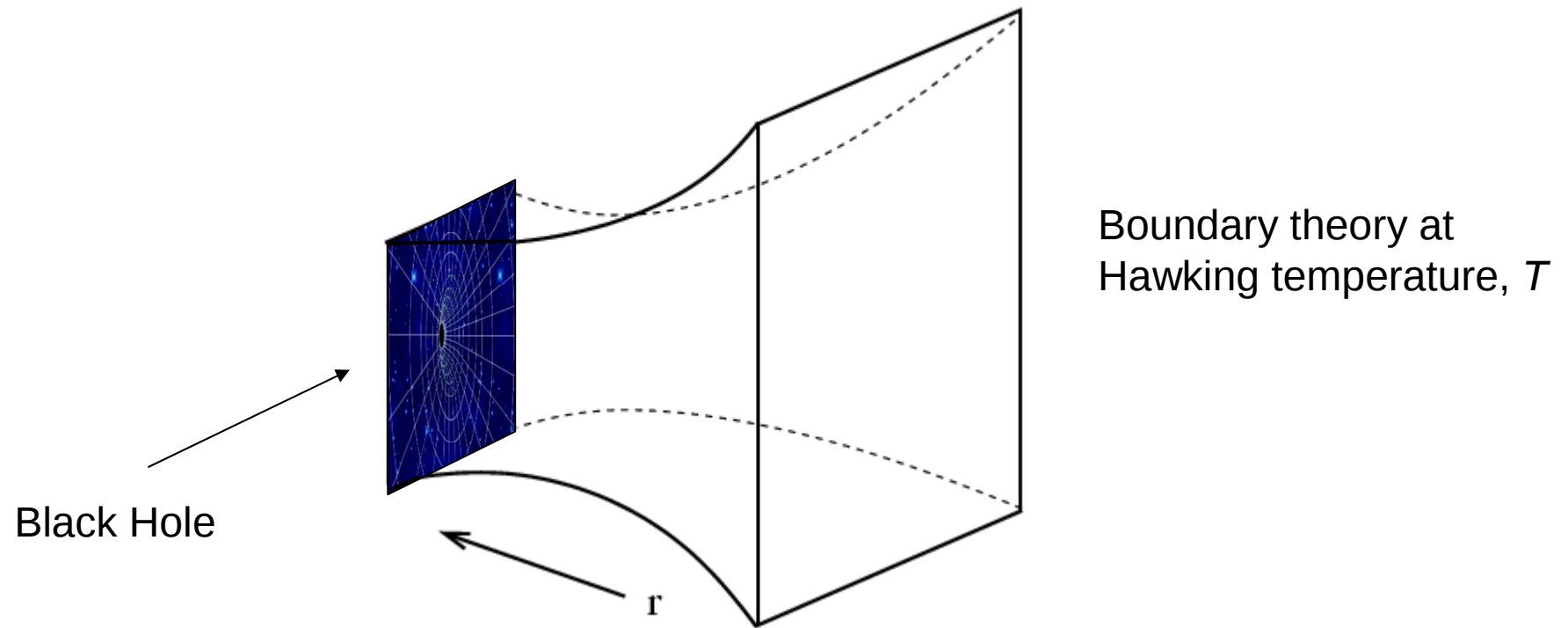
- Fix the source $\phi_0(\vec{x})$
- Fix another boundary condition in the interior of space
 - regularity
 - ingoing boundary condition at horizon
- Compute the response $\phi_1(\vec{x})$



What AdS/CFT is Good For

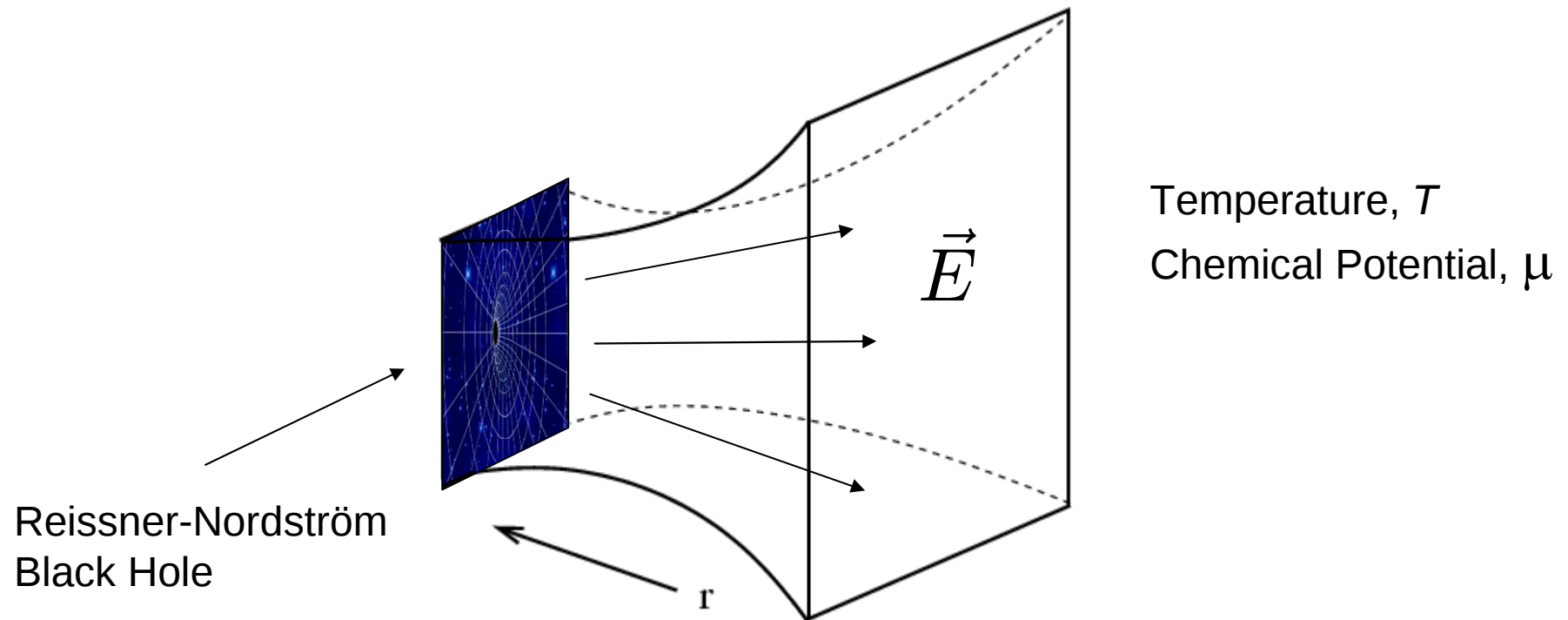


Finite Temperature



Euclidean and *Lorentzian* signatures

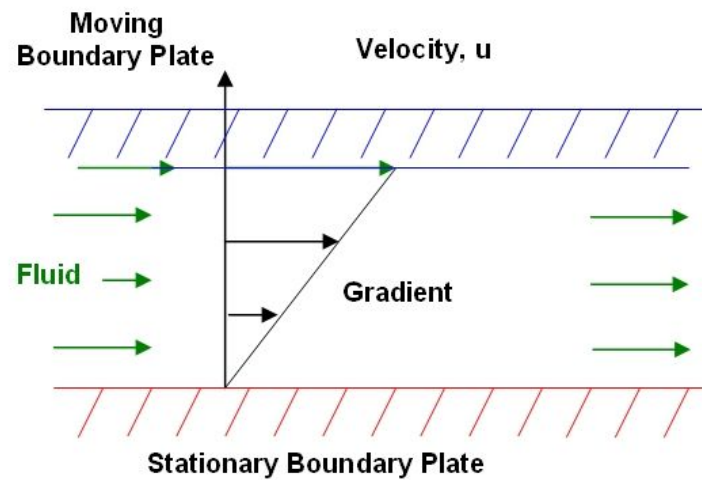
Finite Density



Transport

Kovtun, Policastro, Son, Starinets

e.g. shear viscosity



$$\frac{\eta}{s} = \frac{1}{4\pi}$$

Many Other Phenomena

- Superconductivity
- Non-Fermi Liquids
- Quantum Oscillations
- Quantum Hall Transitions
- Dynamical Lattice Formation
- Band Structure
- ...

ANTI DE SITTER SPACE AND HOLOGRAPHY

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Recently, it has been proposed by Maldacena that large N limits of certain conformal field theories in d dimensions can be described in terms of supergravity (and string theory) on the product of $d+1$ -dimensional AdS space with a compact manifold. Here we elaborate on this idea and propose a precise correspondence between conformal field theory observables and those of supergravity: correlation functions in conformal field theory are given by the dependence of the supergravity action on the asymptotic behavior at infinity. In particular, dimensions of operators in conformal field theory are given by masses of particles in supergravity. As quantitative confirmation of this correspondence, we note that the Kaluza-Klein modes of Type IIB supergravity on $AdS_5 \times S^5$ match with the chiral operators of $\mathcal{N} = 4$ super Yang-Mills theory in four dimensions. With some further assumptions, one can deduce a Hamiltonian version of the correspondence and show that the $\mathcal{N} = 4$ theory has a large N phase transition related to the thermodynamics of AdS black holes.

1. Introduction

To understand the large N behavior of gauge theories with $SU(N)$ gauge group is a longstanding problem [1], and offers perhaps the best hope of eventually understanding the classic strong coupling mysteries of QCD. It has long been suspected that the large N behavior, if accessible at all, should be described by string theory perhaps with Liouville fields and higher dimensions; see [2] for recent discussion. Lately, with hints coming from explorations of the near horizon structure of certain black hole metrics [3-7] and investigations [8-12] of scattering from those metrics, Maldacena has made [13] a remarkable suggestion concerning the large N limit not of conventional $SU(N)$ gauge theories but of some of their conformally invariant cousins. According to this proposal, the large N limit of a conformally invariant theory in d dimensions is governed by supergravity (and string theory) on $d+1$ -dimensional AdS space (often called AdS_{d+1}) times a compact manifold which in the maximally supersymmetric cases is a sphere. There has also been a discussion of the flow to conformal field theory in some cases [14] and many other relevant discussions of branes, field theories, and AdS spaces [15-21].

An important example to which this discussion applies is $\mathcal{N} = 4$ super Yang-Mills theory in four dimensions, with gauge group $SU(N)$ and coupling constant g_{YM} . This theory conjecturally is equivalent to Type IIB superstring theory on $AdS_5 \times S^5$, with string coupling constant g_{st} proportional to g_{YM}^2 , N units of five-form flux on S^5 , and radius of curvature $(g_{YM}^2 N)^{1/4}$. In the large N limit with $x = g_{YM}^2 N$ fixed but large, the string theory is weakly coupled and supergravity is a good approximation to it. So the hope is that for large N and large x , the $\mathcal{N} = 4$ theory in four dimensions is governed by the tree approximation to supergravity. In some other important examples discussed in [13], there is no dimensionless parameter analogous to x , and supergravity should apply simply if N is large.

The discussion in [13] was motivated by consideration of black holes, which are also likely to suggest future generalizations. The black holes in question have near-horizon AdS geometries, and for our purposes it will suffice to work on the AdS spaces. AdS space has many unusual properties. It has a boundary at spatial infinity (for example, see [22], pp. 131-4), as a result of which quantization [23] and analysis of stability [24-26] are not straightforward. As we describe below, the boundary M_d of AdS_{d+1} is in fact a copy of d -dimensional Minkowski space (with some points at infinity added); the symmetry group $SO(2, d)$ of AdS_{d+1} acts on M_d as the conformal group. The fact that $SO(2, d)$ acts on

AdS_{d+1} as a group of ordinary symmetries and on M_d as a group of conformal symmetries means that there are two ways to get a physical theory with $SO(2, d)$ symmetry: in a relativistic field theory (with or without gravity) on AdS_{d+1} , or in a conformal field theory on M_d . Conformal free fields on M_d furnish the “singleton” representations of $SO(2, d)$; these are small representations, originally studied by Dirac (for $d = 3$), and interpreted in terms of free field theory on M_d in [27–29]. The possible relation of field theory on AdS_{d+1} to field theory on M_d has been a subject of long interest; see [19, 21] for discussions motivated by recent developments, and additional references.

The main idea in [13] was not that supergravity, or string theory, on AdS_{d+1} should be *supplemented* by singleton (or other) fields on the boundary, but that a suitable theory on AdS_{d+1} would be *equivalent* to a conformal field theory in d dimensions; the conformal field theory might be described as a generalized singleton theory. A precise recipe for computing observables of the conformal field theory in terms of supergravity on AdS_{d+1} was not given in [13]; obtaining one will be the goal of the present paper. Our proposal is that correlation functions in conformal field theory are given by the dependence of the supergravity action on the asymptotic behavior at infinity. A special case of the proposal is that dimensions of operators in the conformal field theory are determined by masses of particles in string theory. The proposal is effective, and gives a practical recipe for computing large N conformal field theory correlation functions from supergravity tree diagrams, under precisely the conditions proposed in [13] – when the length scale of AdS_{d+1} is large compared to the string and Planck scales.

One of the most surprising claims in [13] was that (for example) to describe the $\mathcal{N} = 4$ super Yang-Mills theory in four dimensions, one should use not just low energy supergravity on AdS_5 but the whole infinite tower of massive Kaluza-Klein states on $AdS_5 \times \mathbf{S}^5$. We will be able to see explicitly how this works. Chiral fields in the four-dimensional $\mathcal{N} = 4$ theory (that is, fields in small representations of the supersymmetry algebra) correspond to Kaluza-Klein harmonics on $AdS_5 \times \mathbf{S}^5$. Irrelevant, marginal, and relevant perturbations of the field theory correspond to massive, massless, and “tachyonic” modes in supergravity. The “tachyonic” modes have negative mass squared, but as shown in [24] do not lead to any instability. The spectrum of Kaluza-Klein excitations of $AdS_5 \times \mathbf{S}^5$, as computed in [30, 31], can be matched precisely with certain operators of the $\mathcal{N} = 4$ theory, as we will see in section 2.6. Stringy excitations of $AdS_5 \times \mathbf{S}^5$ correspond to operators whose dimensions diverge for $N \rightarrow \infty$ in the large x limit.

Now we will recall how Minkowski space appears as the boundary of AdS space. The conformal group $SO(2, d)$ does not act on Minkowski space, because conformal transformations can map an ordinary point to infinity. To get an action of $SO(2, d)$, one must add some “points at infinity.” A compactification on which $SO(2, d)$ does act is the “quadric,” described by coordinates $u, v, x^1, \dots, x^d$, obeying an equation

$$uv - \sum_{i,j} \eta_{ij} x^i x^j = 0, \quad (1.1)$$

and subject to an overall scaling equivalence ($u \rightarrow su, v \rightarrow sv, x^i \rightarrow sx^i$, with real non-zero s). In (1.1), η_{ij} is the Lorentz metric of signature $-++\dots+$. (1.1) defines a manifold that admits the action of a group $SO(2, d)$ that preserves the quadratic form $-du dv + \sum_{i,j} dx^i dx^j$ (whose signature is $--++\dots+$). To show that (1.1) describes a compactification of Minkowski space, one notes that generically $v \neq 0$, in which case one may use the scaling relation to set $v = 1$, after which one uses the equation to solve for u . This leaves the standard Minkowski space coordinates x^i , which parametrize the portion of the quadric with $v \neq 0$. The quadric differs from Minkowski space by containing also “points at infinity,” with $v = 0$. The compactification (1.1) is topologically $(\mathbf{S}^1 \times \mathbf{S}^{d-1})/\mathbf{Z}_2$ (the $\mathbf{Z}_2$ acts by a π rotation of the first factor and multiplication by -1 on the second),¹ and has closed timelike curves, so one may prefer to replace it by its universal cover, which is topologically $\mathbf{S}^{d-1} \times \mathbf{R}$ (where $\mathbf{R}$ can be viewed as the “time” direction).

AdS_{d+1} can be described by the same coordinates u, v, x^i but with a scaling equivalence only under an overall sign change for all variables, and with (1.1) replaced by

$$uv - \sum_{i,j} \eta_{ij} x^i x^j = 1. \quad (1.2)$$

This space is not compact. If u, v, x^i go to infinity while preserving (1.2), then in the limit, after dividing the coordinates by a positive constant factor, one gets a solution of (1.1).² This is why the conformal compactification M_d of Minkowski space is the boundary of

¹ After setting $u = a + b, v = a - b$, and renaming the variables in a fairly obvious way, the equation becomes $a_1^2 + a_2^2 = \sum_{j=1}^d y_j^2$. Scaling by only positive s , one can in a unique way map to the locus $a_1^2 + a_2^2 = \sum_{j=1}^d y_j^2 = 1$, which is a copy of $\mathbf{S}^1 \times \mathbf{S}^{d-1}$; scaling out also the transformation with $s = -1$ gives $(\mathbf{S}^1 \times \mathbf{S}^{d-1})/\mathbf{Z}_2$.

² Because the constant factor here is positive, we must identify the variables in (1.2) under an overall sign change, to get a manifold whose boundary is M_d .

AdS_{d+1} . Again, in [1,2] there are closed timelike curves; if one takes the universal cover to eliminate them, then the boundary becomes the universal cover of M_d .

According to [13], an $\mathcal{N} = 4$ theory formulated on M_4 is equivalent to Type IIB string theory on $AdS_5 \times \mathbf{S}^5$. We can certainly identify the M_4 in question with the boundary of AdS_5 ; indeed this is the only possible $SO(2, 4)$ -invariant relation between these two spaces. The correspondence between $\mathcal{N} = 4$ on M_4 and Type IIB on $AdS_5 \times \mathbf{S}^5$ therefore expresses a string theory on $AdS_5 \times \mathbf{S}^5$ in terms of a theory on the boundary. This correspondence is thus “holographic,” in the sense of [32,33]. This realization of holography is somewhat different from what is obtained in the matrix model of M -theory [34], since for instance it is covariant (under $SO(2, d)$). But otherwise the two are strikingly similar. In both approaches, M -theory or string theory on a certain background is described in terms of a field theory with maximal supersymmetry.

The realization of holography via AdS space is also reminiscent of the relation [35] between conformal field theory in two dimensions and Chern-Simons gauge theory in three dimensions. In each case, conformal field theory on a d -manifold M_d is related to a generally covariant theory on a $d + 1$ -manifold B_{d+1} whose boundary is M_d . The difference is that in the Chern-Simons case, general covariance is achieved by considering a field theory that does not require a metric on spacetime, while in AdS supergravity, general covariance is achieved in the customary way, by integrating over metrics.

After this work was substantially completed, I learned of independent work [36] in which a very similar understanding of the the CFT/ AdS correspondence to what we describe in section 2 is developed, as well as two papers [37,38] that consider facts relevant to or aspects of the Hamiltonian formalism that we consider in section 3.

2. Boundary Behavior

2.1. Euclidean Version Of AdS_{d+1}

So far we have assumed Lorentz signature, but the identification of the boundary of AdS_{d+1} with d -dimensional Minkowski space holds also with Euclidean signature, and it will be convenient to formulate the present paper in a Euclidean language. The Euclidean version of AdS_{d+1} can be described in several equivalent ways. Consider a Euclidean space $\mathbf{R}^{d+1}$ with coordinates $y_0, \dots, y_d$, and let B_{d+1} be the open unit ball, $\sum_{i=0}^d y_i^2 < 1$. AdS_{d+1} can be identified as B_{d+1} with the metric

$$ds^2 = \frac{4 \sum_{i=0}^d dy_i^2}{(1 - |y|^2)^2}. \quad (2.1)$$

We can compactify B_{d+1} to get the closed unit ball $\overline{B}_{d+1}$, defined by $\sum_{i=0}^d y_i^2 \leq 1$. Its boundary is the sphere $\mathbf{S}^d$, defined by $\sum_{i=0}^d y_i^2 = 1$. $\mathbf{S}^d$ is the Euclidean version of the conformal compactification of Minkowski space, and the fact that $\mathbf{S}^d$ is the boundary of $\overline{B}_{d+1}$ is the Euclidean version of the statement that Minkowski space is the boundary of AdS_{d+1} . The metric (2.1) on B_{d+1} does not extend over $\overline{B}_{d+1}$, or define a metric on $\mathbf{S}^d$, because it is singular at $|y| = 1$. To get a metric which does extend over $\overline{B}_{d+1}$, one can pick a function f on $\overline{B}_{d+1}$ which is positive on B_{d+1} and has a first order zero on the boundary (for instance, one can take $f = 1 - |y|^2$), and replace ds^2 by

$$d\tilde{s}^2 = f^2 ds^2. \quad (2.2)$$

Then $d\tilde{s}^2$ restricts to a metric on $\mathbf{S}^d$. As there is no natural choice of f , this metric is only well-defined up to conformal transformations. One could, in other words, replace f by

$$f \rightarrow fe^w \quad (2.3)$$

with w any real function on $\overline{B}_{d+1}$, and this would induce the conformal transformation

$$d\tilde{s}^2 \rightarrow e^{2w} d\tilde{s}^2 \quad (2.4)$$

in the metric of $\mathbf{S}^d$. Thus, while AdS_{d+1} has (in its Euclidean version) a metric invariant under $SO(1, d+1)$, the boundary $\mathbf{S}^d$ has only a conformal structure, which is preserved by the action of $SO(1, d+1)$.³

Alternatively, with the substitution $r = \tanh(y/2)$, one can put the AdS_{d+1} metric (2.1) in the form

$$ds^2 = dy^2 + \sinh^2 y \, d\Omega^2 \quad (2.5)$$

where $d\Omega^2$ is the metric on the unit sphere, and $0 \leq y < \infty$. In this representation, the boundary is at $y = \infty$. Finally, one can regard AdS_{d+1} as the upper half space $x^0 > 0$ in a space with coordinates $x^0, x^1, \dots, x^d$, and metric

$$ds^2 = \frac{1}{x_0^2} \left(\sum_{i=0}^d (dx^i)^2 \right). \quad (2.6)$$

In this representation, the boundary consists of a copy of $\mathbf{R}^d$, at $x^0 = 0$, together with a single point P at $x^0 = \infty$. ($x^0 = \infty$ consists of a single point since the metric in the x^i direction vanishes as $x^0 \rightarrow \infty$). Thus, from this point of view, the boundary of AdS_{d+1} is a conformal compactification of $\mathbf{R}^d$ obtained by adding in a point P at infinity; this of course gives a sphere $\mathbf{S}^d$.

³ This construction is the basic idea of Penrose's method of compactifying spacetime by introducing conformal infinity; see [39].

2.2. Massless Field Equations

Now we come to the basic fact which will be exploited in the present paper. We start with the case of massless field equations, where the most elegant statement is possible. We will to begin with discuss simple and elementary equations, but the idea is that similar properties should hold for the supergravity equations relevant to the proposal in [13] and in fact for a very large class of supergravity theories, connected to branes or not.

The first case to consider is a scalar field ϕ . For such a field, by the massless field equation we will mean the most naive Laplace equation $D_i D^i \phi = 0$. (From some points of view, as for instance in [24, 40, 41], “masslessness” in AdS space requires adding a constant to the equation, but for our purposes we use the naive massless equation.) A basic fact about AdS_{d+1} is that given any function $\phi(\Omega)$ on the boundary $\mathbf{S}^d$, there is a unique extension of ϕ to a function on $\overline{B}_{d+1}$ that has the given boundary values and obeys the field equation. Uniqueness depends on the fact that there is no nonzero square-integrable solution of the Laplace equation (which, if it existed, could be added to any given solution, spoiling uniqueness). This is true because given a square-integrable solution, one would have by integration by parts

$$0 = - \int_{B_{d+1}} d^{d+1}y \sqrt{g} \phi D_i D^i \phi = \int_{B_{d+1}} d^5y \sqrt{g} |d\phi|^2, \quad (2.7)$$

so that $d\phi = 0$ and hence (for square-integrability) $\phi = 0$. Now, in the representation (2.5) of B_{d+1} , the Laplace equation reads

$$\left(-\frac{1}{(\sinh y)^d} \frac{d}{dy} (\sinh y)^d \frac{d}{dy} + \frac{L^2}{\sinh^2 y} \right) \phi = 0, \quad (2.8)$$

where L^2 (the square of the angular momentum) is the angular part of the Laplacian. If one writes $\phi = \sum_{\alpha} \phi_{\alpha}(y) f_{\alpha}(\Omega)$, where the f_{α} are spherical harmonics, then the equation for any ϕ_{α} looks for large y like

$$\frac{d}{dy} e^{dy} \frac{d}{dy} \phi_{\alpha} = 0, \quad (2.9)$$

with the two solutions $\phi_{\alpha} \sim 1$ and $\phi_{\alpha} \sim e^{-dy}$. One linear combination of the two solutions is smooth near $y = 0$; this solution has a non-zero constant term at infinity (or one would get a square-integrable solution of the Laplace equation). So for every partial wave, one gets a unique solution of the Laplace equation with a given constant value at infinity; adding these up with suitable coefficients, one gets a unique solution of the Laplace equation with any desired limiting value $\phi(\Omega)$ at infinity. In section 2.4, we give an alternative proof of

existence of ϕ based on an integral formula. This requires the explicit form of the metric on B_{d+1} while the proof we have given is valid for any spherically symmetric metric with a double pole on the boundary. (With a little more effort, one can make the proof without assuming spherical symmetry.)

In the case of gravity, by the massless field equations, we mean the Einstein equations (with a negative cosmological constant as we are on AdS_{d+1}). Any metric on B_{d+1} that has the sort of boundary behavior seen in (2.1) (a double pole on the boundary) induces as in (2.2) a conformal structure on the boundary $\mathbf{S}^d$. Conversely, by a theorem of Graham and Lee [42] (see [43–46] for earlier mathematical work), any conformal structure on $\mathbf{S}^d$ that is sufficiently close to the standard one arises, by the procedure in (2.2), from a unique metric on B_{d+1} that obeys the Einstein equations with negative cosmological constant and has a double pole at the boundary. (Uniqueness of course means uniqueness up to diffeomorphism.) This is proved by first showing existence and uniqueness at the linearized level (for a first order deformation from the standard conformal structure on $\mathbf{S}^d$) roughly along the lines of the above treatment of scalars, and then going to small but finite perturbations via an implicit function argument.

For a Yang-Mills field A with curvature F , by the massless field equations we mean the usual minimal Yang-Mills equations $D^i F_{ij} = 0$, or suitable generalizations, for instance resulting from adding a Chern-Simons term in the action. One expects an analog of the Graham-Lee theorem stating that any A on $\mathbf{S}^d$ that is sufficiently close to $A = 0$ is the boundary value of a solution of the Yang-Mills equations on B_{d+1} that is unique up to gauge transformation. To first order around $A = 0$, one needs to solve Maxwell’s equations with given boundary values; this can be analyzed as we did for scalars, or by an explicit formula that we will write in the next subsection. The argument beyond that is hopefully similar to what was done by Graham and Lee. For topological reasons, existence and uniqueness of a gauge field on B_{d+1} with given boundary values can only hold for A sufficiently close to zero. ⁴ Likewise in the theorem of Graham and Lee, the restriction to conformal structures

⁴ This is proved as follows. We state the proof for even d . (For odd d , one replaces $\mathbf{S}^2$ by $\mathbf{S}^1$ and $c_{\frac{1}{2}d+1}$ by $c_{\frac{1}{2}(d+1)}$ in the following argument.) Consider on $\mathbf{S}^2 \times \mathbf{S}^d$ a gauge field with a nonzero $\frac{1}{2}d + 1^{th}$ Chern class, so that $\int_{\mathbf{S}^2 \times \mathbf{S}^d} \text{Tr} F \wedge F \wedge \dots \wedge F \neq 0$. If every gauge field on $\mathbf{S}^d$ could be extended over B_{d+1} to a solution of the Yang-Mills equations unique up to gauge transformation, then by making this unique extension fiberwise, one would get an extension of the given gauge field on $\mathbf{S}^2 \times \mathbf{S}^d$ over $\mathbf{S}^2 \times B_{d+1}$. Then one would deduce that $0 = \int_{\mathbf{S}^2 \times B_{d+1}} d\text{Tr} F \wedge F \wedge \dots \wedge F = \int_{\mathbf{S}^2 \times \mathbf{S}^d} \text{Tr} F \wedge F \wedge \dots \wedge F$, which is a contradiction.

that are sufficiently close to the standard one is probably also necessary for most values of d (one would prove this as in the footnote using a family of $\mathbf{S}^d$'s that cannot be extended to a family of $\overline{B}_{d+1}$'s).

2.3. Ansatz For The Effective Action

We will now attempt to make more precise the conjecture [13] relating field theory on the boundary of AdS space to supergravity (and string theory) in bulk. We will make an ansatz whose justification, initially, is that it combines the ingredients at hand in the most natural way. Gradually, further evidence for the ansatz will emerge.

Suppose that, in any one of the examples in [13], one has a massless scalar field ϕ on AdS_{d+1} ,⁵ obeying the simple Laplace equation $D_i D^i \phi = 0$, with no mass term or curvature coupling, or some other equation with the same basic property: the existence of a unique solution on $\overline{B}_{d+1}$ with any given boundary values.

Let ϕ_0 be the restriction of ϕ to the boundary of AdS_{d+1} . We will assume that in the correspondence between AdS_{d+1} and conformal field theory on the boundary, ϕ_0 should be considered to couple to a conformal field $\mathcal{O}$, via a coupling $\int_{\mathbf{S}^d} \phi_0 \mathcal{O}$. This assumption is natural given the relation of the conjectured CFT/ AdS correspondence to analyses [8–11] of interactions of fields with branes. ϕ_0 is conformally invariant – it has conformal weight zero – since the use of a function f as in (2.2) to define a metric on $\mathbf{S}^d$ does not enter at all in the definition of ϕ_0 . So conformal invariance dictates that $\mathcal{O}$ must have conformal dimension d . We would like to compute the correlation functions $\langle \mathcal{O}(x_1) \mathcal{O}(x_2) \dots \mathcal{O}(x_n) \rangle$ at least for distinct points $x_1, x_2, \dots, x_n \in \mathbf{S}^n$. Even better, if the singularities in the operator product expansion on the boundary are mild enough, we would like to extend the definition of the correlation function (as a distribution with some singularities) to allow the points to coincide. Even more optimistically, one would like to define the generating functional $\langle \exp(\int_{\mathbf{S}^d} \phi_0 \mathcal{O}) \rangle_{CFT}$ (the expectation value of the given exponential in the conformal field theory on the boundary), for a function ϕ_0 , at least in an open neighborhood of $\phi_0 = 0$, and not just as a formal series in powers of ϕ_0 . Usually, in quantum field theory one would face very difficult problems of renormalization in defining such objects. We will see that in the present context, as long as one only considers massless fields in bulk, one meets

⁵ In each of the cases considered in [13], the spacetime is really $AdS_{d+1} \times W$ for some compact manifold W . At many points in this paper, we will be somewhat informal and suppress W from the notation.

operators for which the short distance singularities are so restricted that the expectation value of the exponential can be defined nicely.

Now, let $Z_S(\phi_0)$ be the supergravity (or string) partition function on B_{d+1} computed with the boundary condition that at infinity ϕ approaches a given function ϕ_0 . For example, in the approximation of classical supergravity, one computes $Z_S(\phi_0)$ by simply extending ϕ_0 over B_{d+1} as a solution ϕ of the classical supergravity equations, and then writing

$$Z_S(\phi_0) = \exp(-I_S(\phi)), \quad (2.10)$$

where I_S is the classical supergravity action. If classical supergravity is not an adequate approximation, then one must include string theory corrections to I_S (and the equations for ϕ), or include quantum loops (computed in an expansion around the solution ϕ) rather than just evaluating the classical action. Criteria under which stringy and quantum corrections are small are given in [13]. The formula (2.10) makes sense unless there are infrared divergences in integrating over AdS_{d+1} to define the classical action $I(\phi)$. We will argue in section 2.4 that when such divergences arise, they correspond to expected renormalizations and anomalies of the conformal field theory.

Our ansatz for the precise relation of conformal field theory on the boundary to AdS space is that

$$\left\langle \exp \int_{\mathbf{S}^d} \phi_0 \mathcal{O} \right\rangle_{CFT} = Z_S(\phi_0). \quad (2.11)$$

As a preliminary check, note that in the case of $\mathcal{N} = 4$ supergravity in four dimensions, this has the expected scaling of the 't Hooft large N limit, since the supergravity action I_S that appears in (2.10) is of order N^2 .⁶

Gravity and gauge theory can be treated similarly. Suppose that one would like to compute correlation functions of a product of stress tensors in the conformal field theory on $\mathbf{S}^d$. The generating functional of these correlation functions would be the partition function of the conformal field theory as a function of the metric on $\mathbf{S}^d$. Except for a c -number anomaly, which we discuss in the next subsection but temporarily suppress, this partition function only depends on the conformal structure of $\mathbf{S}^d$. Let us denote as

⁶ The action contains an Einstein term, of order $1/g_s^2$. This is of order N^2 as $g_{st} \sim g_{YM}^2 \sim 1/N$. It also contains a term F^2 with F the Ramond-Ramond five-form field strength; this has no power of g_{st} but is of order N^2 as, with N units of RR flux, one has $F \sim N$.

$Z_{CFT}(h)$ the partition function of the conformal field theory formulated on a four-sphere with conformal structure h . We interpret the CFT/ AdS correspondence to be that

$$Z_{CFT}(h) = Z_S(h) \quad (2.12)$$

where $Z_S(h)$ is the supergravity (or string theory) partition function computed by integrating over metrics that have a double pole near the boundary and induce, on the boundary, the given conformal structure h . In the approximation of classical supergravity, one computes $Z_S(h)$ by finding a solution g of the Einstein equations with the required boundary behavior, and setting $Z_S(h) = \exp(-I_S(g))$.

Likewise, for gauge theory, suppose the AdS theory has a gauge group G , of dimension s , with gauge fields A^a , $a = 1, \dots, s$. Then in the scenario of [13], the group G is a global symmetry group of the conformal field theory on the boundary, and there are currents J_a in the boundary theory. We would like to determine the correlation functions of the J_a 's, or more optimistically, the expectation value of the generating function $\exp(\int_{\mathbf{S}^d} J_a A_0^a)$, with A_0 an arbitrary source. For this we make an ansatz precisely along the above lines: we let $Z_S(A_0)$ be the supergravity or string theory partition function with the boundary condition that the gauge field A approaches A_0 at infinity; and we propose that

$$\left\langle \exp\left(\int_{\mathbf{S}^d} J_a A_0^a\right) \right\rangle_{CFT} = Z_S(A_0). \quad (2.13)$$

In the approximation of classical field theory, $Z_S(A_0)$ is computed by extending A over B_{d+1} as a solution of the appropriate equations and setting $Z_S(A_0) = \exp(-I_S(A))$.

These examples hopefully make the general idea clear. One computes the supergravity (or string) partition function as a function of the boundary values of the massless fields, and interprets this as a generating functional of conformal field theory correlation functions, for operators whose sources are the given boundary values. Of course, in general, one wishes to compute the conformal field theory partition function with all massless fields turned on at once – rather than considering them separately, as we have done for illustrative purposes. One also wishes to include massive fields, but this we postpone until after performing some illustrative computations in the next subsection.

A Small Digression

At this point, one might ask what is the significance of the fact that the Graham-Lee theorem presumably fails for conformal structures that are sufficiently far from the round

one, and that the analogous theorem for gauge theory definitely fails for sufficiently strong gauge fields on the boundary. For this discussion, we will be more specific and consider what is perhaps the best understood example considered in [13], namely the $\mathcal{N} = 4$ theory in four dimensions. One basic question one should ask is why, or whether, the partition function of this conformal field theory on $\mathbf{S}^4$ should converge. On $\mathbf{R}^4$ this theory has a moduli space of classical vacua, parametrized by expectation values of six scalars X^a in the adjoint representation (one requires $[X^a, X^b] = 0$ for vanishing energy). Correlation functions on $\mathbf{R}^4$ are not unique, but depend on a choice of vacuum. On $\mathbf{S}^4$, because of the finite volume, the vacuum degeneracy and non-uniqueness do not arise. Instead, one should worry that the path integral would not converge, but would diverge because of the integration over the flat directions in the potential, that is over the noncompact space of constant (and commuting) X^a . What saves the day is that the conformal coupling of the X^a to $\mathbf{S}^4$ involves, for conformal invariance, a curvature coupling $(R/6)\text{Tr}X^2$ (R being the scalar curvature on $\mathbf{S}^4$). If one is sufficiently close to the round four-sphere, then $R > 0$, and the curvature coupling prevents the integral from diverging in the region of large constant X 's. If one is very far from the standard conformal structure, so that R is negative, or if the background gauge field A_0 is big enough, the partition function may well diverge.

Alternatively, it is possible that the Graham-Lee theorem, or its gauge theory counterpart, could fail in a region in which the conformal field theory on $\mathbf{S}^4$ is still well-behaved. For every finite N , the conformal field theory partition function on $\mathbf{S}^4$, as long as it converges sufficiently well, is analytic as a function of the conformal structure and other background fields. But perhaps such analyticity can break down in the large N limit; the failure of existence and uniqueness of the extension of the boundary fields to such a classical solution could correspond to such nonanalyticity. Such singularities that arise only in the large N limit are known to occur in some toy examples [47], and in section 3.2 we will discuss an analogous but somewhat different source of such nonanalyticity.

2.4. Some Sample Calculations

We will now carry out some sample calculations, in the approximation of classical supergravity, to illustrate the above ideas.

First we consider an AdS theory that contains a massless scalar ϕ with action

$$I(\phi) = \frac{1}{2} \int_{B_{d+1}} d^{d+1}y \sqrt{g} |d\phi|^2. \quad (2.14)$$

We assume that the boundary value ϕ_0 of ϕ is the source for a field $\mathcal{O}$ and that to compute the two point function of $\mathcal{O}$, we must evaluate $I(\phi)$ for a classical solution with boundary value ϕ_0 . For this, we must solve for ϕ in terms of ϕ_0 , and then evaluate the classical action (2.14) for the field ϕ .

To solve for ϕ in terms of ϕ_0 , we first look for a “Green’s function,” a solution K of the Laplace equation on B_{d+1} whose boundary value is a delta function at a point P on the boundary. To find this function, it is convenient to use the representation of B_{d+1} as the upper half space with metric

$$ds^2 = \frac{1}{x_0^2} \sum_{i=0}^d (dx_i)^2, \quad (2.15)$$

and take P to be the point at $x_0 = \infty$. The boundary conditions and metric are invariant under translations of the x_i , so K will have this symmetry and is a function only of x_0 . The Laplace equation reads

$$\frac{d}{dx_0} x_0^{-d+1} \frac{d}{dx_0} K(x_0) = 0. \quad (2.16)$$

The solution that vanishes at $x_0 = 0$ is

$$K(x_0) = c x_0^d \quad (2.17)$$

with c a constant. Since this grows at infinity, there is some sort of singularity at the boundary point P . To show that this singularity is a delta function, it helps to make an $SO(1, d+1)$ transformation that maps P to a finite point. The transformation

$$x_i \rightarrow \frac{x_i}{x_0^2 + \sum_{j=1}^d x_j^2}, \quad i = 0, \dots, d \quad (2.18)$$

maps P to the origin, $x_i = 0$, $i = 0, \dots, d$, and transforms K to

$$K(x) = c \frac{x_0^d}{(x_0^2 + \sum_{j=1}^d x_j^2)^d}. \quad (2.19)$$

A scaling argument shows that $\int dx_1 \dots dx_d K(x)$ is independent of x_0 ; also, as $x_0 \rightarrow 0$, K vanishes except at $x_1 = \dots = x_d = 0$. Moreover K is positive. So for $x_0 \rightarrow 0$, K becomes a delta function supported at $x_i = 0$, with unit coefficient if c is chosen correctly. Henceforth, we write $\mathbf{x}$ for the d -tuple $x_1, x_2, \dots, x_d$, and $|\mathbf{x}|^2$ for $\sum_{j=1}^d x_j^2$.

Using this Green's function, the solution of the Laplace equation on the upper half space with boundary values ϕ_0 is

$$\phi(x_0, x_i) = c \int d\mathbf{x}' \frac{x_0^d}{(x_0^2 + |\mathbf{x} - \mathbf{x}'|^2)^d} \phi_0(x'_i). \quad (2.20)$$

($d\mathbf{x}'$ is an abbreviation for $dx'_1 dx'_2 \dots dx'_d$.) It follows that for $x_0 \rightarrow 0$,

$$\frac{\partial \phi}{\partial x_0} \sim dc x_0^{d-1} \int d\mathbf{x}' \frac{\phi_0(x')}{|\mathbf{x} - \mathbf{x}'|^{2d}} + O(x_0^{d+1}). \quad (2.21)$$

By integrating by parts, one can express $I(\phi)$ as a surface integral, in fact

$$I(\phi) = \lim_{\epsilon \rightarrow 0} \int_{T_\epsilon} d\mathbf{x} \sqrt{h} \phi (\vec{n} \cdot \vec{\nabla}) \phi, \quad (2.22)$$

where T_ϵ is the surface $x_0 = \epsilon$, h is its induced metric, and n is a unit normal vector to T_ϵ . One has $\sqrt{h} = x_0^{-d}$, $\vec{n} \cdot \vec{\nabla} \phi = x_0 (\partial \phi / \partial x_0)$. Since $\phi \rightarrow \phi_0$ for $x_0 \rightarrow 0$, and $\partial \phi / \partial x_0$ behaves as in (2.21), (2.22) can be evaluated to give

$$I(\phi) = \frac{cd}{2} \int d\mathbf{x} d\mathbf{x}' \frac{\phi_0(\mathbf{x}) \phi_0(\mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|^{2d}}. \quad (2.23)$$

So the two point function of the operator $\mathcal{O}$ is a multiple of $|\mathbf{x} - \mathbf{x}'|^{-2d}$, as expected for a field $\mathcal{O}$ of conformal dimension d .

Gauge Theory

We will now carry out a precisely analogous computation for free $U(1)$ gauge theory.

The first step is to find a Green's function, that is a solution of Maxwell's equations on B_{d+1} with a singularity only at a single point on the boundary. We use again the description of B_{d+1} as the upper half space $x_0 \geq 0$, and we look for a solution of Maxwell's equations by a one-form A of the form $A = f(x_0) dx^i$ (for some fixed $i \geq 1$). We have $dA = f'(x_0) dx^0 \wedge dx^i$, so

$$*dA = \frac{1}{x_0^{d-3}} f'(x_0) (-1)^i dx^1 dx^2 \dots \widehat{dx^i} \dots dx^d, \quad (2.24)$$

where the notation $\widehat{dx^i}$ means that dx^i is to be omitted from the $d-1$ -fold wedge product. Maxwell's equations $d(*dA) = 0$ give $f(x_0) = x_0^{d-2}$ (up to a constant multiple) and hence we can take

$$A = \frac{d-1}{d-2} x_0^{d-2} dx^i, \quad (2.25)$$

where the constant is for convenience. After the inversion $x_i \rightarrow x_i/(x_0^2 + |\mathbf{x}|^2)$, we have $A = ((d-1)/(d-2)) \left(\frac{x_0}{x_0^2 + |\mathbf{x}|^2} \right)^{d-2} d \left(\frac{x^i}{x_0^2 + |\mathbf{x}|^2} \right)$. We make a gauge transformation, adding to A the exact form obtained as the exterior derivative of $-(d-2)^{-1}(x_0^{d-2}x_i/(x_0^2 + |\mathbf{x}|^2)^{d-1})$, and get

$$A = \frac{x_0^{d-2}dx_i}{(x_0^2 + |\mathbf{x}|^2)^{d-1}} - \frac{x_0^{d-3}x_idx_0}{(x_0^2 + |\mathbf{x}|^2)^{d-1}}. \quad (2.26)$$

Now suppose that we want a solution of Maxwell's equations that at $x_0 = 0$ coincides with $A_0 = \sum_{i=1}^d a_i dx^i$. Using the above Green's function, we simply write (up to a constant multiple)

$$A(x_0, \mathbf{x}) = \int d\mathbf{x}' \frac{x_0^{d-2}}{(x_0^2 + |\mathbf{x} - \mathbf{x}'|^2)^{d-1}} a_i(\mathbf{x}') dx^i - x_0^{d-3} dx_0 \int d\mathbf{x}' \frac{(x - x')^i a_i(\mathbf{x}')}{(x_0^2 + |\mathbf{x} - \mathbf{x}'|^2)^{d-1}}. \quad (2.27)$$

Hence

$$\begin{aligned} F = dA = & (d-1)x_0^{d-3}dx_0 \int d\mathbf{x}' \frac{a_i(\mathbf{x}')dx^i}{(x_0^2 + |\mathbf{x} - \mathbf{x}'|^2)^{d-1}} \\ & - 2(d-1)x_0^{d-1}dx_0 \int d\mathbf{x}' \frac{a_i(\mathbf{x}')dx^i}{(x_0^2 + |\mathbf{x} - \mathbf{x}'|^2)^d} \\ & - 2(d-1)x_0^{d-3}dx_0 \int d\mathbf{x}' \frac{(x_i - x'_i)dx^i a_k(\mathbf{x}')(x^k - (x')^k)}{(x_0^2 + |\mathbf{x} - \mathbf{x}'|^2)^d} + \dots, \end{aligned} \quad (2.28)$$

where the ... are terms with no dx^0 .

Now, by integration by parts, the action is

$$I(A) = \frac{1}{2} \int_{B_{d+1}} F \wedge *F = \frac{1}{2} \lim_{\epsilon \rightarrow 0} \int_{T_\epsilon} A \wedge *F, \quad (2.29)$$

with T_ϵ the surface $x_0 = \epsilon$. Using the above formulas for A and F , this can be evaluated, and one gets up to a constant multiple

$$I = \int d\mathbf{x} d\mathbf{x}' a_i(\mathbf{x}) a_j(\mathbf{x}') \left(\frac{\delta_{ij}}{|\mathbf{x} - \mathbf{x}'|^{2d-2}} - \frac{2(x - x')_i (x - x')_j}{|\mathbf{x} - \mathbf{x}'|^{2d}} \right). \quad (2.30)$$

This is the expected form for the two-point function of a conserved current.

Chern-Simons Term And Anomaly

Now let us explore some issues that arise in going beyond the free field approximation. We consider Type IIB supergravity on $AdS_5 \times \mathbf{S}^5$. On the AdS_5 space there are massless $SU(4)$ gauge fields (which gauge the $SU(4)$ R -symmetry group of the boundary conformal

field theory). The R -symmetry is carried by chiral fermions on $\mathbf{S}^4$ (positive chirality $\mathbf{4}$'s and negative chirality $\overline{\mathbf{4}}$'s), so there is an anomaly in the three-point function of the R -symmetry currents. Since we identify the effective action of the conformal field theory with the classical action of supergravity (evaluated for a classical solution with given boundary values), the classical supergravity action must not be gauge invariant. How does this occur?

The classical supergravity action that arises in $\mathbf{S}^5$ compactification of Type IIB has in addition to the standard Yang-Mills action, also a Chern-Simons coupling (this theory has been described in [48,49]; the Chern-Simons term can be found, for example, in eqn. (4.15) in [49]). The action is thus

$$I(A) = \int_{B_5} \text{Tr} \left(\frac{F \wedge *F}{2g^2} + \frac{iN}{16\pi^2} (A \wedge dA \wedge dA + \dots) \right), \quad (2.31)$$

where the term in parentheses is the Chern-Simons term.

To determine the conformal field theory effective action for a source A_0 , one extends A_0 to a field A on B_{d+1} that obeys the classical equations. Note that A will have to be complex (because of the i multiplying the Chern-Simons term), and since classical equations for strong complex-valued gauge fields will not behave well, this is another reason, in addition to arguments given in section 2.3, that A_0 must be sufficiently small. Because of gauge-invariance, there is no natural choice of A ; we simply pick a particular A .

Any gauge transformation on $\mathbf{S}^4$ can be extended over B_5 ; given a gauge transformation g on $\mathbf{S}^4$, we pick an arbitrary extension of it over B_5 and call it $\hat{g}$. If A_0 is changed by a gauge transformation g , then A also changes by a gauge transformation, which we can take to be $\hat{g}$.

Now we want to see the anomaly in the conformal field theory effective action $W(A_0)$. In the classical supergravity approximation, the anomaly immediately follows from the relation $W(A_0) = I(A)$. In fact, $I(A)$ is not gauge-invariant; under a gauge transformation it picks up a boundary term (because the Chern-Simons coupling changes under gauge transformation by a total derivative) which precisely reproduces the chiral anomaly on the boundary.

Going beyond the classical supergravity approximation does not really change the discussion of the anomaly, because for *any* A whose boundary values are A_0 , the gauge-dependence of $I(A)$ is the same; hence averaging over A 's (in computing quantum loops)

does not matter. Likewise, stringy corrections to $I(A)$ involve integrals over B_5 of gauge-invariant local operators, and do not affect the anomaly.

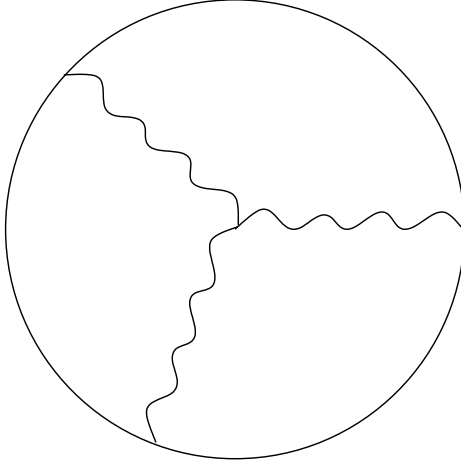


Fig. 1. A contribution of order g to the three point function $\langle J(P_1)J(P_2)J(P_3) \rangle$ of three currents in the boundary conformal field theory. The currents are inserted at points on the boundary of AdS_5 and the interaction takes place in the interior. The boundary is denoted by a solid circle. Wavy lines are gluon propagators, and the vertex in the interior could come from the conventional Yang-Mills action or from the Chern-Simons term.

n-Point Function Of Currents

To actually compute the n -point function of currents in the boundary theory, one would have to compute Feynman diagrams such as those sketched in Figures 1 and 2. In the diagrams, the boundary of AdS space is sketched as a circle; wavy lines are gauge boson propagators. There are two kinds of propagators. Propagation between two interior points is made by the conventional AdS gauge field propagator. Propagation between a boundary point and an interior point is made with the propagator used in (2.28); it expresses the influence in the interior of a “source” on the boundary. By a “source” we mean really a delta function term in the boundary values.

The operator product coefficients for a product of currents would be computed by inserting currents $J_{i_1}, \dots, J_{i_k}$ at boundary points $P_1, \dots, P_k$, and letting the P ’s approach each other, say at Q . Singularities arise only if some vertices in the interior of AdS space approach Q at the same time. After collapsing all propagators that connect points that are approaching each other, one gets a reduced diagram in which some number q of propagators

– with $q \geq 0$ – connect the point Q to an interior point. Intuitively, one expects this configuration to represent the insertion of an operator at Q . The OPE coefficient comes from evaluating the collapsing propagators (and integrating over positions of interaction vertices in the bulk that are approaching Q). For $q = 0$, the operator that is inserted at Q is a multiple of the identity, for $q = 1$ it is a current J , and for general q it appears intuitively that this operator should correspond to a normal ordered product of q currents, with derivatives perhaps acting on some of them. By studying the AdS propagators, it can be shown, at least in simple cases, that the operator product singularities obtained in this way are the expected ones. But we will not try to demonstrate this in the present paper.

A few noteworthy facts are the following. *Any* AdS theory, not necessarily connected with a specific supergravity compactification, appears to give boundary correlation functions that obey the general axioms of conformal field theory. If only massless particles are considered on AdS , one apparently gets an astonishingly simple closed operator product expansion with only currents (or only currents and stress tensors). To obtain the more realistic OPE of, for instance, the $\mathcal{N} = 4$ super Yang-Mills theory in four dimensions, one must include additional fields in the supergravity; we introduce them in sections 2.5 and 2.6.

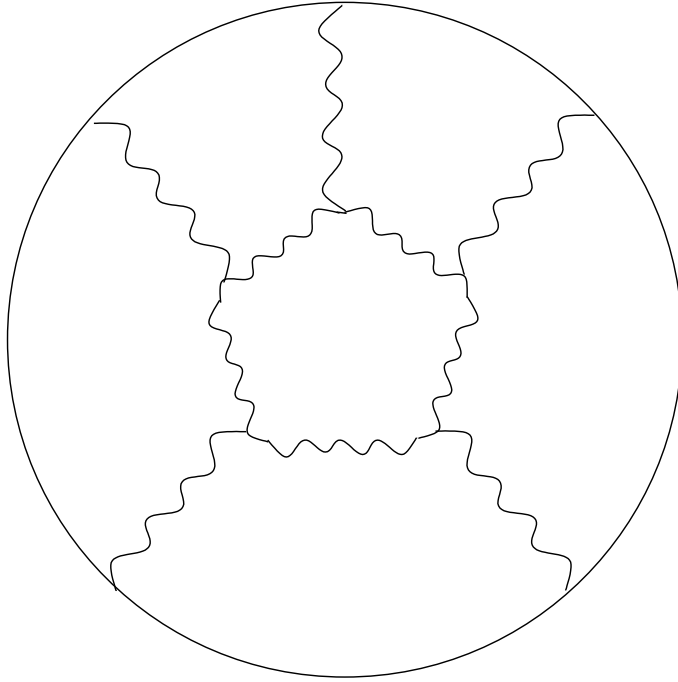


Fig. 2. A one-loop quantum correction to the five-point function of currents in the boundary conformal field theory.

Gravity

Now we will discuss, though only schematically, the gravitational case, that is, the dependence of the effective action of the boundary conformal field theory on the conformal structure on the boundary. We consider a nonstandard conformal structure on $\mathbf{S}^d$ and find, in accord with the Graham-Lee theorem, an Einstein metric on B_{d+1} that induces the given conformal structure on $\mathbf{S}^d$. To compute the partition function of the conformal field theory, we must evaluate the Einstein action on B_{d+1} for this metric. In writing this action, we must remember to include a surface term [50,51] in the Einstein action. The action thus reads

$$I(g) = \int_{B_{d+1}} d^{d+1}x \sqrt{g} \left(\frac{1}{2}R + \Lambda \right) + \int_{\partial B_{d+1}} K, \quad (2.32)$$

where R is the scalar curvature, Λ the cosmological constant, and K is the trace of the second fundamental form of the boundary.

Now, everything in sight in (2.32) is divergent. The Einstein equations $R_{ij} - \frac{1}{2}g_{ij}R = \Lambda g_{ij}$ imply that R is a constant, so that the bulk integral in (2.32) is just a multiple of the volume of B_{d+1} . This is of course infinite. Also, there is a question of exactly what the boundary term in (2.32) is supposed to mean, if the boundary is at infinity. To proceed, therefore, we need to regularize the action. The rough idea is to pick a positive function f on B_{d+1} that has a first order zero near the boundary. Of course, this breaks conformal invariance on the boundary of B_{d+1} , and determines an actual metric h on $\mathbf{S}^d$ via (2.2). We can regularize the volume integral by limiting it to the region $B(\epsilon)$ defined by $f > \epsilon$. Also, the boundary term in the action now gets a precise meaning: one integrates over the boundary of $B(\epsilon)$. Of course, divergences will appear as $\epsilon \rightarrow 0$.

We want to show at least schematically that, with a natural choice of f , the divergent terms are local integrals on the boundary. The strategy for proving this is as follows. Theorems 2.1 and 2.3 of [44] show that the metric of B_{d+1} is determined locally by the conformal structure of $\mathbf{S}^d$ up to very high order. The proof of the theorems in section 5 of that paper involves showing that once one picks a metric on $\mathbf{S}^d$ in its given conformal class, one can pick distinguished coordinates near the boundary of B_{d+1} to very high order. These distinguished coordinates give a natural definition of f to high order and with that definition, the divergent part of the action depends locally on the metric of $\mathbf{S}^d$. The divergent terms are thus local integrals on $\mathbf{S}^d$, of the general form $\int_{\mathbf{S}^d} d^d x \sqrt{h} P(R, \nabla R, \dots)$, where P is a polynomial in the Riemann tensor of $\mathbf{S}^d$ and its derivatives.

The effective action of the conformal field theory on $\mathbf{S}^d$ can thus be made finite by subtracting local counterterms. After the subtractions are made, the resulting finite effective action will not necessarily be conformally invariant, but the conformal anomaly – the variation of the effective action under conformal transformations of the metric – will be given by a local expression. In fact, the conformal anomaly comes as usual from the logarithmically divergent term; all the divergent terms are local.

The structure that we have described is just what one obtains when a conformal field theory is coupled to a background metric. A regularization that breaks conformal invariance is required. After picking a regularization, one encounters divergent terms which are local; on subtracting them, one is left with a local conformal anomaly. The structure of the conformal anomaly in general dimensions is reviewed in [52]. The conformal anomaly arises only for even d ; this is related to the fact that Theorems 2.1 and 2.3 in [44] make different claims for odd and even d . In the even d case, the theorems allow logarithmic terms that presumably lead to the conformal anomaly.

2.5. The Massive Case

We have come about as far as we can considering only massless fields on AdS space. To really make contact with the ideas in [13] requires considering massive excitations as well, since, among other reasons, the AdS compactifications considered in [13] apparently do not have consistent low energy truncations in which only massless fields are included. The reason for this last statement is that in, for instance, the $AdS_5 \times \mathbf{S}^5$ example, the radius of the $\mathbf{S}^5$ is comparable to the radius of curvature of the AdS_5 , so that the inverse radius of curvature (which behaves in AdS space roughly as the smallest wavelength of any excitation, as seen in [24]) is comparable to the masses of the Kaluza-Klein excitations.

For orientation, we consider a scalar with mass m in AdS_{d+1} . The action is

$$I(\phi) = \frac{1}{2} \int d\mu (|d\phi|^2 + m^2 \phi^2) \quad (2.33)$$

with $d\mu$ the Riemannian measure. The wave equation, which we wrote before as (2.8), receives an extra contribution from the mass term, and is now

$$\left(-\frac{1}{(\sinh y)^d} \frac{d}{dy} (\sinh y)^d \frac{d}{dy} + \frac{L^2}{\sinh^2 y} + m^2 \right) \phi = 0. \quad (2.34)$$

The L^2 term is still irrelevant at large y . The two linearly independent solutions of (2.34) behave for large y as $e^{\lambda y}$ where

$$\lambda(\lambda + d) = m^2. \quad (2.35)$$

For reasons that will be explained later, m^2 is limited to the region in which this quadratic equation has real roots. Let λ_+ and λ_- be the larger and smaller roots, respectively; note that $\lambda_+ \geq -d/2$ and $\lambda_- \leq -d/2$. One linear combination of the two solutions extends smoothly over the interior of AdS_{d+1} ; this solution behaves at infinity as $\exp(\lambda_+ y)$.

This state of affairs means that we cannot find a solution of the massive equation of motion (2.34) that approaches a constant at infinity. The closest that we can do is the following. Pick any positive function f on B_{d+1} that has a simple zero on the boundary. For instance, f could be e^{-y} (which has a simple zero on the boundary, as one can see by mapping back to the unit ball with $r = \tanh(y/2)$). Then one can look for a solution of the equation of motion that behaves as

$$\phi \sim f^{-\lambda_+} \phi_0, \quad (2.36)$$

with an arbitrary “function” ϕ_0 on the boundary.

But just what kind of object is ϕ_0 ? The definition of ϕ_0 as a function depends on the choice of a particular f , which was the same choice used in (2.2) to define a metric (and not just a conformal structure) on the boundary of AdS_{d+1} . If we transform $f \rightarrow e^w f$, the metric will transform by $d\tilde{s}^2 \rightarrow e^{2w} d\tilde{s}^2$. At the same time, (2.36) shows that ϕ_0 will transform by $\phi_0 \rightarrow e^{w\lambda_+} \phi_0$. This transformation under conformal rescalings of the metric shows that ϕ_0 must be understood as a conformal density of length dimension λ_+ , or mass dimension $-\lambda_+$. (Henceforth, the word “dimension” will mean mass dimension.)

Computation Of Two Point Function

If, therefore, in a conformal field theory on the boundary of AdS_{d+1} , there is a coupling $\int \phi_0 \mathcal{O}$ for some operator $\mathcal{O}$, then $\mathcal{O}$ must have conformal dimension $d + \lambda_+$. Let us verify this by computing the two point function of the field $\mathcal{O}$.

To begin with, we need to find the explicit form of a function ϕ that obeys the massive wave equation and behaves as $f^{-\lambda_+} \phi_0$ at infinity. We represent AdS space as the half-space $x_0 \geq 0$ with metric $ds^2 = (1/x_0^2)(dx_0^2 + \sum_i dx_i^2)$. As before, the first step is to find a Green’s function, that is a solution of $(-D_i D^i + m^2)K = 0$ that vanishes on the boundary except at one point. Again taking this point to be at $x_0 = \infty$, K should be a function of x_0 only, and the equation reduces to

$$\left(-x_0^{d+1} \frac{d}{dx_0} x_0^{-d+1} \frac{d}{dx_0} + m^2 \right) K(x_0) = 0. \quad (2.37)$$

The solution that vanishes for $x_0 = 0$ is $K(x_0) = x_0^{d+\lambda_+}$. After the inversion $x_i \rightarrow x_i/(x_0^2 + |\mathbf{x}|^2)$, we get the solution

$$K = \frac{x_0^{d+\lambda_+}}{(x_0^2 + |\mathbf{x}|^2)^{d+\lambda_+}}, \quad (2.38)$$

which behaves as $x_0^{d+\lambda_+}$ for $x_0 \rightarrow 0$ except for a singularity at $\mathbf{x} = 0$. Now,

$$\lim_{x_0 \rightarrow 0} \frac{x_0^{d+2\lambda_+}}{(x_0^2 + |\mathbf{x}|^2)^{d+\lambda_+}} \quad (2.39)$$

is a multiple of $\delta(\mathbf{x})$, as one can prove by using a scaling argument to show that

$$\int d\mathbf{x} \frac{x_0^{d+2\lambda_+}}{(x_0^2 + |\mathbf{x}|^2)^{d+\lambda_+}} \quad (2.40)$$

is independent of x_0 , and observing that for $x_0 \rightarrow 0$, the function in (2.39) is supported near $x = 0$. So if we use the Green's function K to construct the solution

$$\phi(x) = c' \int d\mathbf{x}' \frac{x_0^{d+\lambda_+}}{(x_0^2 + |\mathbf{x} - \mathbf{x}'|^2)^{d+\lambda_+}} \phi_0(\mathbf{x}') \quad (2.41)$$

of the massive wave equation, then ϕ behaves for $x_0 \rightarrow 0$ like $x_0^{-\lambda_+} \phi_0(x)$, as expected.

Moreover, given the solution (2.41), one can evaluate the action (2.33) by the same arguments (integration by parts and reduction to a surface term) that we used in the massless case, with the result

$$I(\phi) = \frac{c'(d + \lambda_+)}{2} \int d\mathbf{x} d\mathbf{x}' \frac{\phi_0(\mathbf{x}) \phi_0(\mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|^{2(\lambda_+ + d)}}. \quad (2.42)$$

This is the expected two point function of a conformal field $\mathcal{O}$ of dimension $\lambda_+ + d$.

In sum, the dimension Δ of a conformal field on the boundary that is related to a field of mass m on AdS space is $\Delta = d + \lambda_+$, and is the larger root of

$$\Delta(\Delta - d) = m^2. \quad (2.43)$$

Thus

$$\Delta = \frac{1}{2}(d + \sqrt{d^2 + 4m^2}). \quad (2.44)$$

In section 3.3, we will propose another explanation, using a Hamiltonian approach, for this relation.

If one considers not a scalar field but a field with different Lorentz quantum numbers, then the dimensions are shifted. For instance, a massless p -form C on AdS space could couple to a $d - p$ -form operator $\mathcal{O}$ on the boundary via a coupling $\int_{M_d} C \wedge \mathcal{O}$; conformal invariance (and gauge invariance of the massless C field) requires that the dimension of $\mathcal{O}$ be $d - p$. For $p = 1$, $\mathcal{O}$ is equivalent to a current, and the fact that the dimension is $d - 1$ was verified explicitly in the last subsection. If instead C is a massive field that behaves as $f^{-\lambda_+} C_0$ near the boundary, then as in our discussion of scalars, C_0 is a p -form on the boundary of conformal dimension $p - \lambda_+$, and an operator $\mathcal{O}$ that couples to C_0 has conformal dimension $\Delta = d - p + \lambda_+$. In this case (2.43) is replaced by

$$(\Delta + p)(\Delta + p - d) = m^2. \quad (2.45)$$

Correlation Functions Of Arbitrary Operators

Now we would like to go beyond the two-point function and consider the n -point functions of arbitrary operators $\mathcal{O}_i$ in the conformal field theory. Let us first discuss the situation in field theory. We can hope to define n -point functions $\langle \mathcal{O}_1(x_1) \mathcal{O}_2(x_2) \dots \mathcal{O}_n(x_n) \rangle$, for distinct points $x_1, \dots, x_n$. If one allows arbitrary local operators of any dimension, the singularities for coincident points are extremely complicated and one cannot without a very high degree of complication and arbitrariness define a generating function

$$\left\langle \exp\left(\int d^d x \sum_a \epsilon_a(x) \mathcal{O}_a(x)\right) \right\rangle. \quad (2.46)$$

It is nevertheless convenient to formally have the machinery of generating functionals. To do this one should consider the ϵ_a to have disjoint support and to be infinitesimal, and evaluate (2.46) only to linear order in each ϵ_a . One permits the operators $\mathcal{O}_a(x)$ to be the same for some distinct values of a .

Now let us consider the AdS description. In doing so, for simplicity of exposition we suppose that the $\mathcal{O}_a$ are all scalar operators, of dimension $d + \lambda_{a,+}$. They correspond to fields ϕ_a on AdS which behave near the boundary as $f^{-\lambda_{a,+}}$. Now, ideally we would like to claim that (2.46) equals the supergravity (or string theory) partition function with boundary conditions such that ϕ_a behaves near the boundary as $f^{-\lambda_{a,+}} \epsilon_a$. This works fine if the $\lambda_{a,+}$ are all negative; then the perturbations vanish on the boundary, and the determination of the boundary behavior of the fields using the linearized equations is self-consistent. The case of negative $\lambda_{a,+}$ is the case of relevant perturbations of the conformal

field theory, of dimension $< d$, and it should come as no surprise that this is the most favorable case for defining the functional (2.46) as an honest functional of the ϵ_a (and not just for infinitesimal ϵ_a). One can also extend this to include operators with vanishing $\lambda_{a,+}$, corresponding to marginal perturbations of the conformal field theory. For this, one must go beyond the linearized approximation in describing the boundary fields, and include general boundary values of massless fields, which we introduced in section 2.3.

The trouble arises if one wants to include irrelevant (or unrenormalizable) perturbations of the conformal field theory. This is the case in which in field theory one would run into severe trouble in making sense of the generating functional (2.46), such that one is led to consider the ϵ_a to be infinitesimal and of disjoint support. One runs into a corresponding phenomenon in the *AdS* description. Since the solutions of the linearized equations have boundary behavior $\phi_a \sim \epsilon_a f^{-\lambda_{a,+}}$, they diverge near the boundary if $\lambda_+ > 0$, in which case use of the linearized equations to determine the boundary behavior is not self-consistent. Just as in the conformal field theory, the only cure for this is to consider the ϵ_a to be infinitesimal. To make sense of the idea of doing the path integral with boundary conditions $\phi_a \sim \epsilon_a f^{-\lambda_{a,+}}$, we consider the ϵ_a to be not real-valued functions but infinitesimal variables with $\epsilon_a^2 = 0$, and with different ϵ_a having disjoint support. We write $\phi_a = \phi'_a + \epsilon_a f^{-\lambda_{a,+}}$, where ϕ'_a is to vanish on the boundary, and substitute this in the supergravity action (or string theory effective action). The substitution is well-behaved, to first order in ϵ_a , introducing no divergent terms at infinity, because the shifts are by solutions of the linearized equations. Computing the path integral over the ϕ'_a gives a concrete recipe to compute the generating functional (2.46) to the same extent that it makes sense in field theory. Of course, in the approximation of classical supergravity, the path integral is computed just by solving the classical equations for ϕ'_a .

2.6. Comparison To “Experiment”

Now we will compare this formalism to “experiment,” that is to some of the surprising features of *AdS* supergravity.

One very odd feature of the discussion in section 2.5 is that relevant or superrenormalizable perturbations correspond to operators with $\lambda_+ < 0$. Since $\lambda_+ = (-d + \sqrt{d^2 + 4m^2})/2$, λ_+ is negative precisely if $m^2 < 0$. The conformal field theories considered in [13] do have relevant perturbations. So the corresponding *AdS* theories must have “tachyons,” that is fields with $m^2 < 0$.

This seems worrisome, to say the least. The same worry was faced in early investigations of gauged and *AdS* supergravity, where it was typically found that there were in fact scalar fields of $m^2 < 0$. One at first believes that these lead to instabilities, but this proved not to be the case [24,26]. Because of the boundary conditions at infinity, the kinetic energy of a scalar field in *AdS* space cannot vanish, and stability requires not that m^2 should be positive but that m^2 should be no smaller than a certain negative lower bound.

Here is a quick and nonrigorous Euclidean space derivation of the lower bound. We will certainly run into a pathology if we consider a scalar field ϕ in *AdS* space that has a normalizable zero mode, for by $SO(1, d)$ invariance, there will be infinitely many such modes. Consider a scalar field whose mass is such that solutions of the wave equation behave near the boundary (in the representation of *AdS* by the metric $dy^2 + \sinh^2 y d\Omega^2$) as $e^{\lambda_+ y}$. Such a field has no normalizable zero modes if $\lambda_+ > -d/2$. A more careful analysis shows that there also are no normalizable zero modes if $\lambda_+ = -d/2$. (This is the case that $\lambda_+ = \lambda_-$, and the solutions of the linearized equation that are regular throughout *AdS* space actually behave near infinity as $ye^{-dy/2}$; such solutions are not normalizable.) The case $\lambda_+ = \lambda_- = -d/2$ arises for $m^2 = -d^2/4$, and this is the lower bound on m^2 . If m^2 is smaller, then $\lambda_{\pm}$ become complex, with real part $-d/2$, and the linearized equations have zero modes with plane wave normalizability; this is the same sort of spectrum and gives the same sort of pathology as for tachyon equations in flat Euclidean space. So the lower bound is $m^2 \geq -d^2/4$, $\lambda_+ \geq -d/2$. Note that the dimension $\Delta = d + \lambda_+$ of a scalar field on the boundary obtained by the correspondence between boundary conformal field theory and *AdS* thus obeys

$$\Delta \geq \frac{d}{2}. \quad (2.47)$$

Making contact with “experiment” also requires familiarity with one other feature of *AdS* supergravity that caused puzzlement in the old days. Consider a theory with maximal supersymmetry – 32 supercharges – such as the $AdS_5 \times \mathbf{S}^5$ example studied in [13]. There are infinitely many massive Kaluza-Klein harmonics. Superficially, with 32 supercharges, the spins in any multiplet should range up to 4. But supergravity only has fields of spin ≤ 2 ! The resolution of this puzzle involves a subtle *AdS* analog [53] of the construction [54] of small multiplets of flat space supersymmetry via central charges. *AdS* supergravity has special “small” multiplets with lower than expected spin, and for theories with 32 supercharges and only fields of spin ≤ 2 , *all* Kaluza-Klein harmonics are in such reduced multiplets!

The Kaluza-Klein harmonics are important in the present discussion for the following reason. In the models considered in [13], in a limit in which classical supergravity is valid, the states with masses of order one are precisely the Kaluza-Klein modes. Indeed, in for example the $AdS_5 \times \mathbf{S}^5$ model, the radius of the $\mathbf{S}^5$ is comparable to the radius of curvature of the AdS_5 factor, so the Kaluza-Klein harmonics have masses of order one, in units of the AdS_5 length scale. Stringy excitations are much heavier; for instance, in this model they have masses of order $(g_{YM}^2 N)^{1/4}$ (which is large in the limit in which classical supergravity is valid), as shown in [13].

Since the dimension of a scalar operator in the four-dimensional conformal field theory is

$$\Delta = d + \lambda_+ = \frac{d + \sqrt{d^2 + 4m^2}}{2}, \quad (2.48)$$

the particles of very large mass in AdS space correspond to operators of very high dimension in the conformal field theory. The conformal fields with dimensions of order one correspond therefore precisely to the Kaluza-Klein excitations.

Since the Kaluza-Klein excitations are in “small” representations of supersymmetry, their masses are protected against quantum and stringy corrections. The conformal fields that correspond to these excitations are similarly in “small” representations, with dimensions that are protected against quantum corrections. It is therefore possible to test the conjectured CFT/ AdS correspondence by comparing the Kaluza-Klein harmonics on $AdS_5 \times \mathbf{S}^5$ to the operators in the $\mathcal{N} = 4$ super Yang-Mills theory that are in small representations of supersymmetry; those operators are discussed in [53].

The Kaluza-Klein harmonics have been completely worked out in [30,31]. The operators of spin zero are in five infinite families. We will make the comparison to the $\mathcal{N} = 4$ theory for the three families that contain relevant or marginal operators. Two additional families that contain only states of positive m^2 will not be considered here.

We recall the following facts about the $\mathcal{N} = 4$ theory. This theory has an R -symmetry group $SU(4)$, which is a cover of $SO(6)$. Viewed as an $N = 0$ theory, it has six real scalars X^a in the $\mathbf{6}$ or vector of $SO(6)$ and adjoint representation of the gauge group, and four fermions λ_α^A , also in the adjoint representation of the gauge group, in the $\mathbf{4}$ of $SU(4)$ or positive chirality spinor of $SO(6)$. (Here $a = 1, \dots, 6$ is a vector index of $SO(6)$, $A = 1, \dots, 4$ is a positive chirality $SO(6)$ spinor index, and α is a positive chirality Lorentz spinor index.) From the point of view of an $\mathcal{N} = 1$ subalgebra of the supersymmetry algebra, this theory has three chiral superfields Φ^z , $z = 1, \dots, 3$ in the adjoint representation, and

a chiral superfield W_α , also in the adjoint representation, that contains the gauge field strength. The superpotential is $W = \epsilon_{z_1 z_2 z_3} \text{Tr} \Phi^{z_1} [\Phi^{z_2}, \Phi^{z_3}]$. Viewed as an $\mathcal{N} = 2$ theory, this theory has a vector multiplet and a hypermultiplet in the adjoint representation.

We can now identify the following families of operators in the $\mathcal{N} = 4$ theory that are in small representations:

(1) First we view the theory from an $\mathcal{N} = 1$ point of view. Any gauge-invariant polynomial in chiral superfields is a chiral operator. If it is not a descendant, that is it cannot be written in the form $\{\overline{Q}_{\dot{\alpha}}, \Lambda^{\dot{\alpha}}\}$ for any $\Lambda^{\dot{\alpha}}$, then it is in a “small” representation of supersymmetry. Consider the operators $T^{z_1 z_2 \dots z_k} = \text{Tr} \Phi^{z_1} \Phi^{z_2} \dots \Phi^{z_k}$. We require $k \geq 2$ as the gauge group is $SU(N)$ and not $U(N)$. If one symmetrizes in $z_1, \dots, z_k$, one gets chiral fields that are in “small” representations. If one does not so symmetrize, one gets a descendant, because of the fact that the commutators $[\Phi^{z_i}, \Phi^{z_j}]$ are derivatives of the superpotential W .

From the $\mathcal{N} = 1$ point of view, the theory has a $U(3)$ global symmetry, of which the $U(1)$ acts as a group of R -symmetries. The R -charge of $T^{z_1 z_2 \dots z_k}$ is k times that of Φ . The operator $T^{z_1 z_2 \dots z_k}$ has dimension k in free field theory. Its dimension is determined by the R -charge and so is exactly k , for all values of the coupling. From a $U(3)$ point of view, T transforms in the representation which is the k -fold symmetric product of the $\mathbf{3}$ with itself. Restoring the $SO(6)$ symmetry of the $\mathcal{N} = 4$ theory, this $U(3)$ representation is part of the $SO(6)$ representation containing symmetric traceless tensors $T^{a_1 a_2 \dots a_k}$ of order k . A field of dimension k in four-dimensional conformal field theory has $\lambda_+ = k - 4$ and corresponds to a scalar field in supergravity with a mass $m^2 = \lambda_+(\lambda_+ + 4) = k(k - 4)$. So we expect in the Kaluza-Klein spectrum of the $\mathcal{N} = 4$ theory that for every $k = 2, 3, \dots$ there should be a scalar with mass $m^2 = k(k - 4)$ transforming in the k^{th} traceless symmetric tensor representation of $SO(6)$. These states can be found in eqn. (2.34) and in Fig. 2 and Table III of [30]. (Formulas given there contain a parameter e which we have set to 1.) Note that a $k = 1$ state does not appear in the supergravity, which confirms that the supergravity is dual to an $SU(N)$ gauge theory and not to a $U(N)$ theory.⁷ In fact, the $k = 2$ state saturates the bound (2.47), and a $k = 1$ state would violate it. For $k = 2$ we get the

⁷ Another reason that the $U(1)$ gauge field could not be coded in the AdS_5 theory is that it is free, while in the AdS_5 theory everything couples to gravity and nothing is free. To describe $U(N)$ gauge theory on the boundary, one would have to supplement the AdS_5 theory with an explicit $U(1)$ singleton field on the boundary.

relevant operator $\text{Tr} X^a X^b - (1/6)\delta^{ab}\text{Tr} X^2$ (however, the operator $\sum_a \text{Tr}(X^a)^2$, which is certainly a relevant operator for weak coupling, is not chiral and so is related to a stringy excitation and has a dimension of order $(g_{YM}^2 N)^{1/4}$ for strong coupling).

(2) Likewise, we can make the chiral superfield $V^{z_1 \dots z_t} = \text{Tr} W_\alpha W^\alpha \Phi^{z_1} \Phi^{z_2} \dots \Phi^{z_t}$ for $t \geq 0$. Again, modulo descendants one can symmetrize in the ordering of all $t + 2$ factors. These operators have dimension $t + 3$ in free field theory, and again their dimensions are protected by the R -symmetry. If we set $t = 0$, we get the relevant operator $\text{Tr} W_\alpha W^\alpha$, which is a linear combination of the gluino bilinears $\text{Tr} \lambda_\alpha^A \lambda^{\alpha B}$, which transform in the **10** of $SO(6)$ (one can think of this representation as consisting of self-dual third rank antisymmetric tensors). For higher t , these states transform in the representation obtained by tensoring the **10** with the t^{th} rank symmetric tensors and removing traces. Setting $k = t + 1$, we hence expect for $k = 1, 2, 3, \dots$ a supergravity harmonic in the representation just stated with mass

$$m^2 = (k + 2)(k - 2). \quad (2.49)$$

Such a harmonic is listed in Fig. 2 and in Table III in [30].

(3) The conformal field theory also contains a special marginal operator, the derivative of the Lagrangian density with respect to the Yang-Mills coupling. This operator is in a small representation, since the Lagrangian cannot be written as an integral over a superspace with 16 fermionic coordinates. It transforms in the singlet representation of $SO(6)$, and, being a marginal operator, is related to a supergravity mode with $m^2 = 0$, in fact, the dilaton. To exhibit this operator as the first in an infinite series, view the $\mathcal{N} = 4$ theory as an $\mathcal{N} = 2$ theory with a vector multiplet and an adjoint hypermultiplet. Let a be the complex scalar in the vector multiplet. Possible Lagrangians for the vector multiplets, with $\mathcal{N} = 2$ supersymmetry and the smallest possible number of derivatives, are determined by a “prepotential,” which is a holomorphic, gauge-invariant function of a . A term $\text{Tr} a^r$ in the prepotential leads to an operator $Q_r = \text{Tr} a^{r-2} F_{ij} F^{ij} + \dots$ which could be added to the Lagrangian density while preserving $\mathcal{N} = 2$ supersymmetry. These operators are all in small representations as the couplings coming from the prepotential cannot be obtained by integration over all of $\mathcal{N} = 2$ superspace. Q_r has dimension $2 + r$ in free field theory, and this dimension is protected by supersymmetry. Since a is part of a vector of $SO(6)$, Q_r is part of a set of operators transforming in the $r - 2^{\text{th}}$ symmetric tensor representation of $SO(6)$. (In fact, Q_r can be viewed as a highest weight vector for this representation.) Hence, if we set $k = r - 2$, we have for $k \geq 0$ an operator of dimension

$k+4$ in the k^{th} symmetric tensor representation, corresponding to Kaluza-Klein harmonics in that representation with mass

$$m^2 = k(k+4). \quad (2.50)$$

These states can be found in Fig. 2 and Table III in [30].

3. Other Spacetimes And Hamiltonian Formalism

3.1. General Formalism

The CFT- AdS correspondence relates conformal field theory on $\mathbf{S}^d$ to supergravity on $AdS_{d+1} \times W$, where W is a compact manifold (a sphere in the maximally supersymmetric cases). What happens if we replace $\mathbf{S}^d$ by a more general compact d -manifold N ? The natural intuitive answer is that one should then replace AdS_{d+1} by a $d+1$ -dimensional Einstein manifold X with negative cosmological constant. The relation between X and M should be just like the relation between $\mathbf{S}^d$ and AdS_{d+1} . X should have a compactification consisting of a manifold with boundary $\overline{X}$, whose boundary points are M and whose interior points are X , and such that the metric on X has a double pole near the boundary. Then the metric on X determines a conformal structure on M , just as we reviewed in section 2.1 in the case of AdS_{d+1} and $\mathbf{S}^d$.

More generally, instead of a $d+1$ -manifold X , one might use a manifold Y of dimension 10 or 11 (depending on whether one is doing string theory or M -theory) that looks near infinity like $X \times W$ for some Einstein manifold X . Moreover, Y might contain various branes or stringy impurities of some kind. These generalizations are probably necessary, since M might be, for example, a four-manifold of non-zero signature, which is not the boundary of any five-manifold. (However, it may be that for such M 's, the conformal field theory partition function diverges for reasons discussed at the end of section 2.3.) But since the discussion is non-rigorous anyway, we will keep things simple and speak in terms of a $d+1$ -dimensional Einstein manifold X .

Once X is found, how will we use it to study conformal field theory on M ? As in section 2.3, we propose that the conformal field theory partition function on M equals the supergravity (or string theory) partition function on X , with boundary conditions given by the conformal structure (and other fields) on M . In the approximation of classical supergravity, we simply solve the classical equations on X with the given boundary conditions, and write

$$Z_{CFT}(M) = \exp(-I_S(X)), \quad (3.1)$$

where $Z_{CFT}(M)$ is the conformal field theory partition function on M and I_S is the supergravity action.

In general, there might be several possible X 's. If so, we have no natural way to pick one, so as is usual in Euclidean quantum gravity, we replace (3.1) by a sum. Before writing the sum, it is helpful to note that in the large N limit, $I_S(X)$ is proportional to a positive power of N , in fact $I_S = N^\gamma F(X)$ for some $\gamma > 0$. For example, $\gamma = 2$ for $\mathcal{N} = 4$ super Yang-Mills in four dimensions. So we postulate that in general

$$Z_{CFT}(M) = \sum_i \exp(-N^\gamma F(X_i)), \quad (3.2)$$

where the X_i are the Einstein manifolds of boundary M .

Another refinement involves spin structures. M may admit several spin structures, in which case $Z_{CFT}(M)$ will in general depend on the spin structure on M . If so, we select a spin structure on M and restrict the sum in (3.2) to run over those X_i over which the given spin structure on M can be extended.⁸

Of course, (3.2) should be viewed as the classical supergravity approximation to an exact formula

$$Z_{CFT}(M) = \sum_i Z_S(X_i), \quad (3.3)$$

where $Z_S(X_i)$ is the partition function of string theory on X_i .

Now, as M is compact, there are no conventional phase transitions in evaluating path integrals on M . As long as $Z_{CFT}(M)$ is sufficiently convergent (we discussed obstructions to convergence at the end of section 2.3), $Z_{CFT}(M)$ is a smooth function of the conformal structure of M (and other fields on M). However, in (3.2) we see a natural mechanism for a singularity or phase transition that would arise only in the large N limit. In the large N limit, the sum in (3.2) will be dominated by that X_i for which $F(X_i)$ is smallest. If $F(M) = -\ln Z_{CFT}(M)$ is the conformal field theory free energy, then

$$\lim_{N \rightarrow \infty} \frac{F(M)}{N^\gamma} = F(X_i), \quad (3.4)$$

⁸ There may be further refinements of a similar nature. For example, in the case that the boundary theory is four-dimensional gauge theory with a gauge group that is locally $SU(N)$, it may be that the gauge group is really $SU(N)/\mathbf{Z}_N$, in which case the partition function on M depends on a choice of a “discrete magnetic flux” $w \in H^2(M, \mathbf{Z})$. Perhaps one should restrict the sum in (3.2) to X_i over which w extends.

for that value of i for which $F(X_i)$ is least. At a point at which $F(X_i) = F(X_j)$ for some $i \neq j$, one may well “jump” from one branch to another. This will produce a singularity of the large N theory, somewhat similar to large N singularities found [47] in certain toy models.

3.2. A Concrete Example

We will next describe a concrete example, namely $M = \mathbf{S}^1 \times \mathbf{S}^{d-1}$, in which one can explicitly describe two possible X ’s and a transition between them. In fact, the two solutions were described (in the four-dimensional case) by Hawking and Page [50] who also pointed out the existence of the phase transition (which they of course interpreted in terms of quantum gravity rather than boundary conformal field theory!).

The first solution is obtained as follows. The simplest way to find Einstein manifolds is to take the quotient of a portion of AdS_{d+1} by a group that acts discretely on it. AdS_{d+1} can be described as the quadric

$$uv - \sum_{i=1}^d x_i^2 = 1. \quad (3.5)$$

(The analogous formula for AdS_{d+1} with Lorentz signature is given in (1.2).) We restrict ourselves to the region AdS_{d+1}^+ of AdS space with $u, v > 0$. We consider the action on AdS_{d+1}^+ of a group $\mathbf{Z}$ generated by the transformation

$$u \rightarrow \lambda^{-1}u, \quad v \rightarrow \lambda v, \quad x_i \rightarrow x_i, \quad (3.6)$$

with λ a fixed real number greater than 1. This group acts freely on AdS_{d+1}^+ . Let X_1 be the quotient $X_1 = AdS_{d+1}^+/\mathbf{Z}$. To describe X_1 , we note that a fundamental domain for the action of $\mathbf{Z}$ on v is $1 \leq v \leq \lambda$, with $v = 1$ and $v = \lambda$ identified. v parametrizes a circle, on which a natural angular coordinate is

$$\theta = 2\pi \ln v / \ln \lambda. \quad (3.7)$$

For given v , one can solve for u by $u = (1 + \sum_i x_i^2)/v$. So X_1 is spanned by x_i and θ , and is topologically $\mathbf{R}^4 \times \mathbf{S}^1$. To see what lies at “infinity” in X_1 , we drop the 1 in (3.5), to get

$$uv - \sum_{i=1}^d x_i^2 = 0, \quad (3.8)$$

and regard u, v , and the x_i as homogeneous coordinates, subject to a scaling relation $u \rightarrow su, v \rightarrow sv, x_i \rightarrow sx_i$ with real positive s (s should be positive since we are working in the region $u, v > 0$). (3.8) does not permit the x_i to all vanish (since $u, v > 0$), so one can use the scaling relation to set $\sum_{i=1}^d x_i^2 = 1$. The x_i modulo the scaling relation thus define a point in $\mathbf{S}^{d-1}$. Just as before, v modulo the action of (3.6) defines a point in $\mathbf{S}^1$, and one can uniquely solve (3.8) for u . Hence the boundary of X_1 is a copy of $M = \mathbf{S}^1 \times \mathbf{S}^{d-1}$.

An important property of this example is that it is invariant under an action of $SO(2) \times SO(d)$. Here, $SO(d)$ is the rotation group of the second factor in $M = \mathbf{S}^1 \times \mathbf{S}^{d-1}$; it acts by rotation of the x_i in (3.5). $SO(2)$ acts by rotation of the angle θ defined in (3.7). The $SO(2) \times SO(d)$ symmetry uniquely determines the conformal structure of $\mathbf{S}^1 \times \mathbf{S}^{d-1}$ up to a constant $\beta = \ln \lambda$ which one can think of as the ratio of the radius of $\mathbf{S}^1$ to the radius of $\mathbf{S}^{d-1}$. $M = \mathbf{S}^1 \times \mathbf{S}^{d-1}$ has two spin structures – spinors may be periodic or antiperiodic around the $\mathbf{S}^1$. The partition functions of M with these spin structures can be interpreted respectively as

$$Z_1(M) = \text{Tr } e^{-\beta H} (-1)^F \quad (3.9)$$

or

$$Z_2(M) = \text{Tr } e^{-\beta H}. \quad (3.10)$$

Here H is the Hamiltonian that generates time translations if one quantizes on $\mathbf{S}^{d-1} \times \mathbf{R}$ (with $\mathbf{R}$ as the “time” direction); or it is the dilation generator if the theory is formulated on $\mathbf{R}^d$. Note that $Z_1(M)$ is not really a supersymmetric index, at least not in a naive sense, since compactification on $\mathbf{S}^{d-1}$ breaks supersymmetry, that is, the dilation generator H does not commute with any supersymmetries.

Both spin structures extend over X_1 , so X_1 contributes to both $\text{Tr } e^{-\beta H} (-1)^F$ and $\text{Tr } e^{-\beta H}$ in the conformal field theory. Its contribution is, however, extremely simple in the approximation of classical supergravity. The reason is that $X_1 = \mathbf{S}^1 \times \mathbf{R}^d$, and β enters only as the radius of $\mathbf{S}^1$. This ensures that the supergravity action is linear in β ,

$$I_S(X_1) = \beta E \quad (3.11)$$

for some constant E , since $I_S(X_1)$ is computed from a local integral on X_1 (plus a boundary term that is a local integral at infinity), and β only enters in determining how far one must integrate in the $\mathbf{S}^1$ direction. If one introduces additional background fields on

$M = \mathbf{S}^1 \times \mathbf{S}^{d-1}$ in a way that preserves the time-translation invariance of $\mathbf{S}^1 \times \mathbf{S}^{d-1}$, then E may depend on the background fields, but is still independent of β . E should be interpreted as the ground state energy in quantization of the conformal field theory on $\mathbf{S}^{d-1}$.

If we set $r^2 = \sum_i x_i^2$, $\sum_i (dx_i)^2 = (dr)^2 + r^2 d\Omega^2$, $t = \ln v + \frac{1}{2} \ln(1+r^2)$, then the metric of X_1 , which is just the AdS metric $-du dv + \sum_i (dx_i)^2$ with some global identifications, becomes

$$ds^2 = (1+r^2)(dt)^2 + \frac{(dr)^2}{1+r^2} + r^2 d\Omega^2, \quad (3.12)$$

which is the form in which it is written in [56].

The Schwarzschild Solution

The second solution X_2 is the AdS Schwarzschild solution. Topologically $X_2 = \mathbf{R}^2 \times \mathbf{S}^{d-1}$. The $\mathbf{S}^1$ factor in $M = \mathbf{S}^1 \times \mathbf{S}^{d-1}$ is the boundary (in a sense with which we are by now familiar) of the $\mathbf{R}^2$ factor in X_2 . Being simply-connected, X_2 has a unique spin structure, which restricts on $\mathbf{S}^1$ to the antiperiodic spin structure. (This amounts to the familiar fact in superstring theory that the antiperiodic or Neveu-Schwarz spin structure on a circle is the one that extends over a disc.) Hence X_2 contributes to $\text{Tr } e^{-\beta H}$, but not to $\text{Tr } e^{-\beta H} (-1)^F$. Hence $\text{Tr } e^{-\beta H}$, but not $\text{Tr } e^{-\beta H} (-1)^F$, may have a large N phase transition from a “flop” between X_1 and X_2 . That it does, at least for $d+1 = 4$, is equivalent to facts described by Hawking and Page [56].

The metric of a Euclidean AdS_{d+1} Schwarzschild black hole of mass m , for $d = 3$, as described in [56], eqn. 2.1, is

$$ds^2 = V(dt)^2 + V^{-1}(dr)^2 + r^2 d\Omega^2 \quad (3.13)$$

with $d\Omega^2$ the line element on a round two-sphere, and

$$V = 1 - \frac{2m}{r} + r^2. \quad (3.14)$$

(Notation compares to that in [56] as follows: we set $b = 1$, since the form of the AdS metric that we have used corresponds for AdS_4 to $\Lambda = -3$; we set $m_p = 1$, and write m instead of M for the black hole mass.) Let r_+ be the largest root of the equation $V(r) = 0$. The spacetime is limited to the region $r \geq r_+$. The metric is smooth and complete if t is regarded as an angular variable with period

$$\beta = \frac{12\pi r_+}{1 + 3r_+^2}. \quad (3.15)$$

This has a maximum as a function of r_+ , so the black hole spacetime X_2 contributes to the thermodynamics only for sufficiently small β , that is sufficiently large temperature.

The action difference between X_2 and X_1 was computed by Hawking and Page to be

$$I = \frac{\pi r_+^2(1 - r_+^2)}{1 + 3r_+^2}. \quad (3.16)$$

This is negative for sufficiently large r_+ , that is for sufficiently small β or sufficiently large temperature. Thus (for $d = 3$ and presumably for all d), the boundary conformal field theories that enter the CFT- AdS correspondence have phase transitions as a function of temperature in the large N limit.

Further support for the interpretation of the AdS black hole in terms of conformal field theory at high temperature comes from the fact that the AdS black hole has positive specific heat [56] in the region in which it has lower action. This is needed for the correspondence with conformal field theory (which certainly has positive specific heat), and is a somewhat surprising result, as it does not hold for Schwarzschild black holes in asymptotically flat space.

On the other hand, it is also shown in [56] that very low mass AdS black holes have negative specific heat, like black holes in asymptotically flat space. In fact, very small mass black holes have a size much less than the AdS radius of curvature, and are very similar to flat space black holes. Such black holes should decay by emission of Hawking radiation; the CFT/ AdS correspondence gives apparently a completely unitary, quantum mechanical description of their decay.

Comparison With Gauge Theory Expectations

We will now specialize to the $AdS_5 \times \mathbf{S}^5$ case, and compare to the expected large N behavior of four-dimensional gauge theories.

Actually, we do not know how the $\mathcal{N} = 4$ theory should behave for large N with large and fixed $g_{YM}^2 N$. We will therefore compare to the expected large N thermodynamics [57] of an $SU(N)$ gauge theory with massive quarks in the fundamental representation (that is, ordinary QCD generalized to large N). It is surprising that this comparison will work, since QCD is confining, while the $\mathcal{N} = 4$ theory has no obviously analogous phenomenon. However, the comparison seems to work, as we will see, at least for large $g_{YM}^2 N$.

The whole idea of the $1/N$ expansion as an approach to QCD [1] is that for large N , the Hilbert space of the theory consists of color singlet particles whose masses and

multiplicities are independent of N . Suppose we formulate the theory on some three-manifold K and define $Z(\beta) = \text{Tr } e^{-\beta H}$, and $F = -\ln Z$. The ground state energy E in this theory is of order N^2 (as vacuum diagrams receive contributions from N^2 species of gluon), while excitation energies and multiplicities are of order one. Hence, we expect that

$$\lim_{N \rightarrow \infty} \frac{F(\beta)}{N^2} = \beta \frac{E}{N^2}. \quad (3.17)$$

The idea is that as the excitations have multiplicity of order one, they contribute to F an amount of order one, and make a vanishing contribution to the large N limit of F/N^2 , which hence should simply measure the ground state energy. The formula in (3.17) precisely has the structure of the contribution (3.11) of the manifold X_1 to the free energy.

In large N QCD, such behavior cannot prevail at all temperatures. At sufficiently high temperatures, one “sees” that the theory is made of quarks and gluons. By asymptotic freedom, the gluons are effectively free at high temperatures. As the number of species is of order N^2 , they contribute to the free energy an amount of order N^2 , with a temperature dependence determined by 3 + 1-dimensional conformal invariance. So for high enough temperature, $\lim_{N \rightarrow \infty} (F(\beta)/N^2) \sim T^4 = \beta^{-4}$.

This behavior is precisely analogous to what we have found for the large N behavior of the $\mathcal{N} = 4$ theory. The low temperature phase, in which the N^2 term in the free energy is trivial (as it reflects only the contribution of the ground state), corresponds to the phase in which the manifold X_1 dominates. The high temperature phase, with a non-trivial N^2 term in the free energy, corresponds to the phase in which the manifold X_2 dominates.

3.3. Hamiltonian Interpretation

Our remaining goal will be to develop a Hamiltonian version of the CFT- AdS correspondence and to use it to give another explanation of the relation between masses on AdS space and conformal dimensions on the boundary theory.

We would like to describe in terms of supergravity the Hilbert space of the boundary conformal field theory on $\mathbf{S}^{d-1}$. The partition function of the conformal field theory on $\mathbf{S}^1 \times \mathbf{S}^{d-1}$ can be computed in terms of the states obtained by quantizing on $\mathbf{S}^{d-1}$. We want the Hilbert space relevant to the low temperature phase – the low-lying excitations of the large N vacuum. So we must compare the conformal field theory to the manifold $X_1 = \mathbf{S}^1 \times \mathbf{R}^d$. We recall that X_1 is just AdS with some global identifications, with metric found in (3.12):

$$ds^2 = (1 + r^2)(dt)^2 + \frac{(dr)^2}{1 + r^2} + r^2 d\Omega^2. \quad (3.18)$$

A $t = 0$ slice of X_1 is a copy of $\mathbf{R}^d$ which can be regarded as an initial value hypersurface in AdS_{d+1} . The Hilbert space obtained by quantization on this $t = 0$ slice is hence simply the quantum Hilbert space of the theory on AdS_{d+1} .

So we identify the Hilbert space of the conformal field theory in quantization on $\mathbf{S}^{d-1}$ with the Hilbert space of the supergravity (or string theory) on AdS_{d+1} .

Now, in conformal field theory in d dimensions, the Hilbert space on $\mathbf{S}^{d-1}$ has an important interpretation, which is particularly well-known in the $d = 2$ case. Consider formulating the theory on $\mathbf{R}^d$ and inserting an operator $\mathcal{O}$ at a point in $\mathbf{R}^d$, say the origin. Think of $\mathbf{S}^{d-1}$ as a sphere around the origin. The path integral in the interior of the sphere determines a quantum state $|\psi_{\mathcal{O}}\rangle$ on the boundary, that is on $\mathbf{S}^{d-1}$. Conversely, suppose we are given a quantum state $|\psi\rangle$ on $\mathbf{S}^{d-1}$. Given the conformal invariance, the overall size of the $\mathbf{S}^{d-1}$ does not matter. So we cut out a tiny hole around the origin, whose boundary is a sphere of radius ϵ , and use the state $|\psi\rangle$ to define boundary conditions on the boundary. In the limit as $\epsilon \rightarrow 0$, this procedure defines a local operator $\mathcal{O}_{\psi}$ as seen by an outside observer. These two operations are inverses of each other and give a natural correspondence in conformal field theory between states on $\mathbf{S}^{d-1}$ and local operators.

So we can assert that, in the CFT- AdS correspondence, the local operators in the boundary conformal field theory correspond to the quantum states on AdS_{d+1} .

As an application of this, we will recover the relation claimed in section 2.5 between dimensions of operators in conformal field theory and particle masses on AdS space. Choosing a point at the origin in $\mathbf{R}^d$ breaks the Euclidean signature conformal group $SO(1, d+1)$ to a subgroup that contains $SO(1, 1) \times SO(d)$. $SO(d)$ is the group of rotations around the given point, and $SO(1, 1)$ is the group of dilatations. The $SO(1, 1)$ eigenvalue is the dimension of an operator. When the AdS metric is written as in (3.18), an $SO(d)$ group (which rotates the sphere whose line element is $d\Omega^2$) is manifest, and commutes with the symmetry of time translations. We call the generator of time translations the energy. The $SO(1, 1)$ generator that measures operator dimensions in the conformal field theory is the unique $SO(1, d+1)$ generator that commutes with $SO(d)$, so it is the energy operator of the AdS theory.

Thus, the energy of a mode in AdS space equals the dimension of the corresponding operator in the boundary conformal field theory. Now, in equation (A6) in [24], it is shown that a scalar field of mass m^2 in AdS space has a discrete energy spectrum (with the definition of energy that we have just given), and that the frequency of the mode of lowest energy is $\omega = \frac{1}{2}(d + \sqrt{d^2 - 4m^2})$. Comparing to the formula (2.44) for the dimension Δ

of the conformal field $\mathcal{O}$ in boundary conformal field theory that is related to a scalar of mass m in AdS space, we see that $\Delta = \omega$, as expected. ⁹ The same equation (A6) also shows that the energies of excited modes of the massive field are of the form $\omega + \text{integer}$, and so agree with the dimensions of derivatives of $\mathcal{O}$.

This work was supported in part by NSF Grant PHY-9513835. I am grateful to N. Seiberg for many very helpful comments, to G. Tian for pointing out the work of Graham and Lee, and to D. Freedman and P. van Nieuwenhuizen for discussions of gauged supergravity.

⁹ In [24], the computation is performed only for $d = 3$. Also, the parameter α in [24] is our $-m^2$, and there is a small clash in notation: the quantity λ_+ defined on p. 276 of [24] is $d + \lambda_+$ in our notation.

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AdS/CFT Correspondence And Topological Field Theory

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In $\mathcal{N} = 4$ super Yang-Mills theory on a four-manifold M , one can specify a discrete magnetic flux valued in $H^2(M, \mathbf{Z}_N)$. This flux is encoded in the AdS/CFT correspondence in terms of a five-dimensional topological field theory with Chern-Simons action. A similar topological field theory in seven dimensions governs the space of “conformal blocks” of the six-dimensional $(0, 2)$ conformal field theory.

1. Introduction

The AdS/CFT correspondence [1] relates $\mathcal{N} = 4$ super Yang-Mills theory on $\mathbf{S}^4$, with gauge group $SU(N)$, to Type IIB superstring theory on $AdS_5 \times \mathbf{S}^5$, with N units of five-form flux on $\mathbf{S}^5$. The correspondence expresses gauge theory correlation functions in terms of the dependence of the string theory on the behavior near the conformal boundary of AdS_5 [2,3].

What happens if we replace $\mathbf{S}^4$ with a more general four-dimensional spin manifold M ? (M must be spin since the $\mathcal{N} = 4$ theory contains spinor fields. In addition, M must be endowed with a metric that – after a suitable conformal rescaling – has positive scalar curvature, or the $\mathcal{N} = 4$ theory is unstable.¹) The behavior of the gauge theory on a four-manifold M is believed to be described, roughly speaking, in terms of the behavior of the string theory on $X \times \mathbf{S}^5$, where X is a negatively curved Einstein manifold with M as conformal boundary, and one must sum over all possible choices of X . This description is somewhat rough since, in general, one might need to include branes or stringy singularities on $X \times \mathbf{S}^5$ or more general topologies that are not simply the product of $\mathbf{S}^5$ with some five-manifold. The general prescription is really that one sums over Type IIB spacetimes that near infinity look like $X \times \mathbf{S}^5$, with X a five-manifold of conformal boundary M .

The gauge theory on M that is described in this correspondence has a gauge group that is locally $SU(N)$, rather than $U(N)$. Many arguments show this, beginning with the fact that in $\mathcal{N} = 4$ super Yang-Mills theory of $U(N)$, the $U(1)$ fields would be free, while in string theory on $AdS_5 \times \mathbf{S}^5$, everything couples to gravity and no field is free. As the gauge group is $SU(N)$, its center is $\mathbf{Z}_N$.

One consequence of the fact that the gauge group is $SU(N)$ rather than $U(N)$ is that it is possible to make a baryon vertex linking N external quarks. This can be constructed using a wrapped fivebrane [4] and can also be understood [5] via a certain Chern-Simons interaction that we will describe shortly.

Our interest in the present paper will, however, be in properties that depend on the global structure of M . In $\mathcal{N} = 4$ super Yang-Mills theory, all fields are in the adjoint representation and hence, as a local gauge transformation, the center of the gauge group

¹ The instability arises because of possible runaway behavior of the massless scalars ϕ of the theory. Positivity of the scalar curvature R of M suppresses the instability because of the $R \text{tr} \phi^2$ coupling required by conformal invariance.

acts trivially. It follows that locally, the gauge group could be $SU(N)/\mathbf{Z}_N$. This has two important consequences that depend on the topology of M :

(1) If $H^1(M, \mathbf{Z}_N) \neq 0$, it is possible to consider gauge transformations on M that are not single-valued in $SU(N)$ but are single-valued in $SU(N)/\mathbf{Z}_N$. For example, if $M = \mathbf{S}^1 \times Y$, one can consider a gauge transformation that is constant in the Y direction and whose restriction to the $\mathbf{S}^1$ direction determines any element of $\pi_1(SU(N)/\mathbf{Z}_N) = \mathbf{Z}_N$. This gives an important group $F \cong \mathbf{Z}_N$ of global symmetries of the thermal physics on Y . More generally, $F = H^1(M, \mathbf{Z}_N)$ classifies $SU(N)/\mathbf{Z}_N$ gauge transformations that cannot be lifted to $SU(N)$.

(2) If $H^2(M, \mathbf{Z}_N) \neq 0$, it is possible to consider $SU(N)/\mathbf{Z}_N$ bundles with “discrete magnetic flux” [6]. In fact, $SU(N)/\mathbf{Z}_N$ bundles on M are classified topologically by specifying the instanton number and also a characteristic class w that takes values in $K = H^2(M, \mathbf{Z}_N)$. (For example, if $N = 2$, we have $SU(2)/\mathbf{Z}_2 = SO(3)$, and w coincides with the second Stiefel-Whitney class of the gauge bundle.)

How to see in the AdS/CFT correspondence the thermal symmetry F has been discussed elsewhere [8]. For our present purposes, we need to recall just one point from that discussion. Type IIB superstring theory has two two-form fields B_{NS} and B_{RR} . In compactification on $X \times \mathbf{S}^5$ (with N units of five-form flux on $\mathbf{S}^5$), one gets a low energy effective action on X that contains a Chern-Simons term

$$L_{CS} = NI_{CS}, \quad (1.1)$$

where

$$I_{CS} = -\frac{i}{2\pi} \int_X B_{RR} \wedge dB_{NS} \quad (1.2)$$

is the basic Chern-Simons invariant of the two B -fields. (The action L_{CS} is important in one approach to the baryon vertex [5].) Though the integrand in I_{CS} is not gauge-invariant, I_{CS} is gauge-invariant mod $2\pi i$ on a closed five-manifold X . It can be written more invariantly as follows: if X is the boundary of a six-manifold Y over which the two B -fields extend, and we write the field strength of a B -field as $H = dB$, then one can write I_{CS} in a manifestly gauge-invariant way as

$$I_{CS} = -\frac{i}{2\pi} \int_Y H_{RR} \wedge H_{NS}. \quad (1.3)$$

More generally, if Y does not exist, a more subtle approach is needed to define I_{CS} . For a general Chern-Simons interaction, one can follow a slightly abstract approach explained

in section 2.10 and the end of the introduction to section 4 in [7]. For the particular Chern-Simons theory that we are discussing here, one can define I_{CS} as a cup product in Cheeger-Simons cohomology.²

A theory of the two B -fields with Lagrangian precisely (1.1) is a simple but subtle topological field theory, of a sort first considered in [10]. The actual low energy effective action that arises in compactification of Type IIB superstring theory on $\mathbf{S}^5$ has many couplings beyond the Chern-Simons interaction; the additional couplings have however a larger number of derivatives and so are irrelevant for certain questions.

The goal of the present paper is to understand how to encode in terms of the AdS/CFT correspondence the dependence of the gauge theory on the discrete magnetic flux w . As we will see, the topological field theory with Lagrangian (1.1) plays a starring role in the analysis. Roughly speaking, what in the gauge theory description appears as the discrete magnetic flux w appears in the gravitational description on $X \times \mathbf{S}^5$ as a quantum state of this topological field theory. We state a precise conjecture in section 2 and carry out some computations supporting it in section 3. Then in section 4, we make an extension to the $(2,0)$ superconformal field theory in six dimensions, and in section 5 we discuss 't Hooft and Wilson loops in the four-dimensional $\mathcal{N} = 4$ theory.

Since we will be developing a fairly elaborate theory to understand in terms of gravity the dependence of the gauge theory on the discrete magnetic flux, it is reasonable to ask what examples are known to which this theory can be applied. Actually, as we noted at the outset, there are relatively few M 's for which the AdS/CFT correspondence is expected to work (M must be spin and with positive scalar curvature), and there are presently very few such examples for which anything is known about the possible five-manifolds X with

² In the language of Cheeger and Simons [9], a B -field on a manifold X is an element of $\widehat{H}^2(X, U(1))$, the group of differential two-characters on X with values in $U(1)$. For two elements $B_{RR}, B_{NS} \in \widehat{H}^2(X, U(1))$, Cheeger and Simons describe in Theorem 1.11 a product $B_{RR} * B_{NS} \in \widehat{H}^5(X, U(1))$. For X of dimension five, $\widehat{H}^5(X, U(1)) = U(1)$, and we set $\exp(-I_{CS}(B_{RR}, B_{NS})) = B_{RR} * B_{NS}$. Equation 1.15 in [9] asserts that if B_{RR} is topologically trivial and hence can be defined by an ordinary two-form, I_{CS} can be computed by the integral written in (1.2): $I_{CS} = -(i/2\pi) \int_X B_{RR} \wedge H_{NS}$. Of course, there is a similar formula $I_{CS} = (i/2\pi) \int_X H_{RR} \wedge B_{NS}$ if B_{NS} is topologically trivial. It can also be proved that $B_{RR} * B_{NS}$ can be defined by the formula in the text if X is the boundary of an appropriate six-manifold Y . The usefulness of interpreting the action in terms of differential characters was explained to me by M. Hopkins, who also pointed out facts that we will use in section 3.4.

conformal boundary M . One simple but important example, in which one can verify the importance of summing over different choices of X , is the case $M = \mathbf{S}^1 \times \mathbf{S}^3$. There are [11] two known choices of X , namely $X_1 = \mathbf{S}^1 \times B_4$ and $X_2 = B_2 \times \mathbf{S}^3$ (here B_n denotes an n -dimensional ball); many important properties of Yang-Mills theory at nonzero temperature are reflected in the behavior of these two X 's [12].

This example has $H^2(M, \mathbf{Z}_N) = 0$ and so does not serve as a good illustration of the issues explored in the present paper. But one can readily modify it to give an example with nontrivial discrete magnetic fluxes. Identify $\mathbf{S}^3$ with the $SU(2)$ manifold and let H be a discrete subgroup of $SU(2)$, acting on the right. Set $M_H = \mathbf{S}^1 \times \mathbf{S}^3/H$. M_H is the boundary of $X_{i,H} = X_i/H$. For suitable H , $H^2(M_H, \mathbf{Z}_N) \neq 0$, so this gives a simple example to which our theory can be applied, showing in particular that the theory is nonvacuous.

This example actually has the following interesting property. $X_{2,H}$ has orbifold singularities (because H acts freely on $\mathbf{S}^3$ but not on B_4). The orbifold singularities are harmless in string theory and will be resolved as one varies the metric on M_H . The manifold $X_{2,H}$ will lose and acquire singularities and undergo monodromies as the metric on M_H is varied.

Another example is $M = \mathbf{S}^2 \times \mathbf{S}^2$, where some X 's have been constructed in [13] and investigated independently in [14]. This example has the property that $H^2(M, \mathbf{Z})$ is not a torsion group.

Some Technical Details

We conclude this introduction by summarizing a few useful details. In this paper, the symbol b_i will denote the i^{th} Betti number of the four-manifold M . We also write the order of the finite group $H^i(M, \mathbf{Z}_N)$ as $N^{\tilde{b}_i}$. (In general the $\tilde{b}_i$ are not integers.) If there is no torsion in the integral cohomology of M , then $b_i = \tilde{b}_i$. Using Poincaré and Pontryagin duality, it is possible to prove that $b_{4-i} = b_i$ and likewise $\tilde{b}_{4-i} = \tilde{b}_i$. The last statement arises because there is a nondegenerate pairing $H^i(M, \mathbf{Z}_N) \times H^{4-i}(M, \mathbf{Z}_N) \rightarrow \mathbf{Z}_N$ (given by the cup product); we write the product of $x \in H^i(M, \mathbf{Z}_N)$ with $y \in H^{4-i}(M, \mathbf{Z}_N)$ as $x \cdot y$. Nondegeneracy of the pairing implies that these groups are of the same order (and in fact are isomorphic as abelian groups). One also has $b_0 = \tilde{b}_0 = 1$. We write χ and σ for the Euler characteristic and signature of M ; we recall the definition $\chi = \sum_{i=1}^4 (-1)^i b_i$. Using the long exact sequence of cohomology groups derived from the short exact sequence

of groups $0 \rightarrow \mathbf{Z} \xrightarrow{N} \mathbf{Z} \rightarrow \mathbf{Z}_N \rightarrow 0$ (the first map is multiplication by N and the second is reduction modulo N), one can prove that in fact

$$\sum_{i=0}^4 (-1)^i b_i = \sum_{i=0}^4 (-1)^i \widetilde{b}_i. \quad (1.4)$$

Hence $b_1 - \frac{1}{2}b_2 = \widetilde{b}_1 - \frac{1}{2}\widetilde{b}_2$, a fact we will use later.

2. Role Of The Topological Field Theory

2.1. The Problem

As a prelude to our main subject, let us ask: In studying the $\mathcal{N} = 4$ super Yang-Mills theory on a four-manifold M via the AdS/CFT correspondence, do we expect to see Montonen-Olive S -duality? The answer is, “Not in a naive way,” since $SU(N)$ is not a self-dual group. For example, if

$$\tau = \frac{4\pi i}{g^2} + \frac{\theta}{2\pi} \quad (2.1)$$

is the coupling parameter of the theory, then under $\tau \rightarrow -1/\tau$ we expect $SU(N)$ to be exchanged with $SU(N)/\mathbf{Z}_N$.

The $SU(N)$ and $SU(N)/\mathbf{Z}_N$ theories are equivalent locally, and they are essentially equivalent on a four-manifold such as $M = \mathbf{S}^4$ or $\mathbf{S}^3 \times \mathbf{S}^1$ whose second cohomology group vanishes.^{[3](#)} These two theories really differ in an interesting way when $H^2(M, \mathbf{Z}_N) \neq 0$. The reason is that $SU(N)$ bundles on M are classified just by the instanton number, but $SU(N)/\mathbf{Z}_N$ bundles are classified by an additional topological invariant which is the “discrete magnetic flux” $w \in H^2(M, \mathbf{Z}_N)$ mentioned in the introduction. The $SU(N)$

³ On such a manifold, the $SU(N)$ and $SU(N)/\mathbf{Z}_N$ theories have the same correlation functions; their partition functions differ by an elementary factor described in eqn. (3.17) of [\[15\]](#). This factor arises because the groups of $SU(N)$ and $SU(N)/\mathbf{Z}_N$ gauge transformations differ slightly; the center of $SU(N)$ consists of gauge transformations in $SU(N)$ that are not considered in $SU(N)/\mathbf{Z}_N$, while $F = H^1(M, \mathbf{Z}_N)$ classifies the classes of global $SU(N)/\mathbf{Z}_N$ gauge transformations that cannot be lifted to $SU(N)$. With $N^{\widetilde{b}_1}$ the order of the finite group F , the group of $SU(N)$ gauge transformations has volume $N^{1-\widetilde{b}_1}$ times that for $SU(N)/\mathbf{Z}_N$, and hence the $SU(N)$ partition function is $N^{-1+\widetilde{b}_1}$ times that for $SU(N)/\mathbf{Z}_N$, a fact which is incorporated in the next equation in the text. If, as assumed in [\[15\]](#), there is no torsion in the cohomology of M , then $\widetilde{b}_1 = b_1$, and the factor becomes N^{-1+b_1} , as written in [\[15\]](#).

theory has a partition function $Z(\tau)$ (we suppress M from the notation when this is likely to cause no confusion), but the $SU(N)/\mathbf{Z}_N$ theory has a family of partition functions $Z_w(\tau)$, one for each $w \in H^2(M, \mathbf{Z}_N)$. The relation between the two is that the $SU(N)$ partition function is obtained from the $SU(N)/\mathbf{Z}_N$ partition function by setting $w = 0$, up to an elementary factor (mentioned in the footnote):

$$Z_{SU(N)}(\tau) = N^{-1+\tilde{b}_1} Z_0(\tau). \quad (2.2)$$

The $SU(N)$ theory is thus obtained by setting $w = 0$, but in the $SU(N)/\mathbf{Z}_N$ theory, one sums over all w . Since these operations are supposed to be exchanged under $\tau \rightarrow -1/\tau$, clearly the contribution with a given w cannot be $SL(2, \mathbf{Z})$ -invariant.

Generalizing ideas of 't Hooft [6], it has been argued [15] that the $Z_w(\tau)$ transform as a unitary representation of $SL(2, \mathbf{Z})$. To describe this representation, one must use the fact that the cup product gives a natural pairing $H^2(M, \mathbf{Z}_N) \times H^2(M, \mathbf{Z}_N) \rightarrow H^4(M, \mathbf{Z}_N) = \mathbf{Z}_N$, as mentioned at the end of the introduction. The most interesting part of the action of $SL(2, \mathbf{Z})$ is the behavior under $\tau \rightarrow -1/\tau$, which according to [15] is a sort of discrete Fourier transform:

$$Z_v(-1/\tau) = N^{-\tilde{b}_2/2} \left(\frac{\tau}{i}\right)^{W/2} \left(\frac{\bar{\tau}}{-i}\right)^{\bar{W}/2} \sum_{w \in H^2(M, \mathbf{Z}_N)} \exp(2\pi i v \cdot w/N) Z_w(\tau). \quad (2.3)$$

The intuitive idea of this formula is that $\tau \rightarrow -1/\tau$ exchanges magnetic and electric flux, but the discrete electric flux is defined by a Fourier transform with respect to the magnetic variable. The factor of $N^{-\tilde{b}_2/2}$ is needed for $S^2 = 1$, since the order of the finite group $K = H^2(M, \mathbf{Z}_N)$ is $N^{\tilde{b}_2}$. (In [15], the cohomology was assumed to be torsion-free, and this factor reduced to N^{b_2} .) The modular weights W and $\bar{W}$ are expected to be linear functions of χ and σ (the Euler characteristic and signature of M). They cannot be determined just from gauge theory, as they can be modified by adding gravitational couplings of the general form $f(\tau, \bar{\tau})RR$ (R being the Riemann tensor of M); to predict the modular weights that are observed in computing the $Z_v(\tau)$ using the AdS/CFT correspondence, one would need to know precisely which such couplings are determined by this correspondence. (In [15], $\bar{W}$ was set to zero, since a twisted version of the theory with a holomorphically varying partition function was considered.)

In addition to (2.3), the $SL(2, \mathbf{Z})$ transformation law of the $Z_v(\tau)$ is specified by describing the behavior under $\tau \rightarrow \tau + 1$. This is described in eqn. (3.14) of [15] and

is determined by the fact that the instanton number of an $SU(N)/\mathbf{Z}_N$ bundle is not an integer but is of the form (if M is a spin manifold)

$$\frac{v^2}{2N} \text{ modulo } \mathbf{Z}, \quad (2.4)$$

as in equation (3.13) of [15]. (A term $v^2/2$, which is integral if M is spin, has been omitted.) The transformation $\tau \rightarrow \tau + 1$ amounts to $\theta \rightarrow \theta + 2\pi$ in gauge theory, so the transformation law is

$$Z_v(\tau + 1) = \exp(2\pi i(v^2/2N - s)) Z_v(\tau), \quad (2.5)$$

where s is a constant that reflects the fact that the gravitational couplings $f(\tau, \bar{\tau})RR$ may not be invariant under $\tau \rightarrow \tau + 1$.

It follows from (2.2) and (2.3) that, essentially as in eqn. (3.18) of [15], the $SU(N)$ and $SU(N)/\mathbf{Z}_N$ partition functions are related by

$$Z_{SU(N)}(-1/\tau) = N^{-\chi/2} \left(\frac{\tau}{i}\right)^{W/2} \left(\frac{\bar{\tau}}{-i}\right)^{\bar{W}/2} Z_{SU(N)/\mathbf{Z}_N}(\tau). \quad (2.6)$$

This is essentially the Montonen-Olive formula, saying that $\tau \rightarrow -1/\tau$ exchanges $SU(N)$ and $SU(N)/\mathbf{Z}_N$; the prefactors reflect gravitational couplings not present in the original Montonen-Olive formulation on flat $\mathbf{R}^4$.⁴

2.2. The Partition Function As A Vector In Hilbert Space

Now we will make a change in viewpoint, which is suggested by experience with rational conformal field theory in two dimensions. A rational conformal field theory on a Riemann surface Σ of positive genus generally has, if one considers the chiral degrees of freedom only, not a single partition function, but a collection of partition functions. It is useful [16] to group these together as a vector in a Hilbert space, which one can think about using quantum mechanical intuition [17], and which one can ultimately understand using topological field theory in one dimension higher [18].

⁴ The derivation of (2.6) in [15] assumed no torsion in $H^2(M, \mathbf{Z}_N)$ and gave for the first factor $N^{-1+b_1-b_2/2} = N^{-\chi/2}$. More generally, including the torsion, one gets $N^{-1+\widetilde{b}_1-\widetilde{b}_2/2}$, but as explained at the end of the introduction, these two factors are equal.

So we introduce a Hilbert space $\mathcal{H}$ with one orthonormal basis vector Ψ_w for every $w \in H^2(M, \mathbf{Z}_N)$. We regard the $Z_w(\tau)$ as components of a vector $\Psi(\tau) \in \mathcal{H}$, with $\Psi(\tau) = \sum_w Z_w(\tau) \Psi_w$. Thus

$$Z_w(\tau) = (\Psi_w, \Psi(\tau)). \quad (2.7)$$

Since the gauge theory “partition function” is thus not an ordinary function but takes values in the Hilbert space $\mathcal{H}$, the AdS/CFT correspondence can only make sense if it is similarly true that the Type IIB partition function on a manifold such as $X \times \mathbf{S}^5$ with conformal boundary M is not an ordinary function but takes values in $\mathcal{H}$.

How can this be? At this point, we must recall the general structure of the AdS/CFT correspondence.

A very general class of observables in quantum field theory is the following. Let the $\mathcal{O}_i$ be a basis for the space of local gauge-invariant operators, and let J_i be c -number sources that couple to them. Thus the generating functional of correlation functions is

$$Z(\tau; J_i) = \left\langle \exp \left(\sum_i \int_M J_i \mathcal{O}_i \right) \right\rangle, \quad (2.8)$$

where $\langle \rangle$ denotes the (unnormalized) expectation value. The usual claim in the AdS/CFT correspondence is that to compute via string theory on $X \times \mathbf{S}^5$ this generating functional in the conformal field theory on M , one must fix the values of fields on X – near its boundary – to an asymptotic behavior determined by the J_i . Hence for fixed J_i , one usually claims that the boundary behavior of the fields on $X \times \mathbf{S}^5$ are fixed; the partition function is then a “number,” that is a function of the J_i . To resolve our present conundrum, we must show that if $H^2(M, \mathbf{Z}_N) \neq 0$, then even after specifying the values of all sources J_i for all local operators $\mathcal{O}_i$ on M , the boundary values of the fields on $X \times \mathbf{S}^5$ are not completely determined; and the dependence on the extra data, whatever it is, must be such that the partition function for given J_i is not a number but a vector in $\mathcal{H}$.

The reason that this is so is that fixing the behavior of all of the local gauge-invariant observables near the boundary of $X \times \mathbf{S}^5$ does not completely specify the gauge-invariant data near the boundary. There is global gauge-invariant information that cannot be measured locally, because the Type IIB theory has the two two-form fields B_{RR} and B_{NS} . One can always add to any given B -field a flat B -field, without affecting any local gauge-invariant information. Flat B -fields on a spacetime such as $Y = X \times \mathbf{S}^5$ are classified modulo gauge transformations by $H^2(Y, U(1))$. Near the boundary, this reduces to

$H^2(M, U(1))$. For $H^2(M, \mathbf{Z}_N)$ to be nonzero (with M of dimension four), $H^2(M, U(1))$ must likewise be nonzero. Hence, our puzzle arises only when there are flat B -fields near M , in which case the partition function of the string theory on $X \times \mathbf{S}^5$ depends on additional data and not only on the sources J_i of gauge-invariant fields.

Roughly speaking, the additional data can be measured by “ θ angles”

$$\begin{aligned}\alpha_{NS} &= \int_S B_{NS} \\ \alpha_{RR} &= \int_S B_{RR},\end{aligned}\tag{2.9}$$

with S a homologically nontrivial two-dimensional surface in M . One might think that the partition function of the string theory on $X \times \mathbf{S}^5$ would be a function of these theta angles, as well as the sources J_i . That would be an interesting result, but not quite what we need. We want instead to see the finite group $H^2(M, \mathbf{Z}_N)$. The reason that the string theory partition function should not be regarded simply as a function of α_{NS} and α_{RR} is that these are, in a sense, canonically conjugate variables.

The low energy effective action for B_{RR} , B_{NS} , after reduction on $\mathbf{S}^5$, looks something like

$$L = -\frac{iN}{2\pi} \int_X B_{RR} \wedge dB_{NS} + \frac{1}{2\gamma} \int_X |dB_{RR}|^2 + \frac{1}{2\gamma'} \int_X |dB_{NS}|^2 + \dots\tag{2.10}$$

Here the first term is the Chern-Simons term, whose significance for understanding the role of the center of the gauge group in the AdS/CFT correspondence was already mentioned in the introduction. The second and third terms (with constants γ, γ' that depend on τ) are the conventional kinetic terms for two-form fields. The “...” are additional gauge-invariant terms of higher dimension.

The first term in (2.10) – the Chern-Simons term – is the important one for our present purposes, for several reasons. The obvious reason is that this is the unique term with only one derivative and hence dominates at long distances (recall that the conformal boundary of X is “infinitely far away,” so the behavior near the boundary is a question of long distance physics). Perhaps even more fundamentally, the second and third terms in (2.10) and all higher terms are integrals of gauge-invariant local densities and hence insensitive to the α ’s, which contribute only to the first term. The Chern-Simons term is gauge-invariant but is not the integral of a gauge-invariant local density; it can and in general does change if one shifts the B -fields by a flat B -field (a fact that was crucial in [8]

to enable the expected thermal symmetry of $SU(N)$ gauge theory at nonzero temperature to emerge from the AdS/CFT correspondence).

So to address the question of α -dependence of the partition function, we can use near the boundary of $X \times \mathbf{S}^5$ the simplified Chern-Simons action

$$L_{CS} = NI_{CS} = -\frac{iN}{2\pi} \int_X B_{RR} \wedge dB_{NS}. \quad (2.11)$$

If we write $H = dB$ for the field strength of a B -field, then the equations of motion derived from the Chern-Simons action are $H_{RR} = H_{NS} = 0$, so this action governs only the α -dependence (not the modes of nonzero H , which are massive in spacetime and have been integrated out to reduce to (2.11)); their behavior near the boundary of $X \times \mathbf{S}^5$ is determined in the usual way by the sources J_i of gauge-invariant local operators on the boundary).

The Chern-Simons action L_{CS} does not depend on a metric on X , so the theory governing the α -dependence is a topological field theory, of a familiar kind [10]. From the first-order form of L_{CS} , we see that B_{RR} and B_{NS} are canonically conjugate variables in this topological field theory. After imposing the equations of motion and dividing by gauge transformations, this means that α_{RR} and α_{NS} are canonically conjugate. Hence, we should not attempt to compute the partition function as a function of both α_{RR} and α_{NS} .

What we should do instead follows from general concepts of quantum mechanics. Near M , X looks like $M \times \mathbf{R}$, with $\mathbf{R}$ the “time” direction. By quantizing the Chern-Simons theory on $M \times \mathbf{R}$, we obtain a “quantum Hilbert space” $\mathcal{H}'$ associated with M . We should interpret the partition function of the theory on X as determining, not a number, but a vector in $\mathcal{H}'$. Because $\mathcal{H}'$ is obtained by quantizing the metric-independent Chern-Simons Lagrangian, it depends only on the topology of M , and not on a metric.

Moreover, the topological field theory with action L_{CS} has an $SL(2, \mathbf{Z})$ symmetry, acting on the pair

$$\begin{pmatrix} B_{RR} \\ B_{NS} \end{pmatrix} \quad (2.12)$$

in the standard fashion. (A subtlety concerning this assertion will be explained in section 3.4.) The Hilbert space $\mathcal{H}'$ hence has a natural action of $SL(2, \mathbf{Z})$.

This is almost what we need: to agree with the expectations in the boundary gauge theory, the partition function of the string theory on $X \times \mathbf{S}^5$ should be (once the gauge-invariant boundary data are specified) not a number but a vector in a Hilbert space $\mathcal{H}$.

$\mathcal{H}$ is determined purely topologically by M (by definition, it has an orthonormal basis consisting of vectors Ψ_w for each $w \in H^2(M, \mathbf{Z}_N)$), and has an action of $SL(2, \mathbf{Z})$ that we described in section 2.1.

So all will be well if $\mathcal{H} = \mathcal{H}'$. Actually, we must describe somewhat more precisely in what sense these two spaces should coincide. The description of the space $\mathcal{H}$ via its basis Ψ_w , $w \in H^2(M, \mathbf{Z}_N)$ is not invariant under S -duality (which as seen in (2.3) does not preserve this basis). To make this description, we need to pick a notion of what we mean by “magnetic flux,” as opposed to “electric flux.” Such a choice breaks $SL(2, \mathbf{Z})$. $SL(2, \mathbf{Z})$ is broken in the same way if we introduce a two-dimensional lattice $\Lambda = \mathbf{Z}^2$ on which $SL(2, \mathbf{Z})$ acts in the standard way, and pick a “polarization” of Λ . One can think of a polarization as a choice of one direction in the lattice Λ . Alternatively, one can think of the pair $(\alpha_{RR}, \alpha_{NS})$ as taking values in $\mathbf{R}^2/\Lambda$; a polarization then is just a choice of what integral linear combination of α_{RR} and α_{NS} is the “momentum” variable. So we must show that for every choice of polarization, $\mathcal{H}'$ acquires a basis in one-to-one correspondence with the elements of $H^2(M, \mathbf{Z}_N)$, and that $SL(2, \mathbf{Z})$ acts on $\mathcal{H}'$ as it does on $\mathcal{H}$.

These assertions will be demonstrated in section 3.

3. Quantization Of The Topological Field Theory

3.1. Preliminaries

To quantize the Chern-Simons theory on $M \times \mathbf{R}$, we first work out the gauge-invariant classical phase space. We work in the gauge (analogous to $A_0 = 0$ gauge for gauge theory) in which $i0$ components of B_{RR} and B_{NS} (i and 0 label directions tangent to M and $\mathbf{R}$, respectively) vanish. In quantization, we restrict to time zero; together with the gauge choice, this means that B_{RR} and B_{NS} are B -fields on M . The canonical commutation relations, if written out in detail in components, read

$$[B_{RRij}(x), B_{NSkl}(y)] = -\frac{2\pi i}{N} \epsilon_{ijkl} \delta^4(x, y), \quad (3.1)$$

with $[B_{RR}(x), B_{RR}(y)] = [B_{NS}(x), B_{NS}(y)] = 0$. Here, x and y are points in M , and $i, j, k, l = 1 \dots 4$ are indices tangent to M .

The “Gauss’s law” constraint ($\delta L / \delta B_{i0} = 0$, where B is B_{RR} or B_{NS}) gives $H_{RR} = H_{NS} = 0$. So, modulo gauge transformations, the phase space consists of pairs of flat B -fields.

Flat B -fields are classified by $H^2(M, U(1))$. B -fields in general are classified topologically by the characteristic class $[H] \in H^3(M, \mathbf{Z})$. However, for a flat B -field, this characteristic class vanishes as a differential form, and hence represents a *torsion* element of $H^3(M, \mathbf{Z})$. Let $H^3(M, \mathbf{Z})_{tors}$ be the torsion subgroup of $H^3(M, \mathbf{Z})$. To any given flat B -field we can, without changing the topological type, add a flat and topologically trivial B -field, via a transformation $B \rightarrow B + \beta$, where β is a globally-defined, closed two-form. We should consider β trivial if its periods are multiples of 2π (since the periods of B can be shifted by multiples of 2π by a gauge transformation), so we regard β as an element of $H^2(M, \mathbf{R})/2\pi H^2(M, \mathbf{Z})$. The space $H^2(M, U(1))$ of flat B -fields thus fits into an exact sequence

$$0 \rightarrow H^2(M, \mathbf{R})/2\pi H^2(M, \mathbf{Z}) \rightarrow H^2(M, U(1)) \rightarrow H^3(M, \mathbf{Z})_{tors} \rightarrow 0. \quad (3.2)$$

This says that a flat B -field has a characteristic class in $H^3(M, \mathbf{Z})_{tors}$, and two flat B -fields with the same characteristic class differ by an element of $H^2(M, \mathbf{R})/2\pi H^2(M, \mathbf{Z})$, which classifies topologically trivial flat B -fields.

Topologically trivial flat B -fields can be measured by their periods, which are the “world-sheet theta angles,”

$$\begin{aligned} \alpha_{NS} &= \int_S B_{NS} \\ \alpha_{RR} &= \int_S B_{RR}, \end{aligned} \quad (3.3)$$

where S is a closed two-surface in M .

There are thus two parts of the quantization: to quantize the α ’s, and to take account of the finite group $H^3(M, \mathbf{Z})_{tors}$. These turn out to present quite different problems and a direct treatment of the quantization involves many subtle details. On the other hand, everything we need to know can be deduced from the symmetries of the problem. In section 3.2, we will consider this analysis using the symmetries. Then – for the benefit of curious readers – we enter into a direct analysis of the quantization.

3.2. Symmetries

The most obvious symmetry of the problem is that we can add to B_{RR} or B_{NS} any B -field B' such that NB' is pure gauge. For instance, under

$$B_{RR} \rightarrow B_{RR} + B', \quad (3.4)$$

the Chern-Simons action $L_{CS}(B_{RR}, B_{NS}) = NI_{CS}(B_{RR}, B_{NS})$ changes by $L_{CS} \rightarrow L_{CS} + NI_{CS}(B', B_{NS}) = L_{CS} + I_{CS}(NB', B_{NS})$. But $I_{CS}(NB', B_{NS}) = 0$ as NB' is pure gauge.

For NB' to be pure gauge means necessarily that B' is flat. Thus, B' determines an element of $H^2(M, U(1))$ that is “of order N ,” or in other words is annihilated by multiplication by N .

We can be more explicit about the quantum field operators that generate these symmetries. (They are analogs of operators considered many years ago in two-dimensional rational conformal field theory [17].) We construct them using the two-form counterparts of the familiar Wilson and 't Hooft loop operators of gauge theories.

First we consider the counterparts of Wilson loops. Let S and T be closed two-surfaces in M , and let

$$\begin{aligned}\Phi_{RR}(S) &= \exp\left(i \int_S B_{RR}\right), \\ \Phi_{NS}(T) &= \exp\left(i \int_T B_{NS}\right).\end{aligned}\tag{3.5}$$

Since these are gauge-invariant operators, they act on the classical phase space and map flat B -fields to flat B -fields. From the canonical commutation relations, we can deduce that

$$\Phi_{RR}(S)\Phi_{NS}(T) = \Phi_{NS}(T)\Phi_{RR}(S) \exp\left(\left(\frac{2\pi i}{N}\right) S \cdot T\right),\tag{3.6}$$

where $S \cdot T$ is the intersection number of the oriented surfaces S and T .

In particular, while $\Phi_{RR}(S)$ commutes with B_{RR} , it shifts B_{NS} by a flat B -field that is essentially determined by (3.6) to be “Poincaré dual” to S . In other words, we interpret the relation $\Phi_{RR}(S)\Phi_{NS}(T)\Phi_{RR}(S)^{-1} = \Phi_{NS}(T) \exp(2\pi i S \cdot T/N)$ to mean that conjugation by $\Phi_{RR}(S)$ shifts B_{NS} by a flat B -field with delta-function support on S (this assertion is in any case clear from the canonical commutation relations), thus multiplying $\Phi_{NS}(T)$ by a phase. The $1/N$ in the exponent in (3.6) means that Φ_{NS} is shifted by a flat B -field of order N . Conversely, conjugation by $\Phi_{NS}(T)$ shifts B_{RR} by such a field. Thus, these operators generate the symmetries that we exhibited directly in (3.4). The c -number factor in the commutation relation (3.6) shows that these symmetries do not commute with one another; there is a central extension by the N^{th} roots of unity.

The operator $\Phi_{RR}(S)$, in acting on gauge-invariant states, is trivial if S consists of N copies of another closed surface S' . That is because in such a case, $\Phi_{RR}(S) = \Phi_{RR}(S')^N$; but $\Phi_{RR}(S')$ shifts B_{NS} by a B -field of order N , and hence $\Phi_{RR}(S')^N$ shifts it by a pure gauge. Likewise, $\Phi_{NS}(T)$ is trivial if $T = NT'$ for some T' .

$\Phi_{RR}(S)$ (or $\Phi_{NS}(T)$) is also trivial if S (or T) is the boundary of a three-manifold $U \subset M$. For then

$$\Phi_{RR}(S) = \exp \left(i \int_U H_{RR} \right), \quad (3.7)$$

and this is trivial as an operator on gauge-invariant states because of the Gauss law constraint $H = 0$.

So the operators $\Phi_{RR}(S)$ depend on S modulo boundaries, and are trivial if S is of the form NS' . So far we have assumed that S has no boundary, but we will next see that we can define an operator $\Phi_{RR}(S)$ with similar properties if S has a boundary, provided this boundary is of the form NC for some circle $C \subset M$. This will make possible the following simple description of the symmetry group. The group $H_2(M, \mathbf{Z}_N)$ classifies two-surfaces $S \subset M$ with boundaries of the form NC , modulo those of the form NS' and modulo boundaries. So once we show that S can have a boundary of the claimed kind, we will have shown that the operators $\Phi_{RR}(S)$ (and similarly the operators $\Phi_{NS}(T)$) are classified by $H_2(M, \mathbf{Z}_N)$. Equivalently, by Poincaré duality, they are classified by $H^2(M, \mathbf{Z}_N)$.

't Hooft Loops

To see how S can have a boundary, we begin with a rather different-sounding question. Are there also in this theory symmetry operators that are analogous to the 't Hooft loops of gauge theory? This would mean the following. We consider a circle $C \subset M$. Deleting C from M , we make a “singular gauge transformation” on B_{NS} (or B_{RR}) such that on a small three-sphere that links once around C , the integral of H_{NS} equals 2π . This means that we put on C a “magnetic source” for B_{NS} .

To find such operators, we can go back to Type IIB superstring theory on $X \times \mathbf{S}^5$. In this theory, the NS fivebrane is a magnetic source for B_{NS} , so the object sought in the last paragraph would be a fivebrane wrapped on $C \times \mathbf{S}^5$. However, as explained in [4], such a fivebrane must be the boundary of N D -strings. In other words, in addition to the magnetic loop, a D -string must be present with a world-volume S whose boundary consists of N copies of C .

The operator $\Phi_{RR}(S)$ describes the coupling of B_{RR} to the D -brane worldvolume S . So we have learned that S can have a boundary of the form NC , provided that there is at the same time a magnetic source for B_{NS} on S .

The recourse to string theory to show the existence of mixed Wilson/'t Hooft operators of this kind is a convenient shortcut, but of course it would be desirable to demonstrate their existence directly in the five-dimensional topological field theory. In fact, this has

essentially already been done in section 5 of [5], in the course of explaining a low energy approach to the existence of the baryon vertex in Type IIB on $AdS_5 \times \mathbf{S}^5$.

The Heisenberg Group And The Hilbert Space

The operators $\Phi_{RR}(S)$ (and likewise $\Phi_{NS}(T)$) are thus classified by $H_2(M, \mathbf{Z}_N)$ or equivalently by $K = H^2(M, \mathbf{Z}_N)$. We write these operators now as $\Phi_{RR}(w)$, $\Phi_{NS}(w)$, with $w \in K$. The commutation relation of these operators is still given by (3.6). We can assume that all intersections of S and T occur away from the boundaries, so there is no real modification in the derivation of (3.6) from the canonical commutation relations. Since S and T may have boundaries of the form NC , the intersection number $S \cdot T$ is only well-defined modulo N , but that is good enough to make sense of the phase factor in the commutation relation.

Because of the central factor in the commutation relation (3.6), the group generated by these operators is not just $K \times K = H^2(M, \mathbf{Z}_N) \times H^2(M, \mathbf{Z}_N)$, but is a central extension of $K \times K$ by $\mathbf{Z}_N$, the group of N^{th} roots of unity. We call this central extension W :

$$0 \rightarrow \mathbf{Z}_N \rightarrow W \rightarrow K \times K \rightarrow 0. \quad (3.8)$$

This central extension is nondegenerate (a statement which, informally, means that the first or second factors of K in the product $K \times K$ give maximal commutative subgroups of W) and is a group of a type known as a finite Heisenberg group.

A “polarization” of such a finite Heisenberg group is a choice of a maximal commutative subgroup of $K \times K$, or more exactly a maximal subgroup that remains commutative when lifted to W . For example, the first or second factor of $K \times K$, or any “diagonal” subgroup obtained from then by an $SL(2, \mathbf{Z})$ transformation, is a polarization. There are also, in general, other polarizations. Picking a polarization of a finite Heisenberg group is a discrete version of picking a representation of the canonical commutation relations $[p, x] = -i\hbar$ for bosons.

One of the main theorems about such finite Heisenberg groups is that, up to isomorphism, such a group has a unique irreducible representation R . This is a discrete version of the fact that the canonical commutation relations for bosons have a unique irreducible representation, up to isomorphism. The representation can be constructed as follows. Since the $\Phi_{RR}(w)$ commute, one can pick a basis of R consisting of their joint eigenstates. Using the nondegeneracy, one can show that there is a vector $|\Omega\rangle \in R$ that is invariant

under all of the $\Phi_{RR}(w)$. From irreducibility of R , one can show that $|\Omega\rangle$ is unique (up to multiplication by a scalar) and that R has a basis consisting of the states

$$\Psi_w = \Phi_{NS}(w)|\Omega\rangle, \quad \text{for } w \in H^2(M, \mathbf{Z}_N). \quad (3.9)$$

The action of the group in this basis follows immediately from the commutation relations.

If the quantum Hilbert space $\mathcal{H}'$ of the Chern-Simons theory is an *irreducible* representation of the finite symmetry group W , then it is easy to obtain all the properties promised in section 2.2. For example, we must show that for each polarization of the lattice $\Lambda = \mathbf{Z}^2$ on which $SL(2, \mathbf{Z})$ acts, $\mathcal{H}'$ has a basis Ψ_w , $w \in H^2(M, \mathbf{Z}_N)$. In this context, a polarization is a choice of what we mean by Φ_{RR} and what we mean by Φ_{NS} . The desired basis is constructed in (3.9) for one choice of polarization, and other polarizations are obtained by $SL(2, \mathbf{Z})$ transformations. There is indeed a natural action of $SL(2, \mathbf{Z})$ on $\mathcal{H}'$, since the commutation relation (3.6) by which W is defined is $SL(2, \mathbf{Z})$ -invariant. It is not hard to see that the $SL(2, \mathbf{Z})$ action agrees with what was described in section 2.1. The main point is that the $SL(2, \mathbf{Z})$ transformation

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad (3.10)$$

which acts by $\tau \rightarrow -1/\tau$, corresponds to a change in polarization that maps Φ_{RR} to Φ_{NS}^{-1} and Φ_{NS} to Φ_{RR} . It maps the state $|\Omega\rangle$ invariant under the Φ_{RR} 's to the state

$$|\tilde{\Omega}\rangle = \frac{1}{N^{\widetilde{b_2/2}}} \sum_{w \in K} \Phi_{NS}(w)|\Omega\rangle, \quad (3.11)$$

which is invariant under the Φ_{NS} 's. So it maps $\Psi_v = \Phi_{NS}(v)|\Omega\rangle$ to

$$\Phi_{RR}(v)|\tilde{\Omega}\rangle = \frac{1}{N^{\widetilde{b_2/2}}} \Phi_{RR}(v) \sum_{w \in K} \Phi_{NS}(w)|\Omega\rangle = \frac{1}{N^{\widetilde{b_2/2}}} \sum_{w \in K} \exp(2\pi i v \cdot w/N) \Phi_{NS}(w)|\Omega\rangle. \quad (3.12)$$

This is the discrete Fourier transform familiar from section 2.1. The transformation under $\tau \rightarrow \tau + 1$ can be understood similarly.

Except for the fact that we have not shown that the finite group W acts irreducibly in the quantum theory, this really establishes what we wanted. However, we will go on in the rest of this section, for the benefit of curious or intrepid readers, to show what is involved in quantizing the theory directly. In the process, it will become clear that W acts irreducibly on $\mathcal{H}'$.

In the analysis, it will be useful to know the following more general description of the representation R . First of all, if F is any polarization of $K \times K$, then R has a basis that is in one-to-one correspondence with the cosets of $(K \times K)/F$. This is proved by first showing that R has an F -invariant vector $|\Omega\rangle$, and then setting $\Upsilon_\lambda = \Phi_\lambda|\Omega\rangle$, where λ runs over a set of representatives of the cosets of K/F , and for each λ , Φ_λ is the corresponding operator on R . The Υ_λ are the desired basis vectors.

3.3. The Torsion-Free Case; Elementary Account

Direct quantization of this system is surprisingly subtle, given that it is an abelian free field theory. The reader may in fact wish to jump directly to section 4.

We begin by considering the case that there is no torsion in $H^2(M, \mathbf{Z})$. If $H^2(M, \mathbf{Z})$ is torsion-free, then this group is a lattice Γ . The intersection pairing on $H^2(M, \mathbf{Z})$ gives an integer-valued inner product on Γ ; we write the product of $x, y \in \Gamma$ as $x \cdot y$. This inner product is unimodular by Poincaré duality and is even because (as we noted at the outset) M is spin.

The exact sequence of abelian groups $0 \rightarrow \mathbf{Z} \xrightarrow{N} \mathbf{Z} \rightarrow \mathbf{Z}_N \rightarrow 0$ leads to a long exact sequence of cohomology groups which reads in part

$$\dots H^2(M, \mathbf{Z}) \xrightarrow{N} H^2(M, \mathbf{Z}) \rightarrow H^2(M, \mathbf{Z}_N) \rightarrow H^3(M, \mathbf{Z}) \xrightarrow{N} H^3(M, \mathbf{Z}) \dots \quad (3.13)$$

If there is no torsion in the cohomology of M , then the map $H^3(M, \mathbf{Z}) \xrightarrow{N} H^3(M, \mathbf{Z})$ is injective. Hence exactness of (3.13) says in the torsion-free case that

$$H^2(M, \mathbf{Z}_N) = H^2(M, \mathbf{Z})/NH^2(M, \mathbf{Z}) = \Gamma/N\Gamma. \quad (3.14)$$

In other words, in the absence of torsion, a $\mathbf{Z}_N$ class is the mod N reduction of an integral class.

In the torsion-free case, the world-sheet theta angles α_{NS} and α_{RR} are a complete set of gauge-invariant functions on the classical phase space. They are canonically conjugate variables, a fact that we have already used in the analysis of the symmetries in section 3.2. A precise form of this statement is that if we pick for the lattice Γ a basis in which the inner product is g_{ij} , $1 \leq i, j \leq b_2$, and write $\alpha_{NS}^i, \alpha_{RR}^i$ for the corresponding components of α_{NS} and α_{RR} , then the Poisson brackets are

$$\{\alpha_{RR}^i, \alpha_{NS}^j\} = \frac{2\pi}{N} g^{ij}. \quad (3.15)$$

According to the description at the end of section 2.2, we are to pick a polarization, quantize, and determine the resulting Hilbert space $\mathcal{H}'$. We pick a polarization⁵ by regarding α_{NS} as the “position” variable and

$$\beta_{RR} = \frac{N}{2\pi} \alpha_{RR} \quad (3.16)$$

as the conjugate momentum. Note that as the periods of α_{RR} are defined mod 2π , those of β_{RR} are defined mod N , so classically β_{RR} is an element of $H^2(M, \mathbf{R})/NH^2(M, \mathbf{Z}) = V/N\Gamma$, where $V = H^2(M, \mathbf{R})$ and $\Gamma = H^2(M, \mathbf{Z})$ is regarded as a lattice in V .

Quantum mechanically there is a very severe additional constraint on β_{RR} . Since the “position” variable α_{NS} is a periodic variable that takes values in $H^2(M, \mathbf{Z})/2\pi\Gamma$, the conjugate momentum β_{RR} takes values in the lattice Γ^* dual to Γ . But Poincaré duality tells us that $\Gamma^* = \Gamma$, so β_{RR} takes values in Γ . Since β_{RR} is in addition defined modulo $N\Gamma$, it takes values in $\Gamma/N\Gamma$. According to (3.14), this is the same as $H^2(M, \mathbf{Z}_N)$.

So we get the desired result. There is one quantum state – one momentum state Ψ_w – for every momentum vector $w \in \Gamma/N\Gamma = H^2(M, \mathbf{Z}_N)$.

Moreover, the $SL(2, \mathbf{Z})$ transformation $\tau \rightarrow -1/\tau$ exchanges the position α_{NS} with the momentum α_{RR} . Hence it acts as a Fourier transform, in agreement with the gauge theory result (2.3).

3.4. More Rigorous Approach

The discussion in section 3.3 was actually somewhat informal, and for the interested reader we will now give some hints for a more precise treatment.

Let T be the torus $T = V/\Gamma$ with $V = H^2(M, \mathbf{R})$ and $\Gamma = H^2(M, \mathbf{Z})$. The phase space of the system is $T' = T \times T$. The first step in quantizing the theory is to find the appropriate line bundle $\mathcal{M}$ over the phase space. $\mathcal{M}$ should be endowed with a connection whose curvature equals the symplectic two-form of the theory. Once the right line bundle is found, quantization is carried out by picking a complex structure and taking holomorphic sections of $\mathcal{M}$, or by using a real polarization and making a more precise version of the informal discussion of section 3.3, or by finding and using a finite Heisenberg group as in section 3.2. But any approach to quantization involves finding the line bundle at the outset.

⁵ For some background on quantization of Chern-Simons theories, see [19, 20].

Since N appears in the symplectic form as a multiplicative factor, $\mathcal{M}$ will be of the form $\mathcal{L}^N$, where $\mathcal{L}$ is the line bundle that we would get at $N = 1$. (The relation $\mathcal{M} = \mathcal{L}^N$ will be obvious from the Chern-Simons construction of the line bundle that we give presently.)

The construction of the line bundle is most naturally made directly from the Chern-Simons action [21-24]. Let p be a point in the classical phase space T' given by a pair $(B_{RR}, B_{NS}) = (a, b)$. Let 0 be the origin in T' (the point with $B_{RR} = B_{NS} = 0$). Then we describe the fiber $\mathcal{L}_p$ of $\mathcal{L}$ at p as follows. For any path γ in T' from 0 to p , $\mathcal{L}_p$ has a basis vector ψ_γ , of norm 1. If γ and γ' are two such paths, we declare that

$$\psi_{\gamma'} = e^{iL_{CS}}\psi_\gamma, \quad (3.17)$$

where the Chern-Simons Lagrangian L_{CS} is understood in the following sense. The path γ from 0 to p determines a “time”-dependent pair of B -fields $(B_{RR}(t), B_{NS}(t))$, where $(B_{RR}(0), B_{NS}(0)) = (0, 0)$ and $(B_{RR}(1), B_{NS}(1)) = (a, b)$. We can fit these together to make a pair (B_{RR}, B_{NS}) over $M \times I$, where I is a unit interval. Likewise, γ' determines a pair of B -fields over $M \times I$. By gluing γ' to γ , with opposite orientation for γ' , we get a B -field pair over the closed five-manifold $M \times \mathbf{S}^1$. L_{CS} in (3.17) is the Chern-Simons action evaluated for this field.

It can be shown that the line bundle $\mathcal{L}$, whose fiber at each $p \in T'$ was just described, is endowed with a natural connection, whose curvature is the symplectic form. The monodromy M_C of this line bundle around any circle $C \subset T'$ is as follows. The circle $C \subset T'$ determines a B -field pair over $M \times C$, and the monodromy is

$$Q(C) = \exp(-L_{CS}), \quad (3.18)$$

with L_{CS} the Chern-Simons action of this pair.

Having found the line bundle, we are ready to quantize. Quantum states are suitable sections of $\mathcal{M} = \mathcal{L}^N$, with the details depending on a choice of polarization.

In the present case, we can make the description of $\mathcal{L}$ much more explicit. In general, a line bundle with connection over any manifold Z is determined up to isomorphism by giving its monodromies around arbitrary loops in Z . Once the curvature is specified, it suffices to consider a set of loops generating the fundamental group of Z . In the present instance, Z is the torus $T' = T \times T = (V \times V)/(\Gamma \times \Gamma)$. The fundamental group of T' is generated by straight lines from the origin in $V \times V$ to lattice points in $\Gamma \times \Gamma$. Let (x, y)

be such a lattice point. A straight line from the origin to this lattice point is given by the family of B -field pairs

$$\begin{aligned} B_{RR} &= 2\pi tx \\ B_{NS} &= 2\pi ty, \end{aligned} \tag{3.19}$$

with $0 \leq t \leq 1$. At $t = 1$, $(B_{RR}, B_{NS}) = (2\pi x, 2\pi y)$ are gauge-equivalent to 0. So the family $(B_{RR}(t), B_{NS}(t))$ gives a closed loop in the phase space. The monodromy around this loop can be computed by direct evaluation of $L_{CS} = -(i/2\pi) \int_{M \times \mathbf{S}^1} B_{RR} \wedge dB_{NS}$,⁶ and is

$$Q(x, y) = (-1)^{x \cdot y}. \tag{3.20}$$

The relation

$$Q(x + x', y + y') = Q(x, y)Q(x', y')(-1)^{x \cdot y' + y \cdot x'} \tag{3.21}$$

follows from this, and signals, as explained in connection with eqn. (2.17) of [7], that the line bundle $\mathcal{L}$ has the desired curvature and first Chern class.

The formula for $Q(x, y)$ is clearly invariant under all of the lattice symmetries of Γ , and thus under all diffeomorphisms of M . Let us check that it is also invariant under $SL(2, \mathbf{Z})$, acting in the standard fashion

$$\begin{aligned} x &\rightarrow ax + by \\ y &\rightarrow cx + dy \end{aligned} \tag{3.22}$$

with a, b, c, d integers such that $ad - bc = 1$. A small computation shows that $Q(x, y)$ is invariant under this transformation for all such a, b, c , and d if and only if x^2 and y^2 are both even. But the manifold M is spin, and the intersection form on $H^2(M, \mathbf{Z})$, for M a four-dimensional spin manifold, is always even. So x^2 and y^2 are even, and Q has the expected symmetries. Since Q determines the line bundle $\mathcal{L}$, $\mathcal{L}$ has these symmetries also.

At this point, the reader may wonder precisely why the spin condition on M was needed. The line bundle $\mathcal{L}$ was determined directly from the Chern-Simons action, which appears to be completely $SL(2, \mathbf{Z})$ -invariant without assuming that M is spin; so why did the spin structure enter? The answer will make it clear that up to the present point, we have indulged in a small sleight of hand. There is actually a subtlety in defining the

⁶ To put this “direct evaluation” on a rigorous basis is slightly subtle as the B -fields on $M \times \mathbf{S}^1$ are topologically nontrivial. The basic idea of a rigorous evaluation in this situation is contained in Example 1.16 in [9].

Chern-Simons action for a pair of B -fields (B_{RR}, B_{NS}) on a five-manifold X , even if X has no boundary. If X is the boundary of a six-manifold Y , and the B -fields extend over Y , then the action is readily defined as

$$I_{CS} = -\frac{i}{2\pi} \int_Y H_{RR} \wedge H_{NS}. \quad (3.23)$$

This formula is completely $SL(2, \mathbf{Z})$ -invariant, proving that $SL(2, \mathbf{Z})$ is a symmetry if Y exists. If no such Y exists, then as mentioned in the introduction, one must define the action by $\exp(-I_{CS}) = B_{RR} * B_{NS}$, with $*$ the multiplication in Cheeger-Simons cohomology. From Theorem 1.11 of [9], one can deduce that $I_{CS}(B_{RR}, B_{NS}) = I_{CS}(B_{NS}, -B_{RR})$ and that $I_{CS}(B_{RR}, B_{NS}) = I_{CS}(B_{RR} + 2B_{NS}, B_{NS})$. $SL(2, \mathbf{Z})$ holds if and only if one has the more precise property $I_{CS}(B_{RR}, B_{NS}) = I_{CS}(B_{RR} + B_{NS}, B_{NS})$, but this does not follow from the theorem and is evidently not true in general if X is not spin. The $SL(2, \mathbf{Z})$ invariance of the explicit formula (3.20) that determines the line bundle shows that, at least in canonical quantization, there is no such difficulty in the spin case.

Quantization

Having thus defined the line bundle, we now wish to carry out quantization. This may be done by picking a polarization and quantizing, but we prefer to make contact with the discussion of section 3.2 by exhibiting the discrete Heisenberg group.

Since the phase space T' is a torus, some obvious symmetries of the phase space (as a symplectic manifold) are translations of the torus, say $(B_{RR}, B_{NS}) \rightarrow (B_{RR}, B_{NS}) + 2\pi(a, b)$, with $a, b \in V/\Gamma$. However, such translations generally do not leave fixed the line bundle $\mathcal{L}$. Under the indicated translation, the monodromy $Q(x, y)$ computed above transforms by

$$Q(x, y) \rightarrow \exp(2\pi i(a \cdot y - b \cdot x)) Q(x, y). \quad (3.24)$$

Thus, the monodromies are invariant, for all x, y , only if $a, b \in \Gamma$, in which case the translation by $2\pi(a, b)$ acts trivially on the torus T' .

So if the quantum line bundle is $\mathcal{L}$, there are no nontrivial translation symmetries. We are more generally interested in the “level N ” theory in which the quantum line bundle is $\mathcal{M} = \mathcal{L}^N$. In this case, the monodromies are Q^N , and it is sufficient for Q^N to be invariant. For this, it is enough that Na and Nb should be lattice points. So the translation by $2\pi(a, b)$ is a symmetry of the quantum theory whenever a and b are both of order N . The points of order N are classified by $\frac{1}{N}\Gamma/\Gamma = \Gamma/N\Gamma = H^2(M, \mathbf{Z}_N)$.

Let us try to define as precisely as possible the symmetry $T_{a,b}$ of translation by $2\pi(a, b)$. Such a symmetry exists because the pullback of $\mathcal{M}$ by the translation in question is isomorphic to $\mathcal{M}$. The operator $T_{a,b}$ is uniquely determined up to multiplication by a complex scalar of modulus 1. That scalar can be restricted by requiring (since the translation by $2\pi(Na, Nb)$ is trivial) that $T_{a,b}^N = 1$. This leaves an ambiguity consisting of multiplication by N^{th} roots of unity. There is no natural way to fix that remaining ambiguity, so we make any arbitrary choice.

One might think that the translation operators $T_{a,b}$ would commute. That is not so, because in the presence of a magnetic field (the symplectic form on T' serves as the “magnetic field”) translations do not commute. Including in the standard way the effects of the magnetic field, the actual relation is

$$T_{a,b}T_{a',b'} = T_{a',b'}T_{a,b} \exp(2\pi i(a \cdot b' - b \cdot a')/N). \quad (3.25)$$

This is the Heisenberg group found in section 3.2. The quantum theory can now be analyzed as in section 3.2, with the important difference that we can now show that W acts irreducibly. For instance, this follows by using a complex polarization and Riemann-Roch theorem to determine the dimension of the quantum Hilbert space, which coincides with that of the irreducible representation R of W . For that matter, the informal treatment in section 3.3 was precise enough, at least for large enough N , to count the quantum states within a positive integer factor, and thus to deduce that W acts irreducibly.

3.5. Restriction On $[H]$

Now we want to begin to consider what happens if there is torsion in the cohomology of M .

We first note that B -fields on M are characterized topologically by a characteristic class $[H] \in H^3(M, \mathbf{Z})$. The Gauss law constraint implies that $H = dB$ is zero as a differential form, so $[H]$ must be torsion. At first sight, it appears that there is no other restriction, and that in quantizing the Chern-Simons theory, $[H_{RR}]$ and $[H_{NS}]$ can be arbitrary torsion elements of $H^3(M, \mathbf{Z})$.

Actually, there is a very important further restriction. This is that if the Chern-Simons theory is taken at level N , then $[H_{RR}]$ and $[H_{NS}]$ must be N -torsion, that is $N[H_{RR}] = N[H_{NS}] = 0$.

To obtain this result, we will consider the partition function of the theory on the five-manifold $X = M \times \mathbf{S}^1$. In topological field theory, this partition function equals the

dimension of the space of physical states on M , and so in particular every physical state contributes.

We consider classical configurations on $M \times \mathbf{S}^1$ in which, when restricted to M (that is to $M \times P$ for any $P \in \mathbf{S}^1$), $[H_{NS}]$ has some given value. We want to show that the contribution to the path integral of such configurations is zero unless $N[H_{NS}] = 0$. The same argument, with B_{RR} and B_{NS} exchanged, shows that the contribution is zero unless also $N[H_{RR}] = 0$ when restricted to M .

The vanishing will come upon adding to B_{RR} a flat B -field B' that is topologically trivial if restricted to M , but nontrivial on $M \times \mathbf{S}^1$. (We restrict B' to be trivial on M because we consider a path integral in which $[H_{RR}]$ and $[H_{NS}]$ are both specified on M ; we want to show that summing over what happens in the “time” direction gives the desired restriction on the initial data.) Under $B_{RR} \rightarrow B_{RR} + B'$, the path integrand $e^{-L_{CS}}$ of Chern-Simons theory transforms by

$$\exp(-L_{CS}) \rightarrow \exp(-L_{CS}) \exp\left(\frac{iN}{2\pi} \int_{M \times \mathbf{S}^1} B' \wedge dB_{NS}\right), \quad (3.26)$$

where the exponential factor needs some clarification (which will be given shortly) because the B -fields in question are topologically nontrivial, but is hopefully clear. If the exponential factor in (3.26) does not equal 1, then the path integral will vanish after summing over B' .

We must therefore understand what is meant by the factor that we write symbolically as

$$\exp\left(\frac{i}{2\pi} \int_X B' \wedge dB_{NS}\right), \quad (3.27)$$

if B' and B are flat but topologically nontrivial. This is determined in eqn. 1.14 of [9]; the intuitively natural result can be explained as follows. We can classify flat B -fields in five dimensions either by $H^2(X, U(1))$ or by the characteristic class $[H] \in H^3(X, \mathbf{Z})$. (The first classification describes the B -field up to gauge transformation, and the second only describes its topological type.) There is a natural, nondegenerate pairing

$$H^2(X, U(1)) \times H^3(X, \mathbf{Z}) \rightarrow H^5(X, U(1)) = U(1) \quad (3.28)$$

given by Poincaré and Pontryagin duality. For $B' \in H^2(X, U(1))$ and B_{NS} regarded as an element of $H^3(X, \mathbf{Z})$, we write this pairing as $E(B', B_{NS})$. Cheeger and Simons show that in this situation $B' * B_{NS} = E(B', B_{NS})$, so we must interpret the phase factor in (3.27) as $E(B', B_{NS})$.

Because of the factor of N in the exponent (3.26), the factor that must be 1 for B_{NS} to contribute to the path integral is actually $E(B', B_{NS})^N = E(B', NB_{NS})$. Nondegeneracy of the pairing (3.28) means that for this to equal 1 for all B' , we need $NB_{NS} = 0$, up to gauge transformation. In other words, we need $N[H_{NS}] = 0$. In our case with $X = M \times \mathbf{S}^1$, since we assume $B' = 0$ when restricted to M , the constraint is that $N[H_{NS}] = 0$ when restricted to M . This is the restriction that we aimed to prove.

The restriction that we have found should be viewed as a quantum extension of Gauss's law, to incorporate torsion. Gauss's law enters in path integrals as a constraint that states must obey, in order to propagate in time. Such propagation is precisely what we have been analyzing. The classical Gauss law says that $H_{RR} = H_{NS} = 0$ as a differential form; the quantum Gauss law says that in addition $N[H_{RR}] = N[H_{NS}] = 0$.

3.6. Quantization In The Presence Of Torsion

We will now analyze the quantum Hilbert space $\mathcal{H}'$ of the Chern-Simons theory, allowing for the possible existence of torsion in $H^2(M, \mathbf{Z}_N)$. In fact, to clarify things, we will consider first the case that the cohomology of M contains only torsion – the opposite case from what we considered in sections 3.3 and 3.4.

After imposing Gauss's law, the physical data consists of a pair of flat B -fields B_{RR} , B_{NS} . From what we have seen in section 3.5, we should impose a discrete form of Gauss's law stating that $N[H_{RR}] = N[H_{NS}] = 0$. Let L_N be the subgroup of $H^3(M, \mathbf{Z})$ consisting of points of order N . The two B -fields determine a pair of points in L_N . If the cohomology of M is pure torsion, there are no additional data to quantize. Each pair $(x, y) \in L_N \times L_N$ contributes one quantum state $\Psi_{x,y}$, and the $\Psi_{x,y}$ give a basis of the physical Hilbert space $\mathcal{H}'$.

We want to compare this to the description from section 2: for each choice of polarization, $\mathcal{H}'$ should have a basis Ψ_w , for $w \in K = H^2(M, \mathbf{Z}_N)$. In an equivalent version presented in section 3.2, $\mathcal{H}'$ should furnish an irreducible description of the finite Heisenberg group W :

$$0 \rightarrow \mathbf{Z}_N \rightarrow W \rightarrow K \times K \rightarrow 0. \quad (3.29)$$

For this, we must understand the structure of $H^2(M, \mathbf{Z}_N)$ when there is torsion. From the exact sequence of abelian groups $0 \rightarrow \mathbf{Z} \xrightarrow{N} \mathbf{Z} \rightarrow \mathbf{Z}_N \rightarrow 0$, with the first map being multiplication by N and the second reduction modulo N , we get a long exact sequence

$$\dots \rightarrow H^2(M, \mathbf{Z}) \xrightarrow{N} H^2(M, \mathbf{Z}) \rightarrow H^2(M, \mathbf{Z}_N) \rightarrow H^3(M, \mathbf{Z}) \xrightarrow{N} H^3(M, \mathbf{Z}) \rightarrow \dots \quad (3.30)$$

This gives a short exact sequence

$$0 \rightarrow E_N \rightarrow H^2(M, \mathbf{Z}_N) \rightarrow L_N \rightarrow 0, \quad (3.31)$$

with $E_N = H^2(M, \mathbf{Z})/NH^2(M, \mathbf{Z})$ the subgroup of $H^2(M, \mathbf{Z}_N)$ consisting of classes that are the reduction of an integer class.

We recall that there is a nondegenerate pairing $H^2(M, \mathbf{Z}_N) \times H^2(M, \mathbf{Z}_N) \rightarrow \mathbf{Z}_N$ given by Poincaré duality; it was used in section 3.2 to construct the finite Heisenberg group. If the cohomology of M is purely torsion, then E_N consists entirely of classes that are reductions of torsion classes. In this case, the pairing on $H^2(M, \mathbf{Z}_N) \times H^2(M, \mathbf{Z}_N)$ is zero when restricted to $E_N \times E_N$ (since in integer cohomology the cup product vanishes for torsion classes), and the self-duality of $H^2(M, \mathbf{Z}_N)$ reduces to a duality between E_N and L_N .

The Heisenberg group W in this situation has a maximal commutative subgroup that comes from the subgroup $E_N \times E_N$ of $H^2(M, \mathbf{Z}_N) \times H^2(M, \mathbf{Z}_N)$. According to the remark at the end of section 3.2, the representation R has therefore a basis in 1-1 correspondence with the cosets in $(K \times K)/(E_N \times E_N)$. That quotient is isomorphic to $L_N \times L_N$, so we get one basis vector $\Psi_{x,y}$ for each pair $(x, y) \in L_N \times L_N$. This is the description we found for the physical Hilbert space $\mathcal{H}'$, so we confirm the expected isomorphism of R with $\mathcal{H}'$.

In particular, we see again that $\mathcal{H}'$ is an irreducible representation of the discrete Heisenberg group.

The General Case

The general case in which the cohomology of M contains torsion but is not purely torsion is a mixture of the cases that we have already considered.

In this case, the cohomology classes of $x = [H_{RR}]$, $y = [H_{NS}]$ determine elements $x, y \in L_N$ which are part of the data. In addition, one must quantize the α 's. We can pick a polarization in which the quantum states are regarded as functions of α_{NS} as well as x, y .

As we found in section 3.3, quantization of the α 's gives one state for every topologically trivial flat B_{NS} -field of order N . Including also the dependence on y means that we drop the requirement that B_{NS} should be topologically trivial; we get one state for every flat B_{NS} field of order N whether it is topologically trivial or not. We let M_N be the group of flat B -fields of order N . Including also the choice of x (which is a point in L_N), we see that the quantum Hilbert space $\mathcal{H}'$ has a basis consisting of one vector

for every element of $L_N \times M_N$. This strongly suggests that there might be a polarization of the discrete Heisenberg group determined by a subgroup F of $K \times K$ such that $(K \times K)/F = L_N \times M_N$. In that case, the irreducible representation R of the discrete Heisenberg group can be identified with $\mathcal{H}'$ as desired.

In fact, we can take $F = E_N \times T_N$, where T_N is the subgroup of $H^2(M, \mathbf{Z}_N)$ consisting of elements that are the reduction mod N of torsion classes in $H^2(M, \mathbf{Z})$. F is a commutative subgroup (even when lifted to W) because the pairing of two elements in $H^2(M, \mathbf{Z})$, one of which is torsion, is zero. Since $(K \times K)/F = (K \times K)/(E_N \times T_N) = (K/E_N) \times (K/T_N)$, the claim that $(K \times K)/F = L_N \times M_N$ is equivalent to the existence of two exact sequences:

$$\begin{aligned} 0 \rightarrow E_N \rightarrow H^2(M, \mathbf{Z}_N) \rightarrow L_N \rightarrow 0, \\ 0 \rightarrow T_N \rightarrow H^2(M, \mathbf{Z}_N) \rightarrow M_N \rightarrow 0. \end{aligned} \tag{3.32}$$

The first is in (3.31), and the second can be derived from the sequence

$$0 \rightarrow \mathbf{Z}_N \rightarrow U(1) \xrightarrow{N} U(1) \rightarrow 0 \tag{3.33}$$

(the first map is the inclusion of the points of order N in $U(1)$, and the second is $a \rightarrow a^N$).

From this sequence we get a long exact cohomology sequence

$$0 \rightarrow H^1(M, U(1))/NH^1(M, U(1)) \rightarrow H^2(M, \mathbf{Z}_N) \rightarrow H^2(M, U(1))_N \rightarrow 0. \tag{3.34}$$

Here $H^2(M, U(1))_N$ is the kernel of $H^2(M, U(1)) \xrightarrow{N} H^2(M, U(1))$; this is the group of flat B -fields of order N , or in other words is M_N . Also, $H^1(M, U(1))$ classifies flat $U(1)$ bundles. Such a bundle has a first Chern class which is a torsion element of $H^2(M, \mathbf{Z})$; and the map from $H^1(M, U(1))/NH^1(M, U(1))$ to $H^2(M, \mathbf{Z}_N)$ is given by mod N reduction of the first Chern class. Conversely, every torsion element of $H^2(M, \mathbf{Z})$ is the first Chern class of a flat line bundle. So the image of $H^1(M, U(1))/NH^1(M, U(1))$ in $H^2(M, \mathbf{Z}_N)$ consists precisely of the classes that are mod N reductions of torsion classes, or in other words this image is T_N . This completes the explanation of the second exact sequence in (3.32).

4. The (0, 2) Theory In Six Dimensions

In this section, we will consider the analogous issues for the (0, 2) conformal field theory in six-dimensions. ⁷ We consider only the A_N theory, that is, the theory associated with M -theory on $AdS_7 \times \mathbf{S}^4$, with N units of four-form flux on $\mathbf{S}^4$. We will, in essence, be determining the analog for the A_N case of the (0, 2) theory of the space of conformal blocks in two-dimensional rational conformal field theory. A difference from the problem considered in sections 2 and 3 is that there is no obvious candidate from classical geometry for what the answer should be.

To study the (0, 2) theory on a six-dimensional spin-manifold M , we consider M -theory on spacetimes that look near infinity like $X \times \mathbf{S}^4$, where X has M for conformal boundary. By analogy with gauge theory in four-dimensions, we suspect that for general M , the partition function will not be a number but a vector in some finite-dimensional Hilbert space associated with M .

A mechanism analogous to what we studied in sections 2 and 3 immediately presents itself. The long wavelength theory on X has a three-form field C , whose effective action contains the Chern-Simons term

$$L_{CS} = -i \frac{1}{2} \frac{N}{2\pi} \int_X C \wedge dC. \quad (4.1)$$

If N is odd, then because of the $1/2$ in (4.1), to make sense of this theory requires a spin structure on X . (This is explained in [7], where the case $N = 1$ was considered and the factor of $1/2$ in (4.1), which arises from a similar factor of $1/6$ in M -theory, played an important role. Some additional important details, such as a gravitational correction to (4.1), were also described there.) There is no harm in this, because in any event, M -theory on $X \times \mathbf{S}^4$ requires a spin structure on X . If N is even or a spin structure is picked on X , then quantization is carried out rather as in section 3.2 by introducing a Heisenberg group associated with $H^3(M, \mathbf{Z}_N)$. This Heisenberg group contains an operator Φ_v for each $v \in H^3(M, \mathbf{Z}_N)$, with ⁸

$$\Phi_v \Phi_w = \Phi_w \Phi_v \exp(2\pi i v \cdot w / N). \quad (4.2)$$

⁷ Some related issues involving a discrete flux in the (0, 2) model have been discussed in [25].

⁸ From one point of view, the need for a spin structure arises because the following formula does not quite determine the Heisenberg group. To have a Heisenberg group, we need a multiplication law $\Phi_v \Phi_w = c(v, w) \Phi_{v+w}$, with $c(v, w) c(w, v)^{-1} = \exp(2\pi i v \cdot w / N)$. Since $v \cdot w$ is only well-defined modulo N , there is no elementary formula for $c(v, w)$ except to take an ambiguous square root $c(v, w) = \pm \sqrt{\exp(2\pi i v \cdot w / N)}$. A spin structure determines a consistent set of square roots. In

We thus have a group extension

$$0 \rightarrow \mathbf{Z}_N \rightarrow W \rightarrow H^3(M, \mathbf{Z}_N) \rightarrow 0, \quad (4.3)$$

and the quantum Hilbert space $\mathcal{H}$ is an irreducible representation of W . (Irreducibility can be proved by a detailed analysis along the lines of sections 3.4-6.)

One important difference from the four-dimensional case is that W is defined using only one copy of $H^3(M, \mathbf{Z}_N)$ (rather than two copies of $H^2(M, \mathbf{Z}_N)$ as in four dimensions). So there is no way to pick a polarization and give an explicit description of $\mathcal{H}$ without using some detailed properties of W and breaking the symmetries of the finite group $H^3(M, \mathbf{Z}_N)$. One simple case is $M = \mathbf{S}^1 \times Y$, with Y a five-dimensional spin manifold. In this case, $H^3(M, \mathbf{Z}_N) = H^2(Y, \mathbf{Z}_N) \oplus H^3(Y, \mathbf{Z}_N)$. A polarization of $H^3(M, \mathbf{Z}_N)$ is given by its subgroup $H^3(Y, \mathbf{Z}_N)$. The quotient is $H^3(M, \mathbf{Z}_N)/H^3(Y, \mathbf{Z}_N) = H^2(Y, \mathbf{Z}_N)$. Hence, the Hilbert space $\mathcal{H}$ has a basis with a vector Ψ_w for each $w \in H^2(Y, \mathbf{Z}_N)$. This result has a simple explanation. The $(0, 2)$ conformal field theory, if compactified on $\mathbf{S}^1$, gives a theory that looks at low energies like $SU(N)/\mathbf{Z}_N$ gauge theory in five dimensions. If this is further compactified on the five-manifold Y , the gauge theory has a discrete magnetic flux taking values in $H^2(Y, \mathbf{Z}_N)$, and this leads to the Hilbert space that we found.

5. 't Hooft And Wilson Lines In Four Dimensions

In this concluding section, we will analyze the behavior of the $\mathcal{N} = 4$ super Yang-Mills theory on a four-manifold M with 't Hooft and Wilson loops included.

We begin on a five-manifold X by supplementing the Chern-Simons Lagrangian L_{CS} by couplings of the B -fields to string worldsheets. The strings in question are fundamental strings coupling to B_{NS} and D -strings coupling to B_{RR} . We take the D -string and elementary string worldsheets to be closed surfaces S_{RR} and S_{NS} , which are not necessarily connected. Including the couplings to the strings, the Lagrangian is

$$\widehat{L} = NI_{CS}(B_{RR}, B_{NS}) - \frac{i}{2\pi} \int_{S_{RR}} B_{RR} - \frac{i}{2\pi} \int_{S_{NS}} B_{NS}. \quad (5.1)$$

fact, having a Heisenberg group extension and not just a commutation relation is equivalent – roughly as we saw in section 3.3 – to having a suitable line bundle over the classical phase space. As analyzed in detail in [4], defining this line bundle depends on having a spin structure on M . The Chern-Simons theory that we have studied in this paper in five dimensions likewise needs a spin structure, as we saw in section 3.3.

If X has conformal boundary M , then the boundary of S_{RR} and S_{NS} should be [26,27] one-manifolds C_{RR} and C_{NS} in M on which 't Hooft and Wilson loops are inserted. (The C 's are not necessarily connected.)

The path integral on X will take values in a Hilbert space associated with the boundary. Near the boundary, we have $X = M \times \mathbf{R}$, where we view $\mathbf{R}$ as the “time” direction. Near the boundary, we can take $S_{RR} = C_{RR} \times \mathbf{R}$ and $S_{NS} = C_{NS} \times \mathbf{R}$. Quantization is carried out formally as in section 3.1. The Gauss law constraint contains an extra contribution from the strings and reads

$$\begin{aligned} NH_{NS} + \delta(C_{RR}) &= 0 \\ -NH_{RR} + \delta(C_{NS}) &= 0. \end{aligned} \tag{5.2}$$

Here, for $C = C_{RR}$ or C_{NS} , $\delta(C)$ is a delta-function supported on C , representing a cohomology class in $H^3(M, \mathbf{Z})$ that is dual to C . We will call this class $[C]$.

If the equations (5.2) have solutions, then as these equations are linear, by shifting the B -fields by any solution, we can reduce to the previous case in which the phase space that must be quantized is governed by $H_{RR} = H_{NS} = 0$. Hence, the quantum Hilbert space is isomorphic to the one obtained without the Wilson lines, though not canonically so as some arbitrary choice of a solution of Gauss’s law has to be made in identifying the phase spaces with and without the Wilson lines. However, the condition that equations (5.2) should have any solutions at all is nontrivial. The requirement that these equations have a solution for some B -fields is simply that the elements $[C_{RR}]$, $[C_{NS}]$ should be divisible by N in $H^3(M, \mathbf{Z})$. (When torsion is present, a full justification of this statement requires a torsion extension of the Gauss’s law constraint, along lines presented in section 3.5.)

For example, if $M = Y \times \mathbf{S}^1$, with Y a three-manifold, then the requirement is that the C ’s should wrap around $\mathbf{S}^1$ a number of times divisible by N . This statement has a simple intuitive interpretation. It means that the Wilson line wrapped on C_{NS} (or the magnetic counterpart wrapped on C_{RR}) is invariant under the thermal symmetry group $F = H^1(M, \mathbf{Z}_N)$. This way of seeing the thermal symmetry group is actually a close cousin of the argument in [8].

For the rest of this section, we specialize to the case $M = \mathbf{S}^4$. In this case, since $H^3(M, \mathbf{Z}) = 0$, there is no topological obstruction to solving the Gauss law constraints, regardless of what the C ’s might be. Moreover, the space of solutions mod gauge transformations is a single point (since the cohomology groups that classify flat B -fields are all zero). So the quantum Hilbert space is one-dimensional.

The expectation value of an arbitrary product of Wilson and 't Hooft loops on $\mathbf{S}^4$ thus takes values in a one-dimensional Hilbert space $\mathcal{H}$. However, the expectation value cannot in general be understood as an ordinary complex number, because there is no natural way to pick a basis vector in $\mathcal{H}$ and identify $\mathcal{H}$ with the complex numbers $\mathbf{C}$. Let us explore the matter somewhat more thoroughly and see what is involved in picking a basis vector.

The presence of the Wilson loop on C_{NS} leads to the presence in the path integral on $X \times \mathbf{S}^5$ of a factor

$$\exp \left(i \int_{S_{NS}} B_{NS} \right). \quad (5.3)$$

This factor is well-defined as a complex number if S_{NS} is a closed surface. If not, the factor takes values in a one-dimensional Hilbert space $\mathcal{H}_{NS}$. $\mathcal{H}$ is the tensor product of $\mathcal{H}_{NS}$ with a similar one-dimensional Hilbert space $\mathcal{H}_{RR}$ associated with the D -strings, as well as a factor independent of all 't Hooft and Wilson loops that enters in defining the bulk term in the action. To trivialize $\mathcal{H}_{NS}$, we should give a surface $D_{NS} \subset M$ of boundary $-C_{NS}$ (the minus sign is a reversal of orientation), and replace (5.3) by

$$\exp \left(i \int_{S_{NS} + D_{NS}} B_{NS} \right). \quad (5.4)$$

Here $S_{NS} + D_{NS}$ is a closed surface in X , so (5.4) is well-defined as a number.⁹ We must, however, investigate the dependence on the choice of D_{NS} . Let D'_{NS} be another choice. Then in replacing D_{NS} by D'_{NS} , (5.4) is multiplied by

$$\Psi = \exp \left(i \int_E B_{NS} \right), \quad (5.5)$$

where E is the closed surface in M defined by $E = D'_{NS} - D_{NS}$. Since E is contained in $\mathbf{S}^4$, whose homology vanishes, it is the boundary of a three-manifold $Y \subset M$, and we can write

$$\Psi = \exp \left(i \int_Y H_{NS} \right). \quad (5.6)$$

If there are no D -strings, then H_{NS} vanishes by the equations of motion, and the factor Ψ in the path integral can be eliminated by a redefinition of the fields, in fact, by $B_{RR} \rightarrow B_{RR} + (2\pi/N)\delta(Y)$. (Y is of codimension two in X , and $\delta(Y)$ is a two-form dual to Y .) But if there are D -strings, this is not quite a symmetry. Under this transformation, the D -string

⁹ D_{NS} should lie in M , not just in X , so as to trivialize $\mathcal{H}$ (which only depends on M) once and for all, independent of the choice of X .

factor $\exp(i \int_{S_{RR}} B_{RR})$ in the path integral picks up a phase $\exp((2\pi i/N)Y \cdot C_{RR})$, where $Y \cdot C_{RR}$ denotes the oriented intersection number of Y and C_{RR} . That intersection number is (by definition) the linking number $\ell(E, C_{RR})$. Hence, under a change in trivialization of $\mathcal{H}_{NS}$ coming from a change in D_{NS} , the path integral is multiplied by

$$\exp(2\pi i \ell(E, C_{RR})/N). \quad (5.7)$$

This factor is, of course, 1 if there are no 't Hooft loops ($C_{RR} = 0$). So the expectation value of a product of Wilson loops only (or likewise of 't Hooft loops only) can be viewed as an ordinary number. But the expectation value of a product consisting of both Wilson and 't Hooft loops, though well-defined as a vector in the one-dimensional Hilbert space $\mathcal{H}$, is ambiguous up to multiplication by an N^{th} root of unity if one wishes to interpret it as an ordinary complex number.

An ambiguity of just this nature can be seen by standard gauge theoretic methods. In that context, one begins with the problem of what it means in $SU(N)/\mathbf{Z}_N$ gauge theory to define the expectation value of a Wilson loop, wrapped on a circle C_{NS} , in the fundamental representation of $SU(N)$. For this, we would want a lifting of the $SU(N)/\mathbf{Z}_N$ bundle to $SU(N)$ at least along C_{NS} . It suffices to give a surface D_{NS} with boundary C_{NS} . For, as $H^2(D_{NS}, \mathbf{Z}_N) = 0$, the $SU(N)/\mathbf{Z}_N$ bundle when restricted to D_{NS} can be lifted to $SU(N)$, and though the lifting is not unique, the nonuniqueness does not affect the value of a Wilson loop that wraps around the boundary of D_{NS} . The answer one gets this way does depend on the choice of D_{NS} if 't Hooft loops are present also. Upon changing D_{NS} to D'_{NS} , one gets in the gauge theory, in the presence of an 't Hooft loop on C_{RR} , a change in the Wilson loop expectation value given by the same formula as in (5.7). That is because, if the linking number is nonzero, the liftings of the $SU(N)/\mathbf{Z}_N$ bundle to $SU(N)$ on D_{NS} and D'_{NS} disagree on their common boundary.

Of course, different points of view are possible about the expectation value of a product of Wilson and 't Hooft loops on $\mathbf{S}^4$. The traditional point of view is to reject such correlation functions on the grounds that they are not well-defined as complex numbers. If one chooses to consider such correlation functions, we have shown how they must be interpreted, and more specifically we have shown that they have the same meaning whether considered in gauge theory on $\mathbf{S}^4$ or in string theory on $AdS_5 \times \mathbf{S}^5$.

This work was supported in part by NSF Grant PHY-9513835. I would like to thank P. Deligne, D. Freed, O. Ganor, M. Hopkins, N. Seiberg, and S. Sethi for discussions.

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To cite this article: Edward Witten JHEP12(1998)012

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AdS/CFT correspondence and topological field theory

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ABSTRACT: In $\mathcal{N} = 4$ super Yang-Mills theory on a four-manifold M , one can specify a discrete magnetic flux valued in $H^2(M, \mathbb{Z}_N)$. This flux is encoded in the AdS/CFT correspondence in terms of a five-dimensional topological field theory with Chern-Simons action. A similar topological field theory in seven dimensions governs the space of “conformal blocks” of the six-dimensional $(0, 2)$ conformal field theory.

KEYWORDS: [Topological Field Theories](#), [1/N Expansion](#), [String Duality](#).

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1. Introduction

The AdS/CFT correspondence [\[1\]](#) relates $\mathcal{N} = 4$ super Yang-Mills theory on $\mathbf{S}^4$, with gauge group $SU(N)$, to Type IIB superstring theory on $AdS_5 \times \mathbf{S}^5$, with N units of five-form flux on $\mathbf{S}^5$. The correspondence expresses gauge theory correlation functions in terms of the dependence of the string theory on the behavior near the conformal boundary of AdS_5 [\[2, 3\]](#).

What happens if we replace $\mathbf{S}^4$ with a more general four-dimensional spin manifold M ? (M must be spin since the $\mathcal{N} = 4$ theory contains spinor fields. In addition, M must be endowed with a metric that — after a suitable conformal rescaling — has positive scalar curvature, or the $\mathcal{N} = 4$ theory is unstable.¹) The behavior of the gauge theory on a four-manifold M is believed to be described, roughly speaking, in terms of the behavior of the string theory on $X \times \mathbf{S}^5$, where X is a negatively curved

¹The instability arises because of possible runaway behavior of the massless scalars ϕ of the theory. Positivity of the scalar curvature R of M suppresses the instability because of the $R \operatorname{tr} \phi^2$ coupling required by conformal invariance.

Einstein manifold with M as conformal boundary, and one must sum over all possible choices of X . This description is somewhat rough since, in general, one might need to include branes or stringy singularities on $X \times \mathbf{S}^5$ or more general topologies that are not simply the product of $\mathbf{S}^5$ with some five-manifold. The general prescription is really that one sums over Type IIB spacetimes that near infinity look like $X \times \mathbf{S}^5$, with X a five-manifold of conformal boundary M .

The gauge theory on M that is described in this correspondence has a gauge group that is locally $SU(N)$, rather than $U(N)$. Many arguments show this, beginning with the fact that in $\mathcal{N} = 4$ super Yang-Mills theory of $U(N)$, the $U(1)$ fields would be free, while in string theory on $AdS_5 \times \mathbf{S}^5$, everything couples to gravity and no field is free. As the gauge group is $SU(N)$, its center is $\mathbb{Z}_N$.

One consequence of the fact that the gauge group is $SU(N)$ rather than $U(N)$ is that it is possible to make a baryon vertex linking N external quarks. This can be constructed using a wrapped fivebrane [4] and can also be understood [5] via a certain Chern-Simons interaction that we will describe shortly.

Our interest in the present paper will, however, be in properties that depend on the global structure of M . In $\mathcal{N} = 4$ super Yang-Mills theory, all fields are in the adjoint representation and hence, as a local gauge transformation, the center of the gauge group acts trivially. It follows that locally the gauge group could be $SU(N)/\mathbb{Z}_N$. This has two important consequences depending on the topology of M :

1. If $H^1(M, \mathbb{Z}_N) \neq 0$, it is possible to consider gauge transformations on M that are not single-valued in $SU(N)$ but are single-valued in $SU(N)/\mathbb{Z}_N$. For example, if $M = \mathbf{S}^1 \times Y$, one can consider a gauge transformation that is constant in the Y direction and whose restriction to the $\mathbf{S}^1$ direction determines any element of $\pi_1(SU(N)/\mathbb{Z}_N) = \mathbb{Z}_N$. This gives an important group $F \cong \mathbb{Z}_N$ of global symmetries of the thermal physics on Y . More generally, $F = H^1(M, \mathbb{Z}_N)$ classifies $SU(N)/\mathbb{Z}_N$ gauge transformations that cannot be lifted to $SU(N)$.
2. If $H^2(M, \mathbb{Z}_N) \neq 0$, it is possible to consider $SU(N)/\mathbb{Z}_N$ bundles with “discrete magnetic flux” [6]. In fact, $SU(N)/\mathbb{Z}_N$ bundles on M are classified topologically by specifying the instanton number and also a characteristic class w that takes values in $K = H^2(M, \mathbb{Z}_N)$. (For example, if $N = 2$, we have $SU(2)/\mathbb{Z}_2 = SO(3)$, and w coincides with the second Stiefel-Whitney class of the gauge bundle.)

How to see in the AdS/CFT correspondence the thermal symmetry F has been discussed elsewhere [7]. For our present purposes, we need to recall just one point from that discussion. Type IIB superstring theory has two two-form fields B_{NS} and B_{RR} . In compactification on $X \times \mathbf{S}^5$ (with N units of five-form flux on $\mathbf{S}^5$), one

gets a low energy effective action on X that contains a Chern-Simons term

$$L_{CS} = NI_{CS}, \quad (1.1)$$

where

$$I_{CS} = -\frac{i}{2\pi} \int_X B_{RR} \wedge dB_{NS} \quad (1.2)$$

is the basic Chern-Simons invariant of the two B -fields. (The action L_{CS} is important in one approach to the baryon vertex [5].) Though the integrand in I_{CS} is not gauge-invariant, I_{CS} is gauge-invariant mod $2\pi i$ on a closed five-manifold X . It can be written more invariantly as follows: if X is the boundary of a six-manifold Y over which the two B -fields extend, and we write the field strength of a B -field as $H = dB$, then one can write I_{CS} in a manifestly gauge-invariant way as

$$I_{CS} = -\frac{i}{2\pi} \int_Y H_{RR} \wedge H_{NS}. \quad (1.3)$$

More generally, if Y does not exist, a more subtle approach is needed to define I_{CS} . For a general Chern-Simons interaction, one can follow a slightly abstract approach explained in section 2.10 and the end of the introduction to section 4 in [8]. For the particular Chern-Simons theory that we are discussing here, one can define I_{CS} as a cup product in Cheeger-Simons cohomology.²

A theory of the two B -fields with Lagrangian precisely [1.1] is a simple but subtle topological field theory, of a sort first considered in [10]. The actual low energy effective action that arises in compactification of Type IIB superstring theory on $\mathbf{S}^5$ has many couplings beyond the Chern-Simons interaction; the additional couplings have however a larger number of derivatives and so are irrelevant for certain questions.

The goal of the present paper is to understand how to encode in terms of the AdS/CFT correspondence the dependence of the gauge theory on the discrete magnetic flux w . As we will see, the topological field theory with Lagrangian [1.1] plays a starring role in the analysis. Roughly speaking, what in the gauge theory description appears as the discrete magnetic flux w appears in the gravitational description on $X \times \mathbf{S}^5$ as a quantum state of this topological field theory. We state a precise

²In the language of Cheeger and Simons [4], a B -field on a manifold X is an element of $\hat{H}^2(X, U(1))$, the group of differential two-characters on X with values in $U(1)$. For two elements $B_{RR}, B_{NS} \in \hat{H}^2(X, U(1))$, Cheeger and Simons describe in Theorem 1.11 a product $B_{RR} * B_{NS} \in \hat{H}^5(X, U(1))$. For X of dimension five, $\hat{H}^5(X, U(1)) = U(1)$, and we set $\exp(-I_{CS}(B_{RR}, B_{NS})) = B_{RR} * B_{NS}$. Equation 1.15 in [4] asserts that if B_{RR} is topologically trivial and hence can be defined by an ordinary two-form, I_{CS} can be computed by the integral written in [1.2]: $I_{CS} = -(i/2\pi) \int_X B_{RR} \wedge H_{NS}$. Of course, there is a similar formula $I_{CS} = (i/2\pi) \int_X H_{RR} \wedge B_{NS}$ if B_{NS} is topologically trivial. It can also be proved that $B_{RR} * B_{NS}$ can be defined by the formula in the text if X is the boundary of an appropriate six-manifold Y . The usefulness of interpreting the action in terms of differential characters was explained to me by M. Hopkins, who also pointed out facts that we will use in section [3.4].

conjecture in section 2 and carry out some computations supporting it in section 3. Then in section 4, we make an extension to the $(2,0)$ superconformal field theory in six dimensions, and in section 5 we discuss 't Hooft and Wilson loops in the four-dimensional $\mathcal{N} = 4$ theory.

Since we will be developing a fairly elaborate theory to understand in terms of gravity the dependence of the gauge theory on the discrete magnetic flux, it is reasonable to ask what examples are known to which this theory can be applied. Actually, as we noted at the outset, there are relatively few M 's for which the AdS/CFT correspondence is expected to work (M must be spin and with positive scalar curvature), and there are presently very few such examples for which anything is known about the possible five-manifolds X with conformal boundary M . One simple but important example, in which one can verify the importance of summing over different choices of X , is the case $M = \mathbf{S}^1 \times \mathbf{S}^3$. There are [11] two known choices of X , namely $X_1 = \mathbf{S}^1 \times B_4$ and $X_2 = B_2 \times \mathbf{S}^3$ (here B_n denotes an n -dimensional ball); many important properties of Yang-Mills theory at nonzero temperature are reflected in the behavior of these two X 's [12].

This example has $H^2(M, \mathbb{Z}_n) = 0$ and so does not serve as a good illustration of the issues explored in the present paper. But one can readily modify it to give an example with nontrivial discrete magnetic fluxes. Identify $\mathbf{S}^3$ with the $SU(2)$ manifold and let H be a discrete subgroup of $SU(2)$, acting on the right. Set $M_H = \mathbf{S}^1 \times \mathbf{S}^3/H$. M_H is the boundary of $X_{i,H} = X_i/H$. For suitable H , $H^2(M_H, \mathbb{Z}_N) \neq 0$, so this gives a simple example to which our theory can be applied, showing in particular that the theory is nonvacuous.

This example actually has the following interesting property. $X_{2,H}$ has orbifold singularities (because H acts freely on $\mathbf{S}^3$ but not on B_4). The orbifold singularities are harmless in string theory and will be resolved as one varies the metric on M_H . The manifold $X_{2,H}$ will lose and acquire singularities and undergo monodromies as the metric on M_H is varied.

Another example is $M = \mathbf{S}^2 \times \mathbf{S}^2$, where some X 's have been constructed in [13] and investigated independently in [14]. This example has the property that $H^2(M, \mathbb{Z})$ is not a torsion group.

Some technical details. We conclude this introduction by summarizing a few useful details. In this paper, the symbol b_i will denote the i^{th} Betti number of the four-manifold M . We also write the order of the finite group $H^i(M, \mathbb{Z}_N)$ as $N^{\tilde{b}_i}$. If there is no torsion in the integral cohomology of M , then $b_i = \tilde{b}_i$ (In general the $\tilde{b}_i$ are not integers). Using Poincaré and Pontryagin duality, it is possible to prove that $b_{4-i} = b_i$ and likewise $\tilde{b}_{4-i} = \tilde{b}_i$. The last statement arises because there is a nondegenerate pairing $H^i(M, \mathbb{Z}_N) \times H^{4-i}(M, \mathbb{Z}_N) \rightarrow \mathbb{Z}_N$ (given by the cup product); we write the product of $x \in H^i(M, \mathbb{Z}_N)$ with $y \in H^{4-i}(M, \mathbb{Z}_N)$ as $x \cdot y$. Nondegeneracy of the pairing implies that these groups are of the same order (and

in fact are isomorphic as abelian groups). One also has $b_0 = \tilde{b}_0 = 1$. We write χ and σ for the Euler characteristic and signature of M ; we recall the definition $\chi = \sum_{i=1}^4 (-1)^i b_i$. Using the long exact sequence of cohomology groups derived from the short exact sequence of groups $0 \rightarrow \mathbb{Z} \xrightarrow{N} \mathbb{Z} \rightarrow \mathbb{Z}_N \rightarrow 0$ (the first map is multiplication by N and the second is reduction modulo N), one can prove that in fact

$$\sum_{i=0}^4 (-1)^i b_i = \sum_{i=0}^4 (-1)^i \tilde{b}_i. \quad (1.4)$$

Hence $b_1 - \frac{1}{2}b_2 = \tilde{b}_1 - \frac{1}{2}\tilde{b}_2$, a fact we will use later.

2. Role of the topological field theory

2.1 The problem

As a prelude to our main subject, let us ask: In studying the $\mathcal{N} = 4$ super Yang-Mills theory on a four-manifold M via the AdS/CFT correspondence, do we expect to see Montonen-Olive S -duality? The answer is, “Not in a naive way,” since $SU(N)$ is not a self-dual group. For example, if

$$\tau = \frac{4\pi i}{g^2} + \frac{\theta}{2\pi} \quad (2.1)$$

is the coupling parameter of the theory, then under $\tau \rightarrow -1/\tau$ we expect $SU(N)$ to be exchanged with $SU(N)/\mathbb{Z}_N$.

The $SU(N)$ and $SU(N)/\mathbb{Z}_N$ theories are equivalent locally, and they are essentially equivalent on a four-manifold such as $M = \mathbf{S}^4$ or $\mathbf{S}^3 \times \mathbf{S}^1$ whose second cohomology group vanishes.³ These two theories really differ in an interesting way when $H^2(M, \mathbb{Z}_N) \neq 0$. The reason is that $SU(N)$ bundles on M are classified just by the instanton number, but $SU(N)/\mathbb{Z}_N$ bundles are classified by an additional topological invariant which is the “discrete magnetic flux” $w \in H^2(M, \mathbb{Z}_N)$ mentioned in the introduction. The $SU(N)$ theory has a partition function $Z(\tau)$ (we suppress M from the notation when this is likely to cause no confusion), but the $SU(N)/\mathbb{Z}_N$ theory has a family of partition functions $Z_w(\tau)$, one for each $w \in H^2(M, \mathbb{Z}_N)$. The

³On such a manifold, the $SU(N)$ and $SU(N)/\mathbb{Z}_N$ theories have the same correlation functions; their partition functions differ by an elementary factor described in eqn. (3.17) of [15]. This factor arises because the groups of $SU(N)$ and $SU(N)/\mathbb{Z}_N$ gauge transformations differ slightly; the center of $SU(N)$ consists of gauge transformations in $SU(N)$ that are not considered in $SU(N)/\mathbb{Z}_N$, while $F = H^1(M, \mathbb{Z}_N)$ classifies the classes of global $SU(N)/\mathbb{Z}_N$ gauge transformations that cannot be lifted to $SU(N)$. With $N^{\tilde{b}_1}$ the order of the finite group F , the group of $SU(N)$ gauge transformations has volume $N^{1-\tilde{b}_1}$ times that for $SU(N)/\mathbb{Z}_N$, and hence the $SU(N)$ partition function is $N^{-1+\tilde{b}_1}$ times that for $SU(N)/\mathbb{Z}_N$, a fact which is incorporated in the next equation in the text. If, as assumed in [15], there is no torsion in the cohomology of M , then $\tilde{b}_1 = b_1$, and the factor becomes N^{-1+b_1} , as written in [15].

relation between the two is that the $SU(N)$ partition function is obtained from the $SU(N)/\mathbb{Z}_N$ partition function by setting $w = 0$, up to an elementary factor (mentioned in the footnote):

$$Z_{SU(N)}(\tau) = N^{-1+\tilde{b}_1} Z_0(\tau). \quad (2.2)$$

The $SU(N)$ theory is thus obtained by setting $w = 0$, but in the $SU(N)/\mathbb{Z}_N$ theory, one sums over all w . Since these operations are supposed to be exchanged under $\tau \rightarrow -1/\tau$, clearly the contribution with a given w cannot be $SL(2, \mathbb{Z})$ -invariant.

Generalizing ideas of 't Hooft [6], it has been argued [15] that the $Z_w(\tau)$ transform as a unitary representation of $SL(2, \mathbb{Z})$. To describe this representation, one must use the fact that the cup product gives a natural pairing $H^2(M, \mathbb{Z}_N) \times H^2(M, \mathbb{Z}_N) \rightarrow H^4(M, \mathbb{Z}_N) = \mathbb{Z}_N$, as mentioned at the end of the introduction. The most interesting part of the action of $SL(2, \mathbb{Z})$ is the behavior under $\tau \rightarrow -1/\tau$, which according to [15] is a sort of discrete Fourier transform:

$$Z_v(-1/\tau) = N^{-\tilde{b}_2/2} \left(\frac{\tau}{i}\right)^{W/2} \left(\frac{\bar{\tau}}{-i}\right)^{\bar{W}/2} \sum_{w \in H^2(M, \mathbb{Z}_N)} \exp(2\pi i v \cdot w/N) Z_w(\tau). \quad (2.3)$$

The intuitive idea of this formula is that $\tau \rightarrow -1/\tau$ exchanges magnetic and electric flux, but the discrete electric flux is defined by a Fourier transform with respect to the magnetic variable. The factor of $N^{-\tilde{b}_2/2}$ is needed for $S^2 = 1$, since the order of the finite group $K = H^2(M, \mathbb{Z}_N)$ is $N^{\tilde{b}_2}$. (In [15], the cohomology was assumed to be torsion-free, and this factor reduced to N^{b_2} .) The modular weights W and $\bar{W}$ are expected to be linear functions of χ and σ (the Euler characteristic and signature of M). They cannot be determined just from gauge theory, as they can be modified by adding gravitational couplings of the general form $f(\tau, \bar{\tau}) R R$ (R being the Riemann tensor of M); to predict the modular weights that are observed in computing the $Z_v(\tau)$ using the AdS/CFT correspondence, one would need to know precisely which such couplings are determined by this correspondence. (In [15], $\bar{W}$ was set to zero, since a twisted version of the theory with a holomorphically varying partition function was considered.)

In addition to [2, 3], the $SL(2, \mathbb{Z})$ transformation law of the $Z_v(\tau)$ is specified by describing the behavior under $\tau \rightarrow \tau + 1$. This is described in eqn. (3.14) of [15] and is determined by the fact that the instanton number of an $SU(N)/\mathbb{Z}_N$ bundle is not an integer but is of the form (if M is a spin manifold)

$$\frac{v^2}{2N} \text{ modulo } \mathbb{Z}, \quad (2.4)$$

as in equation (3.13) of [15]. (A term $v^2/2$, which is integral if M is spin, has been omitted.) The transformation $\tau \rightarrow \tau + 1$ amounts to $\theta \rightarrow \theta + 2\pi$ in gauge theory, so the transformation law is

$$Z_v(\tau + 1) = \exp\left(2\pi i(v^2/2N - s)\right) Z_v(\tau), \quad (2.5)$$

where s is a constant that reflects the fact that the gravitational couplings $f(\tau, \bar{\tau})RR$ may not be invariant under $\tau \rightarrow \tau + 1$.

It follows from [2.2] and [2.3] that, essentially as in eqn. (3.18) of [1.5], the $SU(N)$ and $SU(N)/\mathbb{Z}_N$ partition functions are related by

$$Z_{SU(N)}(-1/\tau) = N^{-\chi/2} \left(\frac{\tau}{i}\right)^{W/2} \left(\frac{\bar{\tau}}{-i}\right)^{\bar{W}/2} Z_{SU(N)/\mathbb{Z}_N}(\tau). \quad (2.6)$$

This is essentially the Montonen-Olive formula, saying that $\tau \rightarrow -1/\tau$ exchanges $SU(N)$ and $SU(N)/\mathbb{Z}_N$; the prefactors reflect gravitational couplings not present in the original Montonen-Olive formulation⁴ on flat $\mathbf{R}^4$.

2.2 The partition function as a vector in Hilbert space

Now we will make a change in viewpoint, which is suggested by experience with rational conformal field theory in two dimensions. A rational conformal field theory on a Riemann surface Σ of positive genus generally has, if one considers the chiral degrees of freedom only, not a single partition function, but a collection of partition functions. It is useful [1.6] to group these together as a vector in a Hilbert space, which one can think about using quantum mechanical intuition [1.7], and which one can ultimately understand using topological field theory in one dimension higher [1.8].

So we introduce a Hilbert space $\mathcal{H}$ with one orthonormal basis vector Ψ_w for every $w \in H^2(M, \mathbb{Z}_N)$. We regard the $Z_w(\tau)$ as components of a vector $\Psi(\tau) \in \mathcal{H}$, with $\Psi(\tau) = \sum_w Z_w(\tau) \Psi_w$. Thus

$$Z_w(\tau) = (\Psi_w, \Psi(\tau)). \quad (2.7)$$

Since the gauge theory “partition function” is thus not an ordinary function but takes values in the Hilbert space $\mathcal{H}$, the AdS/CFT correspondence can only make sense if it is similarly true that the Type IIB partition function on a manifold such as $X \times \mathbf{S}^5$ with conformal boundary M is not an ordinary function but takes values in $\mathcal{H}$.

How can this be? At this point, we must recall the general structure of the AdS/CFT correspondence.

A very general class of observables in quantum field theory is the following. Let the $\mathcal{O}_i$ be a basis for the space of local gauge-invariant operators, and let J_i be c -number sources that couple to them. Thus the generating functional of correlation functions is

$$Z(\tau; J_i) = \left\langle \exp \left(\sum_i \int_M J_i \mathcal{O}_i \right) \right\rangle, \quad (2.8)$$

⁴The derivation of [2.6] in [1.5] assumed no torsion in $H^2(M, \mathbb{Z}_N)$ and gave for the first factor $N^{-1+b_1-b_2/2} = N^{-\chi/2}$. More generally, including the torsion, one gets $N^{-1+\tilde{b}_1-\tilde{b}_2/2}$, but as explained at the end of the introduction, these two factors are equal.

where $\langle \rangle$ denotes the (unnormalized) expectation value. The usual claim in the AdS/CFT correspondence is that to compute via string theory on $X \times \mathbf{S}^5$ this generating functional in the conformal field theory on M , one must fix the values of fields on X – near its boundary – to an asymptotic behavior determined by the J_i . Hence for fixed J_i , one usually claims that the boundary behavior of the fields on $X \times \mathbf{S}^5$ are fixed; the partition function is then a “number,” that is a function of the J_i . To resolve our present conundrum, we must show that if $H^2(M, \mathbb{Z}_N) \neq 0$, then even after specifying the values of all sources J_i for all local operators $\mathcal{O}_i$ on M , the boundary values of the fields on $X \times \mathbf{S}^5$ are not completely determined; and the dependence on the extra data, whatever it is, must be such that the partition function for given J_i is not a number but a vector in $\mathcal{H}$.

The reason that this is so is that fixing the behavior of all of the local gauge-invariant observables near the boundary of $X \times \mathbf{S}^5$ does not completely specify the gauge-invariant data near the boundary. There is global gauge-invariant information that cannot be measured locally, because the Type IIB theory has the two two-form fields B_{RR} and B_{NS} . One can always add to any given B -field a flat B -field, without affecting any local gauge-invariant information. Flat B -fields on a spacetime such as $Y = X \times \mathbf{S}^5$ are classified modulo gauge transformations by $H^2(Y, U(1))$. Near the boundary, this reduces to $H^2(M, U(1))$. For $H^2(M, \mathbb{Z}_N)$ to be nonzero (with M of dimension four), $H^2(M, U(1))$ must likewise be nonzero. Hence, our puzzle arises only when there are flat B -fields near M , in which case the partition function of the string theory on $X \times \mathbf{S}^5$ depends on additional data and not only on the sources J_i of gauge-invariant fields.

Roughly speaking, the additional data can be measured by “ θ angles”

$$\begin{aligned}\alpha_{NS} &= \int_S B_{NS}, \\ \alpha_{RR} &= \int_S B_{RR},\end{aligned}\tag{2.9}$$

with S a homologically nontrivial two-dimensional surface in M . One might think that the partition function of the string theory on $X \times \mathbf{S}^5$ would be a function of these theta angles, as well as the sources J_i . That would be an interesting result, but not quite what we need. We want instead to see the finite group $H^2(M, \mathbb{Z}_N)$. The reason that the string theory partition function should not be regarded simply as a function of α_{NS} and α_{RR} is that these are, in a sense, canonically conjugate variables.

The low energy effective action for B_{RR} , B_{NS} , after reduction on $\mathbf{S}^5$, looks something like

$$L = -\frac{iN}{2\pi} \int_X B_{RR} \wedge dB_{NS} + \frac{1}{2\gamma} \int_X |dB_{RR}|^2 + \frac{1}{2\gamma'} \int_X |dB_{NS}|^2 + \cdots\tag{2.10}$$

Here the first term is the Chern-Simons term, whose significance for understanding the role of the center of the gauge group in the AdS/CFT correspondence was already mentioned in the introduction. The second and third terms (with constants γ, γ' that depend on τ) are the conventional kinetic terms for two-form fields. The “...” are additional gauge-invariant terms of higher dimension.

The first term in 2.10 – the Chern-Simons term – is the important one for our present purposes, for several reasons. The obvious reason is that this is the unique term with only one derivative and hence dominates at long distances (recall that the conformal boundary of X is “infinitely far away,” so the behavior near the boundary is a question of long distance physics). Perhaps even more fundamentally, the second and third terms in 2.10 and all higher terms are integrals of gauge-invariant local densities and hence insensitive to the α ’s, which contribute only to the first term. The Chern-Simons term is gauge-invariant but is not the integral of a gauge-invariant local density; it can and in general does change if one shifts the B -fields by a flat B -field (a fact that was crucial in [7] to enable the expected thermal symmetry of $SU(N)$ gauge theory at nonzero temperature to emerge from the AdS/CFT correspondence).

So to address the question of α -dependence of the partition function, we can use near the boundary of $X \times \mathbf{S}^5$ the simplified Chern-Simons action

$$L_{CS} = NI_{CS} = -\frac{iN}{2\pi} \int_X B_{RR} \wedge dB_{NS}. \quad (2.11)$$

If we write $H = dB$ for the field strength of a B -field, then the equations of motion derived from the Chern-Simons action are $H_{RR} = H_{NS} = 0$, so this action governs only the α -dependence (not the modes of nonzero H , which are massive in spacetime and have been integrated out to reduce to 2.11; their behavior near the boundary of $X \times \mathbf{S}^5$ is determined in the usual way by the sources J_i of gauge-invariant local operators on the boundary).

The Chern-Simons action L_{CS} does not depend on a metric on X , so the theory governing the α -dependence is a topological field theory, of a familiar kind [10]. From the first-order form of L_{CS} , we see that B_{RR} and B_{NS} are canonically conjugate variables in this topological field theory. After imposing the equations of motion and dividing by gauge transformations, this means that α_{RR} and α_{NS} are canonically conjugate. Hence, we should not attempt to compute the partition function as a function of both α_{RR} and α_{NS} .

What we should do instead follows from general concepts of quantum mechanics. Near M , X looks like $M \times \mathbf{R}$, with $\mathbf{R}$ the “time” direction. By quantizing the Chern-Simons theory on $M \times \mathbf{R}$, we obtain a “quantum Hilbert space” $\mathcal{H}'$ associated with M . We should interpret the partition function of the theory on X as determining, not a number, but a vector in $\mathcal{H}'$. Because $\mathcal{H}'$ is obtained by quantizing the metric-independent Chern-Simons Lagrangian, it depends only on the topology of M , and not on a metric.

Moreover, the topological field theory with action L_{CS} has an $SL(2, \mathbb{Z})$ symmetry, acting on the pair

$$\begin{pmatrix} B_{RR} \\ B_{NS} \end{pmatrix} \quad (2.12)$$

in the standard fashion. (A subtlety concerning this assertion will be explained in section 3.4.) The Hilbert space $\mathcal{H}'$ hence has a natural action of $SL(2, \mathbb{Z})$.

This is almost what we need: to agree with the expectations in the boundary gauge theory, the partition function of the string theory on $X \times \mathbf{S}^5$ should be (once the gauge-invariant boundary data are specified) not a number but a vector in a Hilbert space $\mathcal{H}$. $\mathcal{H}$ is determined purely topologically by M (by definition, it has an orthonormal basis consisting of vectors Ψ_w for each $w \in H^2(M, \mathbb{Z}_N)$), and has an action of $SL(2, \mathbb{Z})$ that we described in section 2.1.

So all will be well if $\mathcal{H} = \mathcal{H}'$. Actually, we must describe somewhat more precisely in what sense these two spaces should coincide. The description of the space $\mathcal{H}$ via its basis Ψ_w , $w \in H^2(M, \mathbb{Z}_N)$ is not invariant under S -duality (which as seen in 2.3 does not preserve this basis). To make this description, we need to pick a notion of what we mean by “magnetic flux,” as opposed to “electric flux.” Such a choice breaks $SL(2, \mathbb{Z})$. $SL(2, \mathbb{Z})$ is broken in the same way if we introduce a two-dimensional lattice $\Lambda = \mathbb{Z}^2$ on which $SL(2, \mathbb{Z})$ acts in the standard way, and pick a “polarization” of Λ . One can think of a polarization as a choice of one direction in the lattice Λ . Alternatively, one can think of the pair $(\alpha_{RR}, \alpha_{NS})$ as taking values in $\mathbf{R}^2/\Lambda$; a polarization then is just a choice of what integral linear combination of α_{RR} and α_{NS} is the “momentum” variable. So we must show that for every choice of polarization, $\mathcal{H}'$ acquires a basis in one-to-one correspondence with the elements of $H^2(M, \mathbb{Z}_N)$, and that $SL(2, \mathbb{Z})$ acts on $\mathcal{H}'$ as it does on $\mathcal{H}$.

These assertions will be demonstrated in section 3.

3. Quantization of the topological field theory

3.1 Preliminaries

To quantize the Chern-Simons theory on $M \times \mathbf{R}$, we first work out the gauge-invariant classical phase space. We work in the gauge (analogous to $A_0 = 0$ gauge for gauge theory) in which $i0$ components of B_{RR} and B_{NS} (i and 0 label directions tangent to M and $\mathbf{R}$, respectively) vanish. In quantization, we restrict to time zero; together with the gauge choice, this means that B_{RR} and B_{NS} are B -fields on M . The canonical commutation relations, if written out in detail in components, read

$$[B_{RRij}(x), B_{NSkl}(y)] = -\frac{2\pi i}{N} \epsilon_{ijkl} \delta^4(x, y), \quad (3.1)$$

with $[B_{RR}(x), B_{RR}(y)] = [B_{NS}(x), B_{NS}(y)] = 0$. Here, x and y are points in M , and $i, j, k, l = 1 \dots 4$ are indices tangent to M .

The “Gauss’s law” constraint ($\delta L/\delta B_{i0} = 0$, where B is B_{RR} or B_{NS}) gives $H_{RR} = H_{NS} = 0$. So, modulo gauge transformations, the phase space consists of pairs of flat B -fields.

Flat B -fields are classified by $H^2(M, U(1))$. B -fields in general are classified topologically by the characteristic class $[H] \in H^3(M, \mathbb{Z})$. However, for a flat B -field, this characteristic class vanishes as a differential form, and hence represents a *torsion* element of $H^3(M, \mathbb{Z})$. Let $H^3(M, \mathbb{Z})_{tors}$ be the torsion subgroup of $H^3(M, \mathbb{Z})$. To any given flat B -field we can, without changing the topological type, add a flat and topologically trivial B -field, via a transformation $B \rightarrow B + \beta$, where β is a globally-defined, closed two-form. We should consider β trivial if its periods are multiples of 2π (since the periods of B can be shifted by multiples of 2π by a gauge transformation), so we regard β as an element of $H^2(M, \mathbf{R})/2\pi H^2(M, \mathbb{Z})$. The space $H^2(M, U(1))$ of flat B -fields thus fits into an exact sequence

$$0 \rightarrow H^2(M, \mathbf{R})/2\pi H^2(M, \mathbb{Z}) \rightarrow H^2(M, U(1)) \rightarrow H^3(M, \mathbb{Z})_{tors} \rightarrow 0. \quad (3.2)$$

This says that a flat B -field has a characteristic class in $H^3(M, \mathbb{Z})_{tors}$, and that two flat B -fields with the same characteristic class differ by an element of $H^2(M, \mathbf{R})/2\pi H^2(M, \mathbb{Z})$, which classifies topologically trivial flat B -fields.

Topologically trivial flat B -fields can be measured by their periods, which are the “world-sheet theta angles,”

$$\begin{aligned} \alpha_{NS} &= \int_S B_{NS}, \\ \alpha_{RR} &= \int_S B_{RR}, \end{aligned} \quad (3.3)$$

where S is a closed two-surface in M .

There are thus two parts of the quantization: to quantize the α ’s, and to take account of the finite group $H^3(M, \mathbb{Z})_{tors}$. These turn out to present quite different problems and a direct treatment of the quantization involves many subtle details. On the other hand, everything we need to know can be deduced from the symmetries of the problem. In section 3.2, we will consider this analysis using the symmetries. Then – for the benefit of curious readers – we enter into a direct analysis of the quantization.

3.2 Symmetries

The most obvious symmetry of the problem is that we can add to B_{RR} or B_{NS} any B -field B' such that NB' is pure gauge. For instance, under

$$B_{RR} \rightarrow B_{RR} + B', \quad (3.4)$$

the Chern-Simons action $L_{CS}(B_{RR}, B_{NS}) = NI_{CS}(B_{RR}, B_{NS})$ changes by $L_{CS} \rightarrow L_{CS} + NI_{CS}(B', B_{NS}) = L_{CS} + I_{CS}(NB', B_{NS})$. But $I_{CS}(NB', B_{NS}) = 0$ as NB' is pure gauge.

For NB' to be pure gauge means necessarily that B' is flat. Thus, B' determines an element of $H^2(M, U(1))$ that is “of order N ,” or in other words is annihilated by multiplication by N .

We can be more explicit about the quantum field operators that generate these symmetries. (They are analogs of operators considered many years ago in two-dimensional rational conformal field theory [17].) We construct them using the two-form counterparts of the familiar Wilson and 't Hooft loop operators of gauge theories.

First we consider the counterparts of Wilson loops. Let S and T be closed two-surfaces in M , and let

$$\begin{aligned}\Phi_{RR}(S) &= \exp\left(i \int_S B_{RR}\right), \\ \Phi_{NS}(T) &= \exp\left(i \int_T B_{NS}\right).\end{aligned}\tag{3.5}$$

Since these are gauge-invariant operators, they act on the classical phase space and map flat B -fields to flat B -fields. From the canonical commutation relations, we can deduce that

$$\Phi_{RR}(S)\Phi_{NS}(T) = \Phi_{NS}(T)\Phi_{RR}(S) \exp\left(\left(\frac{2\pi i}{N}\right) S \cdot T\right),\tag{3.6}$$

where $S \cdot T$ is the intersection number of the oriented surfaces S and T .

In particular, while $\Phi_{RR}(S)$ commutes with B_{RR} , it shifts B_{NS} by a flat B -field that is essentially determined by §.6 to be “Poincaré dual” to S . In other words, we interpret the relation $\Phi_{RR}(S)\Phi_{NS}(T)\Phi_{RR}(S)^{-1} = \Phi_{NS}(T) \exp(2\pi i S \cdot T/N)$ to mean that conjugation by $\Phi_{RR}(S)$ shifts B_{NS} by a flat B -field with delta-function support on S (this assertion is in any case clear from the canonical commutation relations), thus multiplying $\Phi_{NS}(T)$ by a phase. The $1/N$ in the exponent in §.6 means that Φ_{NS} is shifted by a flat B -field of order N . Conversely, conjugation by $\Phi_{NS}(T)$ shifts B_{RR} by such a field. Thus, these operators generate the symmetries that we exhibited directly in §.4. The c -number factor in the commutation relation §.6 shows that these symmetries do not commute with one another; there is a central extension by the N^{th} roots of unity.

The operator $\Phi_{RR}(S)$, in acting on gauge-invariant states, is trivial if S consists of N copies of another closed surface S' . That is because in such a case, $\Phi_{RR}(S) = \Phi_{RR}(S')^N$; but $\Phi_{RR}(S')$ shifts B_{NS} by a B -field of order N , and hence $\Phi_{RR}(S')^N$ shifts it by a pure gauge. Likewise, $\Phi_{NS}(T)$ is trivial if $T = NT'$ for some T' .

$\Phi_{RR}(S)$ (or $\Phi_{NS}(T)$) is also trivial if S (or T) is the boundary of a three-manifold $U \subset M$. For then

$$\Phi_{RR}(S) = \exp\left(i \int_U H_{RR}\right),\tag{3.7}$$

and this is trivial as an operator on gauge-invariant states because of the Gauss law constraint $H = 0$.

So the operators $\Phi_{RR}(S)$ depend on S modulo boundaries, and are trivial if S is of the form NS' . So far we have assumed that S has no boundary, but we will next see that we can define an operator $\Phi_{RR}(S)$ with similar properties if S has a boundary, provided this boundary is of the form NC for some circle $C \subset M$. This will make possible the following simple description of the symmetry group. The group $H_2(M, \mathbb{Z}_N)$ classifies two-surfaces $S \subset M$ with boundaries of the form NC , modulo those of the form NS' and modulo boundaries. So once we show that S can have a boundary of the claimed kind, we will have shown that the operators $\Phi_{RR}(S)$ (and similarly the operators $\Phi_{NS}(T)$) are classified by $H_2(M, \mathbb{Z}_N)$. Equivalently, by Poincaré duality, they are classified by $H^2(M, \mathbb{Z}_N)$.

't Hooft loops. To see how S can have a boundary, we begin with a rather different-sounding question. Are there also in this theory symmetry operators that are analogous to the 't Hooft loops of gauge theory? This would mean the following. We consider a circle $C \subset M$. Deleting C from M , we make a “singular gauge transformation” on B_{NS} (or B_{RR}) such that on a small three-sphere that links once around C , the integral of H_{NS} equals 2π . This means that we put on C a “magnetic source” for B_{NS} .

To find such operators, we can go back to Type IIB superstring theory on $X \times \mathbf{S}^5$. In this theory, the NS fivebrane is a magnetic source for B_{NS} , so the object sought in the last paragraph would be a fivebrane wrapped on $C \times \mathbf{S}^5$. However, as explained in [4], such a fivebrane must be the boundary of N D -strings. In other words, in addition to the magnetic loop, a D -string must be present with a world-volume S whose boundary consists of N copies of C .

The operator $\Phi_{RR}(S)$ describes the coupling of B_{RR} to the D -brane worldvolume S . So we have learned that S can have a boundary of the form NC , provided that there is at the same time a magnetic source for B_{NS} on S .

The recourse to string theory to show the existence of mixed Wilson/'t Hooft operators of this kind is a convenient shortcut, but of course it would be desirable to demonstrate their existence directly in the five-dimensional topological field theory. In fact, this has essentially already been done in section 5 of [5], in the course of explaining a low energy approach to the existence of the baryon vertex in Type IIB on $AdS_5 \times \mathbf{S}^5$.

The Heisenberg group and the Hilbert space. The operators $\Phi_{RR}(S)$ (and likewise $\Phi_{NS}(T)$) are therefore classified by $H_2(M, \mathbb{Z}_N)$ or equivalently by $K = H^2(M, \mathbb{Z}_N)$. We write these operators now as $\Phi_{RR}(w)$, $\Phi_{NS}(w)$, with $w \in K$. The commutation relation of these operators is still given by 3.6. We can assume that all intersections of S and T occur away from the boundaries, so there is no real modification in the derivation of 3.6 from the canonical commutation relations. Since S and T may have boundaries of the form NC , the intersection number $S \cdot T$ is only

well-defined modulo N , but that is good enough to make sense of the phase factor in the commutation relation.

Because of the central factor in the commutation relation [3.6](#), the group generated by these operators is not just $K \times K = H^2(M, \mathbb{Z}_N) \times H^2(M, \mathbb{Z}_N)$, but is a central extension of $K \times K$ by $\mathbb{Z}_N$, the group of N^{th} roots of unity. We call this central extension W :

$$0 \rightarrow \mathbb{Z}_N \rightarrow W \rightarrow K \times K \rightarrow 0. \quad (3.8)$$

This central extension is nondegenerate (a statement which, informally, means that the first or second factors of K in the product $K \times K$ give maximal commutative subgroups of W) and is a group of a type known as a finite Heisenberg group.

A “polarization” of such a finite Heisenberg group is a choice of a maximal commutative subgroup of $K \times K$, or more exactly a maximal subgroup that remains commutative when lifted to W . For example, the first or second factor of $K \times K$, or any “diagonal” subgroup obtained from then by an $SL(2, \mathbb{Z})$ transformation, is a polarization. There are also, in general, other polarizations. Picking a polarization of a finite Heisenberg group is a discrete version of picking a representation of the canonical commutation relations $[p, x] = -i\hbar$ for bosons.

One of the main theorems about such finite Heisenberg groups is that, up to isomorphism, such a group has a unique irreducible representation R . This is a discrete version of the fact that the canonical commutation relations for bosons have a unique irreducible representation, up to isomorphism. The representation can be constructed as follows. Since the $\Phi_{RR}(w)$ commute, one can pick a basis of R consisting of their joint eigenstates. Using the nondegeneracy, one can show that there is a vector $|\Omega\rangle \in R$ that is invariant under all of the $\Phi_{RR}(w)$. From irreducibility of R , one can show that $|\Omega\rangle$ is unique (up to multiplication by a scalar) and that R has a basis consisting of the states

$$\Psi_w = \Phi_{NS}(w)|\Omega\rangle, \quad \text{for } w \in H^2(M, \mathbb{Z}_N). \quad (3.9)$$

The action of the group in this basis follows immediately from the commutation relations.

If the quantum Hilbert space $\mathcal{H}'$ of the Chern-Simons theory is an *irreducible* representation of the finite symmetry group W , then it is easy to obtain all the properties promised in section [2.2](#). For example, we must show that for each polarization of the lattice $\Lambda = \mathbb{Z}^2$ on which $SL(2, \mathbb{Z})$ acts, $\mathcal{H}'$ has a basis Ψ_w , $w \in H^2(M, \mathbb{Z}_N)$. In this context, a polarization is a choice of what we mean by Φ_{RR} and what we mean by Φ_{NS} . The desired basis is constructed in [3.9](#) for one choice of polarization, and other polarizations are obtained by $SL(2, \mathbb{Z})$ transformations. There is indeed a natural action of $SL(2, \mathbb{Z})$ on $\mathcal{H}'$, since the commutation relation [3.6](#) by which W is defined is $SL(2, \mathbb{Z})$ -invariant. It is not hard to see that the $SL(2, \mathbb{Z})$ action agrees with what was described in section [2.1](#). The main point is that the $SL(2, \mathbb{Z})$

transformation

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad (3.10)$$

which acts by $\tau \rightarrow -1/\tau$, corresponds to a change in polarization that maps Φ_{RR} to Φ_{NS}^{-1} and Φ_{NS} to Φ_{RR} . It maps the state $|\Omega\rangle$ invariant under the Φ_{RR} 's to the state

$$|\tilde{\Omega}\rangle = \frac{1}{N^{\tilde{b}_2/2}} \sum_{w \in K} \Phi_{NS}(w) |\Omega\rangle, \quad (3.11)$$

which is invariant under the Φ_{NS} 's. So it maps $\Psi_v = \Phi_{NS}(v) |\Omega\rangle$ to

$$\begin{aligned} \Phi_{RR}(v) |\tilde{\Omega}\rangle &= \frac{1}{N^{\tilde{b}_2/2}} \Phi_{RR}(v) \sum_{w \in K} \Phi_{NS}(w) |\Omega\rangle \\ &= \frac{1}{N^{\tilde{b}_2/2}} \sum_{w \in K} \exp(2\pi i v \cdot w / N) \Phi_{NS}(w) |\Omega\rangle. \end{aligned} \quad (3.12)$$

This is the discrete Fourier transform familiar from section 2.1. The transformation under $\tau \rightarrow \tau + 1$ can be understood similarly.

Except for the fact that we have not shown that the finite group W acts irreducibly in the quantum theory, this really establishes what we wanted. However, we will go on in the rest of this section, for the benefit of curious or intrepid readers, to show what is involved in quantizing the theory directly. In the process, it will become clear that W acts irreducibly on $\mathcal{H}'$.

In the analysis, it will be useful to know the following more general description of the representation R . First of all, if F is any polarization of $K \times K$, then R has a basis that is in one-to-one correspondence with the cosets of $(K \times K)/F$. This is proved by first showing that R has an F -invariant vector $|\Omega\rangle$, and then setting $\Upsilon_\lambda = \Phi_\lambda |\Omega\rangle$, where λ runs over a set of representatives of the cosets of K/F , and for each λ , Φ_λ is the corresponding operator on R . The Υ_λ are the desired basis vectors.

3.3 The torsion-free case; elementary account

Direct quantization of this system is surprisingly subtle, given that it is an abelian free field theory. The reader may in fact wish to jump directly to section 4.

We begin by considering the case that there is no torsion in $H^2(M, \mathbb{Z})$. If $H^2(M, \mathbb{Z})$ is torsion-free, then this group is a lattice Γ . The intersection pairing on $H^2(M, \mathbb{Z})$ gives an integer-valued inner product on Γ ; we write the product of $x, y \in \Gamma$ as $x \cdot y$. This inner product is unimodular by Poincaré duality and is even because (as we noted at the outset) M is spin.

The exact sequence of abelian groups $0 \rightarrow \mathbb{Z} \xrightarrow{N} \mathbb{Z} \rightarrow \mathbb{Z}_N \rightarrow 0$ leads to a long exact sequence of cohomology groups which reads in part

$$\dots H^2(M, \mathbb{Z}) \xrightarrow{N} H^2(M, \mathbb{Z}) \rightarrow H^2(M, \mathbb{Z}_N) \rightarrow H^3(M, \mathbb{Z}) \xrightarrow{N} H^3(M, \mathbb{Z}) \dots \quad (3.13)$$

If the cohomology of M is torsion-free, then the map $H^3(M, \mathbb{Z}) \xrightarrow{N} H^3(M, \mathbb{Z})$ is injective. Hence exactness of [3.13](#) says in the torsion-free case that

$$H^2(M, \mathbb{Z}_N) = H^2(M, \mathbb{Z})/NH^2(M, \mathbb{Z}) = \Gamma/N\Gamma. \quad (3.14)$$

In other words, in the absence of torsion, a $\mathbb{Z}_N$ class is the mod N reduction of an integral class.

In the torsion-free case, the world-sheet theta angles α_{NS} and α_{RR} are a complete set of gauge-invariant functions on the classical phase space. They are canonically conjugate variables, a fact that we have already used in the analysis of the symmetries in section [3.2](#). A precise form of this statement is that if we pick for the lattice Γ a basis in which the inner product is g_{ij} , $1 \leq i, j \leq b_2$, and write α_{NS}^i , α_{RR}^i for the corresponding components of α_{NS} and α_{RR} , then the Poisson brackets are

$$\{\alpha_{RR}^i, \alpha_{NS}^j\} = \frac{2\pi}{N} g^{ij}. \quad (3.15)$$

According to the description at the end of section [2.2](#), we are to pick a polarization, quantize, and determine the resulting Hilbert space $\mathcal{H}'$. We pick a polarization⁵ by regarding α_{NS} as the “position” variable and

$$\beta_{RR} = \frac{N}{2\pi} \alpha_{RR} \quad (3.16)$$

as the conjugate momentum. Note that, since the periods of α_{RR} are defined mod 2π , then those of β_{RR} are defined mod N , so, classically, β_{RR} is an element of $H^2(M, \mathbb{R})/NH^2(M, \mathbb{Z}) = V/N\Gamma$, where $V = H^2(M, \mathbb{R})$ and $\Gamma = H^2(M, \mathbb{Z})$ is regarded as a lattice in V .

Quantum mechanically there is a very severe additional constraint on β_{RR} . Since the “position” variable α_{NS} is a periodic variable that takes values in $H^2(M, \mathbb{Z})/2\pi\Gamma$, the conjugate momentum β_{RR} takes values in the lattice Γ^* dual to Γ . But Poincaré duality tells us that $\Gamma^* = \Gamma$, so β_{RR} takes values in Γ . Since β_{RR} is in addition defined modulo $N\Gamma$, it takes values in $\Gamma/N\Gamma$. According to [3.14](#), this is the same as $H^2(M, \mathbb{Z}_N)$.

So we get the desired result. There is one quantum state – one momentum state Ψ_w – for every momentum vector $w \in \Gamma/N\Gamma = H^2(M, \mathbb{Z}_N)$.

Moreover, the $SL(2, \mathbb{Z})$ transformation $\tau \rightarrow -1/\tau$ exchanges the position α_{NS} with the momentum α_{RR} . Hence it acts as a Fourier transform, in agreement with the gauge theory result [2.3](#).

3.4 More rigorous approach

The discussion in section [3.3](#) was actually somewhat informal, and for the interested reader we will now give some hints for a more precise treatment.

⁵For some background on quantization of Chern-Simons theories, see [\[19, 20\]](#).

Let T be the torus $T = V/\Gamma$ with $V = H^2(M, \mathbf{R})$ and $\Gamma = H^2(M, \mathbb{Z})$. The phase space of the system is $T' = T \times T$. The first step in quantizing the theory is to find the appropriate line bundle $\mathcal{M}$ over the phase space. $\mathcal{M}$ should be endowed with a connection whose curvature equals the symplectic two-form of the theory. Once the right line bundle is found, quantization is carried out by picking a complex structure and taking holomorphic sections of $\mathcal{M}$, or by using a real polarization and making a more precise version of the informal discussion of section 3.3, or by finding and using a finite Heisenberg group as in section 3.2. But any approach to quantization involves finding the line bundle at the outset.

Since N appears in the symplectic form as a multiplicative factor, $\mathcal{M}$ will be of the form $\mathcal{L}^N$, where $\mathcal{L}$ is the line bundle that we would get at $N = 1$. (The relation $\mathcal{M} = \mathcal{L}^N$ will be obvious from the Chern-Simons construction of the line bundle that we give presently.)

The construction of the line bundle is most naturally made directly from the Chern-Simons action [21, 22]. Let p be a point in the classical phase space T' given by a pair $(B_{RR}, B_{NS}) = (a, b)$. Let 0 be the origin in T' (the point with $B_{RR} = B_{NS} = 0$). Then we describe the fiber $\mathcal{L}_p$ of $\mathcal{L}$ at p as follows. For any path γ in T' from 0 to p , $\mathcal{L}_p$ has a basis vector ψ_γ , of norm 1. If γ and γ' are two such paths, we declare that

$$\psi_{\gamma'} = e^{iL_{CS}} \psi_\gamma, \quad (3.17)$$

where the Chern-Simons Lagrangian L_{CS} is understood in the following sense. The path γ from 0 to p determines a “time”-dependent pair of B -fields $(B_{RR}(t), B_{NS}(t))$, where $(B_{RR}(0), B_{NS}(0)) = (0, 0)$ and $(B_{RR}(1), B_{NS}(1)) = (a, b)$. We can fit these together to make a pair (B_{RR}, B_{NS}) over $M \times I$, where I is a unit interval. Likewise, γ' determines a pair of B -fields over $M \times I$. By gluing γ' to γ , with opposite orientation for γ' , we get a B -field pair over the closed five-manifold $M \times \mathbf{S}^1$. L_{CS} in 3.17 is the Chern-Simons action evaluated for this field.

It can be shown that the line bundle $\mathcal{L}$, whose fiber at each $p \in T'$ was just described, is endowed with a natural connection, whose curvature is the symplectic form. The monodromy M_C of this line bundle around any circle $C \subset T'$ is as follows. The circle $C \subset T'$ determines a B -field pair over $M \times C$, and the monodromy is

$$Q(C) = \exp(-L_{CS}), \quad (3.18)$$

with L_{CS} the Chern-Simons action of this pair.

Having found the line bundle, we are ready to quantize. Quantum states are suitable sections of $\mathcal{M} = \mathcal{L}^N$, with the details depending on a choice of polarization.

In the present case, we can make the description of $\mathcal{L}$ much more explicit. In general, a line bundle with connection over any manifold Z is determined up to isomorphism by giving its monodromies around arbitrary loops in Z . Once the curvature is specified, it suffices to consider a set of loops generating the fundamental

group of Z . In the present instance, Z is the torus $T' = T \times T = (V \times V)/(\Gamma \times \Gamma)$. The fundamental group of T' is generated by straight lines from the origin in $V \times V$ to lattice points in $\Gamma \times \Gamma$. Let (x, y) be such a lattice point. A straight line from the origin to this lattice point is given by the family of B -field pairs

$$\begin{aligned} B_{RR} &= 2\pi tx, \\ B_{NS} &= 2\pi ty, \end{aligned} \tag{3.19}$$

with $0 \leq t \leq 1$. At $t = 1$, $(B_{RR}, B_{NS}) = (2\pi x, 2\pi y)$ are gauge-equivalent to 0. So the family $(B_{RR}(t), B_{NS}(t))$ gives a closed loop in the phase space. The monodromy around this loop can be computed by direct evaluation of $L_{CS} = -(i/2\pi) \int_{M \times \mathbf{S}^1} B_{RR} \wedge dB_{NS}$,⁶ and is

$$Q(x, y) = (-1)^{x \cdot y}. \tag{3.20}$$

The relation

$$Q(x + x', y + y') = Q(x, y)Q(x', y')(-1)^{x \cdot y' + y \cdot x'} \tag{3.21}$$

follows from this, and signals, as explained in connection with eqn. (2.17) of [8], that the line bundle $\mathcal{L}$ has the desired curvature and first Chern class. The formula for $Q(x, y)$ is clearly invariant under all of the lattice symmetries of Γ , and thus under all diffeomorphisms of M . Let us check that it is also invariant under $SL(2, \mathbb{Z})$, acting in the standard fashion

$$\begin{aligned} x &\rightarrow ax + by, \\ y &\rightarrow cx + dy, \end{aligned} \tag{3.22}$$

with a, b, c, d integers such that $ad - bc = 1$. A small computation shows that $Q(x, y)$ is invariant under this transformation for all such a, b, c , and d if and only if x^2 and y^2 are both even. But the manifold M is spin, and the intersection form on $H^2(M, \mathbb{Z})$, for M a four-dimensional spin manifold, is always even. So x^2 and y^2 are even, and Q has the expected symmetries. Since Q determines the line bundle $\mathcal{L}$, $\mathcal{L}$ has these symmetries also.

At this point, the reader may wonder precisely why the spin condition on M was needed. The line bundle $\mathcal{L}$ was determined directly from the Chern-Simons action, which appears to be completely $SL(2, \mathbb{Z})$ -invariant without assuming that M is spin; so why did the spin structure enter? The answer will make it clear that up to the present point, we have indulged in a small sleight of hand. There is actually a subtlety in defining the Chern-Simons action for a pair of B -fields (B_{RR}, B_{NS}) on a

⁶To put this “direct evaluation” on a rigorous basis is slightly subtle as the B -fields on $M \times \mathbf{S}^1$ are topologically nontrivial. The basic idea of a rigorous evaluation in this situation is contained in Example 1.16 in [9].

five-manifold X , even if X has no boundary. If X is the boundary of a six-manifold Y , and the B -fields extend over Y , then the action is readily defined as

$$I_{CS} = -\frac{i}{2\pi} \int_Y H_{RR} \wedge H_{NS}. \quad (3.23)$$

This formula is completely $SL(2, \mathbb{Z})$ -invariant, proving that $SL(2, \mathbb{Z})$ is a symmetry if Y exists. If no such Y exists, then as mentioned in the introduction, one must define the action by $\exp(-I_{CS}) = B_{RR} * B_{NS}$, with $*$ the multiplication in Cheeger-Simons cohomology. From Theorem 1.11 of [9], one can deduce that $I_{CS}(B_{RR}, B_{NS}) = I_{CS}(B_{NS}, -B_{RR})$ and that $I_{CS}(B_{RR}, B_{NS}) = I_{CS}(B_{RR} + 2B_{NS}, B_{NS})$. Moreover $SL(2, \mathbb{Z})$ holds if and only if one has the more precise property $I_{CS}(B_{RR}, B_{NS}) = I_{CS}(B_{RR} + B_{NS}, B_{NS})$, but this does not follow from the theorem and is evidently not true in general if X is not spin. The $SL(2, \mathbb{Z})$ invariance of the explicit formula 3.20 that determines the line bundle shows that, at least in canonical quantization, there is no such difficulty in the spin case.

Quantization. Having thus defined the line bundle, we now wish to carry out quantization. This may be done by picking a polarization and quantizing, but we prefer to make contact with the discussion of section 3.2 by exhibiting the discrete Heisenberg group.

Since the phase space T' is a torus, some obvious symmetries of the phase space (as a symplectic manifold) are translations of the torus, say $(B_{RR}, B_{NS}) \rightarrow (B_{RR}, B_{NS}) + 2\pi(a, b)$, with $a, b \in V/\Gamma$. However, such translations generally do not leave fixed the line bundle $\mathcal{L}$. Under the indicated translation, the monodromy $Q(x, y)$ computed above transforms by

$$Q(x, y) \rightarrow \exp(2\pi i(a \cdot y - b \cdot x))Q(x, y). \quad (3.24)$$

Thus, the monodromies are invariant, for all x, y , only if $a, b \in \Gamma$, in which case the translation by $2\pi(a, b)$ acts trivially on the torus T' .

So if the quantum line bundle is $\mathcal{L}$, there are no nontrivial translation symmetries. We are more generally interested in the “level N ” theory in which the quantum line bundle is $\mathcal{M} = \mathcal{L}^N$. In this case, the monodromies are Q^N , and it is sufficient for Q^N to be invariant. For this, it is enough that Na and Nb should be lattice points. So the translation by $2\pi(a, b)$ is a symmetry of the quantum theory whenever a and b are both of order N . The points of order N are classified by $\frac{1}{N}\Gamma/\Gamma = \Gamma/N\Gamma = H^2(M, \mathbb{Z}_N)$.

Let us try to define as precisely as possible the symmetry $T_{a,b}$ of translation by $2\pi(a, b)$. Such a symmetry exists because the pullback of $\mathcal{M}$ by the translation in question is isomorphic to $\mathcal{M}$. The operator $T_{a,b}$ is uniquely determined up to multiplication by a complex scalar of modulus 1. That scalar can be restricted by requiring (since the translation by $2\pi(Na, Nb)$ is trivial) that $T_{a,b}^N = 1$. This leaves

an ambiguity consisting of multiplication by N^{th} roots of unity. There is no natural way to fix that remaining ambiguity, so we make any arbitrary choice.

One might think that the translation operators $T_{a,b}$ would commute. That is not so, because in the presence of a magnetic field (the symplectic form on T' serves as the “magnetic field”) translations do not commute. Including in the standard way the effects of the magnetic field, the actual relation is

$$T_{a,b}T_{a',b'} = T_{a',b'}T_{a,b} \exp(2\pi i(a \cdot b' - b \cdot a')/N). \quad (3.25)$$

This is the Heisenberg group found in section 3.2. The quantum theory can now be analyzed as in section 3.2, with the important difference that we can now show that W acts irreducibly. For instance, this follows by using a complex polarization and Riemann-Roch theorem to determine the dimension of the quantum Hilbert space, which coincides with that of the irreducible representation R of W . For that matter, the informal treatment in section 3.3 was precise enough, at least for large enough N , to count the quantum states within a positive integer factor, and thus to deduce that W acts irreducibly.

3.5 Restriction on $[H]$

Now we want to begin to consider what happens if there is torsion in the cohomology of M .

We first note that B -fields on M are characterized topologically by a characteristic class $[H] \in H^3(M, \mathbb{Z})$. The Gauss law constraint implies that $H = dB$ is zero as a differential form, so $[H]$ must be torsion. At first sight, it appears that there is no other restriction, and that in quantizing the Chern-Simons theory, $[H_{RR}]$ and $[H_{NS}]$ can be arbitrary torsion elements of $H^3(M, \mathbb{Z})$.

Actually, there is a very important further restriction. This is that if the Chern-Simons theory is taken at level N , then $[H_{RR}]$ and $[H_{NS}]$ must be N -torsion, that is $N[H_{RR}] = N[H_{NS}] = 0$.

To obtain this result, we will consider the partition function of the theory on the five-manifold $X = M \times \mathbf{S}^1$. In topological field theory, this partition function equals the dimension of the space of physical states on M , and so in particular every physical state contributes.

We consider classical configurations on $M \times \mathbf{S}^1$ in which, when restricted to M (that is to $M \times P$ for any $P \in \mathbf{S}^1$), $[H_{NS}]$ has some given value. We want to show that the contribution to the path integral of such configurations is zero unless $N[H_{NS}] = 0$. The same argument, with B_{RR} and B_{NS} exchanged, shows that the contribution is zero unless also $N[H_{RR}] = 0$ when restricted to M .

The vanishing will come upon adding to B_{RR} a flat B -field B' that is topologically trivial if restricted to M , but nontrivial on $M \times \mathbf{S}^1$. (We restrict B' to be trivial on M because we consider a path integral in which $[H_{RR}]$ and $[H_{NS}]$ are both specified

on M ; we want to show that summing over what happens in the “time” direction gives the desired restriction on the initial data.) Under $B_{RR} \rightarrow B_{RR} + B'$, the path integrand $e^{-L_{CS}}$ of Chern-Simons theory transforms by

$$\exp(-L_{CS}) \rightarrow \exp(-L_{CS}) \exp\left(\frac{iN}{2\pi} \int_{M \times \mathbf{S}^1} B' \wedge dB_{NS}\right), \quad (3.26)$$

where the exponential factor needs some clarification (which will be given shortly) because the B -fields in question are topologically nontrivial, but is hopefully clear. If the exponential factor in [3.26](#) does not equal 1, then the path integral will vanish after summing over B' .

We must therefore understand what is meant by the factor that we write symbolically as

$$\exp\left(\frac{i}{2\pi} \int_X B' \wedge dB_{NS}\right), \quad (3.27)$$

if B' and B are flat but topologically nontrivial. This is determined in eqn. 1.14 of [\[9\]](#); the intuitively natural result can be explained as follows. We can classify flat B -fields in five dimensions either by $H^2(X, U(1))$ or by the characteristic class $[H] \in H^3(X, \mathbb{Z})$. (The first classification describes the B -field up to gauge transformation, and the second only describes its topological type.) There is a natural, nondegenerate pairing

$$H^2(X, U(1)) \times H^3(X, \mathbb{Z}) \rightarrow H^5(X, U(1)) = U(1) \quad (3.28)$$

given by Poincaré and Pontryagin duality. For $B' \in H^2(X, U(1))$ and B_{NS} regarded as an element of $H^3(X, \mathbb{Z})$, we write this pairing as $E(B', B_{NS})$. Cheeger and Simons show that in this situation $B' * B_{NS} = E(B', B_{NS})$, so we must interpret the phase factor in [3.27](#) as $E(B', B_{NS})$.

Because of the factor of N in the exponent [3.26](#), the factor that must be 1 for B_{NS} to contribute to the path integral is actually $E(B', B_{NS})^N = E(B', NB_{NS})$. Nondegeneracy of the pairing [3.28](#) means that for this to equal 1 for all B' , we need $NB_{NS} = 0$, up to gauge transformation. In other words, we need $N[H_{NS}] = 0$. In our case with $X = M \times \mathbf{S}^1$, since we assume $B' = 0$ when restricted to M , the constraint is that $N[H_{NS}] = 0$ when restricted to M . This is the restriction that we aimed to prove.

The restriction that we have found should be viewed as a quantum extension of Gauss’s law, to incorporate torsion. Gauss’s law enters in path integrals as a constraint that states must obey, in order to propagate in time. Such propagation is precisely what we have been analyzing. The classical Gauss law says that $H_{RR} = H_{NS} = 0$ as a differential form; the quantum Gauss law says that in addition $N[H_{RR}] = N[H_{NS}] = 0$.

3.6 Quantization in the presence of torsion

We will now analyze the quantum Hilbert space $\mathcal{H}'$ of the Chern-Simons theory, allowing for the possible existence of torsion in $H^2(M, \mathbb{Z}_N)$. In fact, to clarify things,

we will consider first the case that the cohomology of M contains only torsion – the opposite case from what we considered in sections 3.3 and 3.4.

After imposing Gauss's law, the physical data consists of a pair of flat B -fields B_{RR} , B_{NS} . From what we have seen in section 3.5, we should impose a discrete form of Gauss's law stating that $N[H_{RR}] = N[H_{NS}] = 0$. Let L_N be the subgroup of $H^3(M, \mathbb{Z})$ consisting of points of order N . The two B -fields determine a pair of points in L_N . If the cohomology of M is pure torsion, there are no additional data to quantize. Each pair $(x, y) \in L_N \times L_N$ contributes one quantum state $\Psi_{x,y}$, and the $\Psi_{x,y}$ give a basis of the physical Hilbert space $\mathcal{H}'$.

We want to compare this to the description from section 2: for each choice of polarization, $\mathcal{H}'$ should have a basis Ψ_w , for $w \in K = H^2(M, \mathbb{Z}_N)$. In an equivalent version presented in section 3.2, $\mathcal{H}'$ should furnish an irreducible description of the finite Heisenberg group W :

$$0 \rightarrow \mathbb{Z}_N \rightarrow W \rightarrow K \times K \rightarrow 0. \quad (3.29)$$

For this, we must understand the structure of $H^2(M, \mathbb{Z}_N)$ when there is torsion. From the exact sequence of abelian groups $0 \rightarrow \mathbb{Z} \xrightarrow{N} \mathbb{Z} \rightarrow \mathbb{Z}_N \rightarrow 0$, with the first map being multiplication by N and the second reduction modulo N , we get a long exact sequence

$$\dots \rightarrow H^2(M, \mathbb{Z}) \xrightarrow{N} H^2(M, \mathbb{Z}) \rightarrow H^2(M, \mathbb{Z}_N) \rightarrow H^3(M, \mathbb{Z}) \xrightarrow{N} H^3(M, \mathbb{Z}) \rightarrow \dots \quad (3.30)$$

This gives a short exact sequence

$$0 \rightarrow E_N \rightarrow H^2(M, \mathbb{Z}_N) \rightarrow L_N \rightarrow 0, \quad (3.31)$$

with $E_N = H^2(M, \mathbb{Z})/NH^2(M, \mathbb{Z})$ the subgroup of $H^2(M, \mathbb{Z}_N)$ consisting of classes that are the reduction of an integer class.

We recall that there is a nondegenerate pairing $H^2(M, \mathbb{Z}_N) \times H^2(M, \mathbb{Z}_N) \rightarrow \mathbb{Z}_N$ given by Poincaré duality; it was used in section 3.2 to construct the finite Heisenberg group. If the cohomology of M is purely torsion, then E_N consists entirely of classes that are reductions of torsion classes. In this case, the pairing on $H^2(M, \mathbb{Z}_N) \times H^2(M, \mathbb{Z}_N)$ is zero when restricted to $E_N \times E_N$ (since in integer cohomology the cup product vanishes for torsion classes), and the self-duality of $H^2(M, \mathbb{Z}_N)$ reduces to a duality between E_N and L_N .

The Heisenberg group W in this situation has a maximal commutative subgroup that comes from the subgroup $E_N \times E_N$ of $H^2(M, \mathbb{Z}_N) \times H^2(M, \mathbb{Z}_N)$. According to the remark at the end of section 3.2, the representation R has therefore a basis in 1-1 correspondence with the cosets in $(K \times K)/(E_N \times E_N)$. That quotient is isomorphic to $L_N \times L_N$, so we get one basis vector $\Psi_{x,y}$ for each pair $(x, y) \in L_N \times L_N$. This is the description we found for the physical Hilbert space $\mathcal{H}'$, so we confirm the expected isomorphism of R with $\mathcal{H}'$.

In particular, we see again that $\mathcal{H}'$ is an irreducible representation of the discrete Heisenberg group.

The general case. The general case in which the cohomology of M contains torsion but is not purely torsion is a mixture of the cases that we have already considered.

In this case, the cohomology classes of $x = [H_{RR}]$, $y = [H_{NS}]$ determine elements $x, y \in L_N$ which are part of the data. In addition, one must quantize the α 's. We can pick a polarization in which the quantum states are regarded as functions of α_{NS} as well as x, y .

As we found in section 3.3, quantization of the α 's gives one state for every topologically trivial flat B_{NS} -field of order N . Including also the dependence on y means that we drop the requirement that B_{NS} should be topologically trivial; we get one state for every flat B_{NS} field of order N whether it is topologically trivial or not. We let M_N be the group of flat B -fields of order N . Including also the choice of x (which is a point in L_N), we see that the quantum Hilbert space $\mathcal{H}'$ has a basis consisting of one vector for every element of $L_N \times M_N$. This strongly suggests that there might be a polarization of the discrete Heisenberg group determined by a subgroup F of $K \times K$ such that $(K \times K)/F = L_N \times M_N$. In that case, the irreducible representation R of the discrete Heisenberg group can be identified with $\mathcal{H}'$ as desired.

In fact, we can take $F = E_N \times T_N$, where T_N is the subgroup of $H^2(M, \mathbb{Z}_N)$ consisting of elements that are the reduction mod N of torsion classes in $H^2(M, \mathbb{Z})$. F is a commutative subgroup (even when lifted to W) because the pairing of two elements in $H^2(M, \mathbb{Z})$, one of which is torsion, is zero. Since $(K \times K)/F = (K \times K)/(E_N \times T_N) = (K/E_N) \times (K/T_N)$, the claim that $(K \times K)/F = L_N \times M_N$ is equivalent to the existence of two exact sequences:

$$\begin{aligned} 0 \rightarrow E_N \rightarrow H^2(M, \mathbb{Z}_N) \rightarrow L_N \rightarrow 0, \\ 0 \rightarrow T_N \rightarrow H^2(M, \mathbb{Z}_N) \rightarrow M_N \rightarrow 0. \end{aligned} \quad (3.32)$$

The first is in 3.31, and the second can be derived from the sequence

$$0 \rightarrow \mathbb{Z}_N \rightarrow U(1) \xrightarrow{N} U(1) \rightarrow 0 \quad (3.33)$$

(the first map is the inclusion of the points of order N in $U(1)$, and the second is $a \rightarrow a^N$). From this sequence we get a long exact cohomology sequence

$$0 \rightarrow H^1(M, U(1))/NH^1(M, U(1)) \rightarrow H^2(M, \mathbb{Z}_N) \rightarrow H^2(M, U(1))_N \rightarrow 0. \quad (3.34)$$

Here $H^2(M, U(1))_N$ is the kernel of $H^2(M, U(1)) \xrightarrow{N} H^2(M, U(1))$; this is the group of flat B -fields of order N , or in other words is M_N . Also, $H^1(M, U(1))$ classifies flat $U(1)$ bundles. Such a bundle has a first Chern class which is a torsion element of $H^2(M, \mathbb{Z})$; and the map from $H^1(M, U(1))/NH^1(M, U(1))$ to $H^2(M, \mathbb{Z}_N)$

is given by mod N reduction of the first Chern class. Conversely, every torsion element of $H^2(M, \mathbb{Z})$ is the first Chern class of a flat line bundle. So the image of $H^1(M, U(1))/NH^1(M, U(1))$ in $H^2(M, \mathbb{Z}_N)$ consists precisely of the classes that are mod N reductions of torsion classes, or in other words this image is T_N . This completes the explanation of the second exact sequence in [3.32](#).

4. The $(0, 2)$ theory in six dimensions

In this section, we will consider the analogous issues for the $(0, 2)$ conformal field theory in six-dimensions.⁷ We consider only the A_N theory, that is, the theory associated with M -theory on $AdS_7 \times \mathbf{S}^4$, with N units of four-form flux on $\mathbf{S}^4$. We will, in essence, be determining the analog for the A_N case of the $(0, 2)$ theory of the space of conformal blocks in two-dimensional rational conformal field theory. A difference from the problem considered in sections [2](#) and [3](#) is that there is no obvious candidate from classical geometry for what the answer should be.

To study the $(0, 2)$ theory on a six-dimensional spin-manifold M , we consider M -theory on spacetimes that look near infinity like $X \times \mathbf{S}^4$, where X has M for conformal boundary. By analogy with gauge theory in four-dimensions, we suspect that for general M , the partition function will not be a number but a vector in some finite-dimensional Hilbert space associated with M .

A mechanism analogous to what we studied in sections [2](#) and [3](#) immediately presents itself. The long wavelength theory on X has a three-form field C , whose effective action contains the Chern-Simons term

$$L_{CS} = -i \frac{1}{2} \frac{N}{2\pi} \int_X C \wedge dC. \quad (4.1)$$

If N is odd, then because of the $1/2$ in [4.1](#), to make sense of this theory requires a spin structure on X . (This is explained in [8](#), where the case $N = 1$ was considered and the factor of $1/2$ in [4.1](#), which arises from a similar factor of $1/6$ in M -theory, played an important role. Some additional important details, such as a gravitational correction to [4.1](#), were also described there.) There is no harm in this, because in any event, M -theory on $X \times \mathbf{S}^4$ requires a spin structure on X . If N is even or a spin structure is picked on X , then quantization is carried out rather as in section [3.2](#) by introducing a Heisenberg group associated with $H^3(M, \mathbb{Z}_N)$. This Heisenberg group contains an operator Φ_v for each $v \in H^3(M, \mathbb{Z}_N)$, with⁸

$$\Phi_v \Phi_w = \Phi_w \Phi_v \exp(2\pi i v \cdot w / N). \quad (4.2)$$

⁷Some related issues involving a discrete flux in the $(0, 2)$ model have been discussed in [23](#).

⁸From one point of view, the need for a spin structure arises because the following formula does not quite determine the Heisenberg group. To have a Heisenberg group, we need a multiplication law $\Phi_v \Phi_w = c(v, w) \Phi_{v+w}$, with $c(v, w)c(w, v)^{-1} = \exp(2\pi i v \cdot w / N)$. Since $v \cdot w$ is only well-defined modulo N , there is no elementary formula for $c(v, w)$ except to take an ambiguous square root $c(v, w) = \pm \sqrt{\exp(2\pi i v \cdot w / N)}$. A spin structure determines a consistent set of square roots. In

We thus have a group extension

$$0 \rightarrow \mathbb{Z}_N \rightarrow W \rightarrow H^3(M, \mathbb{Z}_N) \rightarrow 0, \quad (4.3)$$

and the quantum Hilbert space $\mathcal{H}$ is an irreducible representation of W . (Irreducibility can be proved by a detailed analysis along the lines of sections 3.3–3.4.)

One important difference from the four-dimensional case is that W is defined using only one copy of $H^3(M, \mathbb{Z}_N)$ (rather than two copies of $H^2(M, \mathbb{Z}_N)$ as in four dimensions). So there is no way to pick a polarization and give an explicit description of $\mathcal{H}$ without using some detailed properties of W and breaking the symmetries of the finite group $H^3(M, \mathbb{Z}_N)$. One simple case is $M = \mathbf{S}^1 \times Y$, with Y a five-dimensional spin manifold. In this case, $H^3(M, \mathbb{Z}_N) = H^2(Y, \mathbb{Z}_N) \oplus H^3(Y, \mathbb{Z}_N)$. A polarization of $H^3(M, \mathbb{Z}_N)$ is given by its subgroup $H^3(Y, \mathbb{Z}_N)$. The quotient is $H^3(M, \mathbb{Z}_N)/H^3(Y, \mathbb{Z}_N) = H^2(Y, \mathbb{Z}_N)$. Hence, the Hilbert space $\mathcal{H}$ has a basis with a vector Ψ_w for each $w \in H^2(Y, \mathbb{Z}_N)$. This result has a simple explanation. The $(0, 2)$ conformal field theory, if compactified on $\mathbf{S}^1$, gives a theory that looks at low energies like $SU(N)/\mathbb{Z}_N$ gauge theory in five dimensions. If this is further compactified on the five-manifold Y , the gauge theory has a discrete magnetic flux taking values in $H^2(Y, \mathbb{Z}_N)$, and this leads to the Hilbert space that we found.

5. 't Hooft and Wilson lines in four dimensions

In this concluding section, we will analyze the behavior of the $\mathcal{N} = 4$ super Yang-Mills theory on a four-manifold M with 't Hooft and Wilson loops included.

We begin on a five-manifold X by supplementing the Chern-Simons Lagrangian L_{CS} by couplings of the B -fields to string worldsheets. The strings in question are fundamental strings coupling to B_{NS} and D -strings coupling to B_{RR} . We take the D -string and elementary string worldsheets to be closed surfaces S_{RR} and S_{NS} , which are not necessarily connected. Including the couplings to the strings, the Lagrangian is

$$\hat{L} = NI_{CS}(B_{RR}, B_{NS}) - \frac{i}{2\pi} \int_{S_{RR}} B_{RR} - \frac{i}{2\pi} \int_{S_{NS}} B_{NS}. \quad (5.1)$$

If X has conformal boundary M , then the boundary of S_{RR} and S_{NS} should be [24, 25] one-manifolds C_{RR} and C_{NS} in M on which 't Hooft and Wilson loops are inserted. (The C 's are not necessarily connected.)

fact, having a Heisenberg group extension and not just a commutation relation is equivalent — roughly as we saw in section 3.3 — to having a suitable line bundle over the classical phase space. As analyzed in detail in [8], defining this line bundle depends on having a spin structure on M . The Chern-Simons theory that we have studied in this paper in five dimensions likewise needs a spin structure, as we saw in section 3.3.

The path integral on X will take values in a Hilbert space associated with the boundary. Near the boundary, we have $X = M \times \mathbf{R}$, where we view $\mathbf{R}$ as the “time” direction. Near the boundary, we can take $S_{RR} = C_{RR} \times \mathbf{R}$ and $S_{NS} = C_{NS} \times \mathbf{R}$. Quantization is carried out formally as in section 3.1. The Gauss law constraint contains an extra contribution from the strings and reads

$$\begin{aligned} NH_{NS} + \delta(C_{RR}) &= 0, \\ -NH_{RR} + \delta(C_{NS}) &= 0. \end{aligned} \tag{5.2}$$

Here, for $C = C_{RR}$ or C_{NS} , $\delta(C)$ is a delta-function supported on C , representing a cohomology class in $H^3(M, \mathbb{Z})$ that is dual to C . We will call this class $[C]$.

If the equations 5.2 have solutions, then as these equations are linear, by shifting the B -fields by any solution, we can reduce to the previous case in which the phase space that must be quantized is governed by $H_{RR} = H_{NS} = 0$. Hence, the quantum Hilbert space is isomorphic to the one obtained without the Wilson lines, though not canonically so as some arbitrary choice of a solution of Gauss’s law has to be made in identifying the phase spaces with and without the Wilson lines. However, the condition that equations 5.2 should have any solutions at all is nontrivial. The requirement that these equations have a solution for some B -fields is simply that the elements $[C_{RR}]$, $[C_{NS}]$ should be divisible by N in $H^3(M, \mathbb{Z})$. (When torsion is present, a full justification of this statement requires a torsion extension of the Gauss’s law constraint, along lines presented in section 3.5.)

For example, if $M = Y \times \mathbf{S}^1$, with Y a three-manifold, then the requirement is that the C ’s should wrap around $\mathbf{S}^1$ a number of times divisible by N . This statement has a simple intuitive interpretation. It means that the Wilson line wrapped on C_{NS} (or the magnetic counterpart wrapped on C_{RR}) is invariant under the thermal symmetry group $F = H^1(M, \mathbb{Z}_N)$. This way of seeing the thermal symmetry group is actually a close cousin of the argument in [7].

For the rest of this section, we specialize to the case $M = \mathbf{S}^4$. In this case, since $H^3(M, \mathbb{Z}) = 0$, there is no topological obstruction to solving the Gauss law constraints, regardless of what the C ’s might be. Moreover, the space of solutions mod gauge transformations is a single point (since the cohomology groups that classify flat B -fields are all zero). So the quantum Hilbert space is one-dimensional.

The expectation value of an arbitrary product of Wilson and ’t Hooft loops on $\mathbf{S}^4$ thus takes values in a one-dimensional Hilbert space $\mathcal{H}$. However, the expectation value cannot in general be understood as an ordinary complex number, because there is no natural way to pick a basis vector in $\mathcal{H}$ and identify $\mathcal{H}$ with the complex numbers $\mathbf{C}$. Let us explore the matter somewhat more thoroughly and see what is involved in picking a basis vector.

The presence of the Wilson loop on C_{NS} leads to the presence in the path integral

on $X \times \mathbf{S}^5$ of a factor

$$\exp \left(i \int_{S_{NS}} B_{NS} \right). \quad (5.3)$$

This factor is well-defined as a complex number if S_{NS} is a closed surface. If not, the factor takes values in a one-dimensional Hilbert space $\mathcal{H}_{NS}$. $\mathcal{H}$ is the tensor product of $\mathcal{H}_{NS}$ with a similar one-dimensional Hilbert space $\mathcal{H}_{RR}$ associated with the D -strings, as well as a factor independent of all 't Hooft and Wilson loops that enters in defining the bulk term in the action. To trivialize $\mathcal{H}_{NS}$, we should give a surface $D_{NS} \subset M$ of boundary $-C_{NS}$ (the minus sign is a reversal of orientation), and replace 5.3 by

$$\exp \left(i \int_{S_{NS} + D_{NS}} B_{NS} \right). \quad (5.4)$$

Here $S_{NS} + D_{NS}$ is a closed surface in X , so 5.4 is well-defined as a number.⁹ We must, however, investigate the dependence on the choice of D_{NS} . Let D'_{NS} be another choice. Then in replacing D_{NS} by D'_{NS} , 5.4 is multiplied by

$$\Psi = \exp \left(i \int_E B_{NS} \right), \quad (5.5)$$

where E is the closed surface in M defined by $E = D'_{NS} - D_{NS}$. Since E is contained in $\mathbf{S}^4$, whose homology vanishes, it is the boundary of a three-manifold $Y \subset M$, and we can write

$$\Psi = \exp \left(i \int_Y H_{NS} \right). \quad (5.6)$$

If there are no D -strings, then H_{NS} vanishes by the equations of motion, and the factor Ψ in the path integral can be eliminated by a redefinition of the fields, in fact, by $B_{RR} \rightarrow B_{RR} + (2\pi/N)\delta(Y)$. (Y is of codimension two in X , and $\delta(Y)$ is a two-form dual to Y .) But if there are D -strings, this is not quite a symmetry. Under this transformation, the D -string factor $\exp(i \int_{S_{RR}} B_{RR})$ in the path integral picks up a phase $\exp((2\pi i/N)Y \cdot C_{RR})$, where $Y \cdot C_{RR}$ denotes the oriented intersection number of Y and C_{RR} . That intersection number is (by definition) the linking number $\ell(E, C_{RR})$. Hence, under a change in trivialization of $\mathcal{H}_{NS}$ coming from a change in D_{NS} , the path integral is multiplied by

$$\exp(2\pi i \ell(E, C_{RR})/N). \quad (5.7)$$

This factor is, of course, 1 if there are no 't Hooft loops ($C_{RR} = 0$). So the expectation value of a product of Wilson loops only (or likewise of 't Hooft loops only) can be viewed as an ordinary number. But the expectation value of a product consisting of both Wilson and 't Hooft loops, though well-defined as a vector in the one-dimensional Hilbert space $\mathcal{H}$, is ambiguous up to multiplication by an N^{th} root of unity if one wishes to interpret it as an ordinary complex number.

⁹ D_{NS} should lie in M , not just in X , so as to trivialize $\mathcal{H}$ (which only depends on M) once and for all, independent of the choice of X .

An ambiguity of just this nature can be seen by standard gauge theoretic methods. In that context, one begins with the problem of what it means in $SU(N)/\mathbb{Z}_N$ gauge theory to define the expectation value of a Wilson loop, wrapped on a circle C_{NS} , in the fundamental representation of $SU(N)$. For this, we would want a lifting of the $SU(N)/\mathbb{Z}_N$ bundle to $SU(N)$ at least along C_{NS} . It suffices to give a surface D_{NS} with boundary C_{NS} . For, as $H^2(D_{NS}, \mathbb{Z}_N) = 0$, the $SU(N)/\mathbb{Z}_N$ bundle when restricted to D_{NS} can be lifted to $SU(N)$, and though the lifting is not unique, the nonuniqueness does not affect the value of a Wilson loop that wraps around the boundary of D_{NS} . The answer one gets this way does depend on the choice of D_{NS} if 't Hooft loops are present also. Upon changing D_{NS} to D'_{NS} , one gets in the gauge theory, in the presence of an 't Hooft loop on C_{RR} , a change in the Wilson loop expectation value given by the same formula as in 5.7. That is because, if the linking number is nonzero, the liftings of the $SU(N)/\mathbb{Z}_N$ bundle to $SU(N)$ on D_{NS} and D'_{NS} disagree on their common boundary.

Of course, different points of view are possible about the expectation value of a product of Wilson and 't Hooft loops on S^4 . The traditional point of view is to reject such correlation functions on the grounds that they are not well-defined as complex numbers. If one chooses to consider such correlation functions, we have shown how they must be interpreted, and more specifically we have shown that they have the same meaning whether considered in gauge theory on S^4 or in string theory on $AdS_5 \times S^5$.

Acknowledgments

This work was supported in part by NSF Grant PHY-9513835. I would like to thank P. Deligne, D. Freed, O. Ganor, M. Hopkins, N. Seiberg, and S. Sethi for discussions.

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Introduction to the AdS/CFT correspondence

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Abstract

This is a pedagogical introduction to the AdS/CFT correspondence, based on lectures delivered by the author at the third IDPASC school. Starting with the conceptual basis of the holographic dualities, the subject is developed emphasizing some concrete topics, which are discussed in detail. A very brief introduction to string theory is provided, containing the minimal ingredients to understand the origin of the AdS/CFT duality. Other topics covered are the holographic calculation of correlation functions, quark-antiquark potentials and transport coefficients.

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1 Introduction and motivation

The AdS/CFT correspondence is a duality relating quantum field theory (QFT) and gravity. More precisely, the correspondence relates the quantum physics of strongly correlated many-body systems to the classical dynamics of gravity in one higher dimension. This duality is also referred to as the holographic duality or the gauge/gravity correspondence. In its original formulation [1–3], the correspondence related a four-dimensional Conformal Field Theory (CFT) to the geometry of an anti-de Sitter (AdS) space in five dimensions.

In the study of collective phenomena in condensed matter physics it is quite common that when a system is strongly coupled it reorganizes itself in such a way that new weakly coupled degrees of freedom emerge dynamically and the system can be better described in terms of fields representing the emergent excitations. The holographic duality is a new example of this paradigm. The new (and surprising!) feature is that the emergent fields live in a space with one extra dimension and that the dual theory is a gravity theory. As we will argue below, the extra dimension is related to the energy scale of the QFT. The holographic description is a geometrization of the quantum dynamics of the systems with a large number of degrees of freedom, which makes manifest that there are deep connections between quantum mechanics and gravity.

The gauge/gravity duality was discovered in the context of string theory, where it is quite natural to realize (gauge) field theories on hypersurfaces embedded in a higher dimensional space, in a theory containing gravity. However, the study of the correspondence has been extended to include very different domains, such as the analysis of the strong coupling dynamics of QCD and the electroweak theories, the physics of black holes and quantum gravity, relativistic hydrodynamics or different applications in condensed matter physics (holographic superconductors, quantum phase transitions, cold atoms, ...). In these lectures we will concentrate on some particular topics, trying to present in clear terms the basic conceptual ideas, as well as the more practical calculational aspects of the subject. For reviews on the different aspects of the duality see [4–10].

We will start by motivating the duality from the Kadanoff-Wilson renormalization group approach to the analysis of lattice systems. Let us consider a non-gravitational system in a lattice with lattice spacing a and hamiltonian given by:

$$H = \sum_{x,i} J_i(x,a) \mathcal{O}^i(x) , \quad (1)$$

where x denotes the different lattice sites and i labels the different operators $\mathcal{O}^i$. The $J_i(x,a)$

are the coupling constants (or sources) of the operators at the point x of the lattice. Notice that we have included a second argument in J^i , to make clear they correspond to a lattice spacing a . In the renormalization group approach we coarse grain the lattice by increasing the lattice spacing and by replacing multiple sites by a single site with the average value of the lattice variables. In this process the hamiltonian retains its form (1) but different operators are weighed differently. Accordingly, the couplings $J_i(x, a)$ change in each step. Suppose that we double the lattice spacing in each step. Then, we would have a succession of couplings of the type:

$$J_i(x, a) \rightarrow J_i(x, 2a) \rightarrow J_i(x, 4a) \rightarrow \cdots . \quad (2)$$

Therefore, the couplings acquire in this process a dependence on the scale (the lattice spacing) and we can write them as $J_i(x, u)$, where $u = (a, 2a, 4a, \cdots)$ is the length scale at which we probe the system. The evolution of the couplings with the scale is determined by flow equations of the form:

$$u \frac{\partial}{\partial u} J_i(x, u) = \beta_i(J_j(x, u), u) , \quad (3)$$

where β_i is the so-called β -function of the i^{th} coupling constant. At weak coupling the β_i 's can be determined in perturbation theory. At strong coupling the AdS/CFT proposal is to consider u as an extra dimension. In this picture the succession of lattices at different values of u are considered as layers of a new higher-dimensional space. Moreover, the sources $J_i(x, u)$ are regarded as fields in a space with one extra dimension and, accordingly we will simply write:

$$J_i(x, u) = \phi_i(x, u) . \quad (4)$$

The dynamics of the sources ϕ_i 's will be governed by some action. Actually, in the AdS/CFT duality the dynamics of the ϕ_i 's is determined by some gravity theory (i. e. by some metric). Therefore, one can consider the holographic duality as a geometrization of the quantum dynamics encoded by the renormalization group. The microscopic couplings of the field theory in the UV can be identified with the values of the bulk fields at the boundary of the extra-dimensional space. Thus, one can say that the field theory lives on the boundary of the higher-dimensional space (see figure 1).

The sources ϕ_i of the dual gravity theory must have the same tensor structure of the corresponding dual operator $\mathcal{O}^i$ of field theory, in such a way that the product $\phi_i \mathcal{O}^i$ is a scalar. Therefore, a scalar field will be dual to a scalar operator, a vector field A_μ will be dual to a current J^μ , whereas a spin-two field $g_{\mu\nu}$ will be dual to a symmetric second-order tensor $T_{\mu\nu}$ which can be naturally identified with the energy-momentum tensor $T_{\mu\nu}$ of the field theory.

The holographic duality raises several conceptual issues which should be addressed in order to fully understand it. The simplest of these issues is the matching of the degrees of freedom on both sides of the correspondence. Let us consider a QFT in a d -dimensional

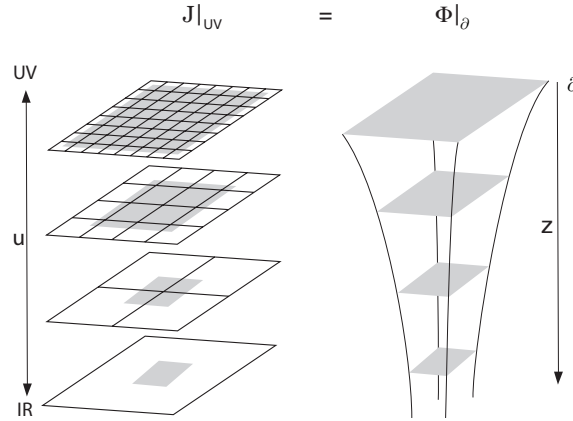


Figure 1: On the left we illustrate the Kadanoff-Wilson renormalization of a lattice system. In the AdS/CFT correspondence the lattices at different scale are considered as the layers of the higher dimensional space represented on the right of the figure.

spacetime (*i.e.* in $d - 1$ spatial dimensions plus time). The number of degrees of freedom of a system is measured by the entropy. On the QFT side the entropy is an extensive quantity. Therefore, if R_{d-1} is $(d - 1)$ -dimensional spatial region, at constant time, its entropy should be proportional to its volume in $d - 1$ dimensions:

$$S_{QFT} \propto \text{Vol}(R_{d-1}) . \quad (5)$$

On the gravity side the theory lives in a $(d + 1)$ -dimensional spacetime. How such a higher dimensional theory can contain the same information as its lower dimensional dual?. The crucial point to answer this question is the fact that the entropy in quantum gravity is subextensive. Indeed, in a gravitational theory the entropy in a volume is bounded by the entropy of a black hole that fits inside the volume and, according to the so-called holographic principle, the entropy is proportional to the surface of the black hole horizon (and not to the volume enclosed by the horizon). More concretely, the black hole entropy is given by the Bekenstein-Hawking formula:

$$S_{BH} = \frac{1}{4G_N} A_H , \quad (6)$$

where A_H is the area of the event horizon and G_N is the Newton constant. In order to apply (6) for our purposes, let R_d be a spatial region in the $(d + 1)$ -dimension spacetime where the gravity theory lives and let us assume that R_d is bounded by a $(d - 1)$ -dimensional manifold R_{d-1} ($R_{d-1} = \partial R_d$). Then, according to (6), the gravitational entropy associated to R_d scales as:

$$S_{GR}(R_d) \propto \text{Area}(\partial R_d) \propto \text{Vol}(R_{d-1}) , \quad (7)$$

which agrees with the QFT behavior (5). In this lectures we will present several refinements

of this argument and we will establish a more precise matching of the degrees of freedom of the two dual theories.

2 The anti-de Sitter space

In general, finding the geometry associated to a given QFT is a very difficult problem. However, if the theory is at a fixed point of the renormalization group flow (with a vanishing β -function) it has conformal invariance (is a CFT) and one can easily find such a metric. Indeed, let us consider a QFT in d spacetime dimensions. The most general metric in $(d + 1)$ -dimensions with Poincaré invariance in d -dimensions is:

$$ds^2 = \Omega^2(z) (-dt^2 + d\vec{x}^2 + dz^2) , \quad (8)$$

where z is the coordinate of the extra dimension, $\vec{x} = (x^1, \dots, x^{d-1})$ and $\Omega(z)$ is a function to be determined. If z represents a length scale and the theory is conformal invariant, then ds^2 must be invariant under the transformation:

$$(t, \vec{x}) \rightarrow \lambda(t, \vec{x}) , \quad z \rightarrow \lambda z . \quad (9)$$

Then, by imposing the invariance of the metric (8) under the transformation (9), we obtain that the function $\Omega(z)$ must transform as:

$$\Omega(z) \rightarrow \lambda^{-1} \Omega(z) , \quad (10)$$

which fixes $\Omega(z)$ to be:

$$\Omega(z) = \frac{L}{z} , \quad (11)$$

where L is a constant. Thus, by plugging the function (11) into the metric (8) we arrive at the following form of ds^2 :

$$ds^2 = \frac{L^2}{z^2} (-dt^2 + d\vec{x}^2 + dz^2) , \quad (12)$$

which is the line element of the AdS space in $(d + 1)$ -dimensions, that we will denote by AdS_{d+1} . The constant L is a global factor in (12), which we will refer to as the anti-de Sitter radius. The (conformal) boundary of the AdS space is located at $z = 0$. Notice that the metric (12) is singular at $z = 0$. This means that we will have to introduce a regularization procedure in order to define quantities in the AdS boundary.

The AdS metric (12) is a solution of the equation of motion of a gravity action of the type:

$$I = \frac{1}{16\pi G_N} \int d^{d+1}x \sqrt{-g} \left[-2\Lambda + R + c_2 R^2 + c_3 R^3 + \dots \right] , \quad (13)$$

with G_N is the Newton constant, the c_i are constants, $g = \det(g_{\mu\nu})$, $R = g^{\mu\nu} R_{\mu\nu}$ is the scalar curvature and Λ is a cosmological constant. In particular, if $c_2 = c_3 = \dots = 0$ the action (13)

becomes the Einstein-Hilbert (EH) action of general relativity with a cosmological constant. In this case the equations of motion are just the Einstein equations:

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = -\Lambda g_{\mu\nu} . \quad (14)$$

Taking the trace on both sides of (14), we get that the scalar curvature is given by:

$$R = g^{\mu\nu} R_{\mu\nu} = 2 \frac{d+1}{d-1} \Lambda . \quad (15)$$

Inserting back this result in the Einstein equation (14) we get that the Ricci tensor and the metric are proportional:

$$R_{\mu\nu} = \frac{2}{d-1} \Lambda g_{\mu\nu} . \quad (16)$$

Therefore the solution of (14) defines an Einstein space. Moreover, the Ricci tensor for the metric (12) can be computed directly from its definition. We get:

$$R_{\mu\nu} = -\frac{d}{L^2} g_{\mu\nu} . \quad (17)$$

By comparing (16) and (17) we get that the AdS_{d+1} space solves the equations of motion (14) of EH gravity with a cosmological constant equal to:

$$\Lambda = -\frac{d(d-1)}{2L^2} , \quad (18)$$

which is negative. It follows from (15) and (18) that the scalar curvature for the AdS_{d+1} space with radius L is given by:

$$R = -\frac{d(d+1)}{L^2} . \quad (19)$$

In these lectures we will be mostly interested in gauge theories in $3+1$ dimensions, which corresponds to taking $d=4$ in our formulas. The dual geometry found above for this case is AdS_5 . This was precisely the system studied in [1] by Maldacena, who conjectured that the dual QFT is super Yang-Mills theory with four supersymmetries ($\mathcal{N}=4$ SYM).

2.1 Counting the degrees of freedom in AdS

After having identified the AdS space as the gravity dual of a field theory with conformal invariance, we can refine the argument of section 1 to match the number of degrees of freedom of both sides of the duality.

Let us consider first the QFT side. To regulate the theory we put both a UV and IR regulator. We place the system in a spatial box of size R (which serves as an IR cutoff) and we introduce a lattice spacing ϵ that acts as a UV regulator. In d spacetime dimensions the

system has R^{d-1}/ϵ^{d-1} cells. Let c_{QFT} be the number of degrees of freedom per lattice site, which we will refer to as the central charge. Then, the total number of degrees of freedom of the QFT is:

$$N_{dof}^{QFT} = \left(\frac{R}{\epsilon}\right)^{d-1} c_{QFT} . \quad (20)$$

The central charge is one of the main quantities that characterize a CFT. If the CFT is a $SU(N)$ gauge field theory, such as the theory with four supersymmetries which will be described below, the fields are $N \times N$ matrices in the adjoint representation which, for large N , contain N^2 independent components. Thus, in these $SU(N)$ CFT's the central charge scales as $c_{SU(N)} \sim N^2$.

Let us now compute the number of degrees of freedom of the AdS_{d+1} solution. According to the holographic principle and to the Bekenstein-Hawking formula (6), the number of degrees of freedom contained in a certain region is equal to the maximum entropy, given by

$$N_{dof}^{AdS} = \frac{A_{\partial}}{4G_N} , \quad (21)$$

with A_{∂} being the area of the region at boundary $z \rightarrow 0$ of AdS_{d+1} . Let us evaluate A_{∂} by integrating the volume element corresponding to the metric (12) at a slice $z = \epsilon \rightarrow 0$:

$$A_{\partial} = \int_{\mathbb{R}^{d-1}, z=\epsilon} d^{d-1}x \sqrt{g} = \left(\frac{L}{\epsilon}\right)^{d-1} \int_{\mathbb{R}^{d-1}} d^{d-1}x . \quad (22)$$

The integral on the right-hand-side of (22) is the the volume of $\mathbb{R}^{d-1}$, which is infinite. As we did on the QFT side, we regulate it by putting the system in a box of size R :

$$\int_{\mathbb{R}^{d-1}} d^{d-1}x = R^{d-1} . \quad (23)$$

Thus, the area of the A_{∂} is given by:

$$A_{\partial} = \left(\frac{RL}{\epsilon}\right)^{d-1} . \quad (24)$$

Let us next introduce the Planck length l_P and the Planck mass M_P for a gravity theory in $d+1$ dimensions as:

$$G_N = (l_P)^{d-1} = \frac{1}{(M_P)^{d-1}} . \quad (25)$$

Then, the number of degrees of freedom of the AdS_{d+1} space is:

$$N_{dof}^{AdS} = \frac{1}{4} \left(\frac{R}{\epsilon}\right)^{d-1} \left(\frac{L}{l_P}\right)^{d-1} . \quad (26)$$

By comparing N_{dof}^{QFT} and N_{dof}^{AdS} we conclude that they scale in the same way with the IR and UV cutoffs R and ϵ and we can identify:

$$\frac{1}{4} \left(\frac{L}{l_P}\right)^{d-1} = c_{QFT} . \quad (27)$$

This gives the matching condition between gravity and QFT that we were looking for. Notice that a theory is (semi)classical when the coefficient multiplying its action is large. In this case the path integral is dominated by a saddle point. The action of our gravity theory in the AdS_{d+1} space of radius L contains a factor L^{d-1}/G_N . Thus, taking into account the definition of the Planck length in (25), we conclude that the classical gravity theory is reliable if:

$$\text{classical gravity in AdS} \rightarrow \left(\frac{L}{l_P}\right)^{d-1} \gg 1, \quad (28)$$

which happens when the AdS radius is large in Planck units. Since the scalar curvature goes like $1/L^2$, the curvature is small in Planck units. Thus, a QFT has a classical gravity dual when c_{QFT} is large, or equivalently if there is a large number of degrees of freedom per unit volume or a large number of species (which corresponds to large N for $SU(N)$ gauge theories).

3 String theory basics

As mentioned in the introduction, the AdS/CFT correspondence was originally discovered in the context of string theory. Although it can be formulated without making reference to its stringy origin, it is however very convenient to know its connection to the theory of relativistic strings in order to acquire a deep understanding of the duality. We will start reviewing qualitatively this connection in this section (refs [11–16] contain systematic expositions of string theory with different levels of technical details).

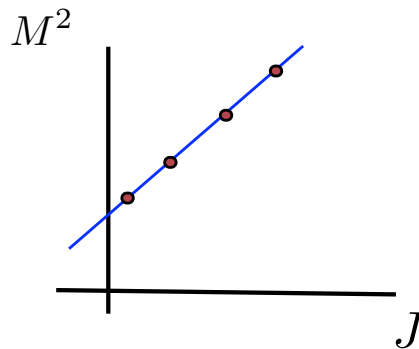


Figure 2: In a Regge trajectory the mass square M^2 of the particles grows linearly with their spin J .

Historically, string theory was introduced in the sixties as an attempt to describe the hadronic resonances of high spin observed in the experiments. Experimentally, the mass

square of these particles is linearly related to its spin J :

$$M^2 \sim J . \quad (29)$$

It is then said that the hadrons are distributed along Regge trajectories (see figure 2). String theory was introduced to reproduce this behavior. Actually, it is not difficult to verify qualitatively that the rotational degree of freedom of the relativistic string gives rise to Regge trajectories as in (29). Indeed, let us suppose that we have an open string with length L and tension T which is rotating around its center of mass. The mass of this object would be $M \sim T L$, whereas its angular momentum J would be $J \sim P L$, with P being its linear momentum. In a relativistic theory $P \sim M$, which implies that $J \sim P L \sim M L \sim T^{-1} M^2$ or, equivalently, $M^2 \sim T J$. Thus, we reproduce the Regge behavior (29) with the slope being proportional to the string tension T .

The basic object of string theory is an object extended along some characteristic distance l_s . Therefore, the theory is non-local. It becomes local in the point-like limit in which the size $l_s \rightarrow 0$. The rotation degree of freedom of the string gives rise to Regge trajectories similar to those observed experimentally. In modern language one can regard a meson as a quark-antiquark pair joined by a string. The energy of such a configuration grows linearly with the length, and this constitutes a model of confinement.

The classical description of a relativistic string is directly inspired from that of a point particle in the special theory of relativity. Indeed, let us consider a relativistic point particle of mass m moving in a flat spacetime with Minkowski metric $\eta_{\mu\nu}$. As it moves the particle describes a curve in spacetime (the so-called worldline), which can be represented by a function of the type:

$$x^\mu = x^\mu(\tau) , \quad (30)$$

where x^μ is the coordinate in the space in which the point particle is moving (the target space) and τ parameterizes the path of the particle (the worldline coordinate). The action of the particle is proportional to the integral of the line element along the trajectory in spacetime, with the coefficient being given by the mass m of the particle:

$$S = -m \int ds = -m \int_{\tau_0}^{\tau_1} d\tau \sqrt{-\eta_{\mu\nu} \dot{x}^\mu \dot{x}^\nu} . \quad (31)$$

Let us now consider a relativistic string. A one-dimensional object moving in spacetime describes a surface (the so-called worldsheet). Let dA be the area element of the worldsheet (see figure 3). Then, the analogue of the action (31) for a string is the so-called Nambu-Goto action:

$$S_{NG} = -T \int dA , \quad (32)$$

where T is the tension of the string, given by:

$$T = \frac{1}{2\pi\alpha'} , \quad (33)$$

and α' is the Regge slope (α' has units of $(length)^2$ or $(mass)^{-2}$). The string length and mass are defined as:

$$l_s = \sqrt{\alpha'} = \frac{1}{M_s} . \quad (34)$$

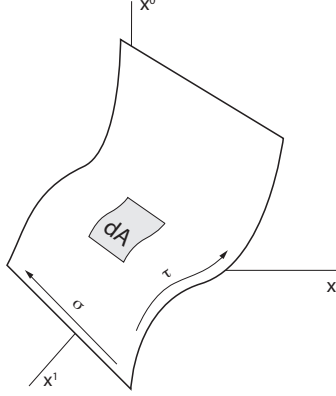


Figure 3: The string sweeps in spacetime a two-dimensional worldsheet.

Then, the string tension is related to l_s and M_s as:

$$T = \frac{1}{2\pi l_s^2} = \frac{M_s^2}{2\pi} . \quad (35)$$

Let us write more explicitly the Nambu-Goto action (32). With this purpose we will take two coordinates ξ^α ($\alpha = 0, 1$) to parameterize the worldsheet Σ ($(\xi^0, \xi^1) = (\tau, \sigma)$). Let us assume that the string moves in a target space $\mathcal{M}$ with metric $G_{\mu\nu}$. The embedding of Σ in the spacetime $\mathcal{M}$ is characterized by a map $\Sigma \rightarrow \mathcal{M}$ with $\xi^\alpha \rightarrow X^\mu(\xi^\alpha)$. The induced metric on Σ is:

$$\hat{G}_{\alpha\beta} \equiv G_{\mu\nu} \partial_\alpha X^\mu \partial_\beta X^\nu . \quad (36)$$

Thus, the Nambu-Goto action of the string is:

$$S_{NG} = -T \int \sqrt{-\det \hat{G}_{\alpha\beta}} d^2\xi . \quad (37)$$

The action (37) depends non-linearly on the embedding functions $X^\mu(\tau, \sigma)$. The classical equations of motion derived from (37) are partial differential equations which, remarkably, can be solved for a flat target spacetime with $G_{\mu\nu} = \eta_{\mu\nu}$, both for Neumann and Dirichlet boundary conditions. Actually, the functions $X^\mu(\tau, \sigma)$ can be represented in a Fourier expansion as an infinite superposition of oscillation modes, much as in the string of a violin.

The quantization of the string can be carried out by using the standard methods in quantum physics. The simplest way is just by canonical quantization, *i.e.* by considering that the X^μ are operators and by imposing canonical commutation relations between coordinates

and momenta. As a result one finds that the different oscillation modes can be interpreted as particles and that the spectrum of the string contains an infinite tower of particles with growing masses and spins that are organized in Regge trajectories, with $1/l_s$ being the mass gap.

The close scrutiny of the consistency conditions of the quantum string reveals many surprises (quantizing the string is like opening Pandora's box!). First of all, the mass spectrum contains tachyons (particles with $m^2 < 0$), which is a signal of instability. In order to avoid this problem one must consider a string which has also fermionic coordinates and require that the system is supersymmetric (*i.e.* that there is a symmetry between bosonic and fermionic degrees of freedom). This generalization of the bosonic string (37) is the so-called superstring. Another consistency requirement imposed by the quantization is that the number D of dimensions of the space in which the string is moving is fixed. For a superstring $D = 10$. This does not mean that the extra dimensions have the same meaning as the ordinary ones of the four-dimensional Minkowski spacetime. Actually, the extra dimensions should be regarded as defining a configuration space (as the phase space in classical mechanics does). We will see below that this is precisely the interpretation that they have in the context of the AdS/CFT correspondence.

Another important piece of information about the nature of string theory is obtained from the analysis of the spectrum of massless particles. Massive particles have a mass which is a multiple of $1/l_s$ and, therefore, they become unobservable in the low-energy limit $l_s \rightarrow 0$. Then, after eliminating the tachyons by using supersymmetry, the massless particles are the low-lying excitations of the spectrum. We must distinguish the case of open and closed strings. The spectrum of open strings contains massless particles of spin one with the couplings needed to have gauge symmetry. These particles can be naturally identified with gauge bosons (photons, gluons,...). The big surprise comes when looking at the spectrum of closed strings, since it contains a particle of spin two and zero mass which can only be interpreted as the graviton (the quantum of gravity). Besides, quantum consistency of the propagation of the string in a curved space implies Einstein equations in ten dimension plus corrections:

$$R_{\mu\nu} + \dots = 0 . \quad (38)$$

Then, one is led to conclude that string theory is not a theory of hadrons but a theory of quantum gravity!! . Thus, the string length l_s should be of the order of the Planck length l_P and not of the order of the hadronic scale ~ 1 fm). Elementary strings without internal structure and zero thickness were born for the wrong purpose. Moreover, in addition to the graviton, the massless spectrum contains antisymmetric tensor fields of the form $A_{\mu_1 \dots \mu_{p+1}}$, which will become very relevant in formulating the AdS/CFT correspondence.

As strings propagate through the target space $\mathcal{M}$, they can suffer interactions by splitting in two or more parts or by joining with other strings. The worldsheet for one of these interactions is just a two-dimensional surface with holes and boundaries which can be thought

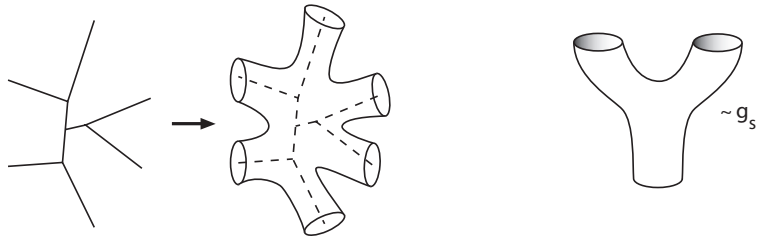


Figure 4: A vertex in field theory corresponds to a two-dimensional surface with boundaries (left). The triple vertex for three closed strings is represented on the right as a pants surface.

as a vertex in QFT perturbation theory in which the lines has been thickened, as shown in figure 4. In closed string theory the basic interaction is a string splitting into two (or the inverse process of two closed strings merging into one). The corresponding worldsheet has the form of pants, which can be obtained by thickening a triple vertex in QFT. A loop in string perturbation theory can be obtained with two triple vertices, which produces a worldsheet which is a Riemann surface with a hole. Higher order terms in perturbation theory correspond to having more holes on the worldsheet (see figure 5). Therefore, the perturbative series in string theory is a topological expansion!. In the topology of two-dimensional surfaces, the number h of holes (or handles) of the surface is called the genus of the surface. In string perturbation theory, h is just the number of string loops.

Let us suppose that we weigh every triple vertex with a string coupling constant coupling g_s . It is clear from the discussion above that string perturbation theory is an expansion of the type:

$$\mathcal{A} = \sum_{h=0}^{\infty} g_s^{2h-2} F_h(\alpha') , \quad (39)$$

where $\mathcal{A}$ is some amplitude. In the next section we will identify a topological expansion in gauge theories which has the same structure as the series (39).

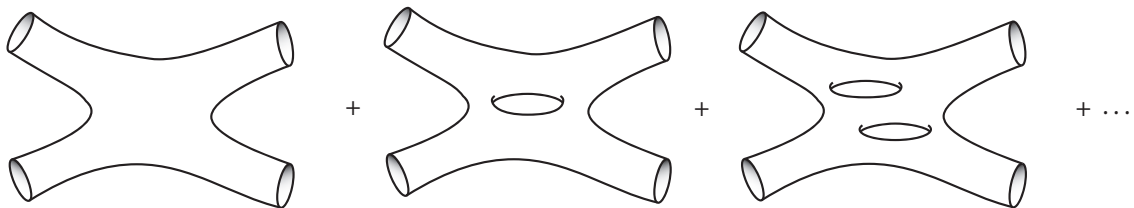


Figure 5: Perturbative expansion of the amplitude corresponding to four closed strings.

4 Large N expansion in gauge theories

Let us consider $U(N)$ Yang-Mills theory with lagrangian

$$\mathcal{L} = -\frac{1}{g^2} \text{Tr}[F_{\mu\nu} F^{\mu\nu}] , \quad (40)$$

where the non-abelian gauge field strength is given by:

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu] . \quad (41)$$

In the equations above A_μ is an $N \times N$ matrix whose elements can be written as $A_{\mu,b}^a$, where a, b are indices that run from 1 to N . Let us rewrite $\mathcal{L}$ as:

$$\mathcal{L} = -\frac{N}{\lambda} \text{Tr}[F_{\mu\nu} F^{\mu\nu}] , \quad (42)$$

where $\lambda = g^2 N$ is the so-called 't Hooft coupling. The 't Hooft expansion [17] corresponds to keeping λ fixed and to perform an expansion of the amplitudes in powers of N . It turns out that the different powers of N correspond to the different topologies. One can prove this by adopting a double line notation for the gauge propagator. Then, one can verify that the color lines form the perimeter of an oriented polygon (a face). Polygons join at a common edges in such a way that every vacuum graph is associated to a triangulated two-dimensional surface formed by sewing all polygons along the edges.

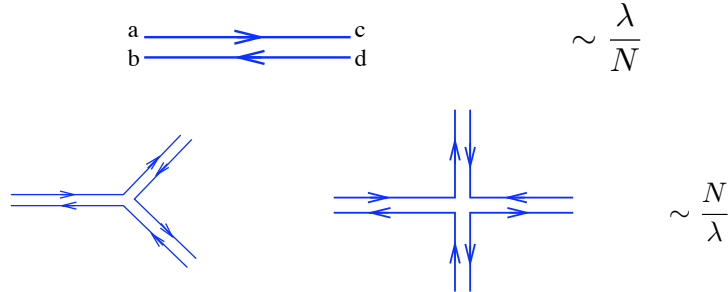


Figure 6: Gauge propagator and vertices in the double line notation of Yang-Mills $U(N)$ theories.

It is not difficult to find the powers of N and λ appearing in a given diagram D with no external lines. The contributions of the gauge propagator and the vertices are displayed in figure 6. Moreover, every index loop contributes with a power of N . Suppose that E is the number of propagators (edges) of D connecting two vertices, V is the number of vertices and F is the number of index loops (faces). Then, one can show that:

$$D \sim \left(\frac{\lambda}{N}\right)^E \left(\frac{N}{\lambda}\right)^V N^F = N^{F-E+V} \lambda^{E-V} . \quad (43)$$

Let us denote by χ the combination of F , E and V that appears in the exponent of N in (43):

$$F - E + V = \chi , \quad (44)$$

which is nothing but the Euler characteristic of the triangulation, which only depends on the topology of the surface. If the diagram triangulates a surface with h handles (an h -torus), χ is given by:

$$\chi = 2 - 2h . \quad (45)$$

Thus the diagrams are weighed with a power of N determined by the number of handles of the surface that they triangulate (see figure 7). The planar diagrams are those that can be drawn on a piece of paper without self-crossing. They correspond to $h = 0$ and their contribution is of the type $N^2 \lambda^n$, for some power n which depends on the diagram considered. Clearly, as the dependence on N goes like N^{2-2h} the diagrams with $h = 0$ are the dominant ones in the large N expansion. For this reason the large N limit is also called the planar limit of the gauge theory (see [18] for more details and examples).

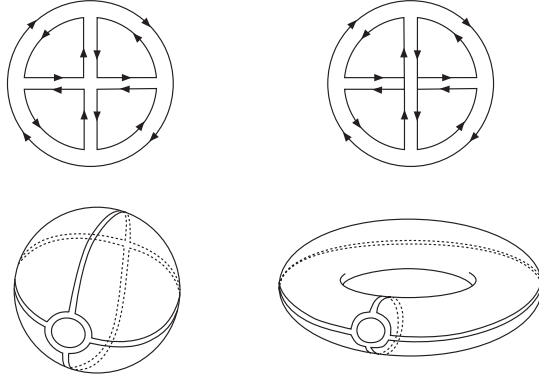


Figure 7: The planar diagram on the left can be drawn on a sphere, whereas the non-planar diagram on the right must be drawn on a torus.

The effective action Z can be obtained by computing the sum over the connected vacuum-to-vacuum diagrams. It has the structure:

$$\log Z = \sum_{h=0}^{\infty} N^{2-2h} \sum_{l=0}^{\infty} c_{l,h} \lambda^l = \sum_{h=0}^{\infty} N^{2-2h} f_h(\lambda) , \quad (46)$$

where $f_h(\lambda)$ is the sum of Feynman diagrams that can be drawn in a surface of genus h . This clearly suggest a connection with the perturbative expansion of string theory written in (39) if one identifies the string coupling constant g_s as:

$$g_s \sim \frac{1}{N} . \quad (47)$$

Heuristically one can say that gauge theory diagrams triangulate the worldsheet of an effective string. The AdS/CFT correspondence is a concrete realization of this connection in the limit $(N, \lambda) \rightarrow \infty$ (*i.e.* for planar theories in the strongly coupled regime). This dual stringy description of gauge theory is nothing but a quantum version of the familiar description of electromagnetism in terms of the string-like lines of force.

5 D-branes

Besides the perturbative structure reviewed above, string theories have a non-perturbative sector, which plays a crucial role in connecting them with gauge theories. The relevant objects in this non-perturbative sector are the solitons, which are extended objects. The most important solitons for the AdS/CFT correspondence are the Dp-branes, that are objects extended in $p + 1$ directions (p spatial + time). The Dp-branes can be defined as hypersurfaces where strings end (see figure [8](#)). They can be obtained by quantizing the string with fixed ends along hyperplanes (Dirichlet boundary conditions). In addition, they can also be understood as objects charged under the antisymmetric tensor fields $A_{\mu_1 \dots \mu_{p+1}}$ of string theory, which naturally couple to the Dp-brane worldvolume as:

$$A_{\mu_1 \dots \mu_{p+1}} \rightarrow \int_{\mathcal{M}_{p+1}} A_{\mu_1 \dots \mu_{p+1}} dx^{\mu_1} \dots dx^{\mu_{p+1}} , \quad (48)$$

with $\mathcal{M}_{p+1}$ being the worldvolume of the Dp-brane. The Dp-branes are dynamical objects that can move and get excited. Schematically, their action takes the form:

$$S_{Dp} = -T_{Dp} \int d^{p+1}x [\dots] , \quad (49)$$

with T_{Dp} being the tension of the Dp-brane, which in terms of the string coupling constant g_s and the string length l_s is given by:

$$T_{Dp} = \frac{1}{(2\pi)^p g_s l_s^{p+1}} . \quad (50)$$

Notice from [\(50\)](#) that $T_{Dp} \sim g_s^{-1}$, which confirms that the Dp-branes are non-perturbative objects in string theory. The dependence of T_{Dp} on l_s in [\(50\)](#) is fixed by dimensional analysis.

The Dp-branes can have two classes of excitations. The first one correspond to rigid motions and deformations of their shape. These degrees of freedom can be parameterized by the $9-p$ coordinates ϕ^i ($i = 1, \dots, 9-p$) transverse to the $(p+1)$ -dimensional worldvolume in the ten-dimensional target space. The ϕ^i 's are just scalar fields on the Dp-brane worldvolume. Besides, the Dp-branes can have internal excitations. To get a clue on how to represent these excitations, let us recall that the endpoint of the string is a charge. When there is a charge,

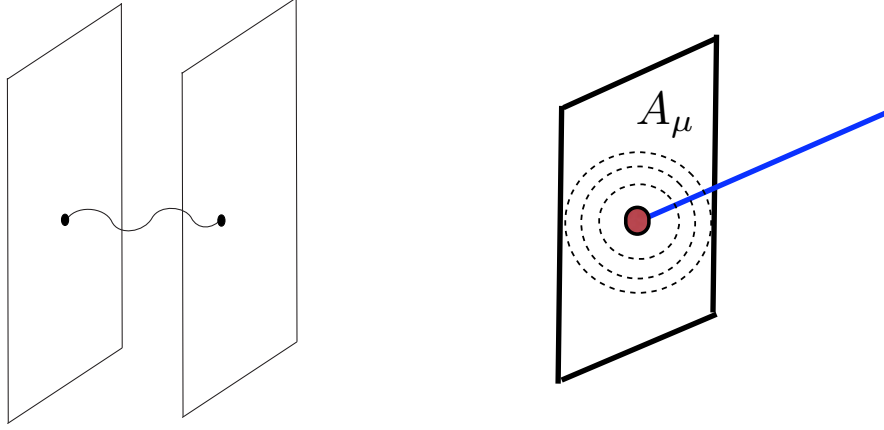


Figure 8: The D-branes are hyperplanes where open strings can end. The endpoints of the strings source gauge fields A_μ on the D-brane worldvolume.

a gauge field is sourced, as illustrated in figure 8. Then, it follows that a Dp-brane has an abelian gauge field A_μ ($\mu = 0, \dots, p$) living in its worldvolume. The action of the Dp-brane which takes into account these two types of excitations is the so-called Dirac-Born-Infeld action, which can be written as:

$$S_{DBI} = -T_{Dp} \int d^{p+1}x \sqrt{-\det(g_{\mu\nu} + 2\pi l_s^2 F_{\mu\nu})} , \quad (51)$$

where $g_{\mu\nu}$ is the induced metric on the worldvolume and $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the field strength of the gauge field A_μ . When the worldvolume gauge field is not excited ($F_{\mu\nu} = 0$) the DBI action (51) becomes the natural generalization of the Nambu-Goto action (37) for an object extended along p spatial directions.

Let us consider a Dp-brane in flat space. The induced metric in this case takes the form:

$$g_{\mu\nu} = \eta_{\mu\nu} + (2\pi l_s^2)^2 \partial_\mu \phi^i \partial_\nu \phi^i , \quad (52)$$

where the ϕ^i are the coordinates that parameterize the embedding of the brane. Let us now expand the square root of (51) in powers of $F_{\mu\nu}$ and $\partial_\mu \phi$. The quadratic terms in this expansion can be written as:

$$S_{DBI}^{(2)} = -\frac{1}{g_{YM}^2} \left(\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{1}{2} \partial_\mu \phi^i \partial^\mu \phi^i + \dots \right) , \quad (53)$$

which is just the ordinary action of a gauge field and $9 - p$ scalar fields. In (53) g_{YM} is the Yang-Mills coupling, which, in terms of l_s and g_s , is given by:

$$g_{YM}^2 = 2(2\pi)^{p-2} l_s^{p-3} g_s . \quad (54)$$

In addition to A_μ and ϕ , in superstring theory the branes have fermionic excitations, which can be represented in terms of fermionic fields.

We have just discovered one of the most important features of the Dp-branes: they contain a gauge theory living in their worldvolume!. In the case of a single Dp-brane the gauge group is $U(1)$, as in (53). However, if we consider a stack of coincident parallel Dp-branes the gauge group gets promoted to $U(N)$. Indeed, one can verify in this case that the fields A_μ and ϕ^i are matrices transforming in the adjoint representation of $U(N)$. The non-diagonal elements of these fields correspond to excitations connecting different branes, whereas the diagonal ones are excitations on a single brane. Actually, the $U(1)$ component of the $U(N)$ gauge theory can be decoupled from the $SU(N)$ fields. Therefore, we conclude that a stack of N Dp-branes realizes a $SU(N)$ gauge theory in $p + 1$ dimensions.

The D-branes have brought a completely new perspective on gauge theories which gives rise to what is called brane engineering. The geometric realization of the gauge symmetry that they provide allows to perform a series of transformations which lead to discover new unexpected properties of gauge theories. For example, one can move the branes, place them in different spaces, etc. In the gauge theory, these transformations lead to dualities, reductions of the amount of supersymmetry, changes of the field content, etc. In general we have a novel geometric insight on gauge dynamics. The AdS/CFT correspondence is an important outcome of this new realization of the gauge symmetry.

The particular case of the D3-branes is specially relevant in what follows. In this case we have a $3 + 1$ worldvolume and $10 - 4 = 6$ scalar fields. The corresponding four-dimensional $SU(N)$ gauge theory can be identified with super Yang-Mills theory with four supersymmetries ($\mathcal{N} = 4$ SYM). This theory is an exact CFT at the quantum level and will be the basic example of the AdS/CFT correspondence. Notice from (54) that the Yang-Mills coupling is dimensionless in this case and is related to the string coupling constant g_s as:

$$g_{YM}^2 = 4\pi g_s , \quad (\text{D3 - branes}) . \quad (55)$$

6 D-branes and gravity

String theory is a gravity theory in which all types of matter distort the spacetime. This distortion is determined by solving Einstein equations which follow from the action:

$$S = \frac{1}{16\pi G} \int d^{10}x \sqrt{-g} R + \dots . \quad (56)$$

The ten-dimensional Newton constant in (56) is related to string parameters as:

$$16\pi G = (2\pi)^7 g_s^2 l_s^8 . \quad (57)$$

The dependence of G on l_s in (57) follows from dimensional analysis. The dependence on g_s follows by comparing two-to-two scattering amplitudes in string theory and in supergravity (see figure 9).

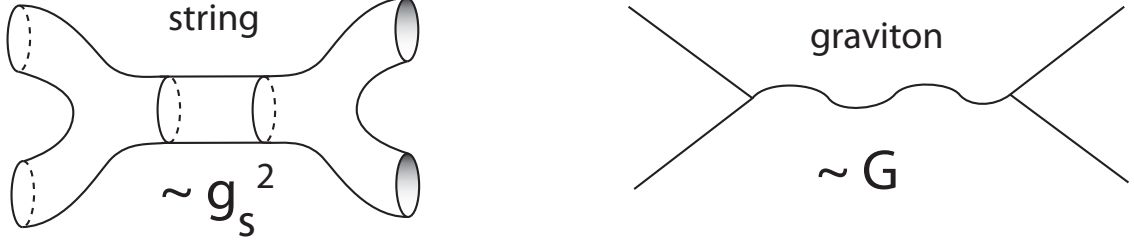


Figure 9: In string theory the exchange of a graviton is obtained as the low-energy limit of the exchange of a closed string.

The Dp-branes (and other extended objects) are solutions of the Einstein equations. Let us consider these solutions at the linearized level of weak gravity (*i.e.* when we are far from the object). The linearized metric for a point-like object in a D -dimensional spacetime is:

$$ds^2 \approx -(1 + 2\varphi) dt^2 + \left(1 - \frac{2}{D-3}\varphi\right) (dx_1^2 + \cdots + dx_{D-1}^2), \quad (58)$$

where the function φ parameterizes the deviation of the metric from the flat Minkowski metric in D dimensions. The specific form (58) can be obtained by solving the linearized Einstein equations and generalizes the standard result in general relativity in four dimensions. As in this latter particular case, one can identify the function φ with the Newtonian gravitational potential, as seen by comparing the motion along a geodesic and Newton's law. Thus:

$$\varphi \sim \frac{GM}{r^{D-3}}, \quad (59)$$

where M is the mass of the particle and $r = \sqrt{x_1^2 + \cdots + x_{D-1}^2}$ is the radial coordinate of the space. The function φ is a solution of the Poisson equation in $D-1$ dimensions. Notice that the power $D-3$ is equal to the number d_T of dimensions transverse to the object (*i.e.* $d_T = D-1$) minus 2:

$$D-3 = d_T - 2. \quad (60)$$

For an extended object (along p spatial directions) the corresponding linearized metric takes the form:

$$ds^2 \approx (1 + 2\varphi) (-dt^2 + dx_1^2 + \cdots + dx_p^2) + \left(1 - \frac{2(p+1)}{D-p-3}\varphi\right) (dx_{p+1}^2 + \cdots + dx_{D-1}^2). \quad (61)$$

Since now the number of transverse directions is $d_T = D-1-p$, the function φ must be:

$$\varphi \sim \frac{GM}{r^{D-p-3}}. \quad (62)$$

The D3-brane solution is a solution of 10d supergravity corresponding to a stack of N coincident D3-branes. The (exact) metric takes the form:

$$ds^2 = H^{-\frac{1}{2}} \left(-dt^2 + dx_1^2 + dx_2^2 + dx_3^2 \right) + H^{\frac{1}{2}} (dr^2 + r^2 d\Omega_5^2) , \quad (63)$$

with $d\Omega_5^2$ is the line element of a unit $\mathbb{S}^5$. Notice that $dr^2 + r^2 d\Omega_5^2$ is the flat metric of $\mathbb{R}^6$. The function $H(r)$ is the so-called warp factor:

$$H = 1 + \frac{L^4}{r^4} , \quad (64)$$

with the constant L being given by:

$$L^4 = 4\pi g_s N l_s^4 . \quad (65)$$

At linearized level (when $r \rightarrow \infty$) the solution (63) corresponds to the general solution in (61) with $D = 10$, $p = 3$ and the function ϕ given by:

$$\varphi = -\frac{1}{4} \frac{L^4}{r^4} . \quad (66)$$

Comparing this expression of φ with the one written in (62) for $D = 10$ and $p = 3$ we conclude that:

$$GM \sim L^4 . \quad (67)$$

Let us find this last relation from a different reasoning. As $G \sim g_s^2 l_s^8$ (see (57)) and since M should be the mass of a stack of N D3-branes with tension $T_{D3} \sim 1/(g_s l_s^4)$, we should have:

$$GM \sim g_s^2 l_s^8 \frac{N}{g_s l_s^4} \sim N g_s l_s^4 , \quad (68)$$

which, apart from a numerical factor is precisely the value of L^4 written in (65).

The geometry of this solution is asymptotically the Minkowski spacetime in 10d with a throat of infinite size (see figure 1). In the throat we can take $r \ll 1$ and neglect the 1 in the function H . This defines the so-called near-horizon limit. Actually, taking

$$H \approx \frac{L^4}{r^4} , \quad (69)$$

the metric (63) becomes:

$$ds^2 = \frac{r^2}{L^2} (-dt^2 + dx_1^2 + dx_2^2 + dx_3^2) + \frac{L^2}{r^2} dr^2 + L^2 d\Omega_5^2 . \quad (70)$$

To identify this near-horizon metric, let us change variable as:

$$r = \frac{L}{z} . \quad (71)$$

Then, the line element (70) can be rewritten as:

$$ds^2 = \frac{L^2}{z^2} (-dt^2 + dx_1^2 + dx_2^2 + dx_3^2 + dz^2) + L^2 d\Omega_5^2, \quad (72)$$

which is precisely the metric of the product space $AdS_5 \times \mathbb{S}^5$, with L being the radius of both factors.

7 The AdS/CFT for $\mathcal{N} = 4$ $SU(N)$ SYM

We are now in a position to formulate the AdS/CFT correspondence, as originally proposed by Maldacena in [1]. We have seen two different descriptions of the D3-brane: as a gauge theory and as a gravity solution. Maldacena proposed that the two descriptions are dual to each other. More generally, it was conjectured in [1] that $\mathcal{N} = 4$ SYM theory with gauge group $SU(N)$ is equivalent to string theory in $AdS_5 \times S^5$.

In order to make the duality more precise, it is quite convenient to find the relation of the two parameters on the two sides of the correspondence. First of all, by combining (65) and (55) we can write the ratio between the AdS radius L and the string length l_s in terms of gauge theory quantities:

$$\left(\frac{L}{l_s}\right)^4 = N g_{YM}^2. \quad (73)$$

Notice that the right-hand side of (73) is just the 't Hooft coupling $\lambda = N g_{YM}^2$. Therefore, we can rewrite (73) as:

$$\frac{l_s^2}{L^2} = \frac{1}{\sqrt{\lambda}}. \quad (74)$$

Moreover, by combining (57) and (55) we can write the ten-dimensional Newton constant G , and the corresponding Planck length l_P , in terms of g_{YM} , namely:

$$G = l_P^8 = \frac{\pi^4}{2} g_{YM}^4 l_s^8. \quad (75)$$

Then, the relation between the Planck length and the AdS radius is given by

$$\left(\frac{l_P}{L}\right)^8 = \frac{\pi^4}{2N^2}. \quad (76)$$

The relations (74) and (76) are essential to delimitate the domain of validity of the dual description in terms of classical gravity. As discussed above, we must require that $l_P/L \ll 1$ if we want to avoid having quantum gravity corrections (this is equivalent to the condition of having small curvature). From (76) is clear that one should require that $N \gg 1$. Notice this conclusion agrees with our general discussion at the end of section 2. Moreover, in order

to avoid stringy corrections due to the full tower of massive states of the string, we must require that $l_s/L \ll 1$ which, in view of (74), means that the 't Hooft coupling must be large. Therefore, we conclude that the gravitational description of $\mathcal{N} = 4$ SYM is reliable if:

$$N \gg 1, \quad \lambda \gg 1. \quad (77)$$

Then, the planar (large N), strongly coupled (large λ) SYM theory can be described as classical gravity. As a first check of the correspondence, let us look at the symmetries on both sides.

7.1 Conformal symmetry

The $\mathcal{N} = 4$ SYM has an exact vanishing β -function and, therefore is a CFT (invariant under the conformal group, which includes dilatations and special conformal transformations). On the gravity side, the space AdS_5 has the four-dimensional conformal group $SO(2, 4)$ as isometry group. The $SO(2, 4)$ group acts on the boundary of AdS_5 as the conformal group in four dimensions. In particular, the dilatations act as:

$$(t, \vec{x}) \rightarrow \lambda(t, \vec{x}), \quad z \rightarrow \lambda z. \quad (78)$$

Therefore, the conformal symmetry is also realized on the gravity side if one identifies the field theory Minkowski spacetime as the boundary of AdS_5 .

It is also interesting to relate the different scales on both theories. From the AdS metric we see (due to the L^2/z^2 factor in front of the metric) that the proper distance d on the bulk and the distance d_{YM} on the Minkowski coordinates are related as:

$$d = \frac{L}{z} d_{YM}. \quad (79)$$

For the energies this relation is inverted, since the energy is conjugated to the time. Then:

$$E = \frac{z}{L} E_{YM}. \quad (80)$$

Thus, the high-energy limit (UV) in the field theory ($E_{YM} \rightarrow \infty$) corresponds to the near-boundary region $z \rightarrow 0$. Conversely, as $E_{YM} \rightarrow 0$ is equivalent to $z \rightarrow \infty$, the low energy limit (IR) corresponds to the near-horizon region $z \rightarrow \infty$.

In a conformal theory, there exist excitations at arbitrary low energies. This corresponds to the fact that in AdS the geometry extends all the way to $z \rightarrow \infty$. On the contrary, in a non-conformal theory there should be a minimal scale and the geometry should end smoothly at some z_0 . This is important when one generalizes the correspondence to non-conformal theories. There are two prominent examples of this generalized duality. The first case corresponds to the confining theories with a mass gap m . In this case $z_0 \sim 1/m$. The second case are the theories at finite temperature. Let T denote the temperature. In this case the geometry is a black hole with a horizon and $z_0 \sim 1/T$.

7.2 Supersymmetry

Let us now see how one can match the supersymmetry on both sides of the correspondence. On the field theory side, the $\mathcal{N} = 4$ SYM theory is maximally supersymmetric. It has 32 fermionic supercharges generated by 4 sets of complex Majoranas $Q_\alpha^A, \bar{Q}_\alpha^A$ ($A = 1 \cdots 4$). This corresponds to having $\mathcal{N} = 4$ supersymmetry in four dimensions. The Q^A can be rotated with the group $SU(4)$, which is the so-called R -symmetry group. Under this group the supercharges transform as:

$$Q_\alpha^A \rightarrow \mathbf{4} , \quad \bar{Q}_\alpha^A \rightarrow \bar{\mathbf{4}} . \quad (81)$$

In addition, the theory has six scalars $\phi_1, \cdots \phi_6$, which transform as the fundamental representation $\mathbf{6}$ of $SO(6)$. Notice the Lie algebra isomorphism $SU(4) \approx SO(6)$.

The $AdS_5 \times S^5$ space is also a maximally supersymmetric solution of ten-dimensional supergravity. It has 32 Killing spinors, which correspond to the supercharges of $\mathcal{N} = 4$ SYM. The rotational symmetry of the five-sphere is $SO(6)$, which can be identified with the R -symmetry of $\mathcal{N} = 4$ SYM. The directions along S^5 correspond to the scalar fields on SYM. Thus, we have a perfect matching with the fields and symmetries of $\mathcal{N} = 4$ SYM. Notice that the scalar fields of the field theory are extra coordinates on the gravity side and the isometries of the compact space are interpreted as internal rotations of the scalar fields and supercharges.

7.3 Reduction on the S^5

Let us now discuss further the role of the five-sphere in the AdS dual of $\mathcal{N} = 4$ SYM. We begin by noticing that any field on $AdS_5 \times S^5$ can be reduced to a tower of fields on AdS_5 by expanding it in terms of the harmonics on S^5 :

$$\phi(x, \Omega) = \sum_l \phi_l(x) Y_l(\Omega) , \quad (82)$$

with x being coordinates of AdS_5 , Ω are coordinates of S^5 and $Y_l(\Omega)$ are spherical harmonics on S^5 . The gravity action, after reducing on S^5 , becomes:

$$S = \frac{1}{16\pi G_5} \int d^5x \left[\mathcal{L}_{grav} + \mathcal{L}_{matter} \right] , \quad (83)$$

where G_5 is the five-dimensional Newton constant, and the gravitational part of the $\mathcal{L}$ is:

$$\mathcal{L}_{grav} = \sqrt{-g} \left[R + \frac{12}{L^2} \right] , \quad (84)$$

which corresponds to a negative cosmological constant $\Lambda = -6/L^2$. The five-dimensional Newton constant can be related to the 10d constant by considering the reduction of the

Einstein-Hilbert term:

$$\frac{1}{16\pi G} \int d^5x d^5\Omega \sqrt{-g_{10}} R_{10} \rightarrow \frac{L^5 \Omega_5}{16\pi G} \int d^5x \sqrt{-g_5} R_5 , \quad (85)$$

where $\Omega_5 = \pi^3$ is the volume of a unit $\mathbb{S}^5$. Then it follows that G_5 and G can be related as:

$$G_5 = \frac{G}{L^5 \Omega_5} = \frac{G}{\pi^3 L^5} . \quad (86)$$

Using the value of the ten-dimensional Newton constant G written in (57), we get that G_5 is given by:

$$G_5 = \frac{\pi}{2N^2} L^3 . \quad (87)$$

From this formula and (27) we can compute the central charge of the SYM theory:

$$c_{SYM} = \frac{1}{4} \frac{L^3}{G_5} = \frac{N^2}{2\pi} . \quad (88)$$

Then, the large N limit corresponds to having a large central charge, in agreement with our previous discussions on the validity domain of the gravitational description.

Let us now consider the reduction (82) of a massless scalar field in $AdS_5 \times \mathbb{S}^5$. The Klein-Gordon equation in ten dimensions is:

$$\nabla^2 \phi = 0 . \quad (89)$$

Since the metric factorizes into a AdS_5 and $\mathbb{S}^5$ part, the D'Alembertian is additive

$$\nabla^2 = \nabla_{AdS_5}^2 + \nabla_{\mathbb{S}^5}^2 , \quad (90)$$

where $\nabla_{\mathbb{S}^5}^2$ is nothing but the quadratic Casimir operator in $SO(6)$. The eigenvalues of $\nabla_{\mathbb{S}^5}^2$ acting on the spherical harmonics are:

$$\nabla_{\mathbb{S}^5}^2 Y_l(\Omega) = -\frac{C_l^{(5)}}{L^2} Y_l(\Omega) , \quad (91)$$

where the $C_l^{(5)}$ are given by:

$$C_l^{(5)} = l(l+4) , \quad l = 0, 1, 2, \dots . \quad (92)$$

Thus, the reduced AdS_5 fields ϕ_l satisfy the massive Klein-Gordon equation

$$\nabla_{AdS_5}^2 \phi_l = m_l^2 \phi_l , \quad m_l^2 = \frac{l(l+4)}{L^2} . \quad (93)$$

Therefore, we have a tower of massive fields ϕ_l , with a particular set of masses, which originate, after dimensional Kaluza-Klein (KK) reduction on the five-sphere, from a single massless scalar field ϕ in ten-dimensions. These fields should have a field theory dual in the $\mathcal{N} = 4$ theory and the mass spectrum (93) should have a counterpart on the field theory side. We will see that this is indeed the case in the next section.

8 A scalar field in AdS

We argued in section [1](#) that fields in AdS correspond to sources of operators on the field theory side and that we can learn about these dual operators by analyzing the dynamics of the sources in the curved space. In this section we study the simplest case of a scalar field in AdS . Accordingly, let us consider the AdS_{d+1} space in euclidean signature with metric:

$$ds^2 = \frac{L^2}{z^2} [dz^2 + \delta_{\mu\nu} dx^\mu dx^\nu] . \quad (94)$$

The action of a scalar field ϕ in the AdS_{d+1} space is:

$$S = -\frac{1}{2} \int d^{d+1}x \sqrt{g} \left[g^{MN} \partial_M \phi \partial_N \phi + m^2 \phi^2 \right] . \quad (95)$$

The equation of motion derived from the action [\(95\)](#) is:

$$\frac{1}{\sqrt{g}} \partial_M \left(\sqrt{g} g^{MN} \partial_N \phi \right) - m^2 \phi = 0 . \quad (96)$$

More explicitly, after using the metric [\(94\)](#), this equation becomes :

$$z^{d+1} \partial_z \left(z^{1-d} \partial_z \phi \right) + z^2 \delta^{\mu\nu} \partial_\mu \partial_\nu \phi - m^2 L^2 \phi = 0 . \quad (97)$$

Let us perform the Fourier transform of ϕ in the x^μ coordinates:

$$\phi(z, x^\mu) = \int \frac{d^d k}{(2\pi)^d} e^{ik \cdot x} f_k(z) . \quad (98)$$

Then, the equation of motion becomes:

$$z^{d+1} \partial_z \left(z^{1-d} \partial_z f_k \right) - k^2 z^2 f_k - m^2 L^2 f_k = 0 . \quad (99)$$

Let us solve [\(99\)](#) near the boundary $z = 0$. We put $f_k \sim z^\beta$ for some exponent β and keep the leading terms near $z = 0$. Then, it is straightforward to find that β must satisfy the following quadratic expression:

$$\beta(\beta - d) - m^2 L^2 = 0 , \quad (100)$$

whose solutions are:

$$\beta = \frac{d}{2} \pm \sqrt{\frac{d^2}{4} + m^2 L^2} . \quad (101)$$

Therefore, near $z \sim 0$ the function $f_k(z)$ behaves as :

$$f_k(z) \approx A(k) z^{d-\Delta} + B(k) z^\Delta , \quad (102)$$

where Δ is given by:

$$\Delta = \frac{d}{2} + \nu, \quad \nu = \sqrt{\frac{d^2}{4} + m^2 L^2}. \quad (103)$$

By performing the inverse Fourier transform we can write the expansion near the boundary in position space:

$$\phi(z, x) \approx A(x) z^{d-\Delta} + B(x) z^\Delta, \quad z \rightarrow 0. \quad (104)$$

Notice that Δ is real if $\nu \in \mathbb{R}$, which happens if the mass m satisfies the so-called Breitenlohner-Freedman (BF) bound:

$$m^2 \geq -\left(\frac{d}{2L}\right)^2. \quad (105)$$

This means that m^2 can be negative (and the field ϕ can be tachyonic) but it must satisfy the BF bound. In what follows we will suppose that the BF bound is satisfied. Moreover, notice that:

$$d - \Delta \leq \Delta \quad \Longleftrightarrow \quad \nu = 2\Delta - d \geq 0, \quad (106)$$

which is obviously satisfied in the mass is above the BF bound. Then, the term behaving as $z^{d-\Delta}$ in (104) is the dominant one as $z \rightarrow 0$. Let us take the boundary as $z = \epsilon$ and neglect the subdominant term. We have:

$$\phi(z = \epsilon, x) \approx \epsilon^{d-\Delta} A(x). \quad (107)$$

As $d - \Delta$ is negative if $m^2 > 0$, the leading term is typically divergent as we approach the boundary at $z = \epsilon \rightarrow 0$. In order to identify the QFT source $\varphi(x)$ from the boundary value of the field $\phi(z, x)$ we have to remove the divergences of the latter. We will simply do it by extracting the divergent multiplicative factor from (107), *i.e.* the QFT source $\varphi(x)$ is identified with $A(x)$. Equivalently, we define:

$$\varphi(x) = \lim_{z \rightarrow 0} z^{\Delta-d} \phi(z, x). \quad (108)$$

Clearly, with this definition $\varphi(x)$ is always finite. Conversely we can write $\phi(z, x) = z^{d-\Delta} \varphi(x)$ at leading order. In order to interpret the meaning of Δ , let us look at the boundary action. If $\mathcal{O}$ is the operator dual to ϕ , this action is given by:

$$S_{bdy} \sim \int d^d x \sqrt{\gamma_\epsilon} \phi(\epsilon, x) \mathcal{O}(\epsilon, x), \quad (109)$$

where $\gamma_\epsilon = \left(\frac{L}{\epsilon}\right)^{2d}$ is the determinant of the induced metric at the $z = \epsilon$ boundary. Then, by plugging $\phi(\epsilon, x) = \epsilon^{d-\Delta} \varphi(x)$ inside S_{bdy} , we get:

$$S_{bdy} \sim L^d \int d^d x \varphi(x) \epsilon^{-\Delta} \mathcal{O}(\epsilon, x). \quad (110)$$

In order to make S_{bdy} finite and independent of ϵ as $\epsilon \rightarrow 0$ we should require:

$$\mathcal{O}(\epsilon, x) = \epsilon^\Delta \mathcal{O}(x) . \quad (111)$$

But, passing from $z = 0$ to $z = \epsilon$ is a scale transformation in the QFT. Thus, Δ must be interpreted as the mass scaling dimension of the dual operator $\mathcal{O}$. In other words, the behavior $\mathcal{O}(\epsilon, x) = \epsilon^\Delta \mathcal{O}(x)$ is the wave function renormalization of $\mathcal{O}$ as we move into the bulk. Similarly, from the relation $\phi(\epsilon, x) = \epsilon^{d-\Delta} \varphi(x)$ it follows that $d - \Delta$ is the mass scaling dimension of the source φ . Remember from (103) that:

$$\Delta = \frac{d}{2} + \sqrt{\left(\frac{d}{2}\right)^2 + m^2 L^2} . \quad (112)$$

Then, if $m^2 \geq -\left(\frac{d}{2L}\right)^2$, the corresponding scaling dimension is real. We consider three different situations, depending on the value of the mass m . The first case we will study is the one in which $m^2 > 0$. In this case $\Delta > d$ and the corresponding operator $\mathcal{O}$ is called an irrelevant operator. By deforming the CFT with $\mathcal{O}$, we have the following term in the action:

$$\Delta S = \int d^d x (mass)^{d-\Delta} \mathcal{O} = \int d^d x (mass)^{\text{negative number}} \mathcal{O} . \quad (113)$$

When computing amplitudes with this action one gets that the effect of the interaction depends on $\left(\frac{Energy}{mass}\right)^\alpha$, for some $\alpha > 0$. Then, the effects of this interaction can be neglected for low energies since they are suppressed by powers. Therefore, the interaction goes away in the IR but changes completely the UV of the theory.

When $m^2 = 0$ we have from (112) that $\Delta = d$ and the corresponding operator is called marginal. Finally, if m^2 is negative and takes value in the range $-(d/2L)^2 < m^2 < 0$, the dual operator it is called relevant, since it has $\Delta < d$ and changes the IR of the theory.

8.1 Application to the KK spectrum in $AdS_5 \times S^5$

We have seen in section 7.3 that a massless scalar in ten-dimensions gives rise to the tower (93) of massive KK scalars in AdS_5 . We will now apply the formula (112) to these five-dimensional scalar fields. First of all, the dimension/mass relation for the case $d = 4$ is:

$$\Delta = 2 + \sqrt{4 + (m L)^2} . \quad (114)$$

Let us first consider a massless scalar, which corresponds to the s-wave ($l = 0$) in (7.3). In this case (114) with $m = 0$ gives $\Delta = 4$. Then, the QFT dual operator should be a scalar operator of dimension 4. Since this s-wave operator is singlet under the $SO(6)$ symmetry of the S^5 , it must not contain the ϕ_i scalars of the dual $\mathcal{N} = 4$ QFT. The only candidate with these characteristics is the glueball operator:

$$\mathcal{O} = \text{Tr}[F_{\mu\nu} F^{\mu\nu}] . \quad (115)$$

Notice that $\dim[\partial]=\dim[A] = 1$, so, indeed, $\dim(\mathcal{O})=4$. For higher order KK modes, the masses are $m^2 L^2 = l(l+4)$ (see (93)). Then, the dimensions are:

$$\Delta_l = 2 + \sqrt{4 + l(l+4)} = 4 + l . \quad (116)$$

In this case, the dual operator should transform under the corresponding representation of $SO(6)$ (a symmetric tensor with l indices). One can construct such a tensor by multiplying $\mathcal{N} = 4$ SYM scalar fields ϕ_i (they transform as vectors of $SO(6)$). Then, the natural operator dual to the l^{th} KK mode is:

$$\mathcal{O}_{i_1, \dots, i_l} = \text{Tr}[\phi_{(i_1, \dots, i_l)} F_{\mu\nu} F^{\mu\nu}] , \quad (117)$$

with $\phi_{(i_1, \dots, i_l)}$ being the traceless symmetric product of l scalar fields ϕ_i of $\mathcal{N} = 4$ SYM. As $\dim[\phi] = 1$, one can check immediately that the dimension of the operator $\mathcal{O}_{i_1, \dots, i_l}$ in (117) is indeed $4 + l$, in agreement with the AdS/CFT result. It has been checked that this agreement can be extended to all the KK modes of 10d supergravity on $AdS_5 \times S^5$ (including fermions, forms,...).

8.2 Normalizable and non-normalizable modes

The natural inner product for solutions of the Klein-Gordon equation in a curved space is:

$$(\phi_1, \phi_2) = -i \int_{\Sigma_t} dz d^d x \sqrt{-g} g^{tt} (\phi_1^* \partial_t \phi_2 - \phi_2 \partial_t \phi_1^*) , \quad (118)$$

with Σ_t being a constant- t slice. Let us consider in particular a field that behaves as $\phi \sim z^\beta$ near $z \approx 0$. In the AdS_{d+1} metric $\sqrt{-g} g^{tt} \sim z^{-d+1}$. Then, $\phi \partial_t \phi \sim z^{2\beta}$ as $z \rightarrow 0$ and the integrand in the norm of ϕ behaves near $z = 0$ as:

$$\sqrt{-g} g^{tt} \phi \partial_t \phi \sim z^{2\beta-d+1} , \quad (119)$$

which leads to a convergent integral if $2\beta - d + 2 > 0$.

Let us now consider the two types of modes in (104). We begin with the subleading modes $B(x)z^\Delta$. In this case the exponent β is $\beta = \Delta = \frac{d}{2} + \nu$ and, therefore, $2\beta - d + 2 = 2\nu + 2$, which is always positive. This mode is normalizable and can be considered as an element of the bulk Hilbert space and, given the equivalence with the dual theory, it should be identified with some state in the boundary theory. Next, we analyze the normalizability of the leading modes $A(x)z^{d-\Delta}$. For these modes $\beta = d - \Delta$ and, thus, $2\beta - d + 2 = 2(1 - \nu)$, which is negative if $\nu \geq 1$. Notice that this condition is equivalent to $m^2 L^2 \geq -\frac{d^2}{4} + 1$. Then, the non-normalizable modes correspond to sources in the boundary theory. Interestingly, for masses in the range $-\frac{d^2}{4} \leq m^2 \leq -\frac{d^2}{4} + 1$ both types of modes are normalizable and they give rise to different Fock spaces of physical states.

8.3 Higher spin fields

The results for the scalar field can be generalized to any p -form field, *i.e.* to any antisymmetric tensor $A_{\mu_1 \dots \mu_p}$ with p indices. For a p -form field of mass m , the dimension Δ of the dual operator is the largest root of the quadratic equation:

$$(\Delta - p)(\Delta + p - d) = m^2 L^2 . \quad (120)$$

This equation can be solved to give the following dimension/mass relation:

$$\Delta = \frac{d}{2} + \sqrt{\left(\frac{d-2p}{2}\right)^2 + m^2 L^2} . \quad (121)$$

In particular, for a massive vector field A_μ , the previous formula (121) with $p = 1$ gives:

$$\Delta = \frac{d}{2} + \sqrt{\left(\frac{d-2}{2}\right)^2 + m^2 L^2} . \quad (122)$$

Taking $m = 0$ in (122), we get $\Delta = d - 1$, which is the dimension of a conserved current j_μ in d dimensions. Finally, one can prove that the dimension/mass relation for a spin 1/2 field is:

$$\Delta = \frac{d}{2} + |m L| . \quad (123)$$

9 Correlation functions

Let us now see how one can compute correlation functions in Euclidean space from gravity. The objective is to obtain Euclidean correlation functions of the type:

$$\langle \mathcal{O}(x_1) \cdots \mathcal{O}(x_n) \rangle . \quad (124)$$

In field theory these correlators can be calculated from a generating function, which is obtained by perturbing the lagrangian by a source term:

$$\mathcal{L} \rightarrow \mathcal{L} + J(x) \mathcal{O}(x) \equiv \mathcal{L} + \mathcal{L}_J . \quad (125)$$

The generating functional is just:

$$Z_{QFT}[J] = \left\langle \exp\left[\int \mathcal{L}_J\right] \right\rangle_{QFT} . \quad (126)$$

The connected correlators are obtained from the functional derivatives of Z :

$$\left\langle \prod_i \mathcal{O}(x_i) \right\rangle = \prod_i \frac{\delta}{\delta J(x_i)} \log Z_{QFT}[J] \Big|_{J=0} \quad (127)$$

Let us consider now any bulk field $\phi(z, x)$ fluctuating in AdS . Let $\phi_0(x)$ be the boundary value of ϕ :

$$\phi_0(x) = \phi(z = 0, x) = \phi|_{\partial AdS}(x) . \quad (128)$$

The field ϕ_0 is related to a source for some dual operator $\mathcal{O}$ in the QFT. As we know the actual source is not the value of ϕ at $z = 0$, which is typically divergent, but the limit:

$$\lim_{z \rightarrow 0} z^{\Delta-d} \phi(z, x) = \varphi(x) . \quad (129)$$

Then, the AdS/CFT prescription for the generating functional is [2, 3]:

$$Z_{QFT}[\phi_0] = \left\langle \exp \left[\int \phi_0 \mathcal{O} \right] \right\rangle_{QFT} = Z_{gravity}[\phi \rightarrow \phi_0] , \quad (130)$$

where $Z_{gravity}[\phi \rightarrow \phi_0]$ is the partition function (*i.e.* the path integral) in the gravity theory evaluated over all functions which have the value ϕ_0 at the boundary of AdS :

$$Z_{gravity}[\phi \rightarrow \phi_0] = \sum_{\{\phi \rightarrow \phi_0\}} e^{S_{gravity}} . \quad (131)$$

In the limit in which classical gravity dominates, one can substitute the sum by the term corresponding to the classical solution. In this case the generating function becomes:

$$Z_{QFT}[\phi_0] \approx e^{S_{gravity}^{on-shell}[\phi \rightarrow \phi_0]} . \quad (132)$$

One should be careful when evaluating the on-shell gravity action because it typically diverges and has to be renormalized following the procedure of holographic renormalization [19, 20] (see below and the review [21]). Thus, the classical action must be substituted by a renormalized version, which will be denoted by S_{grav}^{ren} and the generating functional becomes:

$$\log Z_{QFT} = S_{grav}^{ren}[\phi \rightarrow \phi_0] . \quad (133)$$

Moreover, the n -point function can be obtained by computing the derivatives with respect to $\varphi = z^{\Delta-d} \phi$:

$$\langle \mathcal{O}(x_1) \cdots \mathcal{O}(x_n) \rangle = \frac{\delta^{(n)} S_{grav}^{ren}[\phi]}{\delta \varphi(x_1) \cdots \delta \varphi(x_n)} \Big|_{\varphi=0} . \quad (134)$$

9.1 One-point function

It is also interesting to compute the one-point function of an operator $\mathcal{O}$ in the presence of the source φ :

$$\langle \mathcal{O}(x) \rangle_{\varphi} = \frac{\delta S_{grav}^{ren}[\phi]}{\delta \varphi(x)} . \quad (135)$$

Taking into account the relation between ϕ and φ (eq. (129)), we get:

$$\langle \mathcal{O}(x) \rangle_\varphi = \lim_{z \rightarrow 0} z^{d-\Delta} \frac{\delta S_{grav}^{ren}[\phi]}{\delta \phi(z, x)} . \quad (136)$$

The functional derivative of the classical on-shell action can be computed in closed form. Indeed, let S_{grav} be represented as:

$$S_{grav} = \int_{\mathcal{M}} \int dz d^d x \mathcal{L}[\phi, \partial\phi] , \quad (137)$$

with $\mathcal{M}$ being a $(d+1)$ -dimensional manifold whose boundary is located at $z = 0$. Under a general change $\phi \rightarrow \phi + \delta\phi$, the classical action S_{grav} varies as:

$$\delta S_{grav} = \int_{\mathcal{M}} \int dz d^d x \left[\frac{\partial \mathcal{L}}{\partial \phi} \delta\phi + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \delta(\partial_\mu \phi) \right] . \quad (138)$$

Let us now use in (138) that $\delta[\partial_\mu \phi] = \partial_\mu(\delta\phi)$ and let us integrate by parts. We get:

$$\delta S_{grav} = \int_{\mathcal{M}} \int dz d^d x \left[\left(\frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \right) \right) \delta\phi + \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \delta\phi \right) \right] . \quad (139)$$

The first term in the previous equation vanishes on-shell due to the Euler-Lagrange equations. As the boundary is at $z = \epsilon \rightarrow 0$, we can write:

$$\delta S_{grav}^{on-shell} = \int_\epsilon^\infty \int d^d x \partial_z \left(\frac{\partial \mathcal{L}}{\partial(\partial_z \phi)} \delta\phi \right) = - \int_{\partial M} d^d x \frac{\partial \mathcal{L}}{\partial(\partial_z \phi)} \delta\phi \Big|_{z=\epsilon} . \quad (140)$$

Let us next define Π as:

$$\Pi = - \frac{\partial \mathcal{L}}{\partial(\partial_z \phi)} , \quad (141)$$

which is the canonical momentum if z is taken as time. Then (140) can be rewritten as:

$$\delta S_{grav}^{on-shell} = \int_{\partial M} d^d x \Pi(\epsilon, x) \delta\phi(\epsilon, x) . \quad (142)$$

Thus, it follows that:

$$\frac{\delta S_{grav}^{on-shell}}{\delta \phi(\epsilon, x)} = \Pi(\epsilon, x) = - \frac{\partial \mathcal{L}}{\partial(\partial_z \phi)} . \quad (143)$$

In general the renormalized action can be written as:

$$S^{ren} = S_{grav}^{on-shell} + S_{ct} , \quad (144)$$

where S_{ct} is the action of counterterms, defined at the boundary $z = \epsilon$. Let us define the renormalized momentum as:

$$\Pi^{ren}(z, x) = \frac{\delta S^{ren}}{\delta \phi(z, x)} . \quad (145)$$

Clearly, by taking $z = \epsilon$ in (145) and (144) we have:

$$\Pi^{ren}(\epsilon, x) = -\frac{\partial \mathcal{L}}{\partial(\partial_z \phi(\epsilon, x))} + \frac{\delta S_{ct}}{\delta \phi(\epsilon, x)} . \quad (146)$$

Therefore, from the definition of Π^{ren} in (145) and eq. (136), we obtain the one-point function of $\mathcal{O}$ in presence of the source φ as the following limit in the AdS boundary:

$$\langle \mathcal{O}(x) \rangle_\varphi = \lim_{z \rightarrow 0} z^{d-\Delta} \Pi^{ren}(z, x) . \quad (147)$$

9.2 Linear response theory

The field theory path integral representation of the one-point function with a source is:

$$\langle \mathcal{O}(x) \rangle_\varphi = \int [D\psi] \mathcal{O}(x) e^{S_E[\psi] + \int d^d y \varphi(y) \mathcal{O}(y)} , \quad (148)$$

where ψ denotes the fields of the QFT. Let us expand the exponent of this expression in a power series of the source φ and let us keep the terms up to linear order:

$$\langle \mathcal{O}(x) \rangle_\varphi = \langle \mathcal{O}(x) \rangle_{\varphi=0} + \int d^d y \langle \mathcal{O}(x) \mathcal{O}(y) \rangle \varphi(y) + \dots . \quad (149)$$

Next, we define the euclidean two-point function $G_E(x - y)$ as:

$$G_E(x - y) = \langle \mathcal{O}(x) \mathcal{O}(y) \rangle . \quad (150)$$

Then, equation (149) can be rewritten as:

$$\langle \mathcal{O}(x) \rangle_\varphi = \langle \mathcal{O}(x) \rangle_{\varphi=0} + \int d^d y G_E(x - y) \varphi(y) . \quad (151)$$

We will consider normal-ordered observables such that $\langle \mathcal{O}(x) \rangle_{\varphi=0}$ vanishes. Notice that this always can be achieved by subtracting to $\mathcal{O}$ its vacuum expectation value (VEV) without source. Then, $\langle \mathcal{O}(x) \rangle_\varphi$ measures the fluctuations of the observable away from the expectation value, *i.e.* the linear response of the system to the external perturbation and we can write:

$$\langle \mathcal{O}(x) \rangle_\varphi = \int d^d y G_E(x - y) \varphi(y) . \quad (152)$$

In momentum space this expression can be written as:

$$\langle \mathcal{O}(k) \rangle_\varphi = G_E(k) \varphi(k) , \quad (153)$$

and, thus, we can obtain the two-point function in momentum space by dividing the one-point function by the source:

$$G_E(k) = \frac{\langle \mathcal{O}(k) \rangle_\varphi}{\varphi(k)} . \quad (154)$$

In the framework of the AdS/CFT correspondence, since $\varphi = z^{\Delta-d}\phi(z, x)$ with $z \rightarrow 0$, we have the following formula for the two-point function in momentum space:

$$G_E(k) = \lim_{z \rightarrow 0} z^{2(d-\Delta)} \frac{\Pi^{ren}(z, k)}{\phi(z, k)} . \quad (155)$$

9.3 Two-point function for a scalar field

Let us apply the developments of the previous sections to the case in which the source ϕ is a scalar field in Euclidean AdS_{d+1} . We will assume that the action of ϕ is:

$$S = -\frac{\eta}{2} \int dz d^d x \sqrt{g} \left[g^{MN} \partial_M \phi \partial_N \phi + m^2 \phi^2 \right] , \quad (156)$$

where η is a normalization constant. In order to evaluate the on-shell action of ϕ we rewrite S as follows:

$$S = -\frac{\eta}{2} \int dz d^d x \partial_M \left[\sqrt{g} \phi g^{MN} \partial_N \phi \right] + \frac{\eta}{2} \int dz d^d x \phi \sqrt{g} \left[\frac{1}{\sqrt{g}} \partial_M \left(\sqrt{g} g^{MN} \partial_N \phi \right) - m^2 \phi \right] . \quad (157)$$

The second term in (157) vanishes when the equation of motion of ϕ is used. Then, the on-shell action is:

$$S^{on-shell} = -\frac{\eta}{2} \int dz d^d x \partial_M \left[\sqrt{g} \phi g^{MN} \partial_N \phi \right] . \quad (158)$$

Taking into account that the boundary is at $z = \epsilon$, which is the lower limit of the integration, the on-shell action (158) can be written as:

$$S^{on-shell} = \frac{\eta}{2} \int d^d x \left(\sqrt{g} \phi g^{zz} \partial_z \phi \right)_{z=\epsilon} . \quad (159)$$

Let us define the “canonical momentum” Π as in (141):

$$\Pi \equiv -\frac{\partial \mathcal{L}}{\partial(\partial_z \phi)} = \eta \sqrt{g} g^{zz} \partial_z \phi . \quad (160)$$

Then the on-shell action of ϕ becomes:

$$S^{on-shell} = \frac{1}{2} \int_{z=\epsilon} d^d x \Pi(z, x) \phi(z, x) . \quad (161)$$

Let us work in momentum space and Fourier transform the field ϕ and the canonical momentum Π :

$$\phi(z, x) = \int \frac{d^d k}{(2\pi)^d} e^{ik \cdot x} f_k(z) , \quad \Pi(z, x) = \int \frac{d^d k}{(2\pi)^d} e^{ik \cdot x} \Pi_k(z) \quad (162)$$

Then, Parseval identity allows to rewrite $S^{on-shell}$ as:

$$S^{on-shell} = \frac{1}{2} \int \frac{d^d k}{(2\pi)^d} \Pi_{-k}(z = \epsilon) f_k(z = \epsilon) . \quad (163)$$

Recalling that the function $f_k(z)$ behaves near $z \sim 0$ as in (102) and the definition of Π , we obtain:

$$\Pi(z, x) \approx \eta L^{d-1} \left[(d - \Delta) A(x) z^{-\Delta} + \Delta B(x) z^{\Delta-d} \right] , \quad (z \rightarrow 0) , \quad (164)$$

which, in momentum space becomes:

$$\Pi_{-k}(z) \approx \eta L^{d-1} \left[(d - \Delta) A(-k) z^{-\Delta} + \Delta B(-k) z^{\Delta-d} \right] , \quad (z \rightarrow 0) , \quad (165)$$

Let us use these results to compute the on-shell action, keeping the terms that do not vanish when $\epsilon \rightarrow 0$. We get:

$$S^{on-shell} = \frac{\eta}{2} L^{d-1} \int \frac{d^d k}{(2\pi)^d} \left[\epsilon^{-2\nu} (d - \Delta) A(-k) A(k) + d A(-k) B(k) \right] . \quad (166)$$

Notice that the first term in (166) is divergent. We will now find a counterterm to renormalize this divergence of the on-shell action. It must be a local quadratic functional defined at the boundary of AdS . The natural candidate would be a term proportional to:

$$\int_{\partial AdS} d^d x \sqrt{\gamma} \phi^2(\epsilon, x) , \quad (167)$$

where γ is the determinant of the induced metric $\gamma_{\mu\nu}$ on the boundary:

$$ds_{z=\epsilon}^2 = \gamma_{\mu\nu} dx^\mu dx^\nu = \frac{L^2}{\epsilon^2} \delta_{\mu\nu} dx^\mu dx^\nu . \quad (168)$$

It is easy to prove that:

$$\int_{\partial AdS} d^d x \sqrt{\gamma} \phi^2(\epsilon, x) = L^d \int \frac{d^d k}{(2\pi)^d} \left[\epsilon^{-2\nu} A(-k) A(k) + 2 A(-k) B(k) \right] , \quad (169)$$

Let us adjust the coefficient of the counterterm action in such a way that the leading divergence is cancelled. It is immediate to check that this counterterm action must be:

$$S_{ct} = -\frac{\eta}{2} \frac{d - \Delta}{L} \int_{\partial AdS} d^d x \sqrt{\gamma} \phi^2 , \quad (170)$$

or, equivalently, in momentum space:

$$S_{ct} = -\frac{\eta}{2} (d - \Delta) L^{d-1} \int \frac{d^d k}{(2\pi)^d} \left[\epsilon^{-2\nu} A(-k) A(k) + 2 A(-k) B(k) \right] . \quad (171)$$

The renormalized action $S^{ren} = S^{on-shell} + S_{ct}$ can be obtained by adding (166) and (171), with the result:

$$S^{ren} = \frac{\eta}{2} L^{d-1} (2\Delta - d) \int \frac{d^d k}{(2\pi)^d} A(-k) B(k) . \quad (172)$$

In order to extract the one-point function from S^{ren} we have to compute the functional derivative with respect to $\varphi(x)$ (recall that $\varphi(x) = A(x)$). However, the coefficient $B(x)$ also depends functionally on $A(x)$. To illustrate this point, let us represent $f_k(z)$ for arbitrary z (not necessarily small) as:

$$f_k(z) = A(k) \phi_1(z, k) + B(k) \phi_2(z, k) , \quad (173)$$

where $\phi_1(z, k)$ and $\phi_2(z, k)$ are independent solutions of the equation satisfied by $f_k(z)$, normalized in such a way that for $z \rightarrow 0$ they behave as:

$$\phi_1(z, k) \approx z^{d-\Delta} , \quad \phi_2(z, k) \approx z^\Delta . \quad (174)$$

(the explicit expressions of $\phi_1(z, k)$ and $\phi_2(z, k)$ are given below). To determine completely ϕ we have to impose regularity conditions in the deep IR $z \rightarrow \infty$. As we will see soon, this fixes uniquely the ratio B/A to a value which is independent of the value of the field at the boundary $z = 0$. Let us denote this ratio by χ :

$$\chi = \frac{B}{A} . \quad (175)$$

Clearly, we can write the renormalized action (172) as:

$$S^{ren} = \frac{\eta}{2} L^{d-1} (2\Delta - d) \int \frac{d^d k}{(2\pi)^d} \chi(k) \varphi(k) \varphi(-k) , \quad (176)$$

where we have used the fact that $\phi(k) = A(k)$. Then, as the functional derivative $\frac{\delta}{\delta \varphi(x)}$ is equivalent to $(2\pi)^d \frac{\delta}{\delta \varphi(-k)}$ in momentum space, we have:

$$\langle \mathcal{O}(k) \rangle_\varphi = (2\pi)^d \frac{\delta S^{ren}}{\delta \varphi(-k)} = \eta L^{d-1} (2\Delta - d) \chi(k) \varphi(k) . \quad (177)$$

Taking into account the definition of χ in (175) and that $2\Delta - d = 2\nu$, we can write (177) as:

$$\langle \mathcal{O}(k) \rangle_\varphi = 2\nu \eta L^{d-1} B(k) . \quad (178)$$

Thus, the subleading contribution near the boundary ($B(k)$) determines the VEV of the operator. The two-point function $G_E(k)$ can be obtained by dividing by the source (see (154)):

$$G_E(k) = 2\nu \eta L^{d-1} \frac{B(k)}{A(k)} . \quad (179)$$

Let us now calculate explicitly $A(k)$ and $B(k)$. We first define the function $g_k(z)$ as:

$$f_k(z) = z^{\frac{d}{2}} g_k(z) . \quad (180)$$

Then, one can easily check from (97) that $g_k(z)$ satisfies the equation:

$$z^2 \partial_z^2 g_k + z \partial_z g_k - (\nu^2 + k^2 z^2) g_k = 0 . \quad (181)$$

Eq. (181) is just the modified Bessel equation, whose two independent solutions can be taken to be $g_k = I_{\pm\nu}(kz)$, where $I_{\pm\nu}$ are modified Bessel functions. Thus, the two independent solutions for $f_k(z)$ are:

$$z^{\frac{d}{2}} I_{\pm\nu}(kz) . \quad (182)$$

Notice that for $z \rightarrow 0$ the modified Bessel functions behave as:

$$I_{\pm\nu}(z) \approx \frac{1}{\Gamma(1 \pm \nu)} \left(\frac{z}{2}\right)^{\pm\nu} , \quad (z \rightarrow 0) . \quad (183)$$

Then, we can take $\phi_1(z, k)$ and $\phi_2(z, k)$ in (173) as:

$$\phi_1(z, k) = \Gamma(1-\nu) \left(\frac{k}{2}\right)^\nu z^{\frac{d}{2}} I_{-\nu}(kz) , \quad \phi_2(z, k) = \Gamma(1+\nu) \left(\frac{k}{2}\right)^{-\nu} z^{\frac{d}{2}} I_\nu(kz) . \quad (184)$$

One can check that these functions have, indeed, the correct behavior when $z \rightarrow 0$. Then, by using (173) we get:

$$f_k(z) = z^{\frac{d}{2}} \left[\Gamma(1-\nu) \left(\frac{k}{2}\right)^\nu A(k) I_{-\nu}(kz) + \Gamma(1+\nu) \left(\frac{k}{2}\right)^{-\nu} B(k) I_\nu(kz) \right] . \quad (185)$$

Let us now impose that $f_k(z)$ is finite when $z \rightarrow \infty$. This condition determines a precise relation between the coefficients $A(k)$ and $B(k)$, as already mentioned above. When $z \rightarrow \infty$, the functions $I_{\pm\nu}(z)$ behave as:

$$I_{\pm\nu}(z) \approx \frac{e^z}{\sqrt{2\pi z}} , \quad (z \rightarrow \infty) . \quad (186)$$

Then, after changing $z \rightarrow kz$, we get for large z :

$$f_k(z) \approx \frac{z^{\frac{d}{2}} e^{kz}}{\sqrt{2\pi kz}} \left[\Gamma(1-\nu) \left(\frac{k}{2}\right)^\nu A(k) + \Gamma(1+\nu) \left(\frac{k}{2}\right)^{-\nu} B(k) \right] , \quad (187)$$

which diverges when $z \rightarrow \infty$ unless the coefficient in brackets vanishes. Then, we must require:

$$\frac{B(k)}{A(k)} = -\frac{\Gamma(1-\nu)}{\Gamma(1+\nu)} \left(\frac{k}{2}\right)^{2\nu} = \frac{\Gamma(-\nu)}{\Gamma(\nu)} \left(\frac{k}{2}\right)^{2\nu} . \quad (188)$$

Using this result we can compute the euclidean two-point function. We get:

$$G_E(k) = 2\nu \eta L^{d-1} \frac{\Gamma(-\nu)}{\Gamma(\nu)} \left(\frac{k}{2}\right)^{2\nu} . \quad (189)$$

Let us write this result in position space. The relation between $G_E(x)$ and $G_E(k)$ is:

$$G_E(x) = \int \frac{d^d k}{(2\pi)^d} e^{ikx} G_E(k) . \quad (190)$$

We now use the formula:

$$\int \frac{d^d k}{(2\pi)^d} e^{ikx} k^n = \frac{2^n}{\pi^{\frac{d}{2}}} \frac{\Gamma(\frac{d+n}{2})}{\Gamma(-\frac{n}{2})} \frac{1}{|x|^{d+n}} , \quad (191)$$

to obtain the correlator in momentum space:

$$\langle \mathcal{O}(x) \mathcal{O}(0) \rangle = \frac{2\nu \eta L^{d-1}}{\pi^{\frac{d}{2}}} \frac{\Gamma(\frac{d}{2} + \nu)}{\Gamma(-\nu)} \frac{1}{|x|^{2\Delta}} . \quad (192)$$

The behavior $\langle \mathcal{O}(x) \mathcal{O}(0) \rangle \sim |x|^{-2\Delta}$ in (192) confirms that Δ is, indeed, the scaling dimension of the operator $\mathcal{O}(x)$.

10 Quark-antiquark potential

Let us consider an external charge moving along a closed curve $\mathcal{C}$ in spacetime in QED. The action for such a charge is:

$$S_{\mathcal{C}} = \oint_{\mathcal{C}} A_{\mu} dx^{\mu} . \quad (193)$$

Adding this term to the action is equivalent to insert in the path integral the quantity:

$$e^{iS_{\mathcal{C}}} = e^{i \oint_{\mathcal{C}} A_{\mu} dx^{\mu}} \equiv W(\mathcal{C}) , \quad (194)$$

where $W(\mathcal{C})$ is the so-called Wilson loop, which is just the holonomy of the gauge field A_{μ} along the closed curve $\mathcal{C}$. Notice that $W(\mathcal{C})$ is the Aharonov-Bohm phase factor due to the propagation of a quark along the closed curve $\mathcal{C}$. The non-abelian analogue of the previous formula is:

$$W(\mathcal{C}) = \text{Tr} P \exp \left[i \oint_{\mathcal{C}} A_{\mu} dx^{\mu} \right] , \quad (195)$$

where $A_{\mu} = A_{\mu}^a T^a$, Tr is the trace over the group indices and P is the path ordering operator. Notice that $\langle W(\mathcal{C}) \rangle$ can be regarded as the amplitude for a creation of a $q\bar{q}$ pair which propagates and is annihilated afterwards.

In particular, we can take a rectangular Wilson loop in Euclidean space. Let T and d be the sizes of the rectangle. In this case the Wilson loop is the amplitude for the propagation of a $q\bar{q}$ pair separated a distance d , which can be calculated by means of the hamiltonian formalism. Actually, when $T \rightarrow \infty$ one has:

$$\lim_{T \rightarrow \infty} \langle W(\mathcal{C}) \rangle \sim e^{-T E(d)} , \quad (196)$$

where $E(d)$ is the energy of a $q\bar{q}$ pair separated a distance d . In a confining theory the energy grows linearly with the $q\bar{q}$ distance d :

$$E(d) \approx \sigma d , \quad \sigma \rightarrow \text{constant} . \quad (197)$$

Then, the Wilson loop in a confining theory satisfies the so-called area law:

$$\lim_{T \rightarrow \infty} \langle W(\mathcal{C}) \rangle \sim e^{-\sigma T d} \sim e^{-\sigma (\text{area enclosed by the loop})} . \quad (198)$$

Let us see how we can compute VEVs of Wilson loops in the AdS/CFT correspondence [22, 23]. Recall that the endpoint of an open string ending on a D-brane is dual to a quark. Thus, in string theory, the Wilson loop is represented by an open string whose worldsheet Σ has a boundary $\partial\Sigma$ which is in a D-brane and describes the curve $\mathcal{C}$, as shown in figure 10. Then:

$$\langle W(\mathcal{C}) \rangle = Z_{string}(\partial\Sigma = \mathcal{C}) . \quad (199)$$

To have infinitely heavy (non-dynamical) quarks we push the D-brane to the AdS boundary and, therefore, $\partial\Sigma = \mathcal{C}$ lies within the AdS boundary $z = 0$. Then, in the 't Hooft limit, we have:

$$Z_{string}(\partial\Sigma = \mathcal{C}) = e^{-S(\mathcal{C})} , \quad (200)$$

where $S(\mathcal{C})$ is the on-shell extremal Nambu-Goto action for the string worldsheet satisfying the boundary condition that Σ ends on the curve $\mathcal{C}$. The Wilson loop becomes simply:

$$\langle W(\mathcal{C}) \rangle = e^{-S(\mathcal{C})} . \quad (201)$$

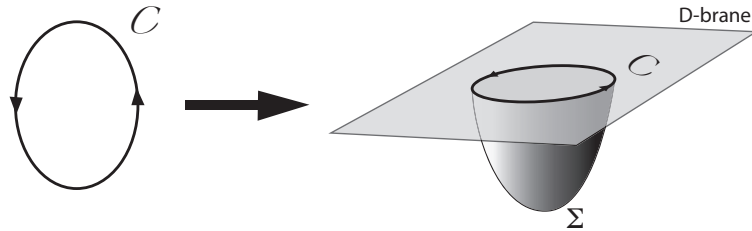


Figure 10: A Wilson loop along a closed curve $\mathcal{C}$ is holographically realized as the boundary of a surface Σ .

We will now apply these ideas to the holographic calculation of the rectangular Wilson loop and the corresponding $q\bar{q}$ potential energy in $\mathcal{N} = 4$ SYM. With this purpose, let us consider a string ending on the boundary of AdS_5 , where it extends along one of the gauge theory coordinates (say $x^1 \equiv x$). We shall parameterize the embedding of the string in AdS_5 by a function $z = z(x)$ and we will denote by z' the derivative of z with respect to x . The induced metric in Euclidean signature on the string worldsheet is:

$$ds^2 = \frac{L^2}{z^2} [dt^2 + (1 + z'^2) dx^2] . \quad (202)$$

Therefore, the Nambu-Goto action for the string is:

$$S = \frac{1}{2\pi\alpha'} \int dt \int dx \sqrt{g} = \frac{TL^2}{2\pi\alpha'} \int dx \frac{\sqrt{1 + z'^2}}{z^2} , \quad (203)$$

where $T = \int dt$. The action (203) corresponds to a problem in classical mechanics where x is the “time” and the lagrangian is given by:

$$\mathcal{L} = \frac{TL^2}{2\pi\alpha'} \frac{\sqrt{1 + z'^2}}{z^2} . \quad (204)$$

Since this lagrangian does not depend on the “time” x , the “energy” is conserved, which means that:

$$z' \frac{\partial \mathcal{L}}{\partial z'} - \mathcal{L} = \text{constant} . \quad (205)$$

By computing the derivative of $\mathcal{L}$ appearing on the left-hand side of (205), the conservation law written above can be proved to be equivalent to the following first integral of the equation of motion:

$$z^2 \sqrt{1 + z'^2} = \text{constant} . \quad (206)$$

For the hanging string configuration we are interested in, the boundary conditions that the function $z(x)$ must satisfy are:

$$z(x = -d/2) = z(x = d/2) = 0 , \quad (207)$$

where d is the quark-antiquark separation. Clearly, there is a value of x for which z is maximal. By symmetry this value is just $x = 0$ and, since $z(x)$ has a maximum implies that $z'(x = 0) = 0$. Let z_* be the maximal value of z , i.e. $z_* = z(x = 0)$. Then, the constant appearing in the conservation law is just z_*^2 and we have:

$$z' = \pm \frac{\sqrt{z_*^4 - z^4}}{z^2} , \quad (208)$$

where the two signs correspond to the two sides of the hanging string. The equation written above can be integrated immediately to give x as a function of z :

$$x = \pm \int_{z_*}^z \frac{\xi^2}{\sqrt{z_*^4 - \xi^4}} d\xi . \quad (209)$$

By introducing a new rescaled variable y by means of the relation $\xi = z_* y$, one can write:

$$x = \pm z_* \int_1^{\frac{z}{z_*}} \frac{y^2}{\sqrt{1-y^4}} dy . \quad (210)$$

By imposing the boundary conditions (207), we can obtain the quark-antiquark separation as a function of z_* :

$$\frac{d}{2} = z_* \int_0^1 \frac{y^2}{\sqrt{1-y^4}} dy . \quad (211)$$

The integral over y in (211) can be computed exactly and one gets:

$$z_* = \frac{d}{2\sqrt{2}\pi^{\frac{3}{2}}} \left(\Gamma\left(\frac{1}{4}\right) \right)^2 . \quad (212)$$

Then, it follows that d and z_* are proportional. Let us now compute the on-shell action for the string, from which we will obtain the quark-antiquark potential. By plugging the conservation law (206) inside the expression of the action in (203), we get:

$$S = \frac{T L^2 z_*^2}{2\pi\alpha'} \int \frac{dx}{z^4} . \quad (213)$$

Let us change variables in the integral (213) from x to z . The jacobian for this change of variables is:

$$\frac{dx}{dz} = \frac{1}{z'} = \frac{z^2}{\sqrt{z_*^4 - z^4}} , \quad (214)$$

where we have taken into account the value of z' written in (208). Taking into account that the variable z is double-valued, the total action is:

$$S = 2 \times \frac{T L^2 z_*^2}{2\pi\alpha'} \int_{\epsilon}^{z_*} \frac{dz}{z^2 \sqrt{z_*^4 - z^4}} . \quad (215)$$

Let us next work on a rescaled variable y , related to z as $z = z_* y$. Then, the on-shell action becomes:

$$S = \frac{T L^2}{\pi\alpha' z_*} I_{\epsilon} , \quad (216)$$

where I_{ϵ} is the following integral:

$$I_{\epsilon} = \int_{\epsilon/z_*}^1 \frac{dy}{y^2 \sqrt{1-y^4}} . \quad (217)$$

The integral I_{ϵ} in (217) diverges when $\epsilon \rightarrow 0$. Indeed, one can prove that, for small ϵ :

$$I_{\epsilon} = -\frac{\pi^{\frac{3}{2}} \sqrt{2}}{\left(\Gamma\left(\frac{1}{4}\right) \right)^2} + \frac{z_*}{\epsilon} . \quad (218)$$

The on-shell action S is just $T E$, where E is the energy of the quark-antiquark. It follows that E is given by:

$$E = -\frac{4\pi^2 L^2}{\left(\Gamma\left(\frac{1}{4}\right)\right)^4} \frac{1}{\alpha'} \frac{1}{d} + \frac{L^2}{\pi\alpha'} \frac{1}{\epsilon} . \quad (219)$$

The divergent term in (219) corresponds to the quark and antiquark masses (which are considered to be infinitely large in the static limit). To check this, let us compute the euclidean action of a single string which goes straight from the boundary $z = \epsilon$ to $z = \infty$ at fixed x . This configuration can be described by using t and z as worldvolume coordinates. The induced metric is obtained by keeping constant the x coordinate and is given by:

$$ds^2 = \frac{L^2}{z^2} (dt^2 + dz^2) . \quad (220)$$

The on-shell Nambu-Goto action for this configuration is:

$$S_{|} = \frac{T}{2\pi\alpha'} \int_{\epsilon}^{\infty} \frac{L^2}{z^2} dz^2 = \frac{TL^2}{2\pi\alpha' \epsilon} . \quad (221)$$

Therefore, the energy for a configuration of two straight parallel strings in AdS_5 is:

$$E_{||} = 2 \times \frac{L^2}{2\pi\alpha' \epsilon} , \quad (222)$$

which is equal to the divergent term in (219), as claimed. The quark-antiquark potential $V_{q\bar{q}}$ is obtained by subtracting the divergent contribution due to the quark masses:

$$V_{q\bar{q}} = E - E_{||} . \quad (223)$$

By subtracting (219) and (222), one gets:

$$V_{q\bar{q}} = -\frac{4\pi^2 L^2}{\left(\Gamma\left(\frac{1}{4}\right)\right)^4} \frac{1}{\alpha'} \frac{1}{d} . \quad (224)$$

Let us write this result in terms of gauge theory quantities. Recall that $L^2 = \sqrt{N^2 g_{YM}} \alpha'$ or, equivalently $L^2 = \sqrt{\lambda} \alpha'$. Then, (224) can be rewritten as:

$$V_{q\bar{q}} = -\frac{4\pi^2 \sqrt{\lambda}}{\left(\Gamma\left(\frac{1}{4}\right)\right)^4} \frac{1}{d} . \quad (225)$$

The Coulombic $1/d$ dependence in (225) is consequence of the conformal invariance of the theory. The non-analytic dependence on the coupling λ is a non-perturbative effect. In field theory this non-analyticity results from the summation of an infinite number of Feynman

diagrams. It is interesting to compare (225) with the perturbative $q\bar{q}$ potential, valid for small λ , which is given by:

$$V_{q\bar{q}} = -\frac{\pi\lambda}{d} . \quad (226)$$

Remarkably we have been able to find a non-perturbative result in a interacting quantum field theory by studying the catenary curve for a hanging string in classical gravity.

In the next two sections we will generalize the result (225) to the case of $\mathcal{N} = 4$ SYM at finite temperature and to the gravity dual of a theory in which the quarks are confined. In the former case we have to study a string hanging from the boundary of an anti-de-Sitter black hole, whereas in the latter we must deal with a geometry obtained from the AdS black hole by a double analytic continuation.

10.1 Quark-antiquark potential at finite temperature

Let us suppose that we put our gauge theory in a thermal bath at a temperature T . In the holographic correspondence the heat bath is dual to a bulk geometry with an event horizon (a black hole) and the Hawking temperature of the black hole is identified as the temperature of heat bath. In particular, the dual of $\mathcal{N} = 4$ SYM at finite temperature is a black hole in AdS_5 , whose metric in euclidean space is (see below):

$$ds^2 = \frac{L^2}{z^2} \left[f(z) dt^2 + d\vec{x}^2 + \frac{dz^2}{f(z)} \right] , \quad (227)$$

with $f(z)$ being the following function:

$$f(z) = 1 - \frac{z^4}{z_0^4} . \quad (228)$$

The parameter z_0 corresponds to the position of the horizon and is related to the black hole temperature T as $T = (\pi z_0)^{-1}$ (see section 11.2 below). In the context of the holographic duality one naturally identifies the black hole temperature T with the temperature of the gauge theory.

As in the zero temperature case, we will parametrize the embedding of the string by a function $z = z(x)$. The induced metric is:

$$ds^2 = \frac{L^2}{z^2} \left[f dt^2 + \left(1 + \frac{z'^2}{f} \right) dx^2 \right] . \quad (229)$$

The Nambu-Goto action for this embedding becomes:

$$S = \frac{\tau L^2}{2\pi\alpha'} \int dx \frac{\sqrt{f(z) + z'^2}}{z^2} , \quad (230)$$

where $\tau = \int dt$. Following the same steps as in the zero temperature case, it is straightforward to derive the following first integral of the equation of motion:

$$\frac{z^2 \sqrt{f(z) + z'^2}}{f(z)} = \text{constant} = \frac{z_*^2}{\sqrt{f(z_*)}}. \quad (231)$$

From (231) we obtain readily the value of z' :

$$z' = \pm \sqrt{\frac{f(z)}{f(z_*)}} \frac{\sqrt{z_*^4 - z^4}}{z^2}. \quad (232)$$

Let us now define the constant ρ as follows:

$$\rho \equiv \left(\frac{z_0}{z_*}\right)^4. \quad (233)$$

Then, we have:

$$x = \pm z_* \sqrt{\rho - 1} \int_1^{\frac{z}{z_*}} \frac{y^2 dy}{\sqrt{(1 - y^4)(\rho - y^4)}}, \quad (234)$$

and the quark-antiquark distance d is equal to:

$$d = 2z_* \sqrt{\rho - 1} \int_0^1 \frac{y^2 dy}{\sqrt{(1 - y^4)(\rho - y^4)}}. \quad (235)$$

Notice that these expressions become the ones at zero temperature as $\rho \rightarrow \infty$, as it should. Moreover, as $z_* \rightarrow z_0$, $\rho \rightarrow 1$ and the distance $d \rightarrow 0$ (see figure 11). Actually, by varying ρ one can see that there is a maximal value of d ($d_{max} \sim z_0$) and the $q\bar{q}$ bound state becomes unbound due to thermal screening, in agreement with the behavior expected from the point of view of the gauge theory at non-zero temperature.

10.2 Quark-antiquark potential in a confining background

Let us consider the AdS black hole in Euclidean signature and let us go back to Minkowski signature by analytic continuation along a different direction, namely by making $x_3 \rightarrow it$. If we call u to the original euclidean time, we arrive at the following metric:

$$ds^2 = \frac{L^2}{z^2} \left[-dt^2 + dx_1^2 + dx_2^2 + f(z)du^2 + \frac{dz^2}{f(z)} \right]. \quad (236)$$

In this case the space ends smoothly at $z = z_0$, which should be though as a IR mass scale. We will now compute the $q\bar{q}$ potential for the metric (236) and we will verify that it corresponds to a confining background. With this purpose, let us consider a fundamental string in the

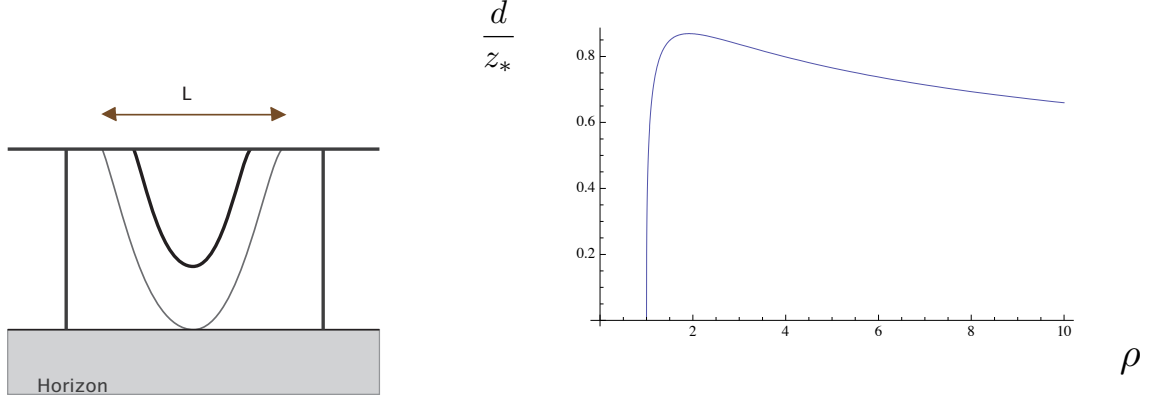


Figure 11: In a black hole metric, the string ends on the horizon when the $q\bar{q}$ separation is large enough. Correspondingly, the distance d reaches a maximum value, as the plot on the right shows.

euclidean version of the above metric and let us use (t, x) as worldsheet coordinates. The induced metric is:

$$ds^2 = \frac{L^2}{z^2} \left[dt^2 + \left(1 + \frac{z'^2}{f} \right) dx^2 \right], \quad (237)$$

and the Nambu-Goto action becomes:

$$S = \frac{\tau L^2}{2\pi\alpha'} \int \frac{dx}{z^2} \sqrt{1 + \frac{z'^2}{f(z)}}, \quad (238)$$

where $\tau = \int dt$. The first integral corresponding to the action (238) is:

$$\frac{z^2}{\sqrt{f(z)}} \sqrt{f(z) + z'^2} = z_*^2, \quad (239)$$

from which we get:

$$z' = \pm \sqrt{f(z)} \frac{\sqrt{z_*^4 - z^4}}{z^2}. \quad (240)$$

This equation can be integrated as:

$$x = \pm z_* \sqrt{\rho} \int_1^{\frac{z}{z_*}} \frac{y^2 dy}{\sqrt{(1-y^4)(\rho-y^4)}}, \quad (241)$$

where the constant ρ is the same as in (233). It follows that the $q\bar{q}$ distance d is:

$$d = 2z_* \sqrt{\rho} \int_0^1 \frac{y^2 dy}{\sqrt{(1-y^4)(\rho-y^4)}}. \quad (242)$$

In this case, d grows without limit as $\rho \rightarrow 1$, or equivalently as the turning point approaches the end of the space (*i.e.* when $z_* \rightarrow z_0$) (see figure [12](#)). In the large d limit the profile of the hanging string is approximately rectangular. The energy due to the vertical parts of the profile can be identified with the masses of the static quarks, which have to be subtracted to get the potential energy. The $q\bar{q}$ potential is just due to the horizontal part of the profile. Since in this part z is approximately constant and equal to z_0 , we get that the contribution to the euclidean action is:

$$S_{horizontal} = \frac{\tau L^2}{2\pi\alpha'} \frac{d}{z_0^2} , \quad (243)$$

which corresponds to an area law (it is proportional to τd) and gives rise to a confining potential of the type:

$$V = \sigma_s d , \quad (244)$$

with σ_s being the effective string tension, given by:

$$\sigma_s = \frac{L^2}{2\pi\alpha'} \frac{1}{z_0^2} . \quad (245)$$

In terms of gauge theory quantities, σ_s can be written as:

$$\sigma_s = \frac{\sqrt{\lambda}}{2\pi z_0^2} . \quad (246)$$

Notice that $M \sim 1/z_0$ is an IR scale that can be identified with the mass gap of the theory (and z_0 with the glueball size). The previous formula for the tension is simply:

$$\sigma_s \sim \sqrt{\lambda} M^2 . \quad (247)$$

11 Black hole thermodynamics

The partition function in statistical mechanics in the canonical ensemble is given by:

$$Z = \text{Tr} e^{-\frac{H}{T}} , \quad (248)$$

where, H is the hamiltonian operator, T is the temperature and we are taking the Boltzmann constant $k_B = 1$. The thermal average of an operator $\mathcal{O}$ at the temperature T is:

$$\langle \mathcal{O} \rangle_T = \frac{\text{Tr} [\mathcal{O} e^{-\frac{H}{T}}]}{Z} . \quad (249)$$

In the path integral approach, the average $\langle \mathcal{O} \rangle_T$ can be written as:

$$\langle \mathcal{O} \rangle_T \sim \int [D\psi] \langle \psi(x), t | \mathcal{O} e^{-\frac{H}{T}} | \psi(x), t \rangle , \quad (250)$$

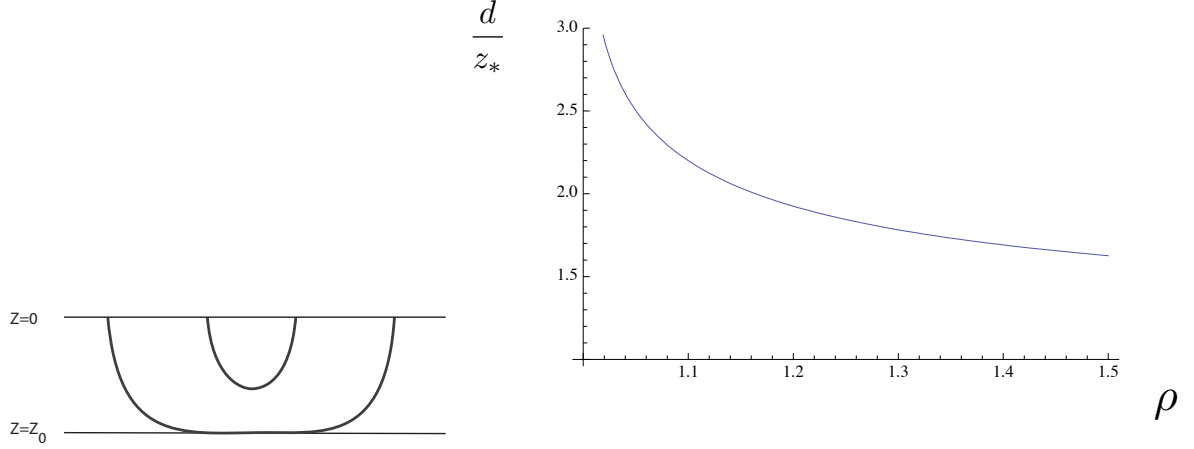


Figure 12: In a confining metric the space ends smoothly and the profile of the hanging string becomes almost rectangular for large $q\bar{q}$ separation (left). The distance d is not bounded, as shown on the plot on the right.

where the expectation value is taken between the same initial and final state $|\psi(x), t\rangle$ (as it corresponds to a trace in Hilbert space). Equivalently, since the hamiltonian operator implements time evolution, we can rewrite (250) as:

$$\langle \mathcal{O} \rangle_T \sim \int [D\psi] \langle \psi(x), t | \mathcal{O} | \psi(x), t + \frac{i}{T} \rangle . \quad (251)$$

Then, to perform thermal averages we have to consider imaginary time evolution and we have to impose periodic boundary conditions in the Hilbert space (antiperiodic for fermions). Then, the euclidean time t_E must be periodically identified

$$t_E \equiv t_E + \frac{1}{T} . \quad (252)$$

Thus, the compactification of Euclidean time is equivalent to having $T \neq 0$.

Let us apply these ideas to obtain the Hawking temperature of a black hole. We will assume that we have an euclidean metric of the type:

$$ds^2 = g(r) [f(r) dt_E^2 + d\vec{x}^2] + \frac{1}{h(r)} dr^2 , \quad (253)$$

where the functions $f(r)$ and $h(r)$ have a first-order zero at $r = r_0$, which is the location of the horizon, and $g(r_0) \neq 0$. Then, for $r \approx r_0$

$$f(r) \approx f'(r_0) (r - r_0) , \quad h(r) \approx h'(r_0) (r - r_0) , \quad (254)$$

while we can take $g(r) = g(r_0)$. Then, the near-horizon euclidean metric can be written as:

$$ds^2 \approx g(r_0) [f'(r_0) (r - r_0) dt_E^2 + d\vec{x}^2] + \frac{1}{h'(r_0)} \frac{dr^2}{r - r_0} . \quad (255)$$

Let us define a new radial variable ρ such that:

$$\frac{1}{h'(r_0)} \frac{dr^2}{r - r_0} = d\rho^2 , \quad (256)$$

which can be integrated to give the following relation between ρ and r :

$$\rho = 2 \sqrt{\frac{r - r_0}{h'(r_0)}} . \quad (257)$$

Next, we define an angular coordinate θ such that

$$g(r_0) f'(r_0) (r - r_0) dt_E^2 = \rho^2 d\theta^2 , \quad (258)$$

which is equivalent to defining θ as:

$$\theta = \frac{1}{2} \sqrt{g(r_0) f'(r_0) h'(r_0)} t_E . \quad (259)$$

In the new variables, the (t_E, r) part of the metric takes the form $d\rho^2 + \rho^2 d\theta^2$, which is locally like the flat metric of a plane. In order to have $\rho = 0$ (*i.e.* the horizon) as a regular point without any curvature singularity, the variable θ must be a periodic variable with period 2π . Otherwise we would have a conical singularity at the origin, due to the defect angle. We have argued above for a general system that the compactification of the euclidean time is equivalent to having a non-zero temperature T . Thus, it follows that we can assign a temperature T to a black hole (the Hawking temperature). In order to find the value of T , let us notice that the periodicity under $\theta \rightarrow \theta + 2\pi$ is equivalent to periodicity under $t_E \rightarrow t_E + \frac{1}{T}$, where T is the Hawking temperature given by:

$$\frac{1}{T} = \frac{4\pi}{\sqrt{g(r_0) f'(r_0) h'(r_0)}} . \quad (260)$$

In the next subsections we will apply this formula to determine the Hawking temperature of a couple of black holes.

11.1 Application to the Schwarzschild black hole

Let us consider the ordinary Schwarzschild metric in four dimensions:

$$ds^2 = -\left(1 - \frac{2GM}{r}\right) dt^2 + \frac{dr^2}{1 - \frac{2GM}{r}} + r^2 d\Omega_2^2 , \quad (261)$$

which is a particular case of the general expression written in (253), with the functions f , h and g being given by:

$$g(r) = 1 , \quad f(r) = h(r) = 1 - \frac{2GM}{r} . \quad (262)$$

In (261) G is the four-dimensional Newton constant and M is the mass of the black hole. The horizon in the geometry (261) is located at $r = r_0 = 2GM$. Since:

$$f'(r_0) = h'(r_0) = \frac{2GM}{r_0^2} \quad (263)$$

it follows that the Hawking temperature for the Schwarzschild black hole is given by:

$$T = \frac{1}{8\pi GM} . \quad (264)$$

Let us use the expression of T in (264) to obtain the black hole entropy S of the Schwarzschild black hole. We will identify the mass M with the internal energy and we will make use of the first law of thermodynamics:

$$dM = TdS = \frac{1}{8\pi GM} dS , \quad (265)$$

which can be integrated to give the entropy S :

$$S = 4\pi G M^2 , \quad (266)$$

The horizon area A_H for the metric (261) is the area of the surface $t = \text{constant}$, $r = r_0$. It is straightforward to demonstrate that A_H is:

$$A_H = 4\pi r_0^2 = 16 G^2 M^2 . \quad (267)$$

It is now immediate to check that the relation between S and A_H is:

$$S = \frac{A_H}{4G} , \quad (268)$$

which is nothing but the celebrated Bekenstein-Hawking entropy formula which relates the entropy of a black hole with the area of its horizon (see (6)).

11.2 AdS black hole

The metric of a black hole in AdS_{d+1} is:

$$ds^2 = \frac{L^2}{z^2} \left[f(z) dt_E^2 + d\vec{x}^2 + \frac{dz^2}{f(z)} \right] , \quad (269)$$

where $f(z)$ is the following function:

$$f(z) = 1 - \frac{z^d}{z_0^d} , \quad (270)$$

with z_0 being a constant ($z = z_0$ is the location of the horizon). This metric is just of our general form (253) with $r \rightarrow z$, f given as above and g and h being:

$$g = \frac{L^2}{z^2} , \quad h = \frac{z^2}{L^2} f . \quad (271)$$

In this case the derivatives of f and h at the horizon are:

$$f'(z_0) = -\frac{d}{z_0} , \quad h'(z_0) = -\frac{dz_0}{L^2} , \quad (272)$$

and, therefore, we have:

$$g(z_0) f'(z_0) h'(z_0) = \frac{d^2}{z_0^2} . \quad (273)$$

Thus, the Hawking temperature is related to z_0 by means of the relation:

$$T = \frac{d}{4\pi z_0} . \quad (274)$$

The horizon is the hypersurface $z = z_0$ and t constant, whose area in the metric (269) is:

$$A_H = \left(\frac{L}{z_0}\right)^{d-1} V_{d-1} , \quad (275)$$

where V_{d-1} is the volume along the Minkowski spacial directions. In terms of the temperature, A_H can be written as:

$$A_H = \left(\frac{4\pi}{d}\right)^{d-1} L^{d-1} T^{d-1} V_{d-1} , \quad (276)$$

and the entropy can be computed from the Bekenstein-Hawking formula, with the result:

$$S = \frac{A_H}{4G_{d+1}} = \frac{1}{4G_{d+1}} \left(\frac{4\pi}{d}\right)^{d-1} L^{d-1} T^{d-1} V_{d-1} . \quad (277)$$

We now define the entropy density s as:

$$s = \frac{S}{V_{d-1}} . \quad (278)$$

Let us write s in terms of the QFT central charge $c_{QFT} = \frac{1}{4}(L/l_P)^{d-1}$. We get:

$$s = \left(\frac{4\pi}{d}\right)^{d-1} c_{QFT} T^{d-1} . \quad (279)$$

In the case of $\mathcal{N} = 4$ SYM, we take $d = 4$ and $c_{SYM} = N^2/2\pi$ and the entropy density is given by:

$$s_{SYM} = \frac{\pi^2}{2} N^2 T^3 . \quad (280)$$

From the expression of the entropy density in (280) we can obtain the value of the pressure by means of the thermodynamic relation:

$$s = \frac{\partial p}{\partial T} . \quad (281)$$

We get:

$$p = \frac{\pi^2}{8} N^2 T^4 . \quad (282)$$

Moreover, the energy density ϵ can be obtained from p and s by means of the standard thermodynamic relation:

$$\epsilon = -p + Ts . \quad (283)$$

Using the values of p and s computed from holography, we arrive at the following value of the energy density:

$$\epsilon = \frac{3\pi^2}{8} N^2 T^4 . \quad (284)$$

11.3 Comparison with field theory

Let us compare the strong coupling values of s , p and ϵ found above with those corresponding the $\mathcal{N} = 4$ SYM at zero coupling. The partition function in the canonical ensemble for a gas of non-interacting relativistic bosons and fermions is

$$\log Z = \mp V_3 \int \frac{d^3 p}{(2\pi)^3} \log \left(1 \mp e^{-\frac{\omega(p)}{T}} \right) , \quad (285)$$

where the upper minus (lower plus) signs corresponds to bosons (respectively, fermions) and $\omega(p) = \sqrt{\vec{p}^2 + m^2}$. In the massless case, we just take $\omega(p) = |\vec{p}|$ and we get:

$$\frac{\log Z}{V_3} = \mp \int_0^\infty \frac{dp}{2\pi^2} p^2 \log \left(1 \mp e^{-\frac{p}{T}} \right) . \quad (286)$$

Let us change variables in the integral (286) from p to $x = p/T$. After integrating by parts we find:

$$\frac{\log Z}{V_3} = \frac{T^3}{6\pi^2} \int_0^\infty dx \frac{x^3}{e^x \mp 1} . \quad (287)$$

To calculate these integrals we use the general results, valid for $n \in \mathbb{Z}$:

$$\int_0^\infty dx \frac{x^{2n-1}}{e^x + 1} = \frac{2^{2n-1} - 1}{2n} \pi^{2n} B_n , \quad \int_0^\infty dx \frac{x^{2n-1}}{e^x - 1} = \frac{(2\pi)^{2n} B_n}{4n} , \quad (288)$$

where B_n denotes the Bernoulli numbers. In particular for $n = 2$, since $B_2 = 1/30$, we have:

$$\int_0^\infty dx \frac{x^3}{e^x - 1} = \frac{\pi^4}{15} , \quad \int_0^\infty dx \frac{x^3}{e^x + 1} = \frac{7\pi^4}{120} . \quad (289)$$

Thus, for bosons:

$$\frac{\log Z}{V_3} = \frac{\pi^2}{90} T^3, \quad (\text{bosons}), \quad (290)$$

while for fermions:

$$\frac{\log Z}{V_3} = \frac{7\pi^2}{720} T^3, \quad (\text{fermions}). \quad (291)$$

The entropy density can be obtained from the partition function by means of the standard statistical mechanics relation:

$$s = \frac{\partial}{\partial T} \left[T \frac{\log Z}{V_3} \right] = 4 \frac{\log Z}{V_3}. \quad (292)$$

Then, it follows that:

$$s_{boson} = \frac{2\pi^2}{45} T^3, \quad s_{fermion} = \frac{7\pi^2}{180} T^3. \quad (293)$$

In $\mathcal{N} = 4$ SYM the number of bosons we have is:

$$\left[2(\text{gauge field}) + 6(\text{scalar field}) \right] N^2 = 8N^2, \quad (294)$$

while the number of fermions is:

$$\left[2 \times 4(\text{Weyl spinors}) \right] N^2 = 8N^2, \quad (295)$$

which, due to the supersymmetry, is the same as the number of bosons. Therefore, the total entropy density of $\mathcal{N} = 4$ SYM when the coupling constant is zero is:

$$s_{\mathcal{N}=4 \text{ free gas}} = 8N^2 \left[\frac{2\pi^2}{45} + \frac{7\pi^2}{180} \right] T^3 = \frac{2\pi^2}{3} N^2 T^3. \quad (296)$$

By using the thermodynamic formulas (283) and (281), we obtain the values of the pressure and energy density:

$$p_{\mathcal{N}=4 \text{ free gas}} = \frac{\pi^2}{6} N^2 T^4, \quad \epsilon_{\mathcal{N}=4 \text{ free gas}} = \frac{\pi^2}{2} N^2 T^4. \quad (297)$$

Then, by comparing with the result given by the black hole, we obtain:

$$s_{black \text{ hole}} = \frac{3}{4} s_{\mathcal{N}=4 \text{ free gas}}, \quad (298)$$

and similarly for the pressure and energy density:

$$p_{black \text{ hole}} = \frac{3}{4} p_{\mathcal{N}=4 \text{ free gas}}, \quad \epsilon_{black \text{ hole}} = \frac{3}{4} \epsilon_{\mathcal{N}=4 \text{ free gas}}. \quad (299)$$

Therefore, the AdS/CFT correspondence predicts that the values of s , p and ϵ at infinite coupling differ from their values at zero coupling by a multiplicative factor $3/4$. These reductions of the entropy, pressure and energy density as the coupling is increased are in very good agreement with the results obtained in lattice simulations for different gauge theories.

12 Transport coefficients

Let us begin by writing the linear response formulas of section 9.2 in real time. We consider a system in QFT to which we couple a source $\varphi(x)$ to a local operator $\mathcal{O}(x)$:

$$S = S_0 + \int d^d x \mathcal{O}(x) \varphi(x) , \quad (300)$$

where the d -dimensional spacetime has Minkowski signature. We will assume that the unperturbed VEV of the operator $\mathcal{O}$ vanishes. Then, the one-point function of the operator $\mathcal{O}$ in the presence of the source is given by:

$$\langle \mathcal{O}(x) \rangle_\varphi = - \int G_R(x - y) \varphi(y) dy , \quad (301)$$

where $G_R(x - y)$ is the retarded Green's function, defined as:

$$i G_R(x - y) \equiv \theta(x^0 - y^0) \langle [\mathcal{O}(x), \mathcal{O}(y)] \rangle . \quad (302)$$

The fact that the linear response is determined by the retarded correlator is a consequence of causality since the source can influence the system only after it has been turned on. In momentum space the relation (301) between the one-point function and the source becomes:

$$\langle \mathcal{O}(\omega, \vec{k}) \rangle_\varphi = -G_R(\omega, \vec{k}) \varphi(\omega, \vec{k}) . \quad (303)$$

We are interested in analyzing the long wavelength hydrodynamic limit. In this case one can take the zero spatial momentum and zero frequency limit of the retarded correlator. In this limit one studies the response of the system to a time varying source $\varphi(t)$, which can be approximately represented as:

$$\langle \mathcal{O} \rangle_\varphi \approx -\chi \partial_t \varphi , \quad (304)$$

where the real constant χ is the so-called transport coefficient. In frequency space, for $\omega \rightarrow 0$, the relation (304) is given by:

$$\langle \mathcal{O} \rangle_\varphi \approx i \omega \chi \varphi(\omega) . \quad (305)$$

The linear response equation for this quantity is obtained from the long wavelength limit $\vec{k} \rightarrow 0$ of (303):

$$\langle \mathcal{O} \rangle_\varphi = -G_R(\omega, \vec{k} = 0) \varphi(\omega) . \quad (306)$$

By comparing (305) and (306) we get:

$$G_R(\omega, \vec{k} = 0) = -i \omega \chi , \quad (\omega \rightarrow 0) , \quad (307)$$

or, taking the imaginary part:

$$\text{Im } G_R(\omega, \vec{k} = 0) = -\omega \chi , \quad (\omega \rightarrow 0) . \quad (308)$$

From (308) we immediately get the so-called Kubo formula for χ :

$$\chi = - \lim_{\omega \rightarrow 0} \lim_{\vec{k} \rightarrow 0} \frac{1}{\omega} \text{Im} G_R(\omega, \vec{k}) . \quad (309)$$

Let us now see how we can make use of (309) to compute the transport coefficient χ by using holographic methods. Let us consider a $(d+1)$ -dimensional metric of the form:

$$ds^2 = g_{tt} dt^2 + g_{zz} dz^2 + g_{xx} \delta_{ij} dx^i dx^j . \quad (310)$$

We will assume the metric (310) has an horizon at $z = z_0$ and that g_{tt} and that g_{zz} behave near $z = z_0$ as:

$$g_{tt} \approx -c_0(z_0 - z) , \quad g_{zz} \approx \frac{c_z}{z_0 - z} , \quad z \rightarrow z_0 , \quad (311)$$

with c_0 and c_z being constants. Let us consider a massless scalar field ϕ in this metric such that its action is:

$$S = -\frac{1}{2} \int d^{d+1}x \sqrt{-g} \frac{\partial_M \phi \partial^M \phi}{q(z)} , \quad (312)$$

where the function $q(z)$ is an effective coupling of the mode. The Euler-Lagrange equation of motion derived from (312) is:

$$\partial_M \left(\frac{\sqrt{-g}}{q} g^{MN} \partial_M \phi \right) = 0 . \quad (313)$$

In terms of the canonical momentum Π , defined as (see (141)):

$$\Pi \equiv -\frac{\partial \mathcal{L}}{\partial(\partial_z \phi)} = \frac{\sqrt{-g}}{q} g^{zz} \partial_z \phi , \quad (314)$$

the equation of motion (313) becomes:

$$\partial_z \Pi = -\frac{\sqrt{-g}}{q} \left(\frac{\partial_t^2 \phi}{g_{tt}} + \frac{\partial_x^2 \phi}{g_{xx}} \right) . \quad (315)$$

Let us now Fourier transform ϕ and Π in the variables $t, \vec{x}$:

$$\begin{aligned} \phi(z, t, \vec{x}) &= \int \frac{d\omega d^{d-1}k}{(2\pi)^d} e^{i(\vec{k} \cdot \vec{x} - \omega t)} \phi(z, \omega, \vec{k}) , \\ \Pi(z, t, \vec{x}) &= \int \frac{d\omega d^{d-1}k}{(2\pi)^d} e^{i(\vec{k} \cdot \vec{x} - \omega t)} \Pi(z, \omega, \vec{k}) . \end{aligned} \quad (316)$$

Since in this massless case the scaling dimension Δ is equal to d (see (112) for $m = 0$), the one-point function is just the limit of the momentum Π at the boundary $z = 0$ and the

boundary field φ is just obtained by taking the limit $z \rightarrow 0$ of the bulk field ϕ , without any multiplicative factor (see eqs. (147) and (129)). It follows from (305) that the transport coefficient χ is obtained as:

$$\chi = \lim_{k_\mu \rightarrow 0} \lim_{z \rightarrow 0} \text{Im} \left[\frac{\Pi(z, k_\mu)}{\omega \phi(z, k_\mu)} \right] = \lim_{k_\mu \rightarrow 0} \lim_{z \rightarrow 0} \frac{\Pi(z, k_\mu)}{i\omega \phi(z, k_\mu)} , \quad (317)$$

where, in the last step, we used the fact that $\text{Re}\Pi/\omega \rightarrow 0$ as $\omega \rightarrow 0$ (see below). It turns out that evaluating $\Pi/(\omega \phi)$ at the boundary is equivalent to evaluate it at the horizon. In order to prove this, let us consider the general equation:

$$\partial_z [A(z) \partial_z \phi] = B(z) \phi(z) . \quad (318)$$

Let us write (318) in hamiltonian form. We first define:

$$P(z) \equiv A(z) \partial_z \phi(z) . \quad (319)$$

Then, (318) can be written as:

$$\partial_z P(z) = B(z) \phi(z) . \quad (320)$$

Then, one can readily prove that (318) and (320) can be combined in the following first-order Riccati equation:

$$\partial_z \left(\frac{P(z)}{\phi(z)} \right) = B(z) - \frac{1}{A(z)} \left(\frac{P(z)}{\phi(z)} \right)^2 . \quad (321)$$

For the equation of motion of the scalar field ϕ in momentum space, the functions $A(z)$ and $B(z)$ are given by:

$$A(z) = \frac{\sqrt{-g}}{q} g^{zz} , \quad B(z) = \frac{\sqrt{-g}}{q} \left[\frac{\omega^2}{g_{tt}} + \frac{\vec{k}^2}{g_{xx}} \right] , \quad (322)$$

and $P(z) = \Pi(z)$ (see (314)). Then, we can write:

$$\partial_z \left[\frac{\Pi}{\omega \phi} \right] = -\omega \left[\frac{q g_{zz}}{\sqrt{-g}} \left(\frac{\Pi}{\omega \phi} \right)^2 - \frac{\sqrt{-g}}{q g_{tt}} \left(1 + \frac{g_{tt}}{g_{xx}} \frac{\vec{k}^2}{\omega^2} \right) \right] . \quad (323)$$

The right-hand side of equation (323) vanishes when the ordered limit $\lim_{\omega \rightarrow 0} \lim_{\vec{k} \rightarrow 0}$ is taken. It follows that $\Pi/(\omega \phi)$ is independent of z in this limit and, as claimed above, it can be evaluated at the horizon. Thus, we can write the transport coefficient χ as:

$$\chi = \lim_{k_\mu \rightarrow 0} \lim_{z \rightarrow z_0} \frac{\Pi(z, k_\mu)}{i\omega \phi(z, k_\mu)} . \quad (324)$$

In order to evaluate the right-hand side of (324), let us study the equation of motion near $z = z_0$, where Π becomes:

$$\Pi \approx \frac{1}{c_z} \frac{\sqrt{-g(z_0)}}{q(z_0)} (z_0 - z) \partial_z \phi , \quad (325)$$

and the equation of motion takes the form:

$$\partial_z \left[(z_0 - z) \partial_z \phi(z, k_\mu) \right] + c_z \left[\frac{\omega^2}{c_0(z_0 - z)} - \frac{k^2}{g_{xx}(z_0)} \right] \phi(z, k_\mu) = 0 . \quad (326)$$

To find an approximate solution of (326) near the horizon $z = z_0$, we neglect the last term in the previous equation and try to find a solution of the type:

$$\phi = (z_0 - z)^\beta . \quad (327)$$

Plugging the ansatz (327) in (326), we get that the exponent β can take the following two values:

$$\beta = \pm i \sqrt{\frac{c_z}{c_0}} \omega , \quad (328)$$

which correspond to the following two solutions:

$$\phi_\pm \sim (z_0 - z)^{\pm i \sqrt{\frac{c_z}{c_0}} \omega} . \quad (329)$$

Only one of the two solutions in (329) is compatible with causality. Indeed, let us define a new variable r as $z_0 - z = e^r$. In the r variable the horizon is located at $r \rightarrow -\infty$. By inserting the t dependence, the two solutions $\phi_\pm$ are:

$$\phi_\pm \sim e^{-i(\omega t \mp \Omega r)} , \quad (330)$$

where $\Omega = \omega \sqrt{c_z/c_0}$. Clearly, ϕ_- is an incoming wave at the horizon since if we increase $t \rightarrow t + \epsilon$, we must decrease r as $r \rightarrow r - \epsilon \Omega / \omega$ to keep ϕ_- constant. Then, the wave ϕ_- moves towards the horizon $r \rightarrow -\infty$ and is an infalling wave at the horizon. Similarly, ϕ_+ is an outgoing wave at the horizon. Causality on the gravity side, is implemented if we choose our solution to be the infalling one ϕ_- . One can show that this corresponds to having retarded Green's functions on the field theory side. This infalling solution near $z = z_0$ satisfies:

$$\partial_z \phi_- = \sqrt{\frac{g_{zz}(z_0)}{-g_{tt}(z_0)}} i\omega \phi_- , \quad (331)$$

and, therefore, one has:

$$\left. \frac{\Pi}{i\omega \phi_-} \right|_{z_0} = \frac{1}{q(z_0)} \sqrt{\frac{g}{g_{zz} g_{tt}}} \Big|_{z_0} . \quad (332)$$

It follows that the transport coefficient χ is given by:

$$\chi = \frac{1}{q(z_0)} \sqrt{\frac{g}{g_{zz} g_{tt}}} \Big|_{z_0} . \quad (333)$$

Notice that the square root in this last equation is just the area of the horizon A_H divided by the spatial volume V . Then, we can alternatively write the transport coefficient χ as:

$$\chi = \frac{1}{q(z_0)} \frac{A_H}{V} . \quad (334)$$

In particular, for an AdS black hole, the previous expression becomes:

$$\chi = \frac{1}{q(z_0)} \left(\frac{L}{z_0} \right)^{d-1} , \quad (\text{AdS black hole}) . \quad (335)$$

More interestingly, one can compare the general formula (334) with the entropy density, as given by the Bekenstein-Hawking formula $s = A_H/(4G_N V)$. By computing the ratio χ/s , we get the simple result:

$$\frac{\chi}{s} = \frac{4G_N}{q(z_0)} . \quad (336)$$

13 Holographic viscosities

Hydrodynamics can be thought as an effective theory describing the dynamics of a continuous system at large distances and time scales. In order to study the dynamics of the system we have to analyze the energy-momentum tensor $T^{\mu\nu}$ which is conserved ($\partial_\mu T^{\mu\nu} = 0$). We will assume that the system is at local thermal equilibrium and that the state of the system at a given time is determined by the local temperature $T(x)$ and the local fluid velocity $u^\mu(x)$, which satisfies the condition $u_\mu u^\mu = -1$.

The relation between $T^{\mu\nu}$ and u^μ and the thermodynamic functions is expressed by means of the so-called constitutive relations, which for isotropic fluids takes the form:

$$T^{\mu\nu} = (\epsilon + p) u^\mu u^\nu + p g^{\mu\nu} - \sigma^{\mu\nu} , \quad (337)$$

where ϵ is the energy density and p is the pressure, while $\sigma^{\mu\nu}$ is the so-called dissipative part of $T^{\mu\nu}$ and depends on the derivatives of $T(x)$ and u^μ . In order to parametrize $\sigma_{\mu\nu}$ at first-order in the derivatives, let us consider a local rest frame in which $u^i(x) = 0$ (and $u^\mu = (1, 0, 0, 0)$). One can choose this frame in such a way that the dissipative corrections to the energy-momentum tensor components $T^{0\mu}$ vanish. Then, $\sigma^{00} = \sigma^{0i} = 0$ or, equivalently, $T^{00} = \epsilon$, $T^{0i} = 0$. The only non-zero elements of the dissipative energy-momentum tensor are σ_{ij} . At first order in derivatives σ_{ij} can be written as:

$$\sigma_{ij} = \eta (\partial_i u_j + \partial_j u_i - \frac{2}{3} \delta_{ij} \partial_k u_k) + \zeta \delta_{ij} \partial_k u_k , \quad (338)$$

where η is the so-called shear viscosity and ζ is the bulk viscosity.

Let us write $\sigma^{\mu\nu}$ in covariant form. We first define the projector onto the directions perpendicular to u^μ , as:

$$P^{\mu\nu} = g^{\mu\nu} + u^\mu u^\nu . \quad (339)$$

Then, in an arbitrary curved metric $\sigma_{\mu\nu}$ can be written as:

$$\sigma^{\mu\nu} = P^{\mu\alpha} P^{\nu\beta} \left[\eta (\nabla_\alpha u_\beta + \nabla_\beta u_\alpha) + \left(\zeta - \frac{2}{3} \eta \right) g_{\alpha\beta} \nabla \cdot u \right] , \quad (340)$$

where the covariant derivatives of the vectors u_β are defined as:

$$\nabla_\alpha u_\beta = \partial_\alpha u_\beta - \Gamma_{\alpha\beta}^\mu u_\mu , \quad (341)$$

with $\Gamma_{\alpha\beta}^\mu$ being the Christoffel symbols, defined as:

$$\Gamma_{\alpha\beta}^\mu = \frac{1}{2} g^{\mu\lambda} \left[\frac{\partial g_{\lambda\alpha}}{\partial x^\beta} + \frac{\partial g_{\lambda\beta}}{\partial x^\alpha} - \frac{\partial g_{\alpha\beta}}{\partial x^\lambda} \right] . \quad (342)$$

We will find $\sigma^{\mu\nu}$ as (minus) the one-point function of $T^{\mu\nu}$ in the presence of a metric perturbation of the type $g^{\mu\nu} \rightarrow \eta^{\mu\nu} + h^{\mu\nu}$. By using linear response theory (eq. (301)), we obtain:

$$\sigma^{\mu\nu}(x) = \int G_R^{\mu\nu,\alpha\beta}(x-y) h_{\alpha\beta}(y) dy , \quad (343)$$

where the retarded correlator is just:

$$i G_R^{\mu\nu,\alpha\beta}(x-y) = \theta(x^0 - y^0) \langle [T^{\mu\nu}(x), T^{\alpha\beta}(y)] \rangle . \quad (344)$$

Let us now consider the following metric perturbation:

$$g_{00}(t, \vec{x}) = -1 , \quad g_{0i}(t, \vec{x}) = 0 , \quad g_{ij}(t, \vec{x}) = \delta_{ij} + h_{ij}(t) . \quad (345)$$

with $h_{ij} \ll 1$ and such that is traceless ($h_{ii} = 0$). We will assume that h_{ij} is a function of t and that this variation with t is slow. The inverse metric is just:

$$g^{00}(t, \vec{x}) = -1 , \quad g^{0i}(t, \vec{x}) = 0 , \quad g^{ij}(t, \vec{x}) = \delta_{ij} - h_{ij}(t) , \quad (346)$$

from which we get the different components of the projector $P^{\mu\nu}$:

$$P^{00} = 0 , \quad P^{0i} = 0 , \quad P^{ij} = \delta_{ij} - h_{ij} . \quad (347)$$

At first order in the perturbation, the Christoffel symbols are given by:

$$\Gamma_{00}^0 = \Gamma_{0i}^0 = 0 , \quad \Gamma_{ij}^0 = \frac{1}{2} \partial_0 h_{ij} . \quad (348)$$

Therefore the covariant derivatives of the velocity are:

$$\nabla_0 u_0 = \nabla_0 u_i = 0 , \quad \nabla_i u_j = \frac{1}{2} \partial_0 h_{ij} . \quad (349)$$

From these values and the hypothesis that the metric perturbation is traceless it follows that the covariant divergence of the velocity vanishes:

$$\nabla \cdot u = \frac{1}{2} \partial_0 h_{ii} = 0 . \quad (350)$$

Let us assume that the only non-zero value of h_{ij} is h_{12} . Then, the linear response value of σ^{12} in frequency space is:

$$\sigma^{12}(\omega) = G_R^{12,12}(\omega, \vec{k} = 0) h_{12}(\omega) . \quad (351)$$

Moreover, by using the values of the covariant derivatives of the velocity we get

$$\sigma^{12}(t) = \eta \partial_0 h_{12}(t) , \quad (352)$$

or, in frequency space (for low ω):

$$\sigma^{12}(\omega) = -i\eta \omega h_{12}(\omega) . \quad (353)$$

By comparing these two expressions of σ^{12} , we obtain again Kubo formula for the shear viscosity, namely:

$$\eta = -\lim_{\omega \rightarrow 0} \left[\frac{1}{\omega} \text{Im} G_R^{12,12}(\omega, \vec{k} = 0) \right] . \quad (354)$$

In order to compute holographically the retarded correlator of T_{12} , let us consider a general $d + 1$ -dimensional diagonal metric and let us perturb it by adding a non-diagonal element along $x^1 x^2$:

$$ds^2 = g_{tt} dt^2 + g_{zz} dz^2 + g_{xx} (\delta_{ij} dx^i dx^j + 2\phi dx^1 dx^2) , \quad (355)$$

where ϕ is small and independent of x^1 and x^2 . Notice that, in the perturbed metric at first order in ϕ , one has:

$$g_{12} = g_{xx} \phi , \quad g^1_2 = \phi . \quad (356)$$

Let us write the $x^1 x^2$ part of the metric (at first order) as:

$$g_{xx} (dx^1)^2 + g_{xx} (dx^2 + \phi dx^1)^2 , \quad (357)$$

which is clearly similar to the ansatz corresponding to a Kaluza-Klein reduction along x^2 with KK gauge field connection:

$$A = \phi dx^1 . \quad (358)$$

Therefore, we can use the known results of the KK reduction to write the quadratic action for ϕ . Indeed, the Einstein-Hilbert action for the metric leads to the following expression for the action of the gauge connection A (or equivalently the metric perturbation ϕ):

$$S_\phi = -\frac{1}{16\pi G_N} \int d^{d+1}x \sqrt{-g} g_{xx} \frac{F^2}{4} , \quad (359)$$

where g is the determinant of the unperturbed metric (with $\phi = 0$), $F = dA$ is the field strength of A and in $F^2 = F_{\mu\nu} F^{\mu\nu}$ the indices are raised with the unperturbed metric. As:

$$F = \partial_t \phi dt \wedge dx^1 + \partial_3 \phi dx^3 \wedge dx^1 + \partial_z \phi dz \wedge dx^1 , \quad (360)$$

we have:

$$F^2 = \frac{2}{g_{xx}} \left[g^{tt} (\partial_t \phi)^2 + g^{xx} (\partial_3 \phi)^2 + g^{zz} (\partial_z \phi)^2 \right] , \quad (361)$$

or, taking into account that $\partial_1 \phi = \partial_2 \phi = 0$:

$$F^2 = \frac{2}{g_{xx}} g^{MN} \partial_M \phi \partial_N \phi . \quad (362)$$

Thus, the action for the perturbation ϕ is:

$$S_\phi = -\frac{1}{16\pi G_N} \int d^{d+1}x \sqrt{-g} \frac{1}{2} g^{MN} \partial_M \phi \partial_N \phi , \quad (363)$$

which is the canonical form of the action for a scalar field, with normalization constant

$$q = 16\pi G_N . \quad (364)$$

It follows from our general calculation of section [12](#) of the transport coefficients for a scalar field that the shear viscosity is given by:

$$\eta = \frac{1}{16\pi G_N} \frac{A_H}{V} , \quad (365)$$

and, therefore, the ratio η/s is just [\[24, 25\]](#):

$$\frac{\eta}{s} = \frac{1}{4\pi} . \quad (366)$$

In ordinary units the ratio η/s is given by:

$$\frac{\eta}{s} = \frac{\hbar}{4\pi k_B} , \quad (367)$$

where k_B is the Boltzmann constant. Notice that this result does not depend on the metric chosen. It is valid for any theory with a gravity dual given by Einstein gravity coupled to

matter fields. In this sense it is a universal result valid at $\lambda \rightarrow \infty$ (infinite coupling limit). The finite coupling corrections can also be calculated. For $\mathcal{N} = 4$ SYM one gets:

$$\frac{\eta}{s} = \frac{1}{4\pi} \left(1 + \frac{15\zeta(3)}{\lambda^{\frac{3}{2}}} + \dots \right), \quad (368)$$

where $\zeta(x)$ is the Riemann zeta function ($\zeta(3) = 1.2020$). In general, η/s at $\lambda \rightarrow \infty$ is very small:

$$\frac{\eta}{s} = 0.07957. \quad (369)$$

The finite coupling corrections make η/s increase. It is interesting to compare with the weak coupling calculation, valid when $\lambda \rightarrow 0$:

$$\frac{\eta}{s} = \frac{A}{\lambda^2 \log\left(\frac{B}{\sqrt{\lambda}}\right)}, \quad (370)$$

where A and B are constant coefficients that depend on the theory. Notice that in (370) $\eta/s \rightarrow \infty$ as $\lambda \rightarrow 0$. This is because a weakly coupled gauge theory is a gas with strong dissipative effects, in which momentum can be transported over long distances due to the long free path. In contrast a strongly coupled plasma is an almost perfect fluid in which momentum is rapidly transferred between layers of sheared fluid.

Kovtun, Son and Starinets (KSS) [24] conjectured that $1/(4\pi)$ is the lower bound for η/s . The lowest values of the η/s ratio found experimentally occur in two physical systems: the quark-gluon plasma created in heavy ion collisions at RHIC and the ultracold atomic Fermi gases at very low temperature. Both systems have η/s which is slightly above $1/(4\pi)$.

Changing the gravity theory the KSS bound is violated. For example, by adding higher curvature terms as in the Gauss-Bonnet gravity, whose action is given by:

$$S_{GB} = \frac{1}{16\pi G_5} \int d^5x \sqrt{-g} \left[R - 2\Lambda - \frac{3}{\Lambda} \lambda_{GB} \left(R^2 - 4R_{\mu\nu}R^{\mu\nu} + R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} \right) \right]. \quad (371)$$

For this gravity theory, the following η/s ratio is obtained:

$$\frac{\eta}{s} = \frac{1 - 4\lambda_{GB}}{4\pi}, \quad (372)$$

which violates the KSS bound if $\lambda_{GB} > 0$.

Final remarks

In these lecture notes we have reviewed the basic features of the AdS/CFT duality, focusing on its conceptual foundations and on some particular applications. We have only scratched

the surface of the subject, which in the last years has become a highly diversified field with many ramifications and connections. For the reader interested in knowing more on some of these applications of the holographic duality, let us quote some review articles, where the reader can find detailed accounts and the original references.

In this review we have mostly dealt with the gravitational description of $\mathcal{N} = 4$ $SU(N)$ gauge theory, which only contains fields transforming in the adjoint representation of the gauge group. In this sense $\mathcal{N} = 4$ SYM is a theory of pure glue. In order to extend the duality to theories closer to particle physics phenomenology one should be able to include flavor fields transforming in the fundamental representation of the gauge group (*i.e.* quarks). This can be done by adding the so-called flavor branes, as reviewed in [8,26]. Moreover, the gauge/gravity duality can be extended to include less supersymmetric theories exhibiting confinement (see [27]) and one can construct holographic duals of quantum chromodynamics ([8,9,28,29]). The holographic methods can also be used to study dynamical electroweak symmetry breaking, in the framework of walking technicolor models [30]. On the other hand, the AdS/CFT correspondence has unveiled the integrable character of planar $\mathcal{N} = 4$ SYM and has allowed to extend the duality beyond the supergravity regime [31].

One of the more interesting recent developments of the gauge/gravity duality is the application of the string theoretical ideas to the down-to-earth problems of condensed matter physics. Indeed, condensed matter physics is full of strongly-coupled systems which display quantum criticality with specific scaling laws. The gauge/gravity duality allows to map this scaling behavior to the general covariance of a gravity theory, for which one can apply the calculational tools and physical intuition of general relativity. In this way one can model the behavior of unusual phases of matter, such as strange metals or unconventional superconductors [6,32–34]. In this context holography is emerging as a new tool to understand the collective quantum behavior not explained by the conventional paradigms, such as the Fermi-liquid theory.

It is also interesting to point out the connection between the AdS/CFT correspondence and quantum information theory. In particular, there is a holographic proposal for the entanglement entropy [35,36], which allows a simple geometrical calculation of the latter as the area of a minimal surface. Let us finally mention that holography has also been applied to the study of strongly-coupled hydrodynamics. This particular version of the duality is called the fluid-gravity correspondence [37,38].

Hopefully these lectures will stimulate the reader to explore some of the topics listed above.

Acknowledgments

I am grateful to Yago Bea, Niko Jokela, Javier Mas and Ricardo Vázquez for their comments and help in the preparation of these lecture notes. I also thank Carlos Merino for his invi-

tation to deliver the course on the AdS/CFT correspondence at the third IDPASC school. This work is funded in part by the Spanish grant FPA2011-22594, by Xunta de Galicia (Consellería de Educación, grant INCITE09 206 121 PR and grant PGIDIT10PXIB206075PR), by the Consolider-Ingenio 2010 Programme CPAN (CSD2007-00042), and by FEDER.

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The AdS/CFT Correspondence

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ABSTRACT: We give a brief review of the AdS/CFT correspondence, which posits the equivalence between a certain gravitational theory and a lower-dimensional non-gravitational one. This remarkable duality, formulated in 1997, has sparked a vigorous research program which has gained in breadth over the years, with applications to many aspects of theoretical (and even experimental) physics, not least to general relativity and quantum gravity. To put the AdS/CFT correspondence in historical context, we start by reviewing the relevant aspects of string theory (of which no prior knowledge is assumed). We then develop the statement of the correspondence, and explain how the two sides of the duality map into each other. Finally, we discuss the implications and applications of the correspondence, and indicate some of the current trends in this subject. The presentation attempts to convey the main concepts in a simple and self-contained manner, relegating supplementary remarks to footnotes.

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1 Prologue: thanks to black holes...

The beautiful theory of general relativity enjoyed many remarkable achievements over the last century and provided a crucial edifice for theoretical physics. The conceptual revolution regarding the nature of space and time has popularized the theory to the extent that it would be hard to find a scientific-minded person who has not heard of Einstein’s theory of gravity. Yet, despite its universally recognized elegance, until recently general relativity has been used by a relatively small subset of the scientific community – much smaller, for instance, than those using quantum mechanics or quantum field theory – mostly¹ restricted to relativists, cosmologists, and a fraction of astrophysicists. This is not so surprising; after all, the effects of general relativity are pretty negligible at the scales relevant in context of condensed matter physics or nuclear physics. It would seem even more audacious to imagine that gravitation could play any interesting role deeper in the quantum world, say for quantum information theory.

Yet over the last decade, general relativity has percolated all of these fields! Nowadays one can find many condensed matter physicists, nuclear physicists, quantum information theorists, and others, who are actively interested in general relativity, not just for idle curiosity, but as a crucial tool in their research. What has nucleated this dramatic transition? Naturally, the overall scientific trend points toward multidisciplinary as the value of cross-fertilization between different areas is becoming more widely appreciated – but that is the effect rather than the cause. The real game changer responsible for this transition was the AdS/CFT correspondence (now more generally known as a gauge/gravity duality), conjectured by Juan Maldacena in 1997 [1]. Indeed, as we will see, the manner in which AdS/CFT related all these subjects to gravity could scarcely have been more spectacular: it turns out that it’s not the weak curvature of our spacetime which comes into play when studying everyday systems, but the most gravitational object there is: the black hole! Incredible as this may sound, it is even more amazing, though in retrospect also demystifying, that the relevant dynamical spacetime is a higher-dimensional one compared to the non-gravitational system it describes. Indeed, the most striking feature of the AdS/CFT duality is its ‘holographic’ nature: it posits full equivalence between certain theories formulated in different numbers of dimensions.

We postpone a more detailed phrasing of the correspondence till §3, after we have put it into historical context and explained the necessary ingredients in §2.² The AdS/CFT correspondence has been arrived at using the framework of string theory, and indeed engendered what might be called a scientific revolution within the subject. This new type of holographic duality not only provided a more complete formulation of the theory, but also profoundly altered our view of the nature of spacetime: the gravitational degrees of freedom emerge as effective classical fields

¹ Of course, there were also (comparatively much smaller numbers of) interested physicists in other fields, amongst them notably string theorists – which, as we’ll soon see, was pivotal in the present context.

² Since the focus throughout this chapter is on explaining the concepts (rather than providing a comprehensive review or a historical account), we will attempt to keep references to a minimum, based mainly on historical importance and suitability for the present audience. For a more comprehensive list of references see the relevant reviews cited below.

from highly quantum gauge theory degrees of freedom. This harks back to earlier expectations motivated by black hole thermodynamics, that spacetime arises as a coarse-grained effective description of some underlying microscopic theory, but with a new twist: the relevant description is lower-dimensional. As we will see, black holes not only motivated this idea, but played a key role in both the derivation and subsequent applications of AdS/CFT.

Over the intervening years, the AdS/CFT correspondence has flowered into a vast and intricate subject; Maldacena’s original paper [1] alone now has well over 10,000 citations. While the AdS/CFT duality as such has not been rigorously ‘proved’ (partly because we do not yet have a complete independent definition of the quantum gravity side of the correspondence), it has successfully withstood such an impressive array of highly nontrivial checks, that a vast majority of the community is now fully convinced the duality holds. Indeed, the evidence has mounted so rapidly, that the mindset soon changed from “can it possibly be true?” to “what does it mean?” or “how does it work?” and “what else can we do with it?”. Hence regardless of its status as a theorem, the utility of AdS/CFT has already been amply demonstrated by the consequent discoveries. In fact, the resulting relations transcend the original construction; just as gauge/gravity duality can actually be formulated without any recourse to string theory, many of its implications in turn hold independently of AdS/CFT. Gratifyingly, the statement of equivalence between two sides of the correspondence stimulates information flow in both directions, teaching us important lessons about each side, as we review in §4. In particular, the gauge/gravity duality engendered a number of fascinating observations about general relativity itself. Conversely, general relativity provides the best means hitherto to calculate certain interesting quantities within strongly coupled field theories – which is the main reason why so many physicists are using general relativity as a tool.

Far-reaching as the implications of the AdS/CFT correspondence have turned out, the lessons of the correspondence are far from exhausted, with new surprises jumping at us from every corner – indeed it is evident that we have barely seen the tip of the iceberg! Particularly enticing is the unquenched promise of ‘solving’ quantum gravity. While this quest motivated the original explorations leading up to AdS/CFT as well as its subsequent developments, we still have a long journey ahead of us. Already, the gauge/gravity duality has given us many crucial insights into quantum gravity; in fact, one review of the gauge/gravity duality [2] starts with the assertion that: *Hidden within every non-Abelian gauge theory, even within the weak and strong nuclear interactions, is a theory of quantum gravity.* While part of the community is still immersed in mining ‘Applied AdS/CFT’ to gain further insight into various systems of interest in condensed matter, nuclear, and particle physics, the overall focus is gradually shifting towards tackling the most mysterious aspects of the correspondence in view of unraveling quantum gravity. Quintessentially quantum notions such as entanglement are gaining in significance, and the links with quantum information theory are now being vigorously explored. Where it will all lead we still don’t know, but for now there are many exciting paths to follow.

2 Strings and branes

With the advent of the singularity theorems [reviewed in ch.8], it became manifest that classical general relativity can break down quite generically: it ceases to provide a valid description of physics near curvature singularities. The natural expectation of course is that quantum mechanical effects ‘resolve’ the singularities, so that the dynamics remains well-defined throughout the entire evolution. However, since quantum mechanics and general relativity by themselves are mutually incompatible, our description of the universe based solely on these would be not just incomplete but actually inconsistent.

In the ongoing quest for quantum gravity, the most useful playground has been provided by black holes,³ for many reasons. Obviously, typical black holes contain curvature singularities, which manifestly require new physics. These might reveal underlying symmetries of the full theory (motivating for instance the ‘cosmological billiards’ program started by [3] based on universal Mixmaster-like behaviour near spacelike singularities), or suggest possibilities of how evolution could remain well-defined through a singularity (as happens e.g. for certain timelike singularities in string theory), but usually direct study suffers from insufficient calculational control. Fortunately, it turns out that we can gain important insight without being quite so ambitious: already the black hole horizon contains many profound clues, as implied by black hole thermodynamics [reviewed in ch.10; see also [4]]. The relations between geometrical properties of the horizon (area and surface gravity) and thermodynamic quantities (entropy and temperature) hint at a statistical mechanical origin. While Hawking [5] famously explained the temperature in terms of semi-classical particle creation, understanding the origin of black hole entropy⁴ in terms of the corresponding number of ‘microstates’ has proved both more difficult and more rewarding. Classically, the black hole ‘no-hair’ theorems imply that these microstates are not simply different spacetime geometries that look like a black hole – we need to go beyond general relativity. Counting the black hole microstates has therefore provided an invaluable benchmark for putative theories of quantum gravity.

To motivate whence one should seek guidance in this endeavor, recall that in units $c = 1$, $\hbar = 1$, the 4-dimensional Newton’s constant is given by $G_N = \ell_p^2$ where the Planck length $\ell_p \approx 1.6 \times 10^{-33} \text{ cm}$ specifies the length scale at which quantum effects become important. Of course, direct access to such scales is many orders of magnitude beyond the reach of accelerators, so the theory must be guided not by experiment but rather by internal consistency. Fortunately, this turns out to be an extremely stringent requirement.

³ We will consider spacetimes with well-defined future null infinity $\mathcal{I}^+$ and use the standard definition of a black hole as the region inside the event horizon $\mathcal{H}^+ \equiv \partial I^-[\mathcal{I}^+]$, i.e. one causally disconnected from the boundary. The horizon is always a spacetime co-dimension 1 null surface, but in higher dimensions it may be extended in one or more directions, giving rise to black strings, black branes, etc.; in the present context the term ‘black hole’ refers to all such objects.

⁴ That entropy of a black hole should be proportional to its area has been deduced by Bekenstein [6] based on various gedanken-experiments, but these did not give a clue to its underlying quantum nature.

Unfortunately, conventional field theory techniques are not well suited to quantizing general relativity directly: The fact that in 4 spacetime dimensions the dimensionless gravitational coupling grows quadratically with energy gives rise to non-renormalizable perturbation theory: the divergences become uncontrollably worse at each order. This indicates that some new physics has to kick in to modify general relativity in the UV (meaning short distances or high energy scales). An ingenious way to tame these divergences is to consider strings as the fundamental degrees of freedom (so that what we previously thought of as particles are simply different excitation modes of the string, a spatially extended 1-dimensional object), which effectively smears out the interactions.

This might at first sight seem rather fanciful, but it works! More than that, it works spectacularly well.⁵ In a consistent Lorentz-invariant quantum theory, a closed string necessarily has a massless spin-two state, the graviton, whose long-wavelength interactions reproduce general relativity. Other states correspond to different particles: gauge bosons, fermions, etc; indeed string theory automatically incorporates the earlier ideas trying to explain the Standard Model such as grand unification, supersymmetry, and Kaluza-Klein theory.⁶ In the course of trying to unravel string theory, it became evident that it in fact encompasses other ‘competing’ constructions as well; indeed, within a few years it loomed into a vast edifice with incredibly rich structure which contains many essential features from other areas of both physics and mathematics.

Given that any string theory is a quantum theory which necessarily includes gravity, it naturally invites the examination of its consequences and implications for quantum gravity. Since spacetime plays a central role in general relativity, the obvious question to ask is how does string theory give rise to the curved dynamical spacetime of general relativity, and what happens to the ‘stringy geometry’ when the classical description breaks down. In the perturbative formulation valid at small string coupling g_s , the spacetime coordinates specifying the position of the string appear as scalar fields in the 2-dimensional sigma model describing the dynamics of the string worldsheet, and the spacetime metric then enters as a coupling constant. But because the strength of gravitational interactions is governed by the string coupling – the d -dimensional Newton’s constant is $G_N = g_s^2 \ell_s^{d-2}$ in units of the string length ℓ_s – it is difficult to directly access the most interesting, strongly gravitational, regime.

Nevertheless, already at this level we encounter several intriguing surprises. Since strings are extended objects, some spacetimes which are singular in general relativity (for instance those with a timelike singularity akin to a conical one) appear regular in string theory. Spacetime

⁵ In fact, strings were originally introduced as a possible theory of the strong interactions already by the early 70s, since they nicely explained certain features of the hadron spectrum. This program however dwindled with the advent of QCD, until it was realized in the mid-70s that string theory is also a theory of gravity [7] (see e.g. [8] for a historical review and [9, 10] for the classic string theory textbooks). It is amusing to note that two decades later the circle got (almost) completed via the AdS/CFT correspondence, wherein string theory turns out to be *dual* to a gauge theory akin to QCD.

⁶ Consistency requires that the (weakly-coupled) theory is formulated in 10 dimensions, but these can be small (and therefore able to accommodate our spacetime looking 4-dimensional). The geometry of these ‘internal’ dimensions then determines the 4-dimensional matter content.

topology-changing transitions can likewise have a completely controlled, non-singular description. Moreover, the so-called ‘T-duality’ equates geometrically distinct spacetimes: because strings can have both momentum and winding modes around compact directions, a spacetime with a compact direction of size R looks the same to strings as spacetime with the compact direction having size ℓ_s^2/R , which also implies that strings can’t resolve distances shorter than the string scale ℓ_s . Indeed this idea is far more general (known as mirror symmetry [11]), and exemplifies why spacetime geometry is not as fundamental as one might naively expect.

However, to learn about the more interesting regime where gravity is strong, we need to go beyond perturbation theory. At first sight this looks extremely challenging, since generically a quantum theory at strong coupling has uncontrollably large fluctuations. But remarkably in string theory the situation is far tamer. It turns out that string theories have a symmetry known as string duality: a strongly coupled limit of any given string theory is equivalent to a weakly coupled limit of another one.⁷ In other words, as we take the coupling large (i.e. in the $g_s \rightarrow \infty$ limit), the theory actually simplifies, and can be described perturbatively in some dual variables. Though the intricate web of such relations was gradually built up through the early 90’s, the crucial insight was provided by Polchinski [12], who showed that so-called Dirichlet-branes,⁸ or D-branes for short, provide the necessary charges⁹ required by the duality. At weak coupling, D-branes behave as topological defects on which open strings can end, but in the full theory they are dynamical objects with mass scaling inversely with string coupling. Hence at strong coupling they become light, providing the natural ‘fundamental’ constituents of the theory.

The advent of string dualities immediately stimulated the program of black hole microstate counting. To explain this, let us take a short detour into specifying a few essential features of D-branes, which will pave the way towards motivating the AdS/CFT correspondence. D-branes can be spatially extended in any number p of dimensions; the world-volume of a Dp -brane is $(p+1)$ -dimensional. Massless modes of open strings ending on a D-brane describe the transverse fluctuations of the brane and give rise to gauge fields along the brane and their fermionic partners; for N coincident D-branes the low energy dynamics is then given by a $U(N)$ gauge theory.¹⁰ A

⁷ There are actually five perturbative 10-dimensional string theories, which differ from each other by the way the supersymmetry acts on the string and whether it contains just closed or both open and closed strings; in the following, when we say ‘string theory’ we mean the ‘Type IIB string theory’ (and similarly for its low-energy limit, supergravity). There is also a sixth corner corresponding to an 11-dimensional theory called M-theory, wherein the size of the 11th dimension may be associated with the strength of the string coupling and whose fundamental excitations are 2-branes and 5-branes.

⁸ Using arguments based on T-duality, the idea of D-branes was in fact introduced already in [13], as extended objects which can couple consistently to strings (and thence conjectured to be in a sense made up of strings). For a technical overview see e.g. [14].

⁹ Namely, the so-called Ramond-Ramond (RR) charges. These are required by duality transformations, but cannot be carried by strings. It was already known that they can be carried by black p -branes [15] (black holes which are extended in p spatial directions), which prompted the speculation that these black p -branes might be related to the dual variables.

¹⁰ Every open string has two endpoints, each lying on one of the branes. When the branes coincide the symmetry gets enhanced since the strings stretching between them become massless.

stack of N coincident Dp -branes has N units of $(p + 1)$ -form charge, and preserves half the supersymmetry. This has the happy consequence that the number of microstates of such a system remains invariant under varying the string coupling g_s , and the mass of such states does not receive quantum corrections. To take advantage of this observation, we need to understand what happens to the system at strong coupling.

The amount by which N coincident D-branes backreact on the spacetime is governed by $G_N M \sim g_s^2 \frac{N}{g_s} = g_s N$ in string units. At sufficiently weak coupling, $g_s N \ll 1$, the branes live in nearly-flat 10-dimensional spacetime, whereas for $g_s N \gg 1$, the branes backreact strongly and source an extremal black brane geometry [15], a 10-dimensional analog of the extremal Reissner-Nordström black hole, extended in p spatial directions.¹¹ In this regime, one can evaluate the horizon area and compare this with the statistical entropy obtained from counting the degeneracy of the system carrying the same charges at weak coupling. This idea was realized in the seminal work of Strominger & Vafa [16] which correctly counts the Bekenstein-Hawking entropy of a 5-dimensional 3-charge extremal black hole (a strongly-coupled description of a dimensionally reduced D1-D5 bound state carrying momentum). Note that this is a far greater achievement than merely matching one number: both the functional dependence on all the charges as well as the overall coefficient come out correctly without any additional input. Since the weakly-coupled calculation looks nothing like the strongly coupled one, this nontrivial success suggests that such black holes are indeed ‘made of’ D-branes.

This new understanding unleashed a flurry of activity which reproduced the microscopic entropy of many other black holes described by more parameters, both in 4 and 5 dimensions and for other charges and angular momentum; see e.g. [17, 18] for early reviews and [19] for a broader overview. One might expect that supersymmetry is essential, but in fact one can reproduce the entropy of some nonextremal black holes as well. Although such black holes are quantum mechanically unstable due to Hawking evaporation, we can describe the Hawking process at weak coupling, as emission of closed strings from the brane. More impressively still, the weakly-coupled D-brane description can even correctly reproduce the full radiation spectrum of the black hole, including the grey-body factors due to the curved spacetime through which the Hawking quanta propagate [20].

As an aside, we should remark that sufficiently far from extremality we do lose control. So the ‘simplest’ black hole from the general relativistic standpoint and one which is perhaps closest to the hearts of most relativists, namely the Schwarzschild solution, still eludes the exact microstate counting. Nevertheless, the scaling of entropy with area (as well as correct dependence on other charges) can be reproduced quite generally, by viewing the black hole at weak coupling as a bound state of strings and D-branes with the same conserved charges. The relation between the two

¹¹ More precisely, the geometry and causal structure depend on p ; the horizon is regular for $p = 3$ and for appropriate bound states of multiple branes, which requires the system to carry more than two types of charge. For purposes of evaluating the entropy, given by quarter the horizon area *in Planck units*, the extra p dimensions however do not play any role because the higher-dimensional Newton’s constant is related to the lower-dimensional one by the same factor (namely the volume of the internal space) that relates the corresponding horizon areas.

descriptions was established by Horowitz & Polchinski [21] and is known as the “correspondence principle”: Consider an excited fundamental string. As we increase the string coupling g_s , the string self-gravitates more strongly and shrinks, while simultaneously the Schwarzschild radius $G_N M$ increases. At high enough coupling the string shrinks to within its own Schwarzschild radius, at which point the object is more appropriately described as a black hole. Conversely, if we start with a black hole and decrease the coupling, the horizon eventually becomes string size (which can however still be macroscopic in Planck units), at which point we can no longer trust the classical black hole solution and the system is more appropriately described as an excited string. Matching the masses¹² on the two sides of the transition, [21] found that the entropies also match, up to a constant of order unity which depends on the precise transition point.

By 1996 string theory has made significant advances towards understanding the quantum aspects of black holes. Cracking the black hole Information Paradox seemed to be just around the corner, and there were many tantalizing hints at a deeper structure underlying the many newly-discovered mysterious connections. Though the former hope was rather too optimistic, little did we expect how substantial a progress the following year will see on the latter front.

3 The AdS/CFT correspondence

In November 1997 Maldacena wrote his groundbreaking paper [1], conjecturing the relation which became known as the AdS/CFT correspondence. In §3.1 we will explain how Maldacena arrived at this remarkable conjecture, indicate its formulation and key features, and mention its immediate sociological impact. Since the statement is perhaps as mystifying as it is profound, we pause in §3.2 to give a more modern perspective which motivates the correspondence without any recourse string theory. We then build up the basics of the AdS/CFT dictionary which simultaneously exemplify some of the initial checks of its validity in §3.3 and §3.4, the latter focusing on the important context of AdS black holes. Finally, we briefly mention several generalizations of AdS/CFT in §3.5. For further details, we refer the reader to the excellent early review by ‘MAGOO’ [23] (see also [24, 25]).

3.1 Maldacena’s derivation

The success in unraveling the D1-D5 system inspired analogous calculations for a D3-brane system, which provided the focal example of Maldacena’s conjecture [1]. Consider N coincident D3-branes (in type IIB string theory). At weak coupling $g_s N \ll 1$, the branes live on a flat

¹² In fact, this resolved a puzzle that previously impeded advancing the suggestion [22] that black holes are really excited string states; namely, the respective entropies don’t scale the same way with the mass: $S_{string} \sim \ell_s M$ while (say in 4 dimensions) $S_{BH} \sim G_N M^2$. The resolution is that both masses cannot be kept fixed simultaneously as we vary the string coupling. Making them match at the transition when the black hole is string size ($g_s \sim (M \ell_s)^{-1/2}$) makes the entropies agree as well. This agreement continues to hold more generally in all dimensions and in the presence of extra charges, angular momentum, etc.

10-dimensional spacetime and we have open strings ending on the D-branes as well as closed strings propagating in the bulk. On the other hand, at strong coupling $g_s N \gg 1$, the branes curve the spacetime substantially, sourcing the extremal black 3-brane geometry [15]:

$$ds^2 = f(r)^{-1/2} \eta_{\mu\nu} dx^\mu dx^\nu + f(r)^{1/2} (dr^2 + r^2 d\Omega_5^2) \quad , \quad f(r) = 1 + \frac{4\pi g_s N \ell_s^4}{r^4} \quad (3.1)$$

where x^μ denote the 4 coordinates along the D3-brane worldvolume and $d\Omega_5^2$ is the metric of a unit S^5 . The solution is supported by a self-dual 5-form field strength, which has flux on the S^5 .

So far, we have specified two distinct regimes of $g_s N$ with no region of overlap. Maldacena's insight was to consider decoupling the theory on the branes from gravity. This can be achieved by taking a low-energy limit, which simplifies the physics enormously. On the one hand, the open string sector decouples from the rest of the theory, so that we end up with a 4-dimensional $SU(N)$ gauge theory (specifically super Yang-Mills) describing the dynamics of the branes. On the other hand, in the black brane spacetime, this limit¹³ focuses on the near-horizon geometry: from the asymptotic viewpoint, any finite-energy excitation near the horizon will be strongly redshifted, while modes which propagate in the asymptotic region of (3.1) decouple from the near-horizon region (since its cross section vanishes in this limit).

Note that just as in the case of extremal Reissner-Nordström black hole, the event horizon $r = 0$ of the black brane solution (3.1) lies at infinite proper distance along spacelike geodesics; its embedding diagram has an infinite ‘throat’. The near-horizon geometry then has an enhanced symmetry, and simplifies to a direct product of a sphere and Anti-de-Sitter spacetime. In case of 4-dimensional extremal Reissner-Nordström black hole, the near-horizon geometry is simply $\text{AdS}_2 \times S^2$, while in the present 10-dimensional case we have $\text{AdS}_5 \times S^5$. In particular, defining $\ell \equiv (4\pi g_s N)^{1/4} \ell_s$ in (3.1), we see that as $r \rightarrow 0$ (zooming near the horizon), $f(r)^{1/2} \rightarrow \ell^2/r^2$, which obtains

$$ds^2 = \frac{r^2}{\ell^2} \eta_{\mu\nu} dx^\mu dx^\nu + \frac{\ell^2}{r^2} dr^2 + \ell^2 d\Omega_5^2 \quad . \quad (3.2)$$

The first two terms here describe the AdS_5 , a maximally symmetric spacetime of constant negative curvature, with radius¹⁴ ℓ . The last term gives the S^5 , likewise with radius ℓ . The D-branes are no longer localized within the geometry, but as in (3.1), their effect manifests itself in the 5-form flux through the S^5 .

The low-energy limit of our D3 brane system is then described by a (10-dimensional) string theory with just closed strings in $\text{AdS}_5 \times S^5$ when $g_s N \gg 1$, and by a (4-dimensional) super Yang-Mills $SU(N)$ gauge theory when $g_s N \ll 1$. But since the gauge theory is well-defined at any coupling, it is natural to conjecture that this description in fact applies even when $g_s N$ is large, i.e. in the *same* regime as where the closed string description holds.

¹³ Actually, [1] implemented this by starting with slightly-separated branes and taking the $\ell_s \rightarrow 0$ limit at fixed (large) N and (small) g_s while simultaneously bringing the branes together, keeping the open string mass fixed. In practice, this can be accomplished by taking $r \rightarrow 0$ in (3.1).

¹⁴ Although AdS is spatially non-compact, it has a characteristic size given by its curvature radius: the Ricci scalar is $R = -20/\ell^2$. Moreover, all timelike geodesics in this spacetime oscillate with period $2\pi\ell$, so AdS acts like a confining box.

This observation led Maldacena [1] to conjecture that

$$\text{String theory on } \text{AdS}_5 \times S^5 \quad \simeq \quad \mathcal{N} = 4, \text{ } SU(N) \text{ gauge theory in } 4D. \quad (3.3)$$

Here the ‘ $\simeq$ ’ sign indicates a full duality: the two sides are simply different languages which describe the *same* physics.

The statement (3.3) quickly became known as the AdS/CFT correspondence, because AdS specifies the sector of spacetimes for the LHS, and the 4-dimensional gauge theory on the RHS is a conformal field theory (CFT). In fact, [1] gave several other examples which we will mention in §3.5, all having the AdS/CFT structure: string or M-theory on AdS_{d+1} (times a compact manifold) is dual to a d -dimensional CFT. Later people started referring to AdS/CFT alternately as gauge/gravity (or sometimes gauge/string) duality to emphasize the emergence of gravity (or string theory) from the gauge theory and to indicate its greater generality.¹⁵

We will discuss the meaning of this statement in greater depth in §3.3 below; for now we make a few remarks to qualify the correspondence (3.3) a bit more precisely. First of all, since the LHS still has dynamical gravity, by “string theory on $\text{AdS}_5 \times S^5$ ” we mean string theory on a 10-dimensional spacetime which is *asymptotically* $\text{AdS}_5 \times S^5$. In particular, all physically sensible gravitational processes are included on the LHS; for instance one can collapse a black hole in AdS (which will provide an important class of examples that we will revisit below).

Secondly, along with the statement (3.3), the AdS/CFT correspondence also specifies how the parameters on the two sides relate to each other. On the string theory side we have two dimensionless parameters, the string coupling g_s and the curvature scale (in string units) of the spacetime on which the theory lives, ℓ/ℓ_s . On the gauge theory side we have the rank of the gauge group N and the Yang-Mills coupling g_{YM} , which is more naturally expressed in terms of the ‘t Hooft coupling¹⁶ $\lambda \equiv g_{\text{YM}}^2 N$. The relation between these is given by

$$4\pi g_s = g_{\text{YM}}^2 \sim \frac{\lambda}{N} \quad \text{and} \quad \frac{\ell}{\ell_s} = (4\pi g_s N)^{1/4} \sim \lambda^{1/4} \quad (3.4)$$

In order to trust the gravity solution (i.e. to suppress stringy corrections of the geometry) we need to keep ℓ large in string units, which translates to $\lambda \gg 1$. On the other hand, to suppress the quantum corrections, we need to keep g_s small. Hence classical gravity is valid in the $N \gg \lambda \gg 1$ regime of the parameter space. The most conservative version of the duality posits (3.3) in this regime, though it is now widely believed that the equivalence holds even at finite N and λ (although direct confirmation is stymied by lack of tools to study the string theory much beyond perturbations in $1/N$ and $1/\lambda$).

¹⁵ As anticipated already by [1], one can modify the boundary conditions away from AdS and correspondingly the field theory need not be a CFT. (Also note that in the latter terminology the order of the two sides has been swapped: “gravity” or “string” refers to the AdS side and “gauge” to the CFT side.)

¹⁶ For $SU(N)$ gauge theories, ‘t Hooft [26] showed that there is a smooth limit (known as the ‘t Hooft limit) which takes $N \rightarrow \infty$ while keeping $\lambda \equiv g_{\text{YM}}^2 N$ fixed; in this limit, only the planar Feynman diagrams contribute and the gauge theory becomes effectively classical when recast in certain other variables (namely, it can be described as a classical string theory).

To summarize, the key features of the AdS/CFT duality are the following.

- There is a mapping between a quantum theory of gravity (namely string theory) and an ordinary (non-gravitational) quantum field theory. Moreover, the dual description of gravity is manifestly background-independent, except for the AdS boundary conditions. This should help us solve many long-standing questions in quantum gravity by recasting them in a non-gravitational language.
- AdS/CFT is a *strong/weak coupling duality*: when the gauge theory is strongly-coupled (and hence all perturbative techniques fail), we can study it using the weakly-coupled string theory context. This has been the main point of [1] (and subsequently the most utilized aspect of the correspondence).
- The AdS/CFT mapping is *holographic*. Indeed, the correspondence provides the most concrete example of the holographic principle, suggested few years earlier by ‘t Hooft [27] and Susskind [28].¹⁷ Albeit partly inspired by related ideas of [30] (see also [31]), [1] did not emphasize holographic nature of the correspondence (in fact the term ‘holographic’ does not appear in the paper); however, it was noted soon thereafter [32, 33].

Though many of the clues hinting at AdS/CFT correspondence (3.3) had existed before, Maldacena’s conjecture [1] had taken most by surprise. Indeed, Juan Maldacena, then a young faculty at Harvard one year after graduating from Princeton, instantly became somewhat of a celebrity in the field. The general excitement was evident the following summer at the Strings ‘98 conference, not only during the talks, but also at the conference dinner when all the participants danced the ‘Maldacena’.¹⁸

The statement of the AdS/CFT duality is so remarkable that one might well wonder whether there is some loophole in this argument: could something have gone wrong along the way? For instance, might there be some sharp transition under varying the coupling $g_s N$ or some non-perturbative effect spoiling the extrapolation? Albeit unexpected, such effects have been vigorously searched for. So far, however, the AdS/CFT correspondence has withstood all tests, and whenever exact calculations are possible on both sides, such as in the maximally supersymmetric case where we can use the tools of integrability [34], a precise match is found. Nevertheless, the search continues as we develop new tools to understand increasingly more general context.

Another (more distanced and vague) level at which the AdS/CFT correspondence has been questioned by its critics refers to the foundations themselves: AdS/CFT has been derived within string theory, but what if string theory itself is incomplete or incorrect? Rather than delving

¹⁷ This bold statement, that in a theory of gravity, we can describe physics in a given region by a theory living on its boundary, was inspired by Bekenstein’s entropy bound [29] based on black hole thermodynamics: since black hole entropy scales with horizon area, the amount of information which can be packed in a spherical region should be bounded by its surface area rather than its volume.

¹⁸ With the lyrics composed by Jeff Harvey who led the dance, this was a take-off on the popular Spanish dance called the Macarena, producing a rather comical effect which made the occasion doubly memorable.

into an involved discussion attempting to dispel such objections here, we will now argue that the gauge/gravity duality actually stands on its own in a self-contained manner, independently of string theory.

3.2 More modern perspective

Maldacena’s derivation notwithstanding, many of us experience some bewilderment at our first encounter of the AdS/CFT duality. Indeed, the assertion that a higher dimensional gravitational theory could be fully described by a lower-dimensional non-gravitational one seems quite preposterous. Moreover, it was generally felt in the string theory community that we understand quantum field theories well, and that they cannot be so well disguised as to actually behave like quantum gravity. In fact, much of the initial effort in this field stemmed from attempts to find a clear ‘counter-example’ which would indicate that the two sides of (3.3) cannot possibly be equivalent. Though all such attempts of course failed, this was a tremendously useful exercise for developing our understanding of how the correspondence works. Before describing some of these checks and entries in the AdS/CFT dictionary, let us pause to see why the correspondence is not obviously wrong, by mentioning a more direct route toward the duality, one which is independent of string theory.

In hindsight, a plausible way to arrive at the gauge/gravity duality is the following.¹⁹ The prospect that a gauge theory might capture a gravitational theory is partly inspired by the matter content: a natural guess, bolstered by representation theory, is that the spin-two graviton could arise as a composite of two spin-one gauge bosons. Of course, this would be precluded by the Weinberg-Witten no-go theorem [35], if the graviton and gauge bosons lived in the same Lorentzian spacetime. However, the holographic principle suggests that the graviton may naturally propagate in higher dimensions, thereby eluding the no-go argument. The gauge theory would then have to contain not just the graviton but also an extra dimension, namely some quantity with respect to which the physics behaves locally. One such quantity in a gauge theory is the energy scale: the renormalization group equation governing the flow of the coupling constants with energy scale is a nonlinear differential equation which is local in the energy scale.

Having identified a gauge theory quantity which could describe an extra direction in the bulk dual, we want to further ensure that this new ‘dimension’ can be macroscopic. Since perturbative gauge theory looks nothing like classical gravity, one is led to suspect that the gauge theory should be highly quantum (i.e. strongly coupled) in order to reproduce gravity.²⁰ Taking this guess seriously, one would then seek a gauge theory where the coupling remains strong over a large range of energies, in order to recover an extra dimension of macroscopic size. This is most easily achieved in a conformal field theory where the coupling does not run, allowing for

¹⁹ The presentation here is largely based on the excellent review of the gauge/gravity duality by Horowitz & Polchinski [2].

²⁰ Anticipating the string theory context, strong coupling can also be motivated by the requirement of keeping only spin-two graviton while removing all higher-spin composite objects.

an infinite holographic direction. Nonetheless, the possibility of having excitations with various energies propagating in higher dimensions still seems to have more degrees of freedom than can be accommodated within a lower-dimensional theory. In order to overcome this obstacle, the gauge theory then has to contain sufficiently large number of degrees of freedom, which one may hope to achieve with a large gauge group; for $SU(N)$ gauge theories, this suggests the 't Hooft limit [26], namely taking $N \rightarrow \infty$ keeping $\lambda \equiv g_{\text{YM}}^2 N$ fixed (but large by the previous argument).

To keep our strongly-coupled gauge theory under control, it is convenient to impose supersymmetry (since this keeps the Hamiltonian bounded from below and thereby precludes many potential instabilities). The most supersymmetric case is $\mathcal{N} = 4$ (which has 4 copies of the minimal 4-dimensional supersymmetry algebra), which simultaneously ensures conformal invariance: The 4-dimensional $\mathcal{N} = 4$ $SU(N)$ gauge theory, super-Yang-Mills,²¹ is a conformal field theory (CFT). When formulated on Minkowski space, $ds_{\text{CFT}}^2 = \eta_{\mu\nu} dx^\mu dx^\nu$, the vacuum is invariant under Poincaré transformations, and by virtue of conformal invariance it is also in particular invariant under a rigid scale transformation $x^\mu \rightarrow \alpha x^\mu$, which simultaneously rescales the energy $E \rightarrow E/\alpha$. Identifying inverse energy with the extra dimension (labeled by a new coordinate z), the most general 5-dimensional bulk metric consistent with these symmetries is AdS_5 :

$$ds^2 = \frac{\ell^2}{z^2} (\eta_{\mu\nu} dx^\mu dx^\nu + dz^2) \quad (3.5)$$

where we've rescaled $z \sim 1/E$ so as to express the metric in terms of one free parameter, the AdS scale ℓ . A trivial change of variables, $z = \ell^2/r$, recasts (3.5) into the form used in (3.2); i.e. r corresponds to energy scale as indeed identified by [1].

So far, the above arguments motivate the equivalence of gravity on 5-dimensional AdS with 4-dimensional $\mathcal{N} = 4$ $SU(N)$ gauge theory, along with the expectation that the number of 'colors' N should be related to the size of AdS , ℓ . But one can go even further. To match the supersymmetry of the gauge theory side, we need to extend gravity to the full supergravity, which is naturally formulated in 10-dimensions and has a solution $\text{AdS}_5 \times S^5$. So we get not just one, but six, extra dimensions, five of which (making up the S^5) are naturally associated with the scalars φ^i of the 4-dimensional super-Yang-Mills. Moreover, if we probe further still, by considering highly boosted bulk states, the corresponding excitations on the gauge theory side exhibit a one-dimensional structure which can be associated with excitations of a string [36]. So by making the full structure self-consistent, one could in principle 'uncover' string theory.²² Hence in hindsight, the AdS/CFT duality (3.3) emerges quite naturally. Moreover, albeit puzzling, it elegantly explains why apparently different results in gauge theory and gravity happen to give

²¹ This theory can be obtained from a dimensional reduction of a 10-dimensional $SU(N)$ gauge theory with 16 supersymmetries with smallest algebra, $\mathcal{L} = \frac{1}{2g_{\text{YM}}^2} \text{Tr}(F_{\mu\nu} F^{\mu\nu}) + i \text{Tr}(\bar{\psi} \gamma^\mu D_\mu \psi)$ (with both the gauge field A_μ and the spinor ψ in the adjoint $N \times N$ representation). Upon dimensional reduction to 4 dimensions, the A_μ decomposes into a 4-dimensional gauge field and 6 scalars φ^i which are symmetric under $SO(6)$ rotation, while the spinor ψ separates into four 4-dimensional Weyl spinors. This 4-dimensional SYM is not only a finite theory, but indeed the simplest interacting 4-dimensional field theory.

²² In fact, the idea of strings arising from large- N (planar) limit of gauge theory was already suggested by 't Hooft [26]; the AdS/CFT correspondence renders this occurrence fully explicit.

the same answers. In a sense, having so many coincidences with no explanation would have been far more perplexing...

Finally, at a broader level, it can be argued [37] that quantum gravity with asymptotically AdS (or even asymptotically flat) boundary conditions is holographic, in the following sense: due to diffeomorphism invariance, the gravitational Hamiltonian is a purely boundary term, expressed as a surface integral at infinity. Since it is the operator which generates time translations, any set of boundary observables which are available at any given time are necessarily available on the boundary at any other time. More technically, Marolf [37] shows that (at least at the perturbative level) there is a complete algebra of boundary observables within any neighborhood of any boundary Cauchy surface. This idea is referred to as boundary unitarity, and has important implications for example about the black hole information paradox.

3.3 Early checks and entries in the AdS/CFT dictionary

Now that we have explained two separate arguments for the gauge/gravity duality (3.3), let us turn to its implications. Of course, even if rigorously proven, the correspondence would be of little use if we didn't know how to relate physically interesting quantities between the two sides. Much the subsequent (and ongoing) effort went into establishing the AdS/CFT dictionary. The basic entries in this dictionary follow immediately from the above arguments. Let us therefore start by specifying these more explicitly.

Symmetries: Perhaps the most immediate check is that the symmetries match between the two sides. Let us start with $\text{AdS}_5 \times S^5$. The isometry group of AdS_5 is $SO(4, 2)$, which is manifest when we write AdS as the embedded hyperboloid

$$-X_{-1}^2 - X_0^2 + X_1^2 + \dots + X_4^2 = -\ell^2 \quad (3.6)$$

in $\mathbb{R}^{4,2}$ with metric $ds^2 = -dX_{-1}^2 - dX_0^2 + dX_1^2 + \dots + dX_4^2$; and similarly, the isometry group of the S^5 is $SO(6)$, so the full (bosonic) symmetry is $SO(4, 2) \times SO(6)$. From the CFT side, the conformal group in four dimensions is $SO(4, 2)$ (which includes Poincaré transformations as well as scale transformations and special conformal transformations), and the six scalar fields φ^i and four fermions are related via a global $SU(4) \simeq SO(6)$ R-symmetry. Both sides also have 32 supersymmetries, which manifest themselves as Killing spinors in $\text{AdS}_5 \times S^5$ on the gravity side, and as superconformal algebra on the gauge theory side.

Spacetime directions: The above symmetry matching immediately suggests how the 10 dimensions of the bulk $\text{AdS}_5 \times S^5$ map to the CFT. From the $SO(6)$ symmetry we see that the S^5 directions are captured by the scalar fields φ^i . The radial bulk direction $r = \ell^2/z$ we have already associated with the energy scale in the gauge theory. Therefore the boundary of AdS, $r = \infty$ or $z = 0$, is naturally identified with the UV of the gauge theory. We will elaborate on this relation further below. What remains are the transverse directions x^μ of AdS, but these

map directly to the Minkowski spacetime dimensions the gauge theory lives on. In particular, the time x^0 on both sides gets naturally identified, which means that the notion of Hamiltonian should likewise match on both sides.

So far, we have indicated how the mapping between bulk spacetime directions and gauge theory quantities works for pure $\text{AdS}_5 \times S^5$. Even for this highly symmetric case, however, it is far from obvious that the relevant gauge theory quantities should be related to each other so as to uphold bulk diffeomorphisms. In other words, to a bulk observer in $\text{AdS}_5 \times S^5$, all 9 spatial directions look locally the same, while in the gauge theory this local symmetry is totally obscure. This becomes much more poignant when the geometry breaks all the symmetries and only asymptotes to $\text{AdS}_5 \times S^5$: in such generic case the bulk-boundary map is no longer easy to specify. For example, even though there is a natural notion of ‘constant time’ slice on the boundary (up to overall boosts), for generic time-dependent bulk geometry there is no correspondingly uniquely defined time-foliation in the bulk. Moreover, while the bulk spacetime is dynamical, governed by 10-dimensional Einstein’s equations with appropriate matter content, the boundary spacetime is fixed: there is no sense of gravitational backreaction in the gauge theory; so in this sense even the x^μ directions ‘emerge’ nontrivially. Indeed, understanding how bulk locality emerges from the gauge theory is one of the important open questions in this field, which would teach us something nontrivial about the gauge theory as well as about the nature of spacetime.

The AdS metric specified by the (x^μ, z) coordinates in (3.5), commonly known as the Poincaré (patch of) AdS, is geodesically incomplete. The Poincaré horizon, $z \rightarrow \infty$, is a regular null surface, and we can extend the spacetime to its global AdS form; in static, spherically symmetric coordinates this can be written as

$$ds^2 = - \left(\frac{\rho^2}{\ell^2} + 1 \right) d\tau^2 + \frac{d\rho^2}{\left(\frac{\rho^2}{\ell^2} + 1 \right)} + \rho^2 d\Omega_3^2 \quad (3.7)$$

Although both Poincaré AdS (3.5) and global AdS (3.7) are explicitly static, the timelike Killing fields $\left(\frac{\partial}{\partial t}\right)^a$ and $\left(\frac{\partial}{\partial \tau}\right)^a$ are distinct. This is indicated in Fig. 1 which presents a sketch of global AdS, showing a Poincaré patch, several geodesics, and an indication of the corresponding coordinates.

We can take (3.3) to apply to string theory on the full global $\text{AdS}_5 \times S^5$, in which case the gauge theory is formulated on the Einstein Static Universe $S^3 \times R$. Notice that in both cases, Poincaré and global, the gauge theory can be naturally thought of as ‘living on the boundary’ of AdS: it is formulated on a spacetime which is in the same conformal class (the relevant structure for a conformal field theory) as the boundary metric induced from the bulk. For example, multiplying the line element (3.7) by ℓ^2/ρ^2 and taking $\rho \rightarrow \infty$ with $d\rho = 0$ yields the Einstein static universe $ds_{\text{CFT}}^2 = -d\tau^2 + \ell^2 d\Omega_3^2$. Moreover, the fact that the conformal compactification of AdS has a timelike boundary allows for a Lorentzian field theory on this background. We will continue using terminology ‘CFT living on the boundary of AdS’ as a conceptual crutch, since it will render many of the other elements of the AdS/CFT dictionary easier to understand intuitively. We should however keep in mind that the gauge theory encodes the entire bulk, not

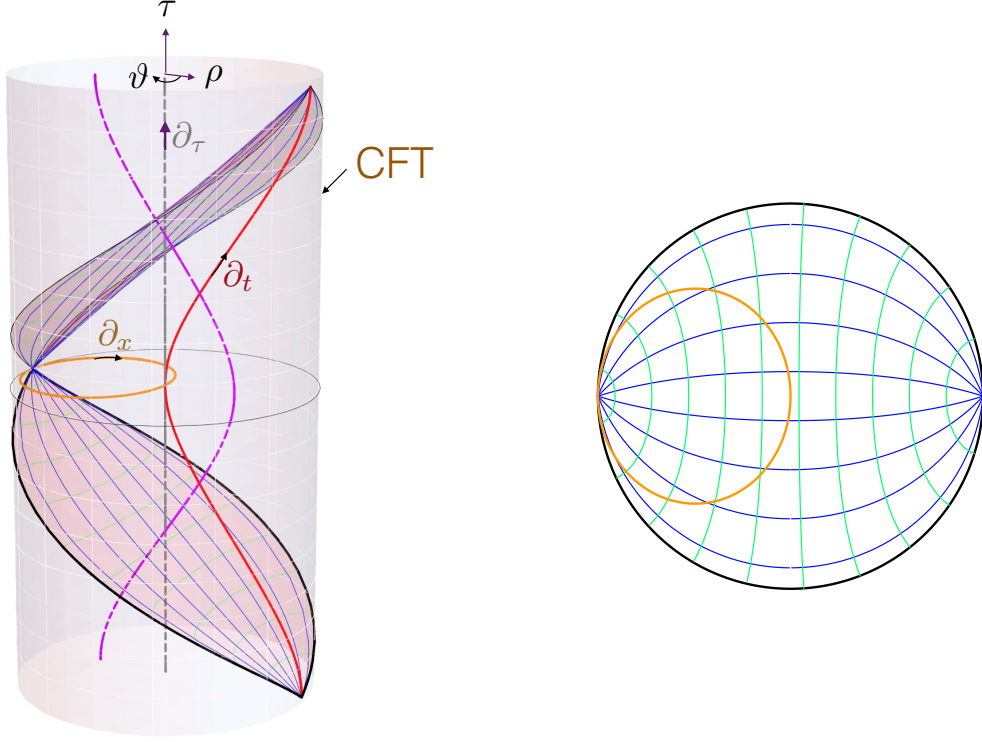


Fig. 1: **Left:** Schematic plot of global AdS_5 , showing the Poincaré patch. The vertical coordinate is τ , the horizontal radial coordinate is $\tan^{-1} \rho$ (so that the spacetime is drawn compactified and radial null geodesics are at 45 degree angle), and the angular coordinate ϑ is one of the S^3 directions of the AdS. Poincaré horizon is a null surface [red] (generated by null geodesics [blue]) whose constant- τ slices are spacelike extremal surfaces anchored on spherical regions or (on equatorial slices of S^3) spacelike geodesics [green]. For orientation we have also plotted two $z = \ell$ (equivalently $r = \ell$) curves, one at constant $x^i = 0$ which describes orbit of $(\frac{\partial}{\partial t})^a$ [red ∂_t], and one at constant $t = 0$ which describes orbit of $(\frac{\partial}{\partial x})^a$ [orange ∂_x] Killing fields. Also shown are two timelike geodesics one static at ‘center’ of global AdS following $(\frac{\partial}{\partial \tau})^a$ [grey ∂_τ] and one [purple] reaching to $\rho = \ell$, to illustrate the confining nature of AdS. **Right:** Poincaré disk of AdS, which is a spatial (constant τ) slice of AdS. Spacelike geodesics and projections of null geodesics, as well as the $z = \ell$, $t = 0$ curve, are drawn [with the same color scheme as in the left panel].

just what is happening asymptotically.

Scale/radius duality: Closely-related to the above remarks is the statement of scale/radius duality, also known as the UV/IR duality. Recall that motion in the radial direction r in (3.2) corresponds to moving in energy scale in the dual field theory. For pure AdS, one can see this at the level of the symmetry, as motivated in §3.2: high energies (or short distance, i.e. UV) on the boundary is associated with large radius (so in this sense IR) in the bulk. In particular, a UV cutoff in the boundary corresponds to an IR cutoff in the bulk. This correspondence was put on a firmer footing in [33], which showed that when suitably regulated, the gauge theory provides a holographic description with one bit of information per Planck area. Operationally, if instead

of energy scales on the boundary one thinks of length scales, in the asymptotic region one can associate the length between endpoints of a given spacelike geodesic with the bulk radial position to which this geodesic penetrates (c.f. the green curves on the right panel of Fig. 1).

The scale/radius duality provides a good guide to our expectations of various physical processes. For example, a bulk particle which falls deeper into AdS (due to the attractive potential caused by the AdS curvature) is described by a localized excitation in the gauge theory which spreads out with time. In the presence of a black hole, the bulk particle nearing the horizon can be regarded as the gauge theory excitation thermalizing [38]. However, the precise form of the scale/radius duality fuzzes out for anything other than pure Poincaré-AdS geometry, so for a generic bulk spacetime it is most useful only asymptotically; deeper in the bulk or for rapidly-evolving processes it need not provide a reliable guide, as indicated below.

Causality: Convenient as the scale/radius duality is for conceptual picture of part of the bulk-boundary mapping, too-naïve applications of it can however lead to apparent contradictions, as instructively exemplified by [39], one of the earliest explorations of AdS/CFT in manifestly Lorentzian context. The authors studied how the CFT manages to reproduce bulk causality, by considering gedanken-experiments involving massless particles in the bulk: Suppose we send two particles radially inward from the same boundary position but at different times. The second particle is always to the future of the first one and therefore cannot influence its dynamics. On the other hand, in the CFT where one expects each particle to be described by some initially localized disturbance which spreads out (and hence the disturbance produced by the second particle to overlap with that of the first one), it appears quite plausible that second can influence the first. If this happened, the CFT would violate bulk causality. One can also imagine sending two particles (non-radially inward) from different positions on the boundary but at the same time. In the CFT, the two disturbances interact on a time scale given by their initial separation, whereas in the bulk, if the particles miss each other, one would expect them to interact on a much longer time scale, if at all.

Both of these puzzles get resolved by correctly accounting for the gravitational backreaction of the particles, upon which the AdS and CFT answers match exactly. In particular, [39] shows that a massless particle produces a gravitational shock wave, which in the CFT gives a ‘light-cone state’, a disturbance which is localized on light cone of the boundary spacetime. So in the first example, the two CFT excitations indeed do not interact, whereas in the second example the shock waves produced by the particles do interact on the same timescale as predicted by the CFT. The lesson from this resolution typifies many of the others: once the physics on both sides is understood correctly, we indeed get a perfect agreement. As a bonus, a physical effect which may be quite subtle on one side is often completely obvious on the other side of the correspondence. In this way, the recasting of a given setup in the dual picture usually elucidates the physics.

Coming back to the specific issue of causality, one might nevertheless wonder how can bulk causality generically coincide with CFT causality, in spite of the bulk having more directions. Since the boundary metric (which determines the causal relations in the CFT) is induced from

the bulk metric, it is clear that any two events which are causally related along the boundary must likewise be causally related in the bulk. However the converse is far less obvious: could one not travel through the bulk faster than around the boundary (as would be the case for a cylinder in flat spacetime), thereby violating CFT causality? It is easy to check that this is not the case for pure global AdS: as evident from Fig. 1 (blue curves), all null geodesics from a given boundary point, whether passing through the bulk or along the boundary, in fact reconverge at the same boundary point (namely antipodally on the sphere and time $\pi \ell$ after the starting point). Given that in pure AdS it takes the same time for a light ray to go through the bulk as along the boundary, one might worry that a small deformation to the bulk geometry could then speed up the bulk geodesic and therefore send signals outside of the CFT light cone. That this does not happen was proved by Gao & Wald [40] who showed that physically sensible²³ deformations of AdS would lead to a gravitational time delay (as opposed to time advance) of bulk geodesics, thereby ensuring that causal processes in the bulk cannot violate CFT causality.

Observables and correlation functions: Now that we have seen how the bulk spacetime as such relates to the gauge theory, what about various bulk fields propagating on this spacetime? At the most basic level, the answer is quite simple: every bulk field ϕ corresponds to an operator $\mathcal{O}$ in the gauge theory. This allows us to extract a useful entry in the AdS/CFT dictionary, namely how the observables of the gauge theory are encoded in the bulk. This has been worked out in the seminal papers [32, 41] within few months after [1] appeared, and the one-to-one identification between the two sides developed therein provided important early checks of the AdS/CFT correspondence.

One natural set of observables is given by expectation values of local gauge invariant operators (constructed as single trace of a local product of super-Yang-Mills fields); they can be generally associated with string states, but a certain subset of these correspond to supergravity fields in the bulk. More specifically, since local operators are associated with the UV of the theory, it is not surprising that expectation values of such operators correspond to the asymptotic behavior of the corresponding bulk fields. In particular, a normalizable bulk field which falls off as $\phi \sim \mu/r^\Delta$ corresponds to an operator $\mathcal{O}$ with scaling dimension Δ (which depends on the mass and type of the field and spacetime dimension), with expectation value $\langle \mathcal{O} \rangle = \mu$. The rate of the bulk field fall off is related to the nature on the field theory perturbation.²⁴

To substantiate this and obtain more general correlation functions, imagine adding a ‘source’ term to the CFT Lagrangian, $\int d^4x \phi_0(x) \mathcal{O}(x)$. Its exponential gives a generating function of correlation functions (i.e. $\langle \mathcal{O}(x) \cdots \mathcal{O}(y) \rangle$ is obtained by taking functional derivatives with respect to ϕ_0 ’s and then setting $\phi_0 = 0$), which one can naturally associate with the partition

²³ More precisely, [40] assume the bulk satisfies the null energy condition (and the null generic condition).

²⁴ More specifically, irrelevant (in the UV) perturbations of the field theory correspond to massive modes in supergravity (which fall off quickly), marginal perturbations correspond to massless modes, and relevant perturbations correspond to modes with negative mass squared; unlike for asymptotically flat spacetime, the latter is allowed in a certain range of masses (above the so-called Breitenlohner-Freedman bound) [42, 43] without producing any instability.

function of the string theory, as a functional of the boundary condition ϕ_0 ,

$$\mathcal{Z}_{\text{string}}[\phi_0] = \langle e^{\int d^4x \phi_0 \mathcal{O}} \rangle \quad (3.8)$$

In effect, one can rephrase the AdS/CFT duality in terms of equivalence between the field theory partition function (viewed as function of sources for each operator) and quantum gravity partition function (viewed as a function of boundary conditions for each bulk field). In the classical limit one can approximate the LHS by the exponential of the classical action for the solution specified by the boundary condition ϕ_0 . One can then think of correlation functions in the gauge theory between operators $\mathcal{O}$ inserted at certain points on the boundary as propagation of corresponding fields in the bulk between these points.

To determine which boundary operators map to which bulk fields, one can use the symmetries. The two objects must have the same Lorentz structure and quantum numbers. For example, conserved currents in the gauge theory are associated to global symmetries, so the corresponding sources act as external background gauge fields, which are boundary values of a dynamical gauge field in the bulk. From the gravitational standpoint, the most crucial example of this correspondence is that bulk gravitons $h_{\mu\nu}$ naturally couple to boundary stress energy momentum tensor $T^{\mu\nu}$. One can think of this as a generalization of the familiar fact that the ADM mass in asymptotically flat spacetime can be extracted from the leading radial fall-off in the metric. (For a more detailed review, see e.g. [44].)

One might naively expect that just knowing the asymptotic values of bulk fields does not tell us much about their structure in the bulk; but it turns out that the situation is in fact much better in $\text{AdS}_5 \times S^5$ than it would be in flat spacetime, because AdS effectively acts like a lens which refocuses some information to be extractible from just the leading fall-offs.²⁵ Nevertheless, we can gain far more information about the bulk physics from less local observables, such as the above-mentioned correlation functions. Roughly-speaking, the further the separation between the insertion points on the boundary, the greater its sensitivity to physics deeper in the bulk, as might indeed be anticipated from the scale/radius duality. For 2-point function of high-dimension operators, one may use a WKB approximation to express the bulk Green's functions in terms of geodesics; as clear from Fig. 1 (green curves), these typically penetrate deeper for greater separation of endpoints. We will mention an interesting application of this in §3.4.

Another important class of nonlocal observables in the gauge theory are Wilson loops, $W(\mathcal{C})$, specified by some closed loop $\mathcal{C}$ on the boundary. Their expectation values for example allow us to compute the quark anti-quark potential (with the quark and anti-quark trajectories given by $\mathcal{C}$). A Wilson loop is defined by a path-ordered integral along $\mathcal{C}$ of a gauge connection A_μ , schematically²⁶ $W(\mathcal{C}) = \text{Tr} [P \exp (i \oint_{\mathcal{C}} A)]$ with the trace taken over some representation of the

²⁵ For example, using operators involving S^5 spherical harmonics, one can read off the size of a small (compared to ℓ) spherical object from local field theory expectation values [45]. This may seem surprising in light of the scale/radius duality which would naively suggest that any information about sub-AdS scales would require highly delocalized observables in the gauge theory, but the operators we use for this task have very high dimension, which gives large dispersion for an actual ‘measurement’ [46].

²⁶ In the AdS/CFT context, this needs to be generalized by also including the gauge theory scalars [47].

gauge group. Though this seems like quite a complicated object in the field theory, it turns out that its bulk dual is remarkably simple: For the fundamental representation, the leading order contribution comes from the proper area of a string worldsheet in the bulk [47], which describes a 2-dimensional²⁷ extremal surface anchored on $\mathcal{C}$. Ultimately this correspondence is rooted in the deep connection between a gauge theory flux tube and a bulk string, which we will revisit in §4. For simple enough cases one can compute $\langle W(\mathcal{C}) \rangle$ exactly in the gauge theory, and confirm precise agreement with the bulk string worldsheet area calculation [50].

The above examples demonstrate that interesting field theory observables can actually be obtained from elementary geometrical bulk constructs, such as areas of extremal surfaces. This has proved invaluable in computing these field theoretic quantities, since the dual calculation is typically much easier than attempting a more direct approach. Conversely, if one were given this ‘data’ in a field theory with a holographic dual, one could in principle use it to probe the corresponding bulk geometry.

Entanglement entropy: Before proceeding with the early checks of AdS/CFT, we pause briefly to mention a more recently-considered (and rather different type of) quantity in the field theory, which is nevertheless thematically related to our story by having a simple geometric dual description: the entanglement entropy. Entanglement is arguably the most non-classical manifestation of quantum mechanics, which can be used as a resource for performing tasks that cannot be accomplished with classical resources, such as quantum teleportation. It is actually used in a wide range of subjects, including quantum information theory, quantum optics, condensed matter physics, etc.. A particularly convenient measure of entanglement is *entanglement entropy*, specified by a subsystem $\mathcal{A}$ and a total state ρ of the system. Formally, it is defined as the Von Neumann entropy $S_{\mathcal{A}} = -\text{Tr}(\rho_{\mathcal{A}} \log \rho_{\mathcal{A}})$ of the reduced density matrix $\rho_{\mathcal{A}}$ obtained by tracing ρ over the complement of $\mathcal{A}$. In a local field theory, the subsystem $\mathcal{A}$ can be specified by a given spatial region, bounded by an ‘entangling surface’ $\partial\mathcal{A}$. Entanglement entropy of $\mathcal{A}$ then contains information about the spatial distribution of quantum correlations in the system.

In all but the simplest systems, entanglement entropy is however extremely difficult to compute, and even harder (if not impossible) to measure, being sensitive to detailed correlations in the state. Remarkably, here too AdS/CFT comes to the rescue. It was proposed by Ryu & Takayanagi [51] that in static situations, the entanglement entropy $S_{\mathcal{A}}$ is given by quarter of the area (in Planck units) of a certain co-dimension 2 bulk surface, analogously to the black hole entropy. The surface in question is a minimal surface at constant time which is anchored on the entangling surface $\partial\mathcal{A}$. This proposal, recently proved [52] using Euclidean path integral techniques, has rendered many non-trivial statements in the field theory (such as the strong subadditivity property of entanglement entropy [53]) beautifully manifest in the geometrical language. Since entanglement entropy is a well-defined quantity even in time-evolving situations

²⁷ In fact, certain other representations can be better characterized using D3-branes, and hence by corresponding 3-dimensional extremal surfaces [48] (though in even more exotic cases, with sufficiently many such branes, the bulk is better described by regular but topologically non-trivial ‘bubbling’ geometries [49]).

(corresponding to time-dependent bulk geometries), the construction of [51] is unnecessarily restrictive. To overcome this limitation, [54] generalized the proposal to arbitrary asymptotically AdS bulk geometries, using the guidance of general covariance: the entanglement entropy is given by the (quarter-)area of a bulk co-dimension 2 *extremal* surface anchored on $\partial\mathcal{A}$.²⁸

There seems to be something quite deep and mysterious about this relation between quantum entanglement on the one hand, and classical geometry on the other, which we will revisit in §4. At a more pragmatic level, we note that as in the case of more conventional CFT observables, entanglement entropy allows us to probe the bulk geometry. In particular, if we know the areas of extremal surfaces anchored on a family of entangling surfaces, we can in certain circumstances invert this relation to determine the bulk metric.

States and geometries: Having discussed how one can probe the geometrical structure of asymptotically $\text{AdS}_5 \times S^5$ spacetimes using various quantities in the gauge theory, let us consider from the gauge theory point of view what *gives rise* to the bulk spacetime deformations in the first place. In the gauge theory, the only state which respects all the symmetries is the vacuum; hence pure $\text{AdS}_5 \times S^5$ corresponds to the vacuum of the field theory. If we consider some excited state, the symmetries will be broken, so correspondingly the bulk geometry should be deformed as well. However, to see this at the classical geometry level, we need the energy of the excitation to scale as N^2 , since the 10-dimensional Newton’s constant in AdS units is $G_N = g_s^2 \ell_s^8 \sim \ell^8/N^2$.

Lower energy (parametrically smaller than $\mathcal{O}(N^2)$) excitations will not backreact on the geometry; instead, they can be described by fields propagating on $\text{AdS}_5 \times S^5$. But these are the quantities we have already considered above. As shown by Witten [32], all linearized supergravity states (perturbations of $\text{AdS}_5 \times S^5$) have corresponding states in the gauge theory; for example the Kaluza-Klein modes of the supergravity fields in the bulk are identified with certain simple operators in the CFT, with excitation spectra matching on both sides. As one increases the energy of the excitations to $\mathcal{O}(N)$, one encounters an interesting effect: massless string states with high angular momentum J blow up into spherical D3-branes whose size grows with J .²⁹ This gives rise to a new class of supersymmetric states, called ‘giant gravitons’, which can wrap inside the internal sphere [59] or the 3-sphere in AdS [60].³⁰ They are stabilized by the angular momentum, but in the former case there is a maximal angular momentum $J \sim N$ for which they can still ‘fit’ into the spacetime. These states were shown to be in one-to-one correspondence with classical supersymmetric solutions in the super-Yang-Mills carrying the same quantum

²⁸ There are two additional specifications which give the holographic entanglement entropy a more non-local flavor: 1) In case of multiple extremal surfaces anchored on $\partial\mathcal{A}$ (which can easily happen in e.g. black hole geometries), the requisite surface is the one with the least area. 2) More intriguingly, the extremal surface should be homologous to the specified region $\mathcal{A}$ (though this property needs further qualifications, as illustrated in [55]). While harder to prove, this covariant prescription has however also passed many non-trivial checks; for example, it implies strong subadditivity [56] and it is non-trivially consistent with field theory causality [57].

²⁹ This is related to the ‘Myers effect’ [58] wherein polarized D-branes blow up into a sphere.

³⁰ Subsequently, [61] provided a unified description on the gauge theory side in terms of gauge invariant operators, conveniently characterized by Young diagrams.

numbers, but additionally they provided an explanation of the previously mysterious stringy exclusion principle [62] which had been derived from the CFT side: because the relevant states are constructed from traces of products of $N \times N$ matrices, there can be at most N independent ones. Since this match is nonperturbative in $1/N$, it provided a nice check that the duality (3.3) remains valid even at finite N .

Let us now return to the higher-energy ($\propto N^2$) gauge theory excitations which do deform the bulk $\text{AdS}_5 \times S^5$ geometry. In fact, since there are $\mathcal{O}(N^2)$ degrees of freedom in the gauge theory, such excitations are rather natural: we can obtain them for instance by exciting each degree of freedom by some small amount. Although the precise identification of a specific $\mathcal{O}(N^2)$ excitation in the field theory in terms of the bulk geometry is in general very difficult, one can in fact retain full control over certain special, but nevertheless rather rich, class of supersymmetric states, describable by free fermions [61, 63]. These turn out to have an explicit one-to-one mapping to the so-called LLM geometries [64], specified by arbitrary closed curves in 2-dimensions (which correspond to the Fermi surface of the fermions, and simultaneously prescribe the boundary conditions which uniquely determine the 10-dimensional bulk geometry). These geometries are smooth and horizonless, but can have interesting topological structure.

For more general excitations, we usually content with considering various coarse-grained features of their bulk duals. Such excited states (in the boundary CFT) have a non-zero expectation value of the (boundary) stress tensor $T^{\mu\nu}$, which can be thought of as arising from the bulk metric being deformed away from pure AdS. The stress tensor, then, provides a useful characterization of the bulk geometry.³¹ We have already encountered an example, namely the light-cone states of [39] being dual to gravitational shock waves in the bulk, but such states are still rather special.

The most *generic* state with energy of $\mathcal{O}(N^2)$, which by the 2nd law of thermodynamics will be in thermodynamical equilibrium, in fact corresponds to a bulk black hole. This is easy to see at the most basic level, as the end state of a generic process: in the gauge theory a generic high energy excitation will thermalize, while in the bulk, the combined effects of backreaction and AdS attractive potential will force a generic excitation to collapse to a black hole. So, roughly-speaking, AdS black holes correspond to thermal states in the gauge theory. This illustrates an important point that although the theories appearing on the two sides of the AdS/CFT correspondence (3.3) are supersymmetric, the actual states we consider need not be. So supersymmetry is *not* a necessary ingredient in mapping between the two sides of the duality. Since apart from the above genericity argument, black holes also provide a particularly useful arena to consider in the context of both elucidating the AdS/CFT correspondence (and hence quantum gravity), as well as for the applications discussed in §4, we will devote a separate subsection to this important topic.

³¹ One might hope that it would actually allow us to determine the bulk metric fully, since by using holographic renormalization group ideas, one can write the metric as a radial series expansion around the asymptotic behavior [65], but in general this series may not converge; in fact it generically leads to naked singularities in the bulk [66].

3.4 Black holes in AdS

We have already evoked (extremal) black branes as a key step in Maldacena’s derivation [1] of the AdS/CFT correspondence; but once in the low-energy limit, the black hole was no longer evident: instead, we were left with a Poincaré patch of AdS (with the black brane horizon becoming the Poincaré horizon), which we could then globally extend to the causally-trivial global AdS geometry.

The black holes we now wish to consider are *within* this asymptotically AdS spacetime, appearing as new objects.³² We will focus on a neutral, static, spherically symmetric black hole in AdS, i.e. the global Schwarzschild-AdS₅ × S⁵ solution, which is the easiest and most illustrative example. We will start by describing the geometry, thermodynamics, and causal structure, then briefly discuss time-dependent context, and finally revisit the previously-discussed issue of whether/how does the gauge theory encode the region behind the horizon.

Geometry: The Schwarzschild-AdS₅ × S⁵ geometry describes a spherical black hole in global AdS, trivially smeared over the S⁵ in a direct product structure. The solution can be obtained either as a 10-dimensional solution to Einstein’s equations with self-dual 5-form field strength with flux given by ℓ , or, by reducing on the S⁵, as a solution to 5-dimensional Einstein’s equations with negative cosmological constant $\Lambda = -\frac{6}{\ell^2}$ but zero bulk stress tensor. The resulting metric is given by

$$ds^2 = -g(r) dt^2 + \frac{dr^2}{g(r)} + r^2 d\Omega_3^2 + \ell^2 d\Omega_5^2, \quad g(r) = \frac{r^2}{\ell^2} + 1 - \frac{r_0^2}{r^2}. \quad (3.9)$$

This describes a 2-parameter family of solutions, characterized by the AdS size ℓ and the black hole size r_+ , related to its mass M by

$$r_0^2 \equiv \frac{8 G_N M}{3 \pi} = r_+^2 \left(\frac{r_+^2}{\ell^2} + 1 \right) \quad (3.10)$$

where G_N is the 5-dimensional Newton’s constant.³³ The relation (3.10) implies that for fixed mass $G_N M$, the black hole size r_+ is smaller than it would be in asymptotically flat case, which is consistent with what one would expect from a confining potential.

Once we have the bulk metric, it is a simple matter to find the induced stress tensor on the boundary.³⁴ In the present situation, this is largely fixed by the symmetries: the stress tensor

³² The large-mass limit of a spherical black hole wherein the horizon becomes translationally invariant, namely the planar Schwarzschild-AdS₅ × S⁵ geometry, can in fact be obtained directly from a D3 brane system, as already described in [1], by suitably exciting the D3 branes and then taking the low energy limit of a near-extremal black 3-brane geometry. As we will see below, such a black hole is most naturally related to Poincaré-AdS.

³³ Note that for notational convenience we have relabeled our coordinates: in the $r_+ \rightarrow 0$ limit we obtain global AdS (3.7), with the coordinate relabeling $\rho \rightarrow r$ and $\tau \rightarrow t$.

³⁴ One can generalize a Brown-York procedure [67] for calculating a quasilocal stress tensor on a cutoff surface in terms of its mean curvature; this was done in a covariant form in [68] (accounting for conformal anomalies and counter-terms), which gave an explicit prescription. Alternately, if one writes the metric in ‘Poincaré coordinates’ generalizing (3.5), then the stress tensor appears as the coefficient of the first subleading (in z) term not specified by the boundary metric [65, 69].

must be static and homogeneous on the S^3 of the boundary Einstein static universe, namely

$$T^{\mu\nu} = \rho u^\mu u^\nu + P h^{\mu\nu} , \quad (3.11)$$

where u^μ is a unit vector in the t direction and $h_{\mu\nu}$ is the metric of the boundary S^3 (with radius ℓ). This describes a perfect fluid stress tensor with energy density ρ and pressure P . Computing $T^{\mu\nu}$ explicitly, we find $\rho = 3P = \frac{M}{2\pi^2 \ell^3}$. Note that the stress tensor is traceless, as befits a state of the CFT, i.e. a conformal fluid.

In the large black hole limit, the horizon becomes planar (i.e. has translational symmetry), and the solution simplifies to a form akin to the Poincaré-AdS geometry (3.2),

$$ds^2 = \frac{r^2}{\ell^2} \left[- \left(1 - \frac{r_+^4}{r^4} \right) dt^2 + dx_i dx^i \right] + \frac{\ell^2}{r^2} \left(1 - \frac{r_+^4}{r^4} \right)^{-1} dr^2 + \ell^2 d\Omega_5^2 \quad (3.12)$$

which indeed reduces to (3.2) for $r_+ = 0$; as mentioned above, this geometry can be obtained as a near-horizon limit of near-extremal black 3-brane [1]. Although (3.12) is written in terms of both r_+ and ℓ , we have an additional symmetry under rescaling r by α and simultaneously t, x_i by α^{-1} , which does not change ℓ or the overall scale but only rescales the horizon radius $r_+ \rightarrow \alpha r_+$. Hence (3.12) really describes a 1-parameter family of solutions, more naturally characterized by r_+/ℓ .

The Schwarzschild-AdS geometry (3.9) can be generalized in various ways. Thanks to the AdS asymptotics, we can have not only spherical and planar black holes, but also hyperbolic ones [70].³⁵ Moreover, these black holes can be charged and/or rotating (see e.g. [71] for a review). In fact, by virtue of being in 5 dimensions, we can even have asymptotically AdS black holes with compact horizon but non-spherical topology such as AdS black rings. The new features compared to the asymptotically flat counter-parts come from the AdS confining potential; thus, taking the size of these black holes to be comparable to, or greater than, the AdS size ℓ , we get a genuinely new behavior, as manifested by the black hole thermodynamics.

Thermodynamics: Once we know how to describe black holes in AdS, we can also elucidate their thermodynamical properties. Note that due to the identification of the Hamiltonians, all the thermodynamic properties such as temperature, energy, entropy, etc., are in direct correspondence between the two sides. In the CFT, a large black hole then corresponds to a hot plasma of the gauge theory degrees of freedom at the Hawking temperature. To describe black hole thermodynamics we use the usual horizon area $\sim$ entropy and surface gravity $\sim$ temperature relations.

Since there are two length scales in the problem, r_+ and ℓ , the temperature (which knows both about the horizon and the asymptotics) is no longer fixed by dimensional analysis, allowing for richer behavior compared to the asymptotically flat case, more akin in fact to a black hole in

³⁵ In fact, one can construct horizons of arbitrary topology by quotienting by discrete isometries; this gives rise to the so-called ‘topological black holes’.

a box [72]. In particular, the Hawking temperature of the Schwarzschild-AdS black hole (3.9) is

$$T = \frac{g'(r_+)}{4\pi} = \frac{2r_+^2 + \ell^2}{2\pi r_+ \ell^2}, \quad (3.13)$$

which interpolates between the asymptotically flat case for $\frac{r_+}{\ell} \ll 1$ and the planar black hole case (3.12) for $\frac{r_+}{\ell} \gg 1$: while the temperature scales as inverse radius for small black holes, it *grows* linearly with horizon radius, $T \sim \frac{3}{4\pi} r_+/\ell^2$, for large ones. This means that a large AdS black hole has positive specific heat, so that it is thermodynamically stable: it cools off as it evaporates and therefore can be in thermal equilibrium with its Hawking radiation. In this sense, AdS black holes are much better suited to thermodynamic analysis than asymptotically flat ones.

Note that (3.13) also implies that there is a minimum-temperature spherical black hole, when $\frac{r_+}{\ell} = \frac{1}{\sqrt{2}}$. Below this temperature, the dominant phase is described by a thermal gas of gravitons in AdS (which, having $\mathcal{O}(1)$ free energy, do not backreact on the pure AdS geometry). This phase in fact dominates in the canonical ensemble till $r_+ = \ell$, above which the thermal state becomes described by the large AdS black hole. This demarcates a first order phase transition, called the Hawking-Page transition [73]. The gauge theory on $S^3 \times R$ can be excited to any temperature, but it likewise exhibits a corresponding phase transition, which can be viewed as a confinement-deconfinement transition when the inverse temperature of the system becomes comparable to its size [74]. At that point, the free energy jumps from $\mathcal{O}(1)$ to $\mathcal{O}(N^2)$, as the color degrees of freedom deconfine. Although sub-AdS size black holes with negative specific heat are not thermodynamically stable within the canonical ensemble, they are still the most entropically favorable states in the microcanonical ensemble.³⁶ However, when we keep the energy rather than the temperature fixed, the corresponding state is not a thermal state in the field theory and consequently the dual of small black holes is less well understood.

Restricting attention to a large AdS black hole, which corresponds to a thermal state in the gauge theory, let us now revisit the question of microscopic accounting for the black hole entropy. Although computing entropy of a strongly coupled hot plasma in the gauge theory is too difficult, one can at least compute the thermal entropy in Yang-Mills perturbatively and compare with the bulk entropy obtained from the black hole area. For a thermal gas with N^2 degrees of freedom at temperature T in $3+1$ dimensional flat spacetime, the entropy density per unit volume scales as $\frac{1}{V} S_{\text{YM}} \sim N^2 T^3$. One might have worried that in the bulk $9+1$ dimensions already a thermal gas has entropy $\sim T^9$ which can be made arbitrarily larger than T^3 at high temperature, but in fact even the most entropic configuration, namely the planar black hole (3.12) has entropy per unit volume (in x^μ directions)

$$\frac{1}{V} S_{\text{BH}} \sim \frac{\ell^2 r_+^3}{g_s^2 \ell_s^8} \sim \frac{N^2 r_+^3}{\ell^6} \sim N^2 T^3. \quad (3.14)$$

³⁶ There are two caveats: First of all, when the black hole becomes sufficiently small, it is no longer favorable for it to be smeared on the S^5 : it becomes dynamically unstable to a Gregory-Laflamme type instability [75] and for $r_+/\ell \approx 0.4$ it localizes on the S^5 [76]. In the gauge theory this corresponds to a much more complicated state with the scalar vevs turned on. Secondly, even a 10-dimensional localized black hole is not stable in the microcanonical ensemble once it becomes parametrically small, namely for $r_+/\ell < N^{-2/17}$ [77].

This gives another non-trivial check of the gauge/gravity duality: the gauge theory does indeed have (parametrically) just enough degrees of freedom to describe a higher-dimensional gravitational theory.

To see the exact match of entropies between the bulk black hole and the thermal gas of the gauge theory is however far more difficult, because unlike in the extremal D-brane system we discussed in §2, here masses change with coupling, so the number of states computed at weak coupling does not agree with the number of states at strong coupling in the gauge theory. If one calculates the number of states at weak coupling including the coefficient, one obtains $S_{\text{BH}} = \frac{3}{4} S_{\text{YM}}|_{g_{\text{YM}}=0}$ [78].³⁷

Causal structure: So far, we have not touched on one important feature of the Schwarzschild-AdS solution (3.9): just like its asymptotically flat counterpart, its global completion has two asymptotic regions, connected by an Einstein-Rosen bridge. The causal structure interpolates between that of Schwarzschild at small r and AdS at large r , so the Penrose diagram looks similar to that of Schwarzschild, but with scri $\mathscr{I}^\pm$ being timelike rather than null.³⁸ This has an interesting implication for the dual description. According to the picture of CFT living on the boundary of AdS, the presence of two disconnected boundaries suggests that the black hole is described by *two* CFTs which do not interact with each other, but are in some entangled state [80]. This can be put on a more formal footing using the *thermofield* formalism [81], which has been very useful in trying to probe the geometry deeper.

The black hole / thermal state correspondence is however too coarse to elucidate the most intriguing quantum gravitational questions. For this purpose we wish to understand the mapping at a much finer level. Ultimately we hope to understand how the gauge theory ‘sees’ the singularity, but as a first step it is useful to understand what perturbing this state in various ways does to the dual description.

Non-equilibrium black holes: Hitherto, we have been talking about static black holes. The easiest departure from this equilibrium context entails small perturbations. For the conventional asymptotically flat black holes, the evolution of a deformed black hole is well-described by so-called quasinormal modes [82], specified by a discrete set of complex frequencies characteristic of the black hole (and independent of the details of the perturbation). Perturbations of asymptotically flat black holes decay exponentially on a timescale inversely proportional to black hole

³⁷ The fact that the two answers do not match precisely is not a contradiction, since they were computed in different regimes. (We indeed expect that the strongly-coupled gauge theory answer is smaller than the weakly-coupled one, since increasing the coupling effectively raises the potential energy of the system, so there are fewer states at a fixed energy.) Moreover, if one computes the leading corrections, the answers are consistent with smooth interpolation in g_{YM} , though why the two limits differ by such a simple factor has not been explained.

³⁸ Strictly speaking, if one fixes a conformal compactification so as to render the boundaries as straight vertical lines, one no longer has the freedom to keep the spacelike singularity drawn as straight horizontal line [79]; instead the singularity bends inward.

mass (with a late-time power-law tail). One can think of the perturbations as fields which either fall through the horizon or escape off to infinity.

In AdS, the confining potential prevents fields from escaping off to infinity, but they can still decay through the horizon. This again gives a discrete set of complex frequencies, characteristic of the AdS black hole. Quasinormal mode frequencies for scalar perturbations were initially computed in [83], which confirmed that the imaginary part (corresponding to the inverse of the decay timescale) grows linearly with temperature (equivalently r_+) for large black hole, but linearly with horizon area (equivalently, scattering cross-section) for small black holes. One might be puzzled as to why an attribute of small AdS black holes does not approach the corresponding attribute of asymptotically flat black holes; the resolution here is that the attribute in question (namely its quasinormal mode frequencies) crucially hinges on the boundary conditions, and since the modes concern late-time decay, there is no causal obstruction to sensing the AdS boundary. Interestingly, the time dependence is simpler in AdS than in asymptotically flat case: the decay remains exponential at arbitrarily late times. In the dual field theory, perturbing the black hole corresponds to perturbing the state away from thermal equilibrium, so quasinormal modes characterize (e.g. predict the timescale for) approach to thermal equilibrium. While this is difficult to verify by direct computation within the strongly-coupled field theory, it is consistent with expectations.

Quasinormal modes for various AdS black holes were studied extensively (for a review see [84]), but perhaps the most interesting point is that for a planar black hole (3.12), there are several ‘massless’ (or ‘hydrodynamic’) quasinormal modes, those with arbitrarily low frequencies (and therefore arbitrarily long timescales for decay) at long wavelengths. This will come to play an important role in our discussion in §4.1, where we associate them with hydrodynamic sound and shear modes of the dual conformal fluid. In describing the long-wavelength excitations of a black hole in terms of a boundary fluid, the fluid speed of sound one expects from (3.11) and conformal invariance is $v_s = 1/\sqrt{3}$. This might at first sight seem unlikely to emerge from the gravity side where the only natural speed is the speed of light, but remarkably, that is just what the bulk physics conspires to predict! Actually quasinormal modes provide a great tool for studying properties of near-equilibrium strongly coupled quantum systems; in particular we can ascertain their response and transport coefficients, which bolstered the connections between horizon dynamics and hydrodynamics [85], eventually culminating in the fluid/gravity correspondence [66] which we will discuss further in §4.1.

Of course, a far more interesting departure from equilibrium is a strongly time-dependent black hole. One crutch to building intuition has been to use mock time-dependence by considering a static black hole in a set-up where the timelike Killing field is not manifest. For example, one can boost a global Schwarzschild-AdS black hole³⁹ or consider it in a restricted region such

³⁹ In fact, this solution was used in [39] to construct the gravitational shock wave by taking an infinite boost limit of a global Schwarzschild-AdS black hole with fixed total energy. In this limit the solution is no longer static.

as the Poincaré patch.⁴⁰ Finally, there do exist explicit genuinely time-dependent black hole solutions which retain sufficient symmetries. The most oft-used one is Vaidya-AdS, describing an imploding spherical null ‘shell’ (of arbitrary density profile) which collapses to a black hole. This solution is sourced by null dust stress tensor, and interpolates between pure AdS before/inside the shell to Schwarzschild-AdS after/outside the shell. When the shell is infinitesimally thin, the collapse occurs maximally suddenly, so this offers a good playground for studying rapidly evolving geometries. In the dual gauge theory, this describes a particular form of global ‘quantum quench’ [88]: namely a sudden disturbance of the system (usually implemented by deforming the Hamiltonian).

From quantum gravitational standpoint, a particularly intriguing time-dependent setup is one in which a black hole collapses and then fully evaporates via Hawking radiation, for understanding the dual description completely would allow us to resolve the black hole Information Paradox. Being described by a unitary gauge theory, such a bulk process would necessarily be unitary, suggesting that ‘information is not lost’ – but of course one also wants to see explicitly where it went (one natural guess being that it remains encoded in subtle correlations in the Hawking quanta). Indeed, the guarantee of unitarity provided an early example of what the gauge theory can teach us about gravity, despite the lack of computational control at strong coupling. However, understanding the detailed process turns out to be extremely difficult. Since large black holes (which are homogeneous on the S^5) are thermodynamically stable, we need to consider sufficiently small black holes; these are not only more complicated geometrically, but also do not have a nice description in the gauge theory – even the encoding of geometry in the vicinity of such small black holes is ill-understood due to the large non-locality of the bulk-boundary map.⁴¹ More importantly, we would also need to understand what happens *inside* the black hole. This is an interesting chapter in itself, to which we turn next.

Inside the horizon: So far, we have discussed physics pertaining to black hole exterior. Since this is visible to an asymptotic bulk observer, it is not so surprising that we can describe these features in the dual gauge theory. On the other hand, the most interesting stuff happens inside the horizon: indeed, much of the motivation for considering black holes in AdS was to gain insight into the strongly quantum-gravitational effects near the singularity through the gauge theory description. It is an important question, then, whether (and how) the gauge theory ‘sees’

⁴⁰ The corresponding CFT dual on $\mathbb{R}^4$ was called ‘conformal soliton’ in [86] and considerations of its causal structure [87] (showing that while the entropy is time-independent, the area of the event horizon of the restricted solution grows and diverges at finite time) indicate that event horizon area may not be a good indicator of entropy in strongly time-dependent situations.

⁴¹ Alternately, one could consider a large black hole, but ‘drain away’ the Hawking radiation by modifying the AdS boundary conditions from reflecting to e.g. transparent, or by deforming the CFT by suitable operators, thereby allowing a large black hole to evaporate. Although complete evaporation would still have to pass through a small black hole stage subject to the above-mentioned problems, to unravel the Information Paradox, one need not consider evaporating the black hole: one can repeat a process of letting a large black hole accrete and evaporate some small fraction of its mass, say in a cyclic manner, as discussed in [89]. However one then has to take into account the fact that one is no longer considering a closed system.

past the horizon. Indeed, it was often suggested in the early days⁴² of AdS/CFT that the dual description stops at the horizon – that the gauge theory cannot encode effects taking place inside the black hole. This supposition, however, is actually more mystifying than the opposite one, since the event horizon is defined globally and thus behaves teleologically, so that the AdS/CFT mapping would have to be maximally temporally nonlocal in order to ‘know where to stop’. It is easy to devise a simple gedanken-experiment [90] wherein we ‘probe’ physics inside AdS via some non-local gauge theory operator (such as a large decorated Wilson loop, dubbed ‘precursor’ in [91]), and *afterwards* excite the state in a way which in the bulk collapses a sufficiently large black hole whose event horizon encompasses the event which we have already probed.

While the previous argument indicates that at least in some situations the gauge theory should encode physics inside the black hole just as well as in causally trivial situations, it leaves the precise nature of such encoding obscure. There have of course been many attempts to reconstruct the geometry (and its breakdown) inside the horizon with a variety of CFT probes,⁴³ but there is a wide realm left unexplored, partly due to insufficient tools to overcome the various limitations of existing techniques. The question of understanding precisely what happens to an observer who falls into a black hole of course has wider appeal and urgency beyond the AdS/CFT context; but in the AdS/CFT setting this question is placed on a more concrete footing. Though better understanding of the precise holographic map is essential, the amount of recent attention this problem has received invites optimism for forthcoming progress.

3.5 Generalizations

So far, we have been describing just one particular case of the AdS/CFT duality, namely (3.3). There are however many ways in which the correspondence can be extended and generalized. The earliest and most immediate one was to consider different number of dimensions, i.e. to describe a dual of a gravitational theory on AdS_{d+1} (times a compact manifold) in terms of a d -dimensional CFT, on which we elaborate below. Another straightforward type of generalization constitutes starting with (3.3) and deforming both sides in a controlled way. If we add extra terms to the Lagrangian, the gauge theory is no longer conformal; it will undergo renormalization group flow (which gives an effective description at a given energy by integrating out the higher-energy degrees of freedom). This is directly mimicked by the behavior of the bulk geometry in the radial

⁴² This belief has recently returned in far more concrete form with the ‘firewall’ proposal, further mention of which we postpone till §4.2.

⁴³ An early idea [92] was to use correlation functions for probing physics behind the horizon. This was explicitly implemented in [79] (see also [93]) which identified the CFT signature of the black hole singularity using analytically continued correlators of high-dimension operators. More direct approach [94] used D-branes as probes, described in terms of the dynamics of rolling scalar fields in the dual gauge theory. A separate set of ideas [95] is to express a local bulk operator (including those inside the horizon) in terms of boundary operators. While this may not be possible in general, the fact that gravitational Hamiltonian is a pure boundary term allows us to relate operators probing a collapsing black hole to earlier boundary ones using boundary evolution [37].

direction.⁴⁴ Depending on the type of deformation, we can get quite a rich set of possibilities, including ones where the low-energy physics is confining, massive, chiral symmetry breaking, etc.. One can also replace the S^5 by any other Einstein manifold or a quotient of the S^5 , which gives rise to more complicated gauge theories. More interestingly, one can even consider different asymptotics.

These extensions provided many further tests of the correspondence as the salient features one expects in the gauge theory were in each case faithfully mimicked by the gravity side. We will elaborate on a few of these developments below, primarily to exemplify the robustness of the gauge/gravity duality, and to give the reader some feel for how far one can wander away from the specific example (3.3). The following is certainly not meant to be an exhaustive list; numerous further examples are reviewed in [23] or more recently in e.g. [25]. The main point is that the gauge/gravity duality really refers to a broad class of dualities. We will see in §4 that these allow for a rich enough structure to cover a vast arena, and in fact bear on much of everyday physics.

Other dimensions: Generalization to other dimensions was apparent from the very outset: [1] explored a number of other examples of the AdS/CFT duality besides (3.3). Instead of starting with string theory, one can start with ‘M-theory’ (an 11-dimensional theory whose low-energy effective action is that of 11-dimensional supergravity). The fundamental objects of M-theory are M2-branes and M5-branes. We can consider the low-energy limit of a stack of N of these, for which much of the development indicated in §3.1 carries through. The near-horizon geometries that we obtain in these cases are $\text{AdS}_4 \times S^7$ with radius $\ell = \frac{1}{2}R_{S^7} \sim \ell_p N^{1/6}$ for M2-branes and $\text{AdS}_7 \times S^4$ with radius $\ell = 2R_{S^4} \sim \ell_p N^{1/3}$ for M5-branes. M-theory on these spacetimes is then dual to a 3-dimensional conformal field theory (the so-called ABJM theory [98]) and a 6-dimensional $(0, 2)$ conformal field theory, respectively. One can also start with string theory compactified (on T^4 or $K3$) down to 6 spacetime dimensions. Take $Q_5 \gg 1$ D5-branes wrapped on the 4 compact dimensions, giving rise to a D-string in the remaining 6 dimensions, and add $Q_1 \gg 1$ D1-branes coincident with this string. In the low-energy limit this system is described by a 2-dimensional $(4, 4)$ superconformal field theory, and the near-horizon geometry of the resulting 6-dimensional black brane is $\text{AdS}_3 \times S^3$ with radii $\ell = R_{S^3} \sim \ell_s (g_6^2 Q_1 Q_5)^{1/4}$. Further possibilities were also mentioned in [1], involving $\text{AdS}_3 \times S^2 \times T^6$ and even $\text{AdS}_2 \times S^2 \times T^6$ (the latter conjectured to be equivalent to certain quantum mechanical system).⁴⁵ Although the dual field theory is not as well understood in all these cases as for the 4-dimensional super-Yang-Mills theory of (3.3), the bulk physics is quite analogous.

One notable feature of the AdS_3 case (keeping the internal modes unexcited) is that the

⁴⁴ One of the initial checks involved RG flows with a conformal fixed point in the IR, whose bulk dual interpolates between the original AdS asymptotics and another (smaller size) AdS, whose radius correctly accounts for the smaller number of IR degrees of freedom [96]. In fact, in static spacetimes one can express the RG flow equation in terms of bulk Einstein’s equations [97].

⁴⁵ One should also mention another nonperturbative realization of the holographic principle, the BFSS matrix model [99], which conjectures duality between M-theory in infinite momentum frame and the large- N limit of a supersymmetric matrix quantum mechanics describing D0-branes.

geometry remains very simple: in 3-dimensions there are no propagating gravitational degrees of freedom, which means that any negatively curved Einstein space is locally AdS_3 . Nevertheless, unlike asymptotically flat 3-dimensional spacetimes, AdS admits (the so-called BTZ) black hole solutions [100], which can be obtained as quotients of AdS_3 but still share many analogous properties with their higher-dimensional cousins. Since the BTZ geometry is algebraically much simpler (and in fact, being locally pure AdS, does not receive any higher-curvature corrections⁴⁶), it has provided quite a useful playground for studying features of black holes in a controlled setting.

Asymptotically locally AdS: A relatively mild, but rather useful, generalization of the standard AdS/CFT correspondence is to put the field theory on a curved background different from the conformally flat ones we have been considering. As is well-appreciated, field theories on curved backgrounds provide a useful step towards elucidating quantum gravity. Even in innocuously mild contexts, we uncover many fascinating effects, such as particle production, vacuum polarization, etc.. However, technical limitations have previously restricted such studies to weakly-coupled field theories, whilst we expect further novel effects at strong coupling. It is particularly interesting to study thermally excited states, and more ambitiously ones out of equilibrium (wherein we cannot take recourse to Euclidean techniques). In all these regimes, AdS/CFT provides an excellent laboratory. Since the background metric in gauge theory is non-dynamical (i.e., it does not need to solve any field equation, and correspondingly the field theory stress tensor does not produce any backreaction), we have a large amount of freedom at our disposal; for example, we can put the field theory on any black hole background. The chosen background then provides boundary conditions for the bulk dynamical spacetime, and by solving bulk Einstein’s equations we can read off the requisite boundary stress tensor (see e.g. [44] for a detailed review), which allows us to extract the essential field theory physics; see [101] for a good overview of recent progress in this direction.

Other asymptotics: For some, the excitement at the profound and far-reaching nature of AdS/CFT was tempered by the regret that (3.3) pertains only to asymptotically AdS bulk spacetimes. We don’t live in AdS, they remarked, but wouldn’t it be nice to describe *our* world holographically? This sentiment motivated the early efforts to obtain a holographic correspondence for spacetimes with asymptotics other than AdS. One obvious idea is to take a further limit within the AdS/CFT correspondence. The earliest attempts involved formulating flat spacetime holography (e.g. defining a scattering matrix) by considering processes happening on length scales much smaller than ℓ [102–104]. However, here we soon lose control over the gauge theory description, as anticipated from the UV/IR duality (though this has been recently revisited in e.g. [105, 106], the latter recasting bulk S-matrix usefully in terms of a Mellin amplitude in CFT).

One ingenious limit which however does allow a more controlled description is the Penrose

⁴⁶ As a consequence, the black hole entropy counting analogous to (3.14) now matches the field theory dual exactly.

limit [107], of zooming in on a null geodesic in the spacetime. This generates a plane wave spacetime, considered by [36], whose boundary is actually one-dimensional and null. In the gauge theory, this limit corresponds to a large-charge sector of the theory, which enables one to use perturbation theory on both sides and thereby retain computational control.⁴⁷ In particular, one can reproduce the complete spectrum of string oscillations in this spacetime. While the contingency of a $d > 2$ dimensional spacetime admitting only 1-dimensional conformal boundary is unusual, the plane wave spacetimes are still perfectly physically sensible geometries in terms of their causal properties: like AdS itself, they are stably causal (but not globally hyperbolic⁴⁸). It is interesting to note that AdS/CFT in fact compels us to relax causality requirements as far as possible in the causal hierarchy: one can obtain physically sensible field theories (such as those with lightlike non-commutativity considered in [108]) whose bulk dual is causal but not distinguishing.⁴⁹ This means that although such spacetimes come arbitrarily close to having closed causal curves (and therefore one might have worried that quantum fluctuations could lead to pathological behavior), the well-posedness of the field theory dual effectively protects the bulk from causal paradoxes.

From the cosmological standpoint, it would be most desirable to obtain holography for spacetimes with positive cosmological constant. This has motivated the so-called dS/CFT correspondence [109] (see also [110]), which attempts to relate dynamical quantum gravity on de Sitter to a Euclidean CFT ‘living on’ de Sitter scri. However, although at a superficial level, de Sitter seems straightforwardly related to AdS, the different causal character of the two situations inhibits the utility of the analogy much beyond a kinematic level. More severely, in string theory, obtaining de Sitter scri is problematic due to quantum instability, manifested in chaotic nucleation of bubbles of metastable vacua with lower cosmological constant [111]. As a result, the dS/CFT correspondence is at a far less solid footing as the AdS/CFT correspondence. One can however utilize properties of a CFT for useful results bearing on cosmology, such as calculating non-gaussianities of primordial fluctuations in single field inflationary models by analytically continued 3-point functions [112]. There have also been many efforts of embedding an inflating geometry *within* AdS such as [113], but here one faces the typical problem of not having sufficient handle on the dual description in the gauge theory.

Though obtaining a precise holographic duality for spacetimes with asymptotics other than AdS remains an ongoing effort, the goal of this endeavor has mostly shifted away from trying to describe ‘our’ universe holographically, instead focusing on elucidating the general structure of the correspondence, to understand the guiding principle behind all gauge/gravity dualities.

⁴⁷ The maximally supersymmetric plane wave solution of supergravity [36] in fact provides the simplest solvable example of a sigma model with Ramond-Ramond background.

⁴⁸ The fact that AdS is not globally hyperbolic is not a problem for the AdS/CFT correspondence, since initial conditions on any ‘Cauchy slice’ are supplemented by reflecting boundary conditions at the boundary, rendering the Cauchy evolution in the bulk well-defined.

⁴⁹ A spacetime is *causal* if it does not contain any closed causal curves, and *distinguishing* if distinct points have distinct causal past and future sets. In the case studied in [108], all points on an entire co-dimension 1 surface have identical causal pasts and identical causal futures, despite maintaining locally Lorentzian structure.

4 Applied AdS/CFT

As might be guessed from its breadth, the AdS/CFT duality has turned out tremendously useful in elucidating both sides of the correspondence: we can use gravity to perform previously intractable calculations within strongly-coupled quantum systems on the one hand, and we can use knowledge of quantum field theories to gain insight into (quantum) gravity on the other. Both directions are interesting from the GR perspective, since the former reveals the surprisingly vast set of applications of general relativity, and the latter teaches us something fascinating about gravity itself.

4.1 Applying general relativity to other real-world systems

On the application front, one may of course object that super-Yang-Mills is a conformal field theory, quite distinct from the ‘real-world’ systems we wish to understand. The remarkable fact is that it nevertheless provides an invaluable toy model for studying universal quantities shared by more familiar systems, as well as for exploring new classes of strongly coupled phenomena. Applying this philosophy to quantum chromodynamics has led to the fruitful synthesis of lattice QCD and heavy ion phenomenology with gauge/string duality via the AdS/QCD program; for a concise overview see [114], and for more extensive reviews see [115, 116]. While the potentiality of connections between particle physics and string theory had been to some extent presaged since the 70’s, it seems rather more surprising that one can likewise apply this philosophy to various condensed matter systems. This program, known as AdS/CMT, has been successful in explaining an impressive array of physical effects raging from superfluid transitions to non-Fermi liquid behavior; for extensive reviews see e.g. [117–119]. In both of these programs, the AdS/CFT correspondence not only provided the best (and often the only) tool to tackle the physical problems of interest, but even more remarkably, these computations actually suggest values which agree with experimentally measured quantities⁵⁰ in the real world! Both of these programs have burgeoned into active and ongoing research areas. Below we highlight only one aspect from each to exemplify their success, referring the reader to the above-mentioned excellent reviews for a more complete overview of the subject.

AdS/QCD: Quantum chromodynamics, the theory of the strong interactions between quarks and gluons, is a gauge theory based on the gauge group $SU(3)$ (i.e. quarks come in $N_c = 3$ colors). Unlike super-Yang-Mills, QCD is neither supersymmetric, nor a conformal theory since the coupling runs: at large energy scales the coupling becomes weak (i.e. QCD is asymptotically

⁵⁰ One of the early successes concern the shear viscosity of the quark-gluon plasma; holographic arguments [120] indicate that in a certain regime (when the holographic dual corresponds to classical general relativity) the dimensionless ratio of viscosity to entropy density has a universal lower bound, $\eta/s \geq 1/4\pi$, although the bound can be violated away from this regime (cf. eg. [121]); gravity side (and therefore the strongly coupled $\mathcal{N} = 4$ SYM) would saturate the bound, while the actual value measured in quark-gluon plasma is only slightly above this lower bound. In contrast, it would diverge at weak coupling.

free), whereas at low energies it is strong.⁵¹ Below a certain ‘confinement’ temperature, the physical degrees of freedom are confined into color singlet hadrons (so thermodynamic quantities scale as N_c^0), whereas for higher temperatures the quarks and gluons deconfine into a ‘quark-gluon plasma’ (with N_c^2 scaling). At temperatures slightly above the deconfinement transition, which are typically the relevant ones for the quark-gluon plasma created in heavy ion colliders, the quarks and gluons are still strongly coupled. This naturally provides an excellent window of opportunity for holographic techniques. More broadly, while at zero temperature, super-Yang-Mills and QCD are very different, one would expect that since finite temperature breaks both supersymmetry and conformal invariance, the two theories become more alike in that regime, as has indeed been observed experimentally.

Understanding confinement remains one of the important problems in QCD, so it is inviting to study it in other theories which serve as toy models for the real world. Confinement indicates that quarks are connected by flux tubes, with energy proportional to length. In the bulk dual, this flux tube (the large- N version of the QCD string) stretching between the quarks is codified by a fundamental string, which ends on the boundary at the position of the quarks.⁵² Hence the thick QCD string in 4 dimensions gets described by an infinitesimally thin fundamental string in the 5-dimensional bulk. Since the fundamental string wants to minimize its worldsheet area (which determines its energy), it ‘hangs’ from the boundary into the bulk. We have already seen an analogous effect manifested in Fig. 1 (green curves, which in this context would represent snapshots of the string at a given time, and whose endpoints would represent the quarks). While the SYM on $\mathbb{R}^4$ does not exhibit confinement (there is no independent length scale in the problem), we have already seen in §3.4 that SYM on Einstein static universe does exhibit a confinement-deconfinement transition [74]. More realistic setup involves a geometry where a spatial circle smoothly caps off, such as the ‘AdS soliton’ constructed by [122], where the string cannot extend beyond the bottom of the geometry. Such a geometry automatically enforces a regime with the confining condition of energy of quark - anti-quark pair growing linearly with their separation: if the string endpoints are much further separated than the scale determining where the geometry caps off, the string extends along the bottom and recovers the scaling (i.e. linear growth with quark separation) characteristic of confinement. In this way, the AdS/CFT correspondence offers a simple picture of quark confinement.

Instead of considering a meson, one could also consider just a single quark. The bulk dual is again a fundamental string ending on the quark. As the quark moves and accelerates, the bulk string trails behind accordingly. One can compute its backreaction on the bulk spacetime, and from this read off the boundary stress tensor $T^{\mu\nu}$, which in turn indicates the energy-momentum distribution in the strongly-coupled plasma through which the quark propagates. This then

⁵¹ This makes direct calculations extremely difficult, and the pre-AdS/CFT state of the art for low-energy calculations was achieved by lattice computations (which however are not well suited to time-dependent context).

⁵² More accurately, since strings are oriented, its endpoints represent a quark + anti-quark pair. The mass of these diverges with the radial position of the string endpoints, which one can however regulate by putting in additional D-brane in a suitable configuration, so as to terminate the string at finite distance radius.

allows us to study interesting features such as the drag the quark experiences in moving through the plasma, its radiation due to acceleration, the propagation and dispersion of this radiation through the plasma, and so on.⁵³ It is rather intriguing that the bulk string codifies both the quark (and its surrounding gluonic cloud) as well as the radiation it produces. In fact, cutting off the same bulk configuration at different radial distances allows us to extract the physics of a ‘dressed quark’, including radiation damping and effects of acceleration on the surrounding gluonic cloud, in very simple way [124]. While from the field theory perspective these are rather complicated and sometimes puzzling effects (such as the well-known pre-acceleration effect), the bulk dual naturally provides neatly-packaged and automatically self-consistent description.

AdS/CMT: The utility of AdS/CFT for studying hot quark-gluon plasmas encouraged people to explore other strongly-coupled systems. Of immediate interest are many of the ones studied in condensed matter physics, not least because understanding strongly-correlated systems at finite temperature is extremely challenging with presently-known condensed matter techniques once a conventional description in terms of weakly-coupled quasiparticles becomes ill-suited. Moreover, these are experimentally accessible systems which we can control, even engineer, and some may have useful technological applications. On the other hand, the large amount of freedom in specifying the material also suggests that we have not yet encountered all of them: there may be novel phases with remarkable physical properties which have been hitherto overlooked. Using a dual description is likely to systematize the enumeration of possibilities, at least in some regime, and therefore point to new phases with novel properties which experimentalists could then look for. In fact, in quantum optics, experimentalists are now able to design an impressive range of cold atom systems with certain specified properties [125]. This might even allow us to do ‘experimental AdS/CFT’ once we design a real system with a gravitational dual. Finally, from the quantum gravity standpoint, condensed matter physics can offer interesting toy models of emergent gravity.

Although there is no obvious analog of large N in condensed matter physics, one can obtain gravitational duals of related systems which are close enough for many purposes. The basic ingredients are as follows: The dimensionality of the system (typically 3 or 2 since some layered materials are effectively 2-dimensional) determines the dimensionality of the bulk; so most studies are conducted in either 3+1 or 4+1 dimensional AdS. As in the case of quark-gluon plasma, to describe the system at finite temperature we consider a black hole in AdS. However, here we also typically want to keep the system at a finite chemical potential; this corresponds to charging up the black hole, which simultaneously lets us tune the temperature independently of the energy density. To model charged condensates, one can include a charged scalar field in the bulk system, which exhibits more diverse behavior. To ‘lattice’ the system, we can either add periodic sources on the boundary which breaks translational invariance, or let the symmetry be broken spontaneously by an instability, resulting in ‘striped phases’ (see e.g. [126, 127] for pioneering efforts in these directions). Such systems naturally implement momentum dissipation and often

⁵³ For another nice review of these effects, see e.g. [123].

allow for new (metallic and insulating) ground states with interesting transitions between them. Once we have set up the gravitational system, we can analyze its properties in the bulk, which translate to the corresponding properties of the boundary dual. For example, to extract transport properties via linear response, we study linear perturbations of the black hole.

One nice example of the power of this technique is the holographic superconductor developed by Hartnoll, Herzog, Horowitz [128]. Real superconductivity occurs when Cooper pairs condense at a critical temperature T_c , leading to infinite DC conductivity. While conventional superconductors are well-described by BCS theory, for the unconventional ones, such as cuprates and organics, the pairing mechanism remained obscure. To describe the basic setup in the bulk, [128] considered an Einstein-Maxwell-scalar system which has Reissner-Nordström-AdS black hole above T_c , but a charged black hole with a scalar hair (corresponding to the condensate) below T_c . One might have naively expected a requisite no-hair theorem would rule this out, but in fact, such a contingency can indeed arise, through an instability of the Reissner-Nordström-AdS solution. As demonstrated by Gubser [129], the effective mass of the scalar field can become negative near the horizon, resulting in the requisite instability.

Fluid/gravity correspondence: Let us now step back from specific systems, and consider them in a more general framework. It is well-known that strongly-coupled quantum systems have an effective description in terms of fluid dynamics. At sufficiently long wavelengths both the quark-gluon plasma as well as many condensed matter systems behave like fluids. They are described by coarse-grained variables, namely the local temperature T and fluid velocity u^μ (with the usual normalization $u_\mu u^\mu = -1$), along with the local densities of all conserved charges. These quantities are slowly-varying (on the microscopic scale) functions of the boundary spacetime coordinates x^μ . The dynamics is captured by the conservation of the fluid stress tensor $\nabla_\mu T^{\mu\nu} = 0$ (and any other conserved currents), supplemented by constitutive relations which allow us to express $T^{\mu\nu}$ in terms of the fluid variables.⁵⁴

To study this system in terms of its gravitational dual, one needs to find bulk solutions which would have the same freedom of specification built in. Finite temperature indicates that we should consider a black hole, and the long-wavelength regime requires the black hole to be planar. When T and u^μ are independent of x^μ , the requisite solution is simply a stationary black hole with temperature T and horizon velocity u^μ , obtained by boosting (3.12). Written more conveniently in ingoing coordinates, we have

$$ds^2 = -2 u_\mu dx^\mu dr + r^2 \left(\eta_{\mu\nu} + \frac{\pi^4 T^4}{r^4} u_\mu u_\nu \right) dx^\mu dx^\nu , \quad (4.1)$$

which indeed gives the induced stress tensor on the boundary,

$$T^{\mu\nu} = \pi^4 T^4 (\eta^{\mu\nu} + 4 u^\mu u^\nu) . \quad (4.2)$$

⁵⁴ It is convenient to organize $T^{\mu\nu}$ in terms of ‘boundary derivatives’ $\frac{\partial}{\partial x^\mu}$; its form in terms of the tensor structures built out of the fluid variables is fixed by symmetries, up to a finite number of undetermined coefficients (functions of T) at each order. These coefficients reflect the microscopic origin of the fluid.

(Although for physical fluids we expect dissipative terms as well, these would not be turned on in this stationary context.) However, to describe a fluid out of global equilibrium, we need to promote T and u^μ to functions of x^μ , subject to satisfying the fluid equations. This might seem very hard since general relativity is a non-linear theory. For example one can't simply replace the parameters T and u^μ in (4.1) by expressions with x^μ -dependence. However, in order to be describable by a fluid, the system we want to consider must be slowly varying in x^μ . This suggests recasting Einstein's equations in a derivative expansion and solving them order by order in x^μ derivatives, subject to regularity.

This strategy was implemented in [66], explicitly up to second order in derivative expansion. We obtain solutions whose radial dependence is solved at the fully non-linear level, written in terms of functions $T(x^\mu)$ and $u^\nu(x^\mu)$. These must satisfy a constraint equation (obtained from the $(r\mu)$ component of Einstein's equations), which precisely reproduces the generalized Navier-Stokes equations describing the fluid. Hence we see that 5-dimensional bulk Einstein's equations (with negative cosmological constant) contain the 4-dimensional fluid Navier-Stokes equations! For each fluid solution we have a corresponding bulk black hole solution whose temperature and horizon velocity mimics that of a fluid. Moreover, by construction the solution remains regular well past the event horizon [130].⁵⁵ This relation is known as the fluid/gravity correspondence; for a review written for GR audience see e.g. [131, 132].

Of course, the idea that a black hole horizon might resemble a fluid is not new; in particular a similar notion appears in the black hole Membrane Paradigm [133, 134]. However, in the fluid/gravity correspondence, the 'membrane' lives on the boundary of the spacetime and is a perfect mirror of the entire bulk physics, not just the horizon. We have already seen that large Schwarzschild-AdS black holes in the bulk correspond to thermal states on the boundary, and that linearized fluctuations, described by quasinormal modes that characterize the black hole, allow us to extract various response and transport properties of the dual thermal state near global equilibrium [135]. The fluid/gravity correspondence [66] takes this relation to the fully non-linear level. We can now read-off the transport and response coefficients directly from the bulk solution; gravity in effect determines the fluid specifically.

The fluid/gravity correspondence has many useful applications. Evidently, by geometrizing the fluid we can gain further insight into its dynamics, which despite much theoretical, experimental, and computational effort, still retains fascinating mysteries. For example, one can try to understand turbulence using the gravitational description, further mentioned below. At a more technical level, another intriguing consequence (in the generalized context of Maxwell-Chern-Simons charged fluid) is the appearance of a new pseudo-vector contribution to the charge current, which has been ignored by Landau & Lifshitz [136], but which may have potentially observable effects [137].

⁵⁵ Despite its global definition, the teleological nature of the event horizon is rather mild in our long-wavelength regime, allowing for explicit determination. Moreover, the pullback of the area form on the horizon gives a natural entropy current in the boundary fluid which is guaranteed to satisfy the 2nd law of thermodynamics by virtue of the black hole area theorem.

Brane worlds: We have touched on applications of AdS/CFT to nuclear physics, condensed matter physics, and even fluid dynamics. Let us conclude this section by mentioning one interesting application of AdS gravity as such, this time to grand unified model building. Known as the ‘brane world’ scenario, this idea was developed in parallel with the AdS/CFT correspondence, also using string theory. Although logically separate, interconnections between the two programs were soon discovered.

Our spacetime appears 3+1 dimensional. This innocuous-sounding statement implies not just that photons seem to propagate in this many dimensions (i.e. at presently-accessible scales we do not directly see any extra dimensions), but also that gravitational interactions satisfy 3+1 dimensional general relativity (approximated by the Newton’s $1/r^2$ force law).⁵⁶ On the other hand, as we mentioned in §2, string theory is naturally formulated in 10-dimensional spacetime, though these dimensions need not be flat or infinite. The conventional way of making this formulation compatible with observation has been to compactify the extra dimensions to be smaller than the present observational bounds on them, analogously to the original Kaluza-Klein idea. Since there is a mass gap in the KK spectrum, the zero mode of the graviton would then look 4-dimensional up to the scale given by the gap. A new possibility arises when we consider gauge fields living on branes, since then only gravity feels the bulk; e.g. two extra millimeter-sized compact dimensions would allow for unification of gravity and gauge interactions at the weak scale [139]. However, this scenario shifts the problem of large hierarchy between the weak scale and fundamental scale of gravity to one of ‘large’ extra dimensions.

Soon after the advent of AdS/CFT, another option was suggested [140]: allowing the spacetime to have a warped product structure of AdS, as opposed to a direct product structure over the internal space, allows for large extra directions which can nevertheless remain consistent with observations. More specifically, while gravity (which is a manifestation of spacetime curvature) of course feels all directions, it can be *effectively* localized on a subspace, as a ‘bound state’ of the higher-dimensional graviton. Although the KK spectrum is continuous, the higher dimensional effects can remain subleading. The explicit construction in [140] involves taking two pieces of AdS_5 and gluing them together (with the junction representing a 3-brane). This obtains a zero mode of the graviton which is localized on the brane and the massive KK mode continuum which corrects the $1/r$ gravitational potential along the brane by $\mathcal{O}(\ell^2/r^3)$ effects.⁵⁷

This intriguing idea generated a large industry of various generalizations and computations of its consequences, both in cosmology and in phenomenology. From the GR standpoint, there are many interesting effects for brane world black holes, whose study was initiated in [142]. To make contact with AdS/CFT, one can consider just one side of the brane, and describe the physics on the brane in terms of a cut-off field theory, weakly-coupled to gravity.

⁵⁶ The experimental bounds on the size of extra dimensions from gravity are of course much weaker; see e.g. [138].

⁵⁷ An earlier construction [141] involved two branes: one with ‘our’ matter fields and another supporting the graviton mode. The apparent hierarchy between the weak and Planck scale is then naturally generated by the warp factor scaling exponentially with compactification radius.

4.2 Lessons for general relativity

The AdS/CFT correspondence has also revealed many new results pertaining to general relativity itself. Although many of these could in principle have been stumbled upon without AdS/CFT, the correspondence provided the incentive or means to look in the right direction. Here we mention a few examples.

New solutions: We have already seen several examples wherein AdS/CFT guided exploration leading to unexpected new solutions.⁵⁸ One example are the hairy black holes uncovered in the process of trying to understand superconductivity. Another is the large set of black hole solutions with no symmetries, arising in the fluid/gravity context (these are described in terms of lower-dimensional fluid solution, in a derivative expansion). Yet another example, in asymptotically locally AdS context, is a ‘flowing funnel’ solution [143, 144] which constructs a stationary black string whose horizon is nevertheless not a Killing horizon.⁵⁹ While these examples all involved black holes, there have also been interesting smooth causally trivial solutions, such as the AdS geons constructed in [146], which are gravitational soliton-like solutions to Einstein’s equation with a helical symmetry.

Instability of AdS: While AdS is known to be linearly stable, the non-linear stability of AdS has been analysed only recently, cf. [147] for a brief review. The program turned out far richer than one might have anticipated. The first surprise came with the numerical study by Bizon & Rostworowski [148], who considered the analog of Choptuik phenomena in AdS. One would expect that collapsing a scalar shell (specified by its amplitude A , with some fixed profile) in AdS will collapse to form a black hole for sufficiently large amplitude $A > A_c$ and disperse for $A < A_c$. Near $A \approx A_c$ we should see the usual critical behavior, at least locally.⁶⁰ These expectations turned out partly correct, but for the unexpected result of [148] that in fact after long enough time, the black forms no matter how small A is, at least for a large class of profiles. For a range of A ’s slightly smaller than A_c , the field disperses after the first implosion but collapses on the second one, for a range of smaller A ’s the field collapses on the third implosion, and so forth, all the way down to vanishing A . This implies that AdS is non-linearly unstable, in the sense of arbitrarily small perturbation evolving to a black hole. Of course, from the field theory standpoint, this might not be so surprising, since we would expect any initial disturbance to eventually thermalize.

The initial expectation following this dramatic result was that any small deformation of AdS will form a black hole. This is however not true either. For example the geons of [146]

⁵⁸ Some can be expressed analytically, some only numerically or approximately in a perturbative expansion; but they can be analysed and studied, often revealing interesting new physics.

⁵⁹ Although this would naively appear to violate the rigidity theorem [145], the horizon in question is non-compact, thereby evading rigidity; in the dual language, one can have entropy production even in steady state.

⁶⁰ Unlike the asymptotically flat case, though, one cannot characterize the final mass scaling in the critical regime as $M \sim (A - A_c)^\gamma$, since the AdS reflecting boundary conditions will cause the initially-formed black hole to accrete the rest of the scalar field as time evolves.

mentioned above provide an explicit example of a near-AdS purely gravitational configuration which does not form a black hole. In the Einstein-scalar system, it was soon discovered that with different starting profiles, one could obtain e.g. time periodic solutions [149], though the issue of characterizing the islands of stability has not been fully settled. So it appears that while most perturbations of AdS eventually form black holes, there are small islands of stability which remain regular. The full story is however yet to be understood.

Turbulence: As mentioned above, fluids display rather rich dynamics, including turbulence. Hence one expects that, in the appropriate regime, event horizons of large black holes in AdS likewise exhibit this kind of instability. From a conventional GR viewpoint, trained by the familiar case where a black hole dissipates perturbations maximally rapidly, the idea that an event horizon could exhibit such a rich behavior may verily seem radical. Nonetheless, this expectation has been confirmed through numerical studies [150], which construct turbulent black holes in asymptotically AdS_4 spacetime by solving Einstein equations numerically.⁶¹ They find the expected inverse cascade, but also point to a novel property of the black hole horizon: steady-state black holes dual to d dimensional turbulent flows have horizons which are approximately fractal, with fractal dimension $d + 4/3$.

This fascinating discovery motivated further exploration of the robustness of this behaviour away from the fluid/gravity regime. This prompted a closer look at dynamics of perturbations of asymptotically flat black holes, which indeed revealed a new type of horizon instability for rapidly rotating Kerr black holes [152]. In particular, near-extremal black holes have long-lived quasinormal modes which can excite other modes and cause them to grow exponentially for a while. The associated energy flow exhibits an inverse cascade, analogously to turbulence for 2-dimensional fluids. This drives the black hole toward richer angular structure, and might even have observational consequences, e.g. for gravitational wave signals and perhaps for spectra of accretion disks.

Firewalls: The path towards the AdS/CFT correspondence was motivated in large measure by the desire to understand black holes: since string theory can resolve singularities and render the entire evolution well-defined, it should be able to shed light on the long-standing puzzles of quantum gravity, such as the black hole Information Paradox: how can the laws of quantum mechanics (specifically unitarity) be upheld by an evaporating black hole with (nearly) thermal Hawking radiation? Direct approach was typically stymied by lack of control over the setup, but with the advent of the AdS/CFT duality, tractability of this question suddenly appeared more promising: the entire gravitational dynamics of collapsing and evaporating black hole (in AdS) is fully captured by a manifestly unitary gauge theory. This means that black hole collapse and evaporation is described by a unitary process, so information cannot be lost. This argument was

⁶¹ An earlier work [151] also used AdS/CFT to study 2-dimensional superfluid turbulence, discovering that the superfluid kinetic energy spectrum actually obeys the Kolmogorov $-5/3$ scaling law, as it would for normal fluids in 3 dimensions.

oft presented as an example of what AdS/CFT can teach us using the CFT side (as opposed to the more usual AdS side) as a starting point.

However, when the earnest effort was made at actually seeing precisely how the information is recovered, the situation turned out to be far more intricate and mystifying: Almheiri, Marolf, Polchinski, and Sully (AMPS) [153] realized that the following three innocuous-sounding assumptions are mutually inconsistent: (i) information is not lost (i.e. black hole formation and evaporation process is unitary), (ii) the radiation is emitted from the region near the horizon, where low energy effective field theory valid, and (iii) the infalling observer encounters nothing unusual at the horizon (i.e. Einstein’s equivalence principle is upheld). Judging that relaxing the last assumption is the least radical, [153] proposed the ‘firewall’ – a place very near the horizon where local effective field theory description breaks down. Naturally, this proposal generated a large amount of controversy, accompanied by a flurry of putative resolutions.⁶² These were however addressed in [156], where the original AMPS argument was sharpened (though dissenting views remain as to whether the proposed resolutions are viable), and another argument was subsequently supplied in [157]. To date, it is still unclear what the final resolution is. Nevertheless, this observation already has deep implications about the encoding of bulk geometry in the dual CFT: whilst we understand a great deal of the spacetime outside the horizon, encoding the region inside is far more subtle.

Entanglement and geometry: The proposals of holographic entanglement entropy [51, 54] suggest a mysterious connection between entanglement and geometry. This has been followed by the bold proposal of [158–160] that entanglement in some sense creates geometry, and conversely, disentangling the degrees of freedom associated with different spatial regions has the geometrical effect of pinching them off from each other. Subsequently, the even more radical notion known as “ER=EPR” [161] posited that the spacetime should admit ‘Einstein-Rosen bridges’ (albeit of possibly Planckian scale) associated with any EPR pairs, present in general entangled states. This was partly inspired by the observation that although both Einstein-Rosen bridge (or wormhole) and EPR correlations appear non-local, they do not violate causality. More recently, there has been mounting attention on how the geometry of an Einstein-Rosen bridge relates to information theoretic constructs. For example, following up on the suggested connections [162] between the distance from the horizon to computational complexity, [163] relates the radial bulk direction to a measure of how well CFT representations of bulk quantum information are protected from local erasures.

Entanglement structure seems to have some deep importance, but its precise nature is far from clear. Even within quantum mechanics, there are many distinct measures of entanglement [164, 165], with varying degrees of computability, depending on what feature one wants to focus on. Entanglement entropy of a given subsystem, defined as the Von Neumann entropy of a

⁶² One alternative is the earlier ‘fuzzball’ proposal; see [154] for review. This describes black hole microstates in terms of smooth horizonless geometries (one early accomplishment being the construction of regular supersymmetric solutions of the D1-D5 system [155]).

reduced density matrix for that subsystem, is well-defined given a corresponding partitioning of the Hilbert space; but for the total system being in a mixed state this quantity counts not just quantum correlations but the classical ones as well. For instance we recover the thermal entropy of the system if we take our subsystem to be the entire system. Some of the other measures characterizing entanglement, such as negativity, robustness, distillable entanglement, entanglement cost, etc., may have more direct link with the genuinely quantum correlations but are often less robustly defined or less easily computable. Nevertheless, apart from having a good thermodynamic limit, the entanglement entropy has certain ‘nice’ properties such as strong subadditivity, and correspondingly it has a simple holographic description: an extremal surface anchored on the boundary of a given region is just about the simplest geometrical construct associated with that boundary region.⁶³

Finally, let us close by mentioning another intriguing recent development suggestive of the deep relation between entanglement entropy and geometry. Entanglement entropy obeys a ‘first law’ which can be thought of as a quantum generalization of the ordinary first law of thermodynamics. Under certain special conditions, [167] have been able to use the holographic entanglement entropy proposal to derive linearized Einstein equations in the bulk. Whether one can obtain the full nonlinear Einstein’s equations from known laws governing the behavior of entanglement entropy in the CFT dual remains a fascinating open problem.

5 Epilogue: onward to quantum gravity...

Our story started with black holes, and black holes have been the star player throughout, appearing at various stages in different guises. Black hole entropy counting motivated the advances in string theory which provided the basic framework. Extremal black branes supplied the specific context from which AdS/CFT was derived. Large black holes in AdS turned out to encode the dynamics of fluids and describe a vast array of experimentally accessible systems. Black hole information paradox has prompted the recent exploration of connections between quantum information and geometry. And, of course, black holes still present a crucial playground for trying to unravel quantum gravity.

While this has been the much sought-for Holy Grail of theoretical physics, justifiably approached from many directions, the context of AdS/CFT correspondence seems particularly well-suited to this endeavor. One hint is that a local formulation of quantum gravity is necessarily problematic, as recently explained in [168]: any gravitational theory with universal coupling to energy is characterized by a Hamiltonian which lives on the boundary (more precisely is a purely boundary term on shell) – indeed, there are no local observables in quantum gravity. Since all bulk dynamics freezes out for such candidate local theory allowing for an effective grav-

⁶³ In fact, an even simpler construct is the ‘causal wedge’ (which only depends on the bulk causal structure) and associated quantities such as the ‘causal holographic information’ [166]. However, the corresponding objects in the CFT dual are yet to be identified.

itational description, bulk gravity must then be encoded in purely boundary dynamics in the original theory. In fact, this emergence entails further non-locality (on top of any non-locality manifest in the original dynamics), characteristic of holographic systems. As we have seen, the AdS/CFT duality implements this emergence quite naturally.

From a historical perspective, it is amusing to note how much black holes had risen in prominence over the last century, starting from presenting almost an embarrassment to general relativity, to scarcely-believed esoteric objects, to astrophysically relevant and mathematically fascinating objects, to key constructs underlying profound dualities connecting far reaches of theoretical physics, and finally to becoming virtually ubiquitous, in describing almost every-day systems. This has been quite an amazing climb, but we're not near the summit yet, and the best is still to come! The glimpse at profound connections between quantum information and geometry hint at more fundamental structures to be uncovered. The resulting scientific revolution may be just around the corner, or might still take years. At present, though, novel vistas have been revealed, and there is much to explore and relate. Perhaps the most intriguing aspect of AdS/CFT is that it interconnects so many previously disconnected and apparently disparate ideas. In 100 years of general relativity, our revelations regarding the nature of space and time have brought us to perhaps the most exciting era to be a physicist discovering the mysteries of the Universe.

Acknowledgments

It is a pleasure to thank Gary Horowitz, Juan Maldacena, Don Marolf, and Mukund Rangamani for useful comments on the presentation. I would also like to thank the Institute for Advanced Study and the Aspen Center for Physics for hospitality during early stages of this work. I thank the STFC Consolidated Grant ST/J000426/1 and ST/L000407/1, FQXi Grant RFP3-1334, and the Ambrose Monell Foundation for support.

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AdS/CFT Correspondence and string/gauge duality

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October 31, 2013

Abstract

The AdS/CFT correspondence is an exact duality between string theory in anti-de Sitter space and conformal field theories on its boundary. Inspired in this correspondence some relations between strings and non conformal field theories have been found. Exact dualities in the non conformal case are intricate but approximations can reproduce important physical results. A simple approximation consists in taking just a slice of the AdS space with a size related to an energy scale. Here we will discuss how this approach can be used to reproduce the scaling of high energy QCD scattering amplitudes. Also we show that very simple estimates for glueball mass ratios can emerge from such an approximation.

1 Introduction

The AdS/CFT correspondence is a duality between the large N limit of $SU(N)$ superconformal field theories and string theory in a higher dimensional anti-de Sitter spacetime[1]. In this correspondence the AdS space shows up both as a near horizon geometry of a set of coincident D-branes or as a solution of supergravity. Precise prescriptions for the realization of the AdS/CFT correspondence were formulated[2, 3] by considering Poincare patches of AdS space. The Poincare coordinate system allows a simple definition for the boundary where the conformal field theory is defined. In this correspondence, bulk fields act as classical sources for boundary correlation functions.[4]–[7] One of the striking features[3] of this correspondence is that it is a realization of the holographic principle: “The degrees of freedom of a quantum theory with gravity can be mapped on the corresponding boundary”. [8]–[12]

String theory, presently the main candidate to describe the fundamental interactions in a unified way, was originally proposed as a model for strong interactions with the successful results of Veneziano amplitude in the Regge regime[13]–[15]. However this model was unable to reproduce the experimental hard and deep inelastic scatterings. These behaviours were latter explained by QCD which is today

the theory accepted to describe strong interactions. Despite of its success in high energies, QCD is non perturbative in the low energy regime, where one needs involved lattice calculations. In fact QCD and strings may hopefully be viewed as complementary theories. A strong indication found by 't Hooft [16] is the relation between large N $SU(N)$ gauge theories and strings [17]. A recent remarkable result in the direction of understanding this duality was the AdS/CFT correspondence found by Maldacena.

One of the long standing puzzles for the string description of strong interactions is the high energy scattering at fixed angles. For string theory in flat space such a process is soft (the amplitudes decay exponentially with energy [18] while both experimental data and QCD theoretical predictions [19, 20] indicate a hard behavior (amplitudes decaying with a power of energy). A solution for this puzzle was proposed recently by Polchinski and Strassler [21], inspired in the AdS/CFT correspondence, considering strings not in flat space but rather in a 10 dimensional space which asymptotically tends to an AdS_5 times a compact space. This space is considered as an approximation for the space dual to a confining gauge theory which can be associated with QCD. An energy scale was introduced and identified with the lightest glueball mass. Then they found the glueball high energy scattering amplitude at fixed angles (with the correct QCD scaling) integrating the string amplitude over the warped AdS extra dimension weighted by the dilaton wave function. Other approaches to this problem have also been discussed recently [22]–[25]. Important related discussions on deep inelastic scattering of hadrons [26] and on exclusive processes in QCD [27], both inspired in the AdS/CFT correspondence appeared recently.

In this paper we are going to review recent results regarding an approximate relation between strings and QCD. In particular we find the scaling of high energy glueball scattering at fixed angles and their masses. This paper is organized in the following way. In section 2 we will review the relation between AdS space and D-brane spaces coming from string theory. In section 3 we first present a mapping between AdS bulk and boundary states for scalar fields. Then we show in section 4 how this mapping can be used to find the same high energy scaling as that of glueball scattering, from the bulk theory. In section 5 we use this mapping to obtain an estimate for the ratio of scalar glueball masses.

2 Branes and AdS space

An $n + 1$ dimensional AdS space can be defined as the hyperboloid ($R = \text{constant}$)

$$X_0^2 + X_{n+1}^2 - \sum_{i=1}^n X_i^2 = R^2 \quad (1)$$

in a flat $n + 2$ dimensional space with measure

$$ds^2 = -dX_0^2 - dX_{n+1}^2 + \sum_{i=1}^n dX_i^2. \quad (2)$$

Usually the AdS space is represented by the so called global coordinates ρ, τ, Ω_i given by [4, 5]

$$\begin{aligned} X_0 &= R \sec \rho \cos \tau \\ X_i &= R \tan \rho \Omega_i ; & \left(\sum_{i=1}^n \Omega_i^2 = 1 \right) \\ X_{n+1} &= R \sec \rho \sin \tau . \end{aligned} \quad (3)$$

These coordinates are defined in the ranges $0 \leq \rho < \pi/2$ and $0 \leq \tau < 2\pi$ and the line element reads

$$ds^2 = \frac{R^2}{\cos^2(\rho)} \left(-d\tau^2 + d\rho^2 + \sin^2(\rho) d\Omega^2 \right) . \quad (4)$$

Note that the timelike coordinate τ in the above metric has a finite range. So in order to identify it with the usual time it is necessary to unwrap it. This is done taking copies of the original AdS space that together represent the AdS covering space [28]. From now on we take this covering space and call it simply as AdS space.

A consistent quantum field theory in AdS space [29, 30] requires the addition of a boundary at spatial infinity: $\rho = \pi/2$. This compactification of the space makes it possible to impose appropriate conditions and find a well defined Cauchy problem. Otherwise massless particles could go to or come from spatial infinity in finite times. Possible energy definitions in AdS spaces, relevant for the AdS/CFT correspondence have also been discussed [31].

Poincaré coordinates $z, \vec{x}, t$ are more useful for the study of the AdS/CFT correspondence. These coordinates are defined by

$$\begin{aligned} X_0 &= \frac{1}{2z} \left(z^2 + R^2 + \vec{x}^2 - t^2 \right) \\ X_j &= \frac{R x^j}{z} ; & (j = 1, \dots, n-1) \\ X_n &= -\frac{1}{2z} \left(z^2 - R^2 + \vec{x}^2 - t^2 \right) \\ X_{n+1} &= \frac{Rt}{z} , \end{aligned} \quad (5)$$

where $\vec{x}$ has $n-1$ components and $0 \leq z < \infty$. In this case the AdS_{n+1} metric reads

$$ds^2 = \frac{R^2}{(z)^2} \left(dz^2 + (d\vec{x})^2 - dt^2 \right) . \quad (6)$$

The AdS boundary $\rho = \pi/2$ in global coordinates corresponds in Poincaré coordinates to the surface $z = 0$ plus a “point” at infinity ($z \rightarrow \infty$). The surface $z = 0$ defines a Minkowski space with coordinates $\vec{x}, t$. The connectedness of the AdS boundary was discussed by Witten and Yau [32].

The point at infinity can not be accommodated in the original Poincare chart [33, 34] so that we have to introduce a second coordinate system to represent it properly. The consistency of the addition of a second chart requires ending the first chart at a finite $z = z_{max}$. At this surface one needs to impose boundary conditions to guarantee the analyticity of fields. The imposition of boundary conditions will be essential to the discussion of the next section.

Finally it is interesting to see how AdS space shows up when we study D-brane systems in string theory. The ten dimensional metric generated by N coincident D3-branes can be written as [35, 2]

$$ds^2 = \left(1 + \frac{R^4}{r^4}\right)^{-1/2} (-dt^2 + d\vec{x}^2) + \left(1 + \frac{R^4}{r^4}\right)^{1/2} (dr^2 + r^2 d\Omega_5^2) \quad (7)$$

where $R^4 = N/2\pi^2 T_3$ and T_3 is the tension of a single D3-brane. Note that there is a horizon at $r = 0$.

Now looking at the near horizon region $r \ll R$ the metric (7) can be approximated as:

$$ds^2 = \frac{r^2}{R^2} (-dt^2 + d\vec{x}^2) + \frac{R^2}{r^2} dr^2 + R^2 d\Omega_5^2. \quad (8)$$

Changing the axial coordinate according to: $z = R^2/r$ this metric takes the form

$$ds^2 = \frac{R^2}{z^2} (dz^2 + (d\vec{x})^2 - dt^2) + R^2 d\Omega_5^2. \quad (9)$$

corresponding to a five dimensional AdS space (in Poincaré coordinates) times a five sphere: $\text{AdS}_5 \times S^5$.

3 Bulk boundary mapping

In the AdS/CFT framework there is a correspondence between (on shell) string theory in the AdS bulk and (off shell) conformal field theory on the boundary. This represents a realisation of the holographic principle which asserts that the degrees of freedom of a quantum theory with gravity in some space can be represented on the corresponding boundary. Based on this principle one can speculate on the possibility of finding a map between quantum states of theories defined in the bulk and on the boundary of a given space. Here we will find a map between AdS bulk and boundary states considering the simple situation of free scalar fields [36]. We will later identify the bulk scalar field with the dilaton which is a massless string excitation.

A free scalar field, like the dilaton, in a Poincaré AdS_5 chart $0 \leq z \leq z_{max}$ has the form [33]

$$\begin{aligned} \Phi(z, \vec{x}, t) &= \sum_{p=1}^{\infty} \int \frac{d^3 k}{(2\pi)^3} \frac{z^2 J_2(u_p z)}{z_{max} w_p(\vec{k}) J_3(u_p z_{max})} \\ &\times \{ \mathbf{a}_p(\vec{k}) e^{-i w_p(\vec{k}) t + i \vec{k} \cdot \vec{x}} + h.c. \}, \end{aligned} \quad (10)$$

with $w_p(\vec{k}) = \sqrt{u_p^2 + \vec{k}^2}$ and u_p defined by

$$u_p z_{max} = \chi_{2,p}, \quad (11)$$

where $\chi_{2,p}$ are the zeros of the Bessel function $J_2(y)$.

The operators $\mathbf{a}_p, \mathbf{a}_p^\dagger$ satisfy the commutation relations

$$[\mathbf{a}_p(\vec{k}), \mathbf{a}_{p'}^\dagger(\vec{k}')] = 2(2\pi)^3 w_p(\vec{k}) \delta_{pp'} \delta^3(\vec{k} - \vec{k}'). \quad (12)$$

As the coordinate range $z = z_{max}$ is arbitrary we can take it as large as we want so that most of the AdS space is described by one Poincaré chart. This way we will not need to use the second Poincaré chart explicitly. The size $z = z_{max}$ corresponds to an energy scale as we will discuss bellow. From now on we call the Poincaré chart together with boundary conditions at $z = z_{max}$ as the AdS slice.

On the four dimensional boundary ($z = 0$) of the AdS slice we consider massive scalar fields $\Theta(\vec{x}, t)$ whose corresponding creation-annihilation operators are assumed to satisfy the algebra

$$[\mathbf{b}(\vec{K}), \mathbf{b}^\dagger(\vec{K}')] = 2(2\pi)^3 w(\vec{K}) \delta^3(\vec{K} - \vec{K}'), \quad (13)$$

where $w(\vec{K}) = \sqrt{\vec{K}^2 + \mu^2}$ and μ is the mass of the field Θ .

Note that if the above bulk and boundary field theories had continuous momenta it would be impossible to find a one to one mapping between the corresponding quantum states since they are defined in different dimensions. However, as we consider just a slice of AdS, the spectrum of the momentum u_p associated with the axial direction z is discrete. Then naturally the continuous part of bulk and boundary momenta $\vec{k}$ and $\vec{K}$ have the same dimensionality. So this discretization makes it possible to establish a one to one mapping between bulk and boundary momenta $(\vec{k}, u_p)$ and $\vec{K}$. Further we assume a trivial mapping between their angular parts and then look for a relation between their moduli $K \equiv |\vec{K}|, k \equiv |\vec{k}|$. It is important to mention that the complete one to one bulk boundary mapping is only possible in the conformal limit $\mu \rightarrow 0$ of the Θ field. In the discussion bellow we will be interested in an approximate map between the dilaton and massive fields so we will let $\mu \neq 0$. Naturally the massless limit is closest to the original AdS/CFT correspondence.

We may introduce a sequence of energy scales $\mathcal{E}_1, \mathcal{E}_2, \mathcal{E}_3 \dots$ in order to map each interval of the boundary momentum modulus $\mathcal{E}_{p-1} < K \leq \mathcal{E}_p$ with $p = 1, 2, \dots$ into the entire range of the transverse bulk momentum modulus k , corresponding to some fixed axial momentum u_p . This mapping between bulk and boundary momenta allow us to map the corresponding creation-annihilation operators [\(12,13\)](#).

For the first energy interval, corresponding to u_1 , defined as $0 \leq K \leq \mathcal{E}_1$, we can write [\[36\]](#)

$$\begin{aligned} k \mathbf{a}_1(\vec{k}) &= K \mathbf{b}(\vec{K}) \\ k \mathbf{a}_1^\dagger(\vec{k}) &= K \mathbf{b}^\dagger(\vec{K}). \end{aligned} \quad (14)$$

This mapping must preserve the physical consistency of both theories. In particular Poincare invariance should not be broken neither for the boundary theory nor for the bulk theory at a fixed z . This is guaranteed imposing that the canonical commutation relations (12,13) are preserved by the mapping (14). Then substituting eq. (14) into relation (12) and using eq. (13) we find an equation in terms of the moduli of bulk and boundary momenta which solution can be written as [36]

$$k = \frac{u_1}{2} \left[\frac{\mathcal{E}_1 + \sqrt{\mathcal{E}_1^2 + \mu^2}}{K + \sqrt{K^2 + \mu^2}} - \frac{K + \sqrt{K^2 + \mu^2}}{\mathcal{E}_1 + \sqrt{\mathcal{E}_1^2 + \mu^2}} \right]. \quad (15)$$

Similar relations can be obtained for the other intervals $\mathcal{E}_{p-1} < K \leq \mathcal{E}_p$ with $p = 2, 3, \dots$. This way we found a one to one map between bulk and boundary creation-annihilation operators. This allows us to construct a similar map for quantum states. This will be used in the next section in order to relate bulk and boundary scattering amplitudes.

One might wonder if the trivial mapping $\vec{k} = \vec{K}$ would also be a solution. However this would not provide a one to one mapping between the entire bulk momenta $(\vec{k}, u_p)$ and the boundary momenta $\vec{K}$ for all different values of p as long as there is only one boundary field with mass μ .

The momentum operators in the bulk and boundary theories are respectively:

$$(\vec{P}, u) = \sum_p \int \frac{d^3k}{2(2\pi)^3} \frac{\mathbf{a}_p^\dagger(\vec{k}) \mathbf{a}_p(\vec{k})}{\sqrt{k^2 + u_p^2}} (\vec{k}, u_p) \quad (16)$$

$$\vec{\Pi} = \int \frac{d^3K}{2(2\pi)^3} \frac{\mathbf{b}^\dagger(\vec{K}) \mathbf{b}(\vec{K})}{\sqrt{K^2 + \mu^2}} \vec{K} \quad (17)$$

Note that Poincare invariance in the $\vec{x}$ directions holds both in the boundary and bulk theories since the canonical commutation relations (12) and (13) are preserved by the mapping.

4 High energy scalling

In this section we are going to study the scattering of scalar particles at high energy using the map found in the previous section. This is inspired in the gauge /string duality suggested by the AdS/CFT correspondence, where the scaling of the high energy scattering of glueballs was obtained [21] from string theory using the duality between glueballs and dilatons [37]. Glueballs in QCD correspond to composite operators. So we do not expect the free boundary scalar field of the previous section to give a complete description of their dynamics. However, we will see that for high energies the map leads to the same scaling as that of QCD glueballs. This may be an indication that for this regime the asymptotic states of the glueballs may be approximated by free scalar fields.

So we will consider the mapping of the previous section in the following way: on the boundary we will approximate the asymptotic behaviour of the glueball operators by free massive scalar fields in four dimensions. In the bulk we take the

massless scalar fields in the AdS slice as representing the dilaton. A high energy scattering on the four dimensional AdS boundary will be mapped into a scattering process of dilaton states in the bulk [23].

The equation (15) is understood as the relation between bulk and boundary momenta for the particles involved in the scatterings. Identifying μ with the mass of the lightest glueball and choosing the AdS size as

$$z_{max} \sim \frac{1}{\mu} \quad (18)$$

we find that $u_1 \sim \mu$, once $z_{max} \sim 1/u_1$ according to eq. (11). Note that the size z_{max} can then be interpreted as an infrared cutoff for the boundary theory.

Further, we can take $\mathcal{E}_1$ large enough so that the momenta associated with the high energy glueball scattering can fit into the region $\mu \ll K \ll \mathcal{E}_1$. Then we can approximate relation (15) as

$$k \approx \frac{\mathcal{E} \mu}{2K}, \quad (19)$$

where we defined $\mathcal{E}_1 \equiv \mathcal{E}$ and disregarded the other energy intervals associated with higher axial momenta u_p , $p \geq 2$. Note that this mapping together with the conditions $\mu \ll K \ll \mathcal{E}$ imply that $\mu \ll k \ll \mathcal{E}$.

Choosing the string scale to be of the same order of the high energy cut off of the boundary theory, i.e., $\mathcal{E} \sim 1/\sqrt{\alpha'}$, we find that the momenta k associated with string theory correspond to a low energy regime well described by supergravity approximation.

The equations (14) and (19) represent a one to one holographic mapping between bulk dilaton and boundary glueball states. Now we are going to use these equations to relate the corresponding scattering amplitudes.

Let us consider, in the bulk string theory, a scattering of 2 particles in the initial state and m particles in the final state, with all particles having axial momentum u_1 . The S matrix reads

$$\begin{aligned} S_{Bulk} &= \langle \vec{k}_3, u_1; \dots; \vec{k}_{m+2}, u_1; out | \vec{k}_1, u_1; \vec{k}_2, u_1; in \rangle \\ &= \langle 0 | \mathbf{a}_{out}(\vec{k}_3) \dots \mathbf{a}_{out}(\vec{k}_{m+2}) \mathbf{a}_{in}^+(\vec{k}_1) \mathbf{a}_{in}^+(\vec{k}_2) | 0 \rangle, \end{aligned} \quad (20)$$

where $\mathbf{a} \equiv \mathbf{a}_1$ and the *in* and *out* states are defined as $|\vec{k}, u_1\rangle = \mathbf{a}^+(\vec{k})|0\rangle$.

Now using the mapping between creation-annihilation operators (14) one can rewrite the above S matrix in terms of boundary operators. Considering fixed angle scattering, we take the bulk momenta to be of the form $k_i = \gamma_i k$ and the boundary momenta $K_i = \Gamma_i K$, where γ_i and Γ_i are constants with $i = 1, 2, \dots, m+2$. Then

$$\begin{aligned} S_{Bulk} &\sim \langle 0 | \mathbf{b}_{out}(\vec{K}_3) \dots \mathbf{b}_{out}(\vec{K}_{m+2}) \mathbf{b}_{in}^+(\vec{K}_1) \mathbf{b}_{in}^+(\vec{K}_2) | 0 \rangle \left(\frac{K}{k} \right)^{m+2} \\ &\sim \langle \vec{K}_3, \dots, \vec{K}_{m+2}, out | \vec{K}_1, \vec{K}_2, in \rangle \left(\frac{K}{k} \right)^{m+2} K^{(m+2)(d-1)}, \end{aligned} \quad (21)$$

where the composite operators on the boundary have scaling dimension d and then their *in* and *out* states are $|\vec{K}\rangle \cong K^{1-d} \mathbf{b}^+(\vec{K})|0\rangle$, within the regime $K \gg \mu$.

Using the relation (19) between bulk and boundary momenta we get

$$S_{Bulk} \sim S_{Bound.} \left(\frac{\sqrt{\alpha'}}{\mu} \right)^{m+2} K^{(m+2)(1+d)} \quad (22)$$

As the scattering amplitudes $\mathcal{M}$ are related to the corresponding S matrices (for non equal *in* and *out* states) by

$$\begin{aligned} S_{Bulk} &= \mathcal{M}_{Bulk} \delta^4(k_1^\rho + k_2^\rho - k_3^\rho - \dots - k_{m+2}^\rho) \\ S_{Bound.} &= \mathcal{M}_{Bound.} \delta^4(K_1^\rho + K_2^\rho - K_3^\rho - \dots - K_{m+2}^\rho) \end{aligned} \quad (23)$$

we find a relation between bulk and boundary scattering amplitudes [23]

$$\begin{aligned} \mathcal{M}_{Bound.} &\sim \mathcal{M}_{Bulk} S_{Bound.} (S_{Bulk})^{-1} \left(\frac{K}{k} \right)^4 \\ &\sim \mathcal{M}_{Bulk} K^{8-(m+2)(d+1)} \left(\frac{\sqrt{\alpha'}}{\mu} \right)^{2-m}. \end{aligned} \quad (24)$$

Now we must evaluate the bulk amplitude from the string low energy effective action. At energies much lower than the string scale $1/\sqrt{\alpha'}$ type IIB string theory can be described by the supergravity action [2, 38]

$$S = \frac{1}{2\kappa^2} \int d^{10}x \sqrt{G} e^{-2\Phi} \left[\mathcal{R} + G^{MN} \partial_M \Phi \partial_N \Phi + \dots \right] \quad (25)$$

where G^{MN} is the ten dimensional metric, $\mathcal{R}$ is the Ricci scalar curvature, Φ is the dilaton field and $\kappa \sim g(\alpha')^2$. We identify this ten dimensional space with an $\text{AdS}_5 \times S^5$ with radius R and measure (9). Furthermore we take an AdS slice with "size" z_{max} representing an infrared cut off associated with the mass of the lightest glueball. Note that we are also considering the dilaton to be in the s -wave state, so we will not take into account variations with respect to S^5 coordinates. Thus, the action becomes

$$\begin{aligned} S &= \frac{\pi^3 R^8}{4\kappa^2} \int d^4x \int_0^{z_{max}} \frac{dz}{z^3} e^{-2\Phi} \\ &\times \left[\mathcal{R} + (\partial_z \Phi)^2 + \eta^{\mu\nu} \partial_\mu \Phi \partial_\nu \Phi + \dots \right] \end{aligned} \quad (26)$$

where $\eta^{\mu\nu}$ is the four dimensional Minkowski metric.

Then the momentum dependence of the bulk amplitude can be determined using dimensional arguments. Note that the global constant R^8/κ^2 associated with this action is dimensionless and the only dimensionfull parameters are $z_{max} \sim 1/\mu$ and the Ricci scalar $\mathcal{R} \sim 1/R^2$. As $\mu \ll k$ the relevant contribution to the bulk amplitude will not involve z_{max} . Further, choosing the condition $1/R \ll k$ we can disregard the term involving the Ricci scalar. This condition does not fix completely the AdS radius R and we additionally impose that $\mu \ll 1/R$. This implies that $z_{max} \gg R$. Then, if one regularizes [2] the divergence ($z = 0$) of the bulk action

by cutting the axial coordinate z at R , one still has a large portion of the original AdS space: $R \leq z \leq z_{max}$. This guarantees that we keep the interesting AdS region which is an approximation for the near horizon geometry of N coincident $D3$ -branes, as in the Maldacena duality.

Taking into account the normalization of the states $|k, u_1\rangle$ one sees that $\mathcal{M}_{Bulk}$ is dimensionally $[\text{Energy}]^{4-n}$, where n is the total number of scattered particles. As k is the only dimensionfull quantity that is relevant at leading order for the bulk scattering in the regime considered we find:

$$\mathcal{M}_{Bulk} \sim k^{2-m}. \quad (27)$$

Using again the relation between bulk and boundary momenta (19) and inserting this result in the boundary amplitude (24) we get

$$\mathcal{M}_{Boundary} \sim K^{4-\Delta}, \quad (28)$$

where $\Delta = (m+2)d$ is the total scaling dimension of the scattering particles associated with glueballs on the four dimensional boundary. Considering $K \sim \sqrt{s}$ we find the expected QCD scaling behavior [19, 20]

$$\mathcal{M}_{Boundary} \sim s^{2-\Delta/2}. \quad (29)$$

This shows that the bulk/boundary one to one mapping (14), (19) can be used to obtain the hard scattering behavior of high energy glueballs at fixed angles, from a low energy approximation of string theory.

It is interesting to relate the different energy scales used in the above derivation of the scattering amplitudes and check their consistency. The scales we discussed are

$$\mu \ll \frac{1}{R} \ll \frac{1}{\sqrt{\alpha'}}. \quad (30)$$

Note that the relation between the AdS radius R , the number of coincident branes N , the string coupling constant g and scale α' is $R^4 \sim gN(\alpha')^2$. Then the above condition between R and α' corresponds to the 't Hooft limit [16]. Assuming that the dimensionless quantity μR is the parameter that relates the energy scales we find $\sqrt{\alpha'} = \mu R^2$ so that the lightest glueball mass is

$$\mu^2 = \frac{1}{gN\alpha'}. \quad (31)$$

This result is in agreement with Maldacena and Nunez [39]. Further, the above relation between the energy scales together with the condition that $k \gg 1/R$ and the mapping between bulk and boundary momenta (19) imply that

$$\mu \ll K \ll \frac{1}{R} \ll k \ll \frac{1}{\sqrt{\alpha'}}, \quad (32)$$

so that the absolute values of k are greater than those of K , although the boundary scattering is a high energy process (with respect to μ) while the bulk scattering is a low energy process (with respect to $1/\sqrt{\alpha'}$). Furthermore K and k in this regime are inversely proportional showing a kind of infrared-ultraviolet duality as expected from holography [10].

5 Glueball masses

We can use the mapping between bulk and boundary scalar fields of the previous section to estimate the masses of scalar glueballs [40]. In contrast to the previous case now we will not consider just one kind of glueball field. Rather we consider, on the boundary of the AdS slice, a sequence of different massive fields $\Theta_i(\vec{x}, t)$ representing the states of scalar glueballs with masses μ_i . Correspondingly we have a sequence of creation-annihilation operators $\mathbf{b}_i, \mathbf{b}_i^\dagger$ satisfying equation (13) with $w_i(\vec{K}) = \sqrt{\vec{K}^2 + \mu_i^2}$.

Introducing momentum operators $\tilde{\Theta}_i(K_j)$ with momentum K_j in the interval $\mathcal{E}_{j-1} \leq K \leq \mathcal{E}_j$ they would not be mapped in a one to one relation with bulk operators $\tilde{\Phi}(k, u_p)$ unless j is limited, since i and p are unlimited. Then such a mapping is possible if we introduce a restriction on the index j . The simplest choice is to take just one value for j . This is obtained taking $\mathcal{E}_1 \equiv \mathcal{E}$ large enough, which means that now $j = 1$. This recovers the one to one mapping of the previous section and in this case it reads

$$\tilde{\Theta}_i(K) \leftrightarrow \tilde{\Phi}(k, u_i),$$

where we have dropped the index of K_1 since this is the only relevant boundary momentum.

This mapping can be written explicitly in terms of bulk and boundary creation-annihilation operators. We will impose the same relation (14) for each pair $\mathbf{a}_i, \mathbf{b}_i$.

Requiring again that equations (14) preserve the canonical commutation relations (12, 13) one finds that the moduli of the momenta are related by (15) where $0 \leq K \leq \mathcal{E}$ as before.

Now we associate the size z_{max} of the AdS space with the mass of the lightest glueball which we choose to be μ_1

$$z_{max} = \frac{\chi_{2,1}}{\mu_1}, \quad (33)$$

so that from equation (11) we have

$$u_i = \frac{\chi_{2,i}}{\chi_{2,1}} \mu_1. \quad (34)$$

An approximate expression for the mapping (15) can be obtained choosing appropriate energy scales. We take $\mathcal{E}$ to be the string scale $1/\sqrt{\alpha'}$ assuming that $K \ll \mathcal{E}$. On the other side we restrict the momenta K associated with glueballs to be much larger than their masses μ_i . Then we have

$$\mu_i \ll K \ll \frac{1}{\sqrt{\alpha'}}. \quad (35)$$

In this regime the mapping (15) reduces to

$$k \approx \frac{u_i}{2\sqrt{\alpha'}K}. \quad (36)$$

This approximate mapping gives a high energy scaling similar to QCD[36]. Using the conditions (35) together with the above mapping we see that the bulk momenta satisfy

$$u_i \ll k \ll \left(\frac{u_i}{\mu_i}\right) \frac{1}{\sqrt{\alpha'}}. \quad (37)$$

Note that the supergravity approximation holds for $k \ll 1/\sqrt{\alpha'}$. So in order to keep this approximation valid for all glueball operators Θ_i the factor u_i/μ_i should be nearly constant. We then impose that

$$\frac{u_i}{\mu_i} = \text{constant}. \quad (38)$$

So the glueball masses are related to the zeros of the Bessel functions by

$$\frac{\mu_i}{\mu_1} = \frac{\chi_{2,i}}{\chi_{2,1}}. \quad (39)$$

Using the values of these zeros one finds the ratio of the glueball masses for the state 0^{++} and its excitations. We are using the conventional notation for these states with spin zero and positive parity and charge conjugation. In order to compare our results from bulk/boundary holographic mapping with those coming from lattice we adopt the mass of the first state as an input. Our results[40] are in good agreement with lattice[41, 42] and AdS-Schwarzschild black hole supergravity[43]–[49] calculations.

It is interesting to mention that an approach to estimate glueball masses in Yang Mills* from a deformed AdS space was discussed very recently[50].

We can generalize the above results to AdS_{n+1} . In this case massless bulk fields are expanded in terms of the Bessel functions $J_{n/2}$ and the mass ratios for the n dimensional "glueballs" are given in terms of their zeros. In particular for AdS_4 , where one expects to recover results from QCD_3 , we find

$$\frac{\mu_i}{\mu_1} = \frac{\chi_{3/2,i}}{\chi_{3/2,1}}. \quad (40)$$

Using this relation we obtain[40] the ratio of masses again in good agreement with lattice[41, 42] and AdS-Schwarzschild black hole supergravity calculations [43]–[49].

It is interesting to see if the AdS slice considered here can be related to the AdS-Schwarzschild black hole metric proposed by Witten[51]. For the case of QCD_3 the metric reads

$$\begin{aligned} ds^2 &= R^2 \left(\rho^2 - \frac{b^4}{\rho^2} \right)^{-1} d\rho^2 + R^2 \left(\rho^2 - \frac{b^4}{\rho^2} \right) d\tau^2 \\ &+ R^2 \rho^2 (d\vec{x})^2 + R^2 d\Omega_5^2, \end{aligned} \quad (41)$$

where $\rho \geq b$, $R^2 = l_s^2 \sqrt{4\pi g_s N}$ and b is inversely proportional to the compactification radius of S_1 where the τ variable is defined.

If we qualitatively neglect the τ contribution to the metric in the limit of very little compactification radius and then take the limit $\rho \gg b$ this metric is approximated by

$$ds^2 = \frac{R^2}{\rho^2} d\rho^2 + R^2 \rho^2 d\tau^2 + R^2 \rho^2 (d\vec{x})^2 + R^2 d\Omega_5^2. \quad (42)$$

That is an $\text{AdS}_4 \times S^5$ space that takes a form similar to eq. (6) if we change the axial coordinate to $z = 1/\rho$. In Witten's framework one must impose regularity conditions at $\rho = b$ because of the presence of the horizon at this position. In our approximation in order to retain this physical condition we impose boundary conditions there and associate it to the cut of our slice ($b = 1/z_{\max}$). This AdS_4 slice is the one used to estimate the glueball mass ratios related to the three dimensional gauge theory (40). So we can think of our AdS_4 slice as a naive approximation to Witten's proposal.

An analogous situation could also be considered for the Witten's proposal to QCD_4 . In that case the situation is more involved because of the form of the metric coming from the compactification of $\text{AdS}_7 \times S^4$.

In conclusion we have seen that the bulk/boundary holographic mapping which reproduces the high energy scaling of QCD like theories can also be applied to estimate glueball mass ratios. We hope that this mapping can be used to describe other particle states that may be related to some properties of QCD.

It is important to remark that one can obtain a similar result for the ratio of the glueball masses considering other mappings between bulk and boundary creation-annihilation operators instead of eq. (14). For example one could take $\mathbf{a}_i = \mathbf{b}_i$. This would contain the solution $k = K$ implying that the masses of the glueballs are identically equal to the values of axial bulk momenta u_i . However such a trivial mapping does not seem to reproduce the high energy QCD scaling.

Let us mention that we used a solution for the dilaton field corresponding to Dirichlet boundary conditions at $z = 0$ and $z = z_{\max}$. This allows the existence of Bessel functions but not the divergent Neumann solutions. Other boundary conditions can also be considered in the same context.

We have also obtained these mass ratios for scalar glueballs starting with the same AdS slice as discussed here without using the holographic mapping of ref [36] but assuming the stronger condition of relating directly the dilaton modes with the glueball masses [52]. The consistency between these results seems to indicate that the holographic mapping found before may indeed be valid within the approximations and the energy region considered.

Acknowledgments

The authors are partially supported by CNPq, FINEP, CAPES (Procad program), FAPERJ - Brazilian research agencies and Fundação José Pelúcio Ferreira (CCMN-UFRJ).

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DEPARTMENT PHYSICS

Review of AdS/CFT Correspondence

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September 25, 2020

SUBMITTED IN PARTIAL FULFILMENT OF THE REQUIREMENTS FOR THE DEGREE
OF MASTER OF SCIENCE OF IMPERIAL COLLEGE LONDON

Abstract

Here I give a basic introduction on the AdS/CFT correspondence by considering a specific case of the duality between $\mathcal{N} = 4$ super-Yang Mills in 3+1 spacetime dimensions and type IIB superstring theory on $AdS_5 \times S^5$. The duality is considered in its weak form - low energy limit where the string theory is approximated by supergravity. I explain the necessary prerequisites to arrive at the conjectured duality, which includes an introduction on conformal field theory, supersymmetry, AdS spacetime and string theory. The duality is motivated by considering D-branes and their decoupled effective theories. The symmetries and representations are matched on both sides of proposed correspondence and a two point correlation function on the field theory side is calculated using supergravity partition function. The results are compared to those calculated from conformal field theory side.

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Chapter 1

Introduction

AdS/CFT is an exciting new theory that relates two major parts of theoretical physics, which are quantum field theories and gravitational theories (usually string theories). On one hand we have gravity theories that in certain limits reduce to classical theory of relativity. On the other hand we have a quantum theory or should I say quantum framework which, by its nature is entirely different. The notions of classical and quantum theory has always been well separated and who could have thought that these seemingly unrelated theories have a very intimate connection. The AdS/CFT correspondence led to huge popularity ¹ in the physics community which in turn led to its applications in different areas of physics, such as strong interactions and condensed matter physics. The reason I find it so exciting is because it gives new insights into a well defined theory of quantum gravity, one of the biggest mysteries of modern theoretical physics.

AdS/CFT correspondance states that a gravity theory on anti-de Sitter spacetime is dual to a conformal field theory [27]. Generally the gravity theories are string theories and CFT's are gauge theories but this is not necessarily true for all cases [15]. Which is why the theory is also referred to as gauge/gravity duality. The main idea of the duality is that AdS spacetime in $d + 1$ dimensions has the same symmetry group as a conformal field theory in d -dimensions. So the CFT "lives" on the boundary of AdS spacetime. The correspondence was first conjectured by Juan Maldacena in 1997 [19] when he looked at different perspectives on D-branes in type IIB superstring theory at low energies. But even before that in 1974 Gerard 't Hooft has noticed a few hints that there exists similarities between quantum field theory and string theory calculations when he considered QCD theories with arbitrary number of colour charges [13]. Another clue for the motivation of the duality was the paper by the same author in 1993 where he expanded on the work of Stephen Hawking on black hole thermodynamics. 't Hooft argued that the degrees of freedom of our world on Planck scale can be described on two dimensional lattice evolving in time [35]. This was motivated by the fact that entropy of a black hole is proportional to the surface

$$S = \frac{A_d}{4G} \tag{1.1}$$

where A_d is the area in Plank units and G is the Newtons gravitational constant. This idea was developed later by Leonard Susskind and is now known as the holographic principle [33], which states that the information in a $d + 1$ dimensional volume is encoded onto a d dimensional area of the surface of the volume. So it's not just about one paper, physicists have been seeing evidence of this duality for a long time but I think Maldacena was the one who made it concrete since he gave a specific example².

The useful aspect of the correspondence is that one can substitute often difficult calculations in one theory, with easier calculations in the theory on the other side of the duality. To compute entropy in a strongly coupled field theory is a very hard computation. However, if this is dual to a gravitational theory, the calculations becomes much more straightforward, since the calculations reduces to computing the entropy of a black hole [21]. A wide area of physics where AdS/CFT correspondence shows much promise is strong coupling theories. There is not one way of dealing with strongly coupled physical systems and perturbation theory works only for weakly coupled ones. usually when a field theory is strongly coupled the gravitational theory is classical and

¹Maldacena's original paper on AdS/CFT correspondence was cited almost 12,000 times in 2015.

²This was a proposed duality between string theory on $AdS_5 \times S^5$ and $\mathcal{N} = 4$ super-Yang-Mills theory in four dimensions.

weakly coupled. Hence, using a dual theory to do calculations brings interesting insights into theories with strong coupling [22]. A considerable amount of studies have been carried out on the applications for quantum chromodynamics and particularly for quark-gluon plasma [16]. Strong coupling is also a familiar feature in many interesting systems in condensed matter physics and it is subject to the use of duality principle as well [30]. A standard example is a study of quantum phase transitions [10], which are a consequence of quantum fluctuations rather than changes in temperature. Generally we get these transitions when varying the coupling and in many cases it is not possible to use perturbation theory.

Despite the popularity and many uses of the correspondence, one might be surprised that there is no formal proof of the conjecture. The reason for this is the same reason why this theory is so popular. Mainly for strong coupling. We do not have a method of dealing with these systems. For interacting theories we can only describe them using perturbation theory, but this is valid only for small couplings. We use the correspondence to gain new insights on how we could deal with large couplings, but to actually prove that the conjecture is true we would have to understand theories on both sides. Which we do not. Hence, until that is the case it is not likely we will have a definitive proof anytime soon [23]. Regardless, the conjecture is believed to be true since many non-trivial checks have been made. By tests I mean the computation and comparison of observables, such as correlation functions on both sides of the correspondence [8].

In this thesis I aim to give a general introduction on the AdS/CFT by considering the original example of Maldacena between superstring theory on $AdS_5 \times S^5$ and $\mathcal{N} = 4$ super-Yang-Mills. In first chapter I give a basic introduction of conformal field theory, starting with Poincaré symmetry group and extending it to include conformal transformations. At the end of this part I explain how to calculate the correlation functions which are later used for comparison of the ones calculated by a dual theory. In the succeeding chapter I expand the Poincaré symmetry to include fermionic generators and introduce supersymmetry. Then I connect the two extensions of the Poincaré into a superconformal group. Chapter 4 is a very brief overview of anti-de Sitter spacetime followed by a chapter on string theory. This was the part of the thesis I spent most time on, since the topic is very broad and condensing it into a few pages was a challenging task. I started by introducing bosonic string theory then I introduced supersymmetry into the picture and considered its low-energy effective action - supergravity. At the end I gave an overview of D-branes which formed a large part of the discussion of chapter 6. This was a chapter on the correspondence itself and the motivation for it, which stems from two different perspectives on D-branes. I conclude this chapter by talking about the map between operators on the CFT side and fields on AdS and checking that the form of the correlation functions matches on both sides using the results from the chapter on CFT.

Writing this dissertation I mostly followed the book by M. Ammon and J. Erdmenger [1]. This was supplemented by many other reviews such as [22] and [17] for a more basic introduction on the subject together with [8] for a more detailed discussion. The online lecture recordings on CFT by Tobias Osbourne was helpful for the first part of section 2. For the introduction on supersymmetry I mostly followed [3] together with [18]. Section on string theory was mainly complimented by [36]. In general I tried giving a coherent and self contained introduction of the original example of AdS/CFT. Although skipping some (important) parts was inevitable as I felt that the condensed version of particular aspects would only contribute to the confusion rather than clarity of an inexperienced reader.

Chapter 2

Conformal field theory

I think that theoretical physics is mainly based by a reduction principle - trying to look at the problems at smaller and smaller scales. But what happens when a theory is exactly the same on all scales? This is what we are about to explore in this chapter.

Conformal field theory is a field theory that is invariant under scale transformations and something called special conformal transformations in addition to the plain vanilla Poincaré symmetries. It has gained popularity around 1984 due to its role in string theory [11]. It is also an important part of statistical physics for solving critical phenomenon [6] and is a very useful tool to study interactions in field theory. In two-dimensions some of CFT's are solvable exactly. [28]. The recent increase in interest of CFT's is due to the main topic of this thesis - AdS/CFT correspondence.

In this chapter I will give a basic outline of conformal field theory and since it is a very broad field and I will only focus on the main points that are crucial for the AdS/CFT correspondence. For the majority of this section I followed Chapters 3.1 and 3.2 of [1].

Let us start with a review of the Poincaré group of spacetime symmetries and work our way towards expanding it to include conformal transformations. Then we will look at calculating some correlation functions of the theory.

2.1 Poincaré algebra

Poincaré group $ISO(d-1, 1)$ is the group of spacetime symmetries that transforms the coordinates by

$$x^\mu \rightarrow \Lambda^\mu{}_\nu x^\nu + a^\mu \quad (2.1)$$

where $\Lambda^\mu{}_\nu$ is an element of Lorentz group of rotations and boosts satisfying

$$\Lambda^\mu{}_\rho \Lambda^\nu{}_\sigma \eta_{\mu\nu} = \eta_{\rho\sigma} \quad (2.2)$$

and a^μ is an element of translation subgroup. An infinitesimal Lorentz transformation can be expressed in terms of the generators as $\Lambda^\mu{}_\nu \approx \delta^\mu_\nu - \frac{i}{2} \omega_{\alpha\beta} (J^{\alpha\beta})^\mu{}_\nu$ where $\omega_{\alpha\beta}$ are infinitesimal and $(J^{\alpha\beta})^\mu{}_\nu$ are the elements of the Lie algebra $\mathfrak{o}(d-1, 1)$ that generates the Lorentz group $O(d-1, 1)$. We may take the explicit representation of the generators to be

$$(J^{\alpha\beta})^\mu{}_\nu = i(\eta^{\mu\alpha} \delta^\beta_\nu - \eta^{\mu\beta} \delta^\alpha_\nu) \quad (2.3)$$

which satisfy the commutation relation

$$[J^{\alpha\beta}, J^{\rho\sigma}]^\mu{}_\nu = -i(\eta^{\rho\alpha} J^{\beta\sigma} - \eta^{\beta\rho} J^{\alpha\sigma} - \eta^{\alpha\sigma} J^{\beta\rho} + \eta^{\beta\sigma} J^{\alpha\rho})^\mu{}_\nu. \quad (2.4)$$

The expression above defines the Lie algebra of the Lorentz group [40]. We can extend this group by including the generators of translations $\mathfrak{t} \subset \mathfrak{iso}(d-1, 1)$ which would define the full group of Minkowski isometries. In field theory the generators of Poincaré group are realised as differential operators acting on field space. Let us derive this representation for a scalar field $\phi(x)$. First consider infinitesimal Lorentz transformation which in the active picture can be expressed as

$$\phi(x) \rightarrow \phi'(x) = \phi(\Lambda^{-1}x) = \phi(x) - \frac{i}{2} \omega_{\alpha\beta} (J^{\alpha\beta})^\mu{}_\nu x^\nu \partial_\mu \phi(x) + \dots \quad (2.5)$$

Plugging in (2.3) into (2.5) and defining the differential operator as $(J^{\alpha\beta})^\mu{}_\nu x^\nu \partial_\mu \equiv J^{\alpha\beta}$ we can show that on a scalar the Lorentz transformation is generated by

$$J^{\alpha\beta} = i(x^\alpha \partial^\beta - x^\beta \partial^\alpha). \quad (2.6)$$

Now consider an infinitesimal translation ϵ^μ . The field transforms in the active picture as

$$\phi(x) \rightarrow \phi'(x) = \phi(x - \epsilon) = \phi(x) - \epsilon^\mu \partial_\mu \phi(x). \quad (2.7)$$

So we can define the translation generator to be $P_\mu = i\partial_\mu$. Together with $J^{\alpha\beta}$ they generate the Poincaré group. We can see that these indeed satisfy the Poincaré algebra

$$[J^{\alpha\beta}, J^{\rho\sigma}]^\mu{}_\nu = -i(\eta^\rho J^{\beta\sigma} - \eta^{\beta\rho} J^{\alpha\sigma} - \eta^{\alpha\sigma} J^{\beta\rho} + \eta^{\beta\sigma} J^{\alpha\rho})^\mu{}_\nu, \quad (2.8)$$

$$[P_\mu, P_\nu] = 0, \quad (2.9)$$

$$[P^\mu, J^{\alpha\beta}] = i(\eta^{\mu\alpha} P^\beta - \eta^{\mu\beta} P^\alpha). \quad (2.10)$$

2.2 Conformal algebra

We can extend the Poincaré group to include two more generators of conformal transformations that changes the distances between points but preserve angles [12]. Under these transformations the metric transforms as

$$g_{\mu\nu}(x) \rightarrow \frac{\partial x'^\rho}{\partial x^\mu} \frac{\partial x'^\sigma}{\partial x^\nu} g_{\rho\sigma}(x) = \Omega(x) g_{\mu\nu}(x). \quad (2.11)$$

What transformations can we do such that the variation of the metric is proportional to itself? Well, consider a general infinitesimal transformation

$$x^\mu \rightarrow x^\mu + \epsilon(x)^\mu. \quad (2.12)$$

Then we can calculate the variation in the metric

$$\delta g(x)_{\mu\nu} = \partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu \quad (2.13)$$

and demand that this must be proportional to the metric $\delta g_{\mu\nu}(x) \propto g_{\mu\nu}(x)$. The result is an equation for $\epsilon^\mu(x)$

$$\partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu = g_{\mu\nu}(x) \frac{2}{d} \partial^\rho \epsilon_\rho \quad (2.14)$$

$$\implies (g_{\mu\nu}(x) \partial_\rho \partial^\rho + (d-2) \partial_\mu \partial_\nu) \partial^\sigma \epsilon_\sigma = 0. \quad (2.15)$$

We can see that there is something special when the number of dimensions $d = 2$. Actually in this case the equations simplify to Cauchy-Riemann equations and it is an interesting case [1]. However, I will not discuss this case further and focus only on $d > 2$. The most general form of the infinitesimal transformation can be written as

$$\epsilon_\mu = a_\mu + \omega_{\mu\nu} x^\nu + \lambda x_\mu + b_\mu x^2 - 2(b \cdot x) x_\mu. \quad (2.16)$$

The first two terms in the expression corresponds to translations and Lorentz transformations and the generators of these make up the familiar Poincaré group that we all know and love. But the rest belong to conformal group which is an extension of the Poincaré. Lets first look at

$$x^\mu \rightarrow x^\mu + \lambda x^\mu. \quad (2.17)$$

This corresponds to **dilation** and is related to **scale** symmetry. A scalar transforms as

$$\phi(x) \rightarrow \phi'(x) = \phi(x - \lambda x) = \phi(x) - \lambda x^\mu \partial_\mu \phi(x). \quad (2.18)$$

The generator of this transformation is $D = ix^\mu \partial_\mu$. The second form of infinitesimal transformation is called the **special conformal transformation**

$$x^\mu \rightarrow x^\mu + b^\mu x^2 - 2(b \cdot x) x_\mu. \quad (2.19)$$

Conformal transformations			
Transformation	Infinitesimal	Finite	Generator
Translation	$x^\mu + a^\mu$	$x^\mu + a^\mu$	$P_\mu = i\partial_\mu$
Lorentz	$x^\mu + \omega_{\mu\nu}x^\nu$	$\Lambda^\mu{}_\nu x^\nu$	$J^{\alpha\beta} = i(x^\alpha\partial^\beta - x^\beta\partial^\alpha)$
Dilation	$x^\mu + \lambda x^\mu$	λx^μ	$D = ix^\mu\partial_\mu$
Special	$x^\mu + b^\mu x^2 - 2(b \cdot x)x_\mu$	$\frac{x^\mu - x^2 b^\mu}{1 - 2(b \cdot x) + b^2 x^2}$	$K_\mu = i(x^2\partial_\mu - x_\mu x^\nu\partial_\nu)$

Table 2.1: Summary of conformal transformations [26]

Similarly a scalar transforms under this transformation as

$$\phi(x) \rightarrow \phi'(x) = \phi(x - bx^2 + 2(b \cdot x)x) = \phi(x) - b^\mu(x^2\partial_\mu - x_\mu x^\nu\partial_\nu)\phi(x) \quad (2.20)$$

So the generator of this transformation is $K_\mu = i(x^2\partial_\mu - x_\mu x^\nu\partial_\nu)$. The summary of these transformations is given in table 2.1. Together with Poincaré, these generators satisfy the **conformal algebra**

$$\begin{aligned}
[J_{\mu\nu}, K_\rho] &= i(\eta_{\mu\rho}K_\nu - \eta_{\nu\rho}K_\mu) \\
[D, P_\mu] &= iP_\mu \\
[K_\mu, K_\nu] &= 0 \\
[D, K_\mu] &= -iK_\mu \\
[K_\mu, P_\nu] &= 2i(\eta_{\mu\nu}D - J_{\mu\nu}) \\
[D, J_{\mu\nu}] &= 0
\end{aligned} \quad (2.21)$$

In d spacetime dimensions this group has $(d+2)(d+1)/2$ generators, which is exactly the same number of generators of the group $SO(d, 2)$. In fact if we combine these generators in the following way:

$$\begin{aligned}
\bar{J}_{\mu\nu} &= J_{\mu\nu} \\
\bar{J}_{\mu d} &= \frac{1}{2}(K_\mu - P_\mu) \\
\bar{J}_{\mu(d+1)} &= \frac{1}{2}(K_\mu + P_\mu) \\
\bar{J}_{(d+1)d} &= D
\end{aligned} \quad (2.22)$$

then we see that these are in fact the generators of the Lie group $SO(d, 2)$ [1] satisfying

$$[\bar{J}_{AB}, \bar{J}_{CD}] = i(\eta_{BC}\bar{J}_{AD} - \eta_{AC}\bar{J}_{BD} - \eta_{BD}\bar{J}_{AC} + \eta_{AD}\bar{J}_{BC}). \quad (2.23)$$

This group leaves the metric $\eta_{AB} = \text{diag}(- + \dots + -)$ invariant. Where $\mu, \nu = 0, 1, \dots, d-1$ and $A, B = 0, 1, \dots, d+1$.

2.3 Conformal action on fields

In the previous section we figured out the algebra of the conformal group by considering the action on scalar fields. The algebra does not depend on a particular representation so let us also consider fields that transform non trivially under Lorentz transformations [1]

$$\Phi(x) \rightarrow e^{-\frac{i}{2}\omega_{\mu\nu}\mathcal{J}^{\mu\nu}}\Phi(\Lambda^{-1}x) \quad (2.24)$$

where $\mathcal{J}^{\mu\nu}$ forms a representation of the Lorentz algebra and is related to the spin of the field. The point $x = 0$ is invariant under these transformations, so the variation at the origin is

$$\delta\Phi(0) = -\frac{i}{2}\omega_{\mu\nu}\mathcal{J}^{\mu\nu}\Phi(0). \quad (2.25)$$

In quantum field theory the action of the group on fields is determined by the commutator since the quantum field transforms as

$$\Phi(0) \rightarrow e^{\frac{-i}{2}\omega_{\mu\nu}\hat{J}^{\mu\nu}}\Phi(0)e^{\frac{i}{2}\omega_{\mu\nu}\hat{J}^{\mu\nu}} = \Phi(0) - \frac{i}{2}\omega_{\mu\nu}[\hat{J}^{\mu\nu}, \Phi(0)] + \dots \quad (2.26)$$

Where $\hat{J}^{\mu\nu}$ is now an operator on Hilbert space. So we see that this field satisfies

$$[\hat{J}^{\mu\nu}, \Phi(0)] = \mathcal{J}^{\mu\nu} \Phi(0). \quad (2.27)$$

Then we can use the translation operator $e^{-i\hat{P}^\mu x_\mu}$ to translate the field to a general point x . This results in the following commutator

$$[\hat{J}_{\mu\nu}, \Phi(x)] = i(x_\mu \partial_\nu - x_\nu \partial_\mu) \Phi(x) + \mathcal{J}_{\mu\nu} \Phi(x). \quad (2.28)$$

This method of obtaining transformations is called **induced representations** [1]. Following a similar logic we can find the action of the conformal group on a general field $\Phi(x)$. Then (omitting the hats on the operators)

$$\begin{aligned} [P_\mu, \Phi(x)] &= i\partial_\mu \Phi(x) \\ [D, \Phi(x)] &= -i\Delta \Phi(x) - ix^\mu \partial_\mu \Phi(x) \\ [J_{\mu\nu}, \Phi(x)] &= i(x_\mu \partial_\nu - x_\nu \partial_\mu) \Phi(x) + \mathcal{J}_{\mu\nu} \Phi(x) \\ [K_\mu, \Phi(x)] &= i(x^2 \partial_\mu - 2x_\mu x^\nu \partial_\nu + 2x_\mu \Delta) \Phi(x) - 2x^\nu \mathcal{J}_{\mu\nu} \Phi(x) \end{aligned} \quad (2.29)$$

Notice that the commutator $[D, \Phi(x)]$ implies that the field transforms non-trivially under dilations. Specifically, the field transforms as

$$\Phi(x) \rightarrow \Phi'(x) = \lambda^\Delta \Phi(\lambda x) \quad (2.30)$$

where Δ is called the **scaling dimension**. And the field $\Phi(x)$ is an eigenstate of this operator with eigenvalue $-i\Delta$. We can also see from the conformal algebra that the operator P_μ raises the scaling dimension by 1 and the operator K_μ lowers the scaling dimension by 1. We can see this from the Jacobi identity. For example

$$\begin{aligned} [D, [P_\mu, \Phi(0)]] + [\Phi(0), [D, P_\mu]] + [P_\mu, [\Phi(0), D]] &= 0 \\ \implies [D, [P_\mu, \Phi(0)]] &= -i(\Delta + 1)[P_\mu, \Phi(0)]. \end{aligned} \quad (2.31)$$

We define a special kind of field called the conformal **primary field**. It is an eigenfunction of D and has the lower bound on Δ , i.e. it is annihilated by K_μ

$$[K_\mu, \mathcal{O}] = 0, \quad [D, \mathcal{O}] = -i\Delta \mathcal{O}. \quad (2.32)$$

A general field at the origin $\Phi(0)$ is primary since it satisfy $[K_\mu, \Phi(0)] = 0$ and $[D, \Phi(0)] = -i\Delta \Phi(0)$. All the other fields are obtained by the action of P_μ on primary fields and they are called **descendant fields** [1]. Consider a primary field with dimension Δ and spin j . Other spin states are obtain by the action of $J^{\mu\nu}$, which fills the spin representation space. Then acting with P_μ would give us an operator with dimension $\Delta + 1$ but this descendant state would be of different spin since it would carry another Lorentz index. Then the other spin states are again obtain by the action of $J^{\mu\nu}$. Hence, the primary operators define the full representation of the algebra which is classified by Δ and j .

2.4 Energy-momentum tensor

According to Noethers theorem we get a conserved current for every continuous symmetry. For spacetime translations the current is the **energy-momentum** tensor $T_{\mu\nu}$. We can use a slightly different definition of the energy momentum tensor which can be written as the functional derivative of the action with respect to metric

$$T^{\mu\nu} = -\frac{2}{\sqrt{-g}} \frac{\delta S}{\delta g^{\mu\nu}}. \quad (2.33)$$

This definition leads to same results, however it has a few advantages such as it is automatically symmetric, it can easily be extended to curved spacetimes and for gauge theories it automatically is gauge invariant [36]. Scale symmetry puts an additional constraint on the tensor. To see this consider the variation of the action

$$\delta S = \int d^d x T^{\mu\nu} \delta g_{\mu\nu}. \quad (2.34)$$

Now recall that scale symmetry transformations leaves the metric proportional to itself, hence

$$\delta S \propto \int d^d x T^\mu{}_\mu. \quad (2.35)$$

But the variation of the action must vanish. So we have that the energy-momentum tensor must be traceless $T^\mu{}_\mu = 0$. We would arrive to the same result by considering the conserved currents for the dilation which is given by [1]

$$J_{(D)\mu} = x^\nu T_{\mu\nu}. \quad (2.36)$$

Then since it is conserved it would lead to the same result

$$\partial^\nu J_{(D)\mu} = \partial^\nu (x^\rho T_{\mu\rho}) = (\partial^\nu x^\rho) T_{\mu\rho} + x^\rho (\partial^\nu T_{\mu\rho}) = T^\rho{}_\rho = 0. \quad (2.37)$$

2.5 Correlation functions

Important objects in CFT are correlation functions and they will be relevant for us when checking if the AdS/CFT correspondence holds. Naturally conformal symmetry puts restrictions on the form of the correlators. Consider a two-point correlation function of scalar fields $\langle 0 | \phi_1(x) \phi_2(y) | 0 \rangle$ with definite scaling dimensions Δ_1 and Δ_2 . Then under dilation the correlator transforms as

$$\langle 0 | \phi(x) \phi(y) | 0 \rangle \rightarrow \lambda^{\Delta_1} \lambda^{\Delta_2} \langle 0 | \phi(\lambda x) \phi(\lambda y) | 0 \rangle. \quad (2.38)$$

Scalar fields are functions of spacetime coordinates and the fact that the correlator is invariant under translations implies that it only depends on the difference of the two coordinate points $x^\mu - y^\mu$. Also the correlator should be Lorentz invariant so it must be a function of $(x - y)^2$ [25]

$$\langle 0 | \phi_1(x) \phi_2(y) | 0 \rangle = f((x - y)^2). \quad (2.39)$$

By translating to the origin

$$\langle 0 | \phi_1(x) \phi_2(0) | 0 \rangle = f(x^2). \quad (2.40)$$

Another important fact is that the vacuum states are invariant under conformal symmetry [12], which means that the generators of conformal algebra annihilates the vacuum state $|0\rangle$. Then we can use the conformal algebra to see the restriction on the two point correlator

$$\begin{aligned} \langle 0 | \phi_1(x) \phi_2(0) D | 0 \rangle &= 0 \\ \implies \langle 0 | \phi_1(x) [\phi_2(0), D] | 0 \rangle + \langle 0 | \phi_1(x) D \phi_2(0) | 0 \rangle &= 0 \\ \implies \phi_1(x) [\phi_2(0), D] | 0 \rangle + \langle 0 | [\phi_1(x), D] \phi_2(0) | 0 \rangle &= 0 \\ \implies i(\Delta_1 + \Delta_2 + x^\mu \partial_\mu) f(x^2) &= 0. \end{aligned} \quad (2.41)$$

This differential equation has a solution of the form

$$f(x^2) = \frac{C}{x^{\Delta_1 + \Delta_2}} \quad (2.42)$$

where C is an integration constant. Or more generally

$$\langle 0 | \phi_a(x) \phi_b(y) | 0 \rangle = f((x - y)^2) = \frac{C_{ab}}{(x - y)^{\Delta_a + \Delta_b}} \quad (2.43)$$

Where C_{ab} is symmetric. If the two fields are primary, then C_{ab} vanishes unless $\Delta_a = \Delta_b$. Similarly one would find a three-point correlator to be

$$\langle 0 | \phi_a(x) \phi_b(y) \phi_c(z) | 0 \rangle = \frac{C_{abc}}{(x - y)^{\Delta_a + \Delta_b - \Delta_c} (x - z)^{\Delta_a + \Delta_c - \Delta_b} (y - z)^{\Delta_b + \Delta_c - \Delta_a}} \quad (2.44)$$

The two and three-point correlation functions are completely determined (up to an integration constant) by conformal algebra. The four and more point functions are less constrained due to crossing ratios [1].

Chapter 3

Supersymmetry

The beginning of supersymmetry can be traced back to 1966 when Hiranaro Miyazawa tried relating baryons and mesons [20]. This was based on internal symmetries and the idea was largely ignored. It was later considered in quantum field theory by Volkov and Akulov [38] and was further developed by Julius Wess and Bruno Zumino in 1974 [39] where they considered spacetime supersymmetries and the possible applications in particle physics. In simple words supersymmetry is a connection between bosonic and fermionic degrees of freedom. The supersymmetry generators Q act on quantum states by turning a boson into a fermion and vice versa

$$Q |\text{fermion}\rangle = |\text{boson}\rangle, \quad Q |\text{boson}\rangle = |\text{fermion}\rangle. \quad (3.1)$$

This is another non-trivial expansion of the Poincaré algebra, which bypasses the Coleman-Mandula¹ theorem by considering graded Lie algebra with fermionic generators. Although there is a bit of scepticism lurking around supersymmetry and whether it has something to do with reality at all, it solves few of the great mysteries in physics. Some of these are the following [3]:

- *Hierarchy problem.* Generally this is a problem with the scale of some parameters of a theory. Where there is a huge discrepancy between the effective value measured by the experiments and predicted value by the theory. Such as the requirement for fine-tuning for Higgs field to account for quadratic radiative contributions at high energies. In supersymmetry these quadratic contributions gets cancelled by the corrections of the supersymmetric particle.
- *Gauge coupling unification.* Grand unification theorems state that the three gauge couplings of the standard model is expected to meet at some higher energy scale. If one only considers the standard model as it stands now the three coupling only meet approximately. However if one considers a supersymmetric extension of the standard model, the gauge couplings meet exactly.
- *Dark matter.* Supersymmetry provides good candidates for dark matter.

These and many more reasons motivate people to study supersymmetry. If supersymmetry is correct it must be broken by the vacuum, which is why we do not observe it in the energy scales accessible to us. However, there is no indication on which energy scale the supersymmetric particles should exist and whether we can expect to confirm them experimentally in the future.

In this chapter I first introduce the supersymmetry algebra and supersymmetric field theory. Then I define the $\mathcal{N} = 4$ super-Yang-Mills theory, which is a supersymmetric version of Yang-Mills theory. I end the discussion by including conformal invariance and defining superconformal group. Writing this chapter I mostly followed [3] together with chapter 3.4 of [1]. A reader not familiar with supersymmetry and spinor notation is advised to consult Appendix A.

3.1 Supersymmetry algebra

In the previous chapter we gave a review of the Poincaré symmetry and then we extended it to include scaling symmetry. Now we will introduce another non-trivial extension of the Poincaré group

¹Coleman-Mandula theorem states the only possible symmetries of quantum field theory S-matrix is a combination of Poincaré and internal symmetries. By relaxing some assumptions we are able to add non-trivial extensions to the allowed symmetry transformations. In supersymmetry case we consider fermionic generators.

by considering graded Lie algebra and fermionic generators. Together with Poincaré generators these make up the Supersymmetry algebra.

A **graded Lie algebra** of grade 1

$$L = L_0 \oplus L_1 \quad (3.2)$$

is defined to be a vector space just as an ordinary Lie algebra and it has the product defined by

$$[\quad, \quad] : L \times L \rightarrow L \quad (3.3)$$

whether the product is a commutator or an anticommutator depends on the number of bosonic and fermionic generators. For operators O_a we would have

$$[O_a, O_b] = O_a O_b - (-1)^{\eta_a \eta_b} O_b O_a \quad (3.4)$$

where $\eta_a = 1$ if it is a fermionic generator and $\eta_a = 0$ if it is bosonic. In supersymmetry L_0 is the Poincaré algebra and L_1 consists of a set of fermionic generators Q_α^I and $\bar{Q}_{\dot{\alpha}}^I$ where $I = 1, \dots, \mathcal{N}$ is the number of generators. Together with the Poincaré generators they have the following commutation relations [3]

$$\begin{aligned} \{Q_\alpha^I, \bar{Q}_{\dot{\alpha}}^J\} &= 2\sigma_{\alpha\dot{\alpha}}^\mu P_\mu \delta^{IJ}, \quad \{Q_\alpha^I, Q_\beta^J\} = \epsilon_{\alpha\beta} Z^{IJ} \\ \{\bar{Q}_{\dot{\alpha}}^I, \bar{Q}_{\dot{\beta}}^J\} &= \epsilon_{\dot{\alpha}\dot{\beta}} (Z^{IJ})^*, \quad [P_\mu, Q_\alpha^I] = 0 \\ [P_\mu, \bar{Q}_{\dot{\alpha}}^I] &= 0, \quad [J_{\mu\nu}, Q_\alpha^I] = i(\sigma_{\mu\nu})_\alpha{}^\beta Q_\beta^I \\ [J_{\mu\nu}, \bar{Q}_{\dot{\alpha}}^I] &= i(\bar{\sigma}_{\mu\nu})^{\dot{\alpha}}{}_{\dot{\beta}} \bar{Q}_{\dot{\beta}}^I \end{aligned} \quad (3.5)$$

where Z^{IJ} are called central charges (i.e. they commute with every other generator) and are antisymmetric $Z^{IJ} = -Z^{JI}$. One should mention the significance of the first commutation relation. Two supersymmetry transformations results in a translation, which implies that the theory is independent of the choice of coordinates which is typical for gravitational theories. If one considers local supersymmetry the result would be a theory of gravity called **supergravity** or SUGRA. Together with (2.21) these commutation relations make up the supersymmetry algebra. The values of Z^{IJ} puts constraints on an additional symmetry that leaves the algebra invariant called the **R-symmetry** which transforms the fermionic generators as

$$Q_\alpha^I \rightarrow R^I{}_J Q_\alpha^J, \quad \bar{Q}_{\dot{\alpha}}^I \rightarrow \bar{Q}_{\dot{\alpha}}^J R^{\dagger J}{}_I. \quad (3.6)$$

If the central charges vanish then this symmetry group is $U(\mathcal{N})$. In other case the symmetry is a subgroup of $U(\mathcal{N})$.

3.2 Representations of supersymmetry algebra

The irreducible representations of the Poincaré group are what we call particles. The two operators that commute with all the generators of the algebra (a.k.a **Casimir operators**) are $P_\mu P^\mu$ and $W_\mu W^\mu$ where $W^\mu = 1/2\epsilon^{\mu\nu\rho\sigma} P_\nu J_{\rho\sigma}$. Hence the massive particles can be labelled by their mass and spin and massless by their energy and helicity.

Superparticles are irreducible representations of supersymmetry algebra where Poincaré is just a subalgebra. Here $P_\mu P^\mu$ is still a Casimir and superparticles have definite mass. However, particles within the same multiplet have different spin so $W_\mu W^\mu$ is no longer suitable to label representations.

A very important note that I would like to make is the language commonly used by physicists. Technically speaking a representation ρ of a group G is a homomorphism between an abstract group element and a general linear operator. i.e. $\rho : G \rightarrow GL(n, \mathbb{C})$. However, physicists are often sloppy and use imprecise language to describe states or fields being representations. They are not! These objects belong to a representation space or a module that the representation acts on. This is often a source of confusion in representation theory and I felt that it is worth pointing this out.

3.2.1 Massless representations

For massless representations it turns out that the central charges Z^{IJ} vanish [18]. So this slightly simplifies our commutation relations (3.5). Our goal is to figure out how the supersymmetry operators are realized and how they act on a representation space. So let's start by taking a particular frame in which a massless particle has a momentum $P_\mu = (E, 0, 0, E)$, then

$$\{Q_\alpha^I, \bar{Q}_{\dot{\alpha}}^J\} = 2\sigma_{\alpha\dot{\alpha}}^\mu P_\mu \delta^{IJ} = \begin{pmatrix} 0 & 0 \\ 0 & 4E \end{pmatrix}_{\alpha\dot{\alpha}} \delta^{IJ}. \quad (3.7)$$

Looking at the first row and column entry of the matrix above and using the supersymmetry algebra we can write the following expression for generators Q_1^I and $\bar{Q}_1^I$

$$\langle \phi | \{Q_1^I, \bar{Q}_1^J\} | \phi \rangle = 0. \quad (3.8)$$

But this is just a sum of two norms $\|Q_1^I | \phi \rangle\|^2$ and $\|\bar{Q}_1^J | \phi \rangle\|^2$. We know these are positive definite, hence

$$Q_1^I = \bar{Q}_1^I = 0. \quad (3.9)$$

Now from (3.7) we see that Q_2^I and $\bar{Q}_2^I$ generators must satisfy

$$\{Q_2^I, \bar{Q}_2^J\} = 4E\delta^{IJ}, \quad \{Q_2^I, Q_2^J\} = 0, \quad \{\bar{Q}_2^I, \bar{Q}_2^J\} = 0. \quad (3.10)$$

These look like our familiar creation and annihilation operators except for the factor of $4E$. So we can define

$$a_I \equiv \frac{1}{\sqrt{4E}} Q_2^I, \quad a_I^\dagger \equiv \frac{1}{\sqrt{4E}} \bar{Q}_2^I \quad (3.11)$$

to get rid of the unnecessary factor and then get the commutation relations for creation and annihilation operators

$$\{a_I, a_J^\dagger\} = \delta^{IJ}, \quad \{a_I, a_J\} = 0, \quad \{a_I^\dagger, a_J^\dagger\} = 0. \quad (3.12)$$

Now to build up the representation space we must define some sort of a vacuum state that is annihilated by the annihilation operator (commonly referred to as the **Clifford vacuum**) and then act on it with the creation operator to build up the rest of the spectrum. But what do these states represent? Well let's look at the following commutators

$$[J_{12}, a_I] = -\frac{1}{2}a_I, \quad [J_{12}, a_I^\dagger] = \frac{1}{2}a_I^\dagger. \quad (3.13)$$

So if we have a state $|p^\mu, \lambda\rangle$ of a massless particle with helicity λ , then $a_I |p^\mu, \lambda\rangle$ lowers the helicity by $\frac{1}{2}$ since

$$J_{12}(a_I |p^\mu, \lambda\rangle) = (a_I J_{12} + [J_{12}, a_I]) |p^\mu, \lambda\rangle = (\lambda - \frac{1}{2})a_I |p^\mu, \lambda\rangle. \quad (3.14)$$

Similarly $J_{12}(a_I^\dagger |p^\mu, \lambda\rangle) = (\lambda + \frac{1}{2})a_I^\dagger |p^\mu, \lambda\rangle$. So we see that

$$a_I |p^\mu, \lambda\rangle = |p^\mu, \lambda - 1/2\rangle, \quad a_I^\dagger |p^\mu, \lambda\rangle = |p^\mu, \lambda + 1/2\rangle. \quad (3.15)$$

To get the full representation space let's define a lowest helicity state $|\Omega\rangle$ such that $a_I |\Omega\rangle = 0$, then act on this state with $a_I^\dagger$ to reach maximum helicity. Since these creation and annihilation operators are fermionic $a_I^2 = 0$ the number of states are finite. Also CPT invariance dictates that for every state we must also add a state of opposite helicity $|p^\mu, \pm\lambda\rangle$. For example for $\mathcal{N} = 1$ and a minimum helicity of $\lambda = 0$ we would get a $(0, \frac{1}{2})$ multiplet. We must also add the CPT conjugate $(-\frac{1}{2}, 0)$. Then this multiplet corresponds to one fermionic degree of freedom (Weyl fermion) and one complex scalar. Different multiplets can be constructed by considering different numbers of fermionic generators $\mathcal{N}$ and different minimum helicity states. More examples of different multiplets for $\mathcal{N} = 1$ is given in table 3.1.

Minimum helicity λ	Multiplet	Name
$\lambda = 0$	$(0, +\frac{1}{2}) \oplus^{\text{CPT}} (-\frac{1}{2}, 0)$	Chiral multiplet
$\lambda = \frac{1}{2}$	$(+\frac{1}{2}, +1) \oplus^{\text{CPT}} (-1, -\frac{1}{2})$	Gauge multiplet
$\lambda = 1$	$(+1, +\frac{3}{2}) \oplus^{\text{CPT}} (-\frac{3}{2}, -1)$	Spin 3/2 multiplet
$\lambda = \frac{3}{2}$	$(+\frac{3}{2}, 2) \oplus^{\text{CPT}} (-2, -\frac{3}{2})$	Graviton multiplet

Table 3.1: $\mathcal{N} = 1$ massless supersymmetry multiplets. Each containing two fermionic and two bosonic degrees of freedom.

3.2.2 Massive representations

In the massive case we have more work to do. This is because the central charges Z^{IJ} may not necessarily vanish and secondly because none of the supersymmetry generators vanish. Let us consider a massive particle with mass m and boost to its rest frame in which the momentum is $P^\mu = (m, 0, 0, 0)$. In this case we get the commutator

$$\{Q_\alpha^I, \bar{Q}_{\dot{\alpha}}^J\} = 2\sigma_{\alpha\dot{\alpha}}^\mu P_\mu \delta^{IJ} = \begin{pmatrix} 2m & 0 \\ 0 & 2m \end{pmatrix}_{\alpha\dot{\alpha}} \delta^{IJ}. \quad (3.16)$$

First let's look at the case where the central charges vanish $Z^{IJ} = 0$.

Vanishing central charges

We can define again creation and annihilation operators as before (only now we will have twice more)

$$a_\alpha^I \equiv \frac{1}{\sqrt{4E}} Q_\alpha^I, \quad a_{\dot{\alpha}}^{I\dagger} \equiv \frac{1}{\sqrt{4E}} \bar{Q}_{\dot{\alpha}}^I. \quad (3.17)$$

Recall that massive states are labeled by their mass m and spin j and a state $|m, j\rangle$ has degeneracy of $2j + 1$ since we can have states with projected spin taking values $j_3 = -j, \dots, +j$ (where j_3 is the eigenvalue of the operator J_{12}). We have again a very similar story where we can lower and raise the spin of the particle by acting with creation and annihilation operators to build up the representation space. The commutation relations for these operators are as before

$$\{a_\alpha^I, a_{\dot{\alpha}}^{J\dagger}\} = \delta^{IJ} \delta_{\alpha\dot{\alpha}}, \quad \{a_\alpha^I, a_\beta^J\} = 0, \quad \{a_{\dot{\alpha}}^{I\dagger}, a_{\dot{\beta}}^{J\dagger}\} = 0. \quad (3.18)$$

We can again use the commutation relations for J_{12} to deduce that the action on a state with spin j_3 is (ignoring other labels)

$$a_1^I |j_3\rangle = |j_3 - 1/2\rangle, \quad a_1^{I\dagger} |j_3\rangle = |j_3 + 1/2\rangle \quad (3.19)$$

when $\alpha, \dot{\alpha} = 1$. However for $\alpha, \dot{\alpha} = 2$ we have

$$a_2^I |j_3\rangle = |j_3 + 1/2\rangle, \quad a_2^{I\dagger} |j_3\rangle = |j_3 - 1/2\rangle. \quad (3.20)$$

Again define a Clifford vacuum with some spin j such that

$$a_\alpha^I |\Omega\rangle = 0. \quad (3.21)$$

Acting on this state with the creation operators is analogous to adding two spin quantum systems of spin- j and spin- $\frac{1}{2}$ [18]. Looking at the j and j_3 labels of the Clifford vacuum $|\Omega\rangle = |j, j_3\rangle$ we have

$$\begin{aligned} a_1^{I\dagger} |j, j_3\rangle &= c_1 |j + 1/2, j_3 + 1/2\rangle + c_2 |j - 1/2, j_3 + 1/2\rangle, \\ a_2^{I\dagger} |j, j_3\rangle &= c_3 |j + 1/2, j_3 - 1/2\rangle + c_4 |j - 1/2, j_3 - 1/2\rangle. \end{aligned} \quad (3.22)$$

Where c_i are the Clebsch-Gordon coefficients. so we have a superposition of different total spins since $j \otimes \frac{1}{2} = (j - \frac{1}{2}) \oplus (j + \frac{1}{2})$. By acting with other raising operators we can build the rest of the representation space. Note that here we no longer have to add a CPT conjugate since these states are automatically CPT invariant because of the values of projected spin can take. For example $N = 1$ and $j = 0$ the multiplet we construct is $(-\frac{1}{2}, 0, 0, \frac{1}{2})$ with the degrees of freedom of one massive complex scalar and one massive Weyl fermion and is CPT invariant.

Non-vanishing central charges

Now consider the case where $Z^{IJ} \neq 0$. making use of the R-symmetry mentioned earlier it is possible to rotate the generators Q_α^I and $\bar{Q}_\alpha^J$ so that the central charge matrix takes the following form [4]

$$Z^{IJ} = \begin{pmatrix} 0 & Z_1 & 0 & 0 & \cdots \\ -Z_1 & 0 & 0 & 0 & \\ 0 & 0 & 0 & Z_2 & \\ 0 & 0 & -Z_2 & 0 & \\ \vdots & & & & \ddots \end{pmatrix}. \quad (3.23)$$

Then we want to define the creation and annihilation operators in such a way that we get our familiar harmonic oscillator algebra. This is achieved with

$$\begin{aligned} a_\alpha^1 &= \frac{1}{\sqrt{2}}(Q_\alpha^1 + \epsilon_{\alpha\beta} Q_\beta^{2\dagger}) \\ b_\alpha^1 &= \frac{1}{\sqrt{2}}(Q_\alpha^1 - \epsilon_{\alpha\beta} Q_\beta^{2\dagger}) \\ a_\alpha^2 &= \frac{1}{\sqrt{2}}(Q_\alpha^3 + \epsilon_{\alpha\beta} Q_\beta^{4\dagger}) \\ b_\alpha^2 &= \frac{1}{\sqrt{2}}(Q_\alpha^3 - \epsilon_{\alpha\beta} Q_\beta^{4\dagger}) \\ &\vdots \end{aligned} \quad (3.24)$$

Then one can check that these indeed give us the desired commutation relations

$$\begin{aligned} \{a_\alpha^i, a_\beta^{j\dagger}\} &= (2m + Z_i)\delta_{ij}\delta_{\alpha\beta}, \quad \{b_\alpha^i, b_\beta^{j\dagger}\} = (2m - Z_i)\delta_{ij}\delta_{\alpha\beta}, \\ \{a_\alpha^i, b_\beta^{j\dagger}\} &= \{a_\alpha^i, a_\beta^j\} = \cdots = 0. \end{aligned} \quad (3.25)$$

Given a particular Clifford vacuum state one can use these operators to construct the representation space. Note that in order to avoid negative norm states we must have $|Z_i| \leq 2m$. We can see that if some $|Z_i|$'s are equal to $2m$ then the corresponding multiplets would be shorter and are called **short multiplets**². If its an equality for all i 's then the multiplets are said to be **ultrashort**.

3.3 Supersymmetric field theory

In field theory we have fields $\Phi(x)$ that transforms in some irreducible representation of the Poincaré group. In this section we will try to generalize this to fields that transform under an irreducible representation of the super Poincaré group (Poincaré + supersymmetry). From these fields we want to construct an invariant Lagrangian $\mathcal{L}$ under supersymmetry transformations. We could start by doing something similar as before by defining a field ϕ that acts as a Clifford vacuum and demand that $[\bar{Q}_\alpha, \phi] = 0$. Then build the rest of the representation space by acting on it with Q_α . However, constructing supersymmetric field theories this way is quite difficult in general. To greatly simplify things we can introduce the notion of **superspace** [1].

3.3.1 Superspace and superfields

In the Poincaré symmetry case we exponentiate the elements of the algebra to get the group element g

$$g = e^{i(\frac{1}{2}\omega_{\mu\nu}J^{\mu\nu} + a_\mu P^\mu)}. \quad (3.26)$$

Is there an analogous way of getting the supersymmetry group elements? For that we need to introduce an extension of the ordinary Minkowski coordinates with (for $\mathcal{N} = 1$ case) four anticommuting coordinates θ_α and $\bar{\theta}_{\dot{\alpha}}$. Then fields (or more precisely superfields) are functions of superspace, a total of eight variables $(x^\mu, \theta_\alpha, \bar{\theta}_{\dot{\alpha}})$. The supersymmetry group elements can be written as

$$g_{SUSY} = e^{i(\frac{1}{2}\omega_{\mu\nu}J^{\mu\nu} + a_\mu P^\mu + \theta^\alpha Q_\alpha + \bar{\theta}^{\dot{\alpha}} \bar{Q}_{\dot{\alpha}})}. \quad (3.27)$$

²Sometimes called BPS states after Bogomolnyi-Prasad-Sommerfeld. If n of the $|Z_i|$'s are equal to $2m$ then the multiplet is called $1/2^n$ BPS multiplet.

The new coordinates are fermionic and their commutation relations are trivial along with every other generator

$$\{\theta^\alpha, \theta^\beta\} = \{\bar{\theta}_{\dot{\alpha}}, \bar{\theta}_{\dot{\beta}}\} = \{\theta^\alpha, \bar{\theta}_{\dot{\beta}}\} = 0. \quad (3.28)$$

Then the anti-commutators between the Q 's can be realized as a commutator

$$[Q\theta, \bar{Q}\bar{\theta}] = 2\theta\sigma^\mu\bar{\theta}P_\mu\delta^{IJ}. \quad (3.29)$$

Where $\theta Q \equiv \theta^\alpha Q_\alpha$. Anti-commuting numbers (also called Grassmann variables) such as the one we have introduced have interesting albeit weird properties that are explained in appendix B. We know that any terms of third order in θ or $\bar{\theta}$ will vanish $\theta_\alpha\theta_\beta\theta_\rho = 0$ since the indices only take the values $\alpha = 1, 2$. So we can write a general expansion of the superfield in terms of these coordinates [3]

$$Y(x, \theta, \bar{\theta}) = \varphi(x) + \theta\psi(x) + \bar{\theta}\bar{\chi}(x) + \theta\theta m(x) + \bar{\theta}\bar{\theta} n(x) + \theta\sigma^\mu\bar{\theta}v_\mu(x) + \theta\theta\bar{\theta}\bar{\lambda}(x) + \theta\theta\bar{\theta}\rho(x) + \theta\theta\bar{\theta}\bar{\theta}d(x) \quad (3.30)$$

where $\varphi(x)$, $m(x)$, $n(x)$, $d(x)$ are scalars, $\psi(x)$, $\bar{\chi}(x)$, $\bar{\lambda}(x)$, $\rho(x)$ are Weyl spinors and $v_\mu(x)$ is a vector field. Under translations the adjoint action of the group on this field is

$$Y(x + \delta, \theta + \delta\theta, \bar{\theta} + \delta\bar{\theta}) = e^{-i(\epsilon\theta + \bar{\epsilon}\bar{\theta})} Y(x, \theta, \bar{\theta}) e^{i(\epsilon\theta + \bar{\epsilon}\bar{\theta})}. \quad (3.31)$$

Note that we did include the spacetime translation operator $e^{-iP_\mu a^\mu}$. This is a consequence of the supersymmetry algebra (3.5). For two subsequent supersymmetry transformations we get a translation. The explicit form of the translations are

$$\delta\theta^\alpha = \epsilon^\alpha, \quad \delta\bar{\theta}^{\dot{\alpha}} = \bar{\epsilon}^{\dot{\alpha}}, \quad \delta x^\mu = i\theta\sigma^\mu\bar{\epsilon} - i\epsilon\sigma^\mu\bar{\theta}. \quad (3.32)$$

We can Taylor expand the field and get

$$Y(x + \delta, \theta + \delta\theta, \bar{\theta} + \delta\bar{\theta}) = Y(x, \theta, \bar{\theta}) + (\epsilon^\alpha\partial_\alpha + \bar{\epsilon}^{\dot{\alpha}}\bar{\partial}_{\dot{\alpha}} + i(\theta\sigma^\mu\bar{\epsilon} - \epsilon\sigma^\mu\bar{\theta})\partial_\mu + \dots)Y(x, \theta, \bar{\theta}). \quad (3.33)$$

On the other hand we can expand 3.31 and get

$$Y(x + \delta, \theta + \delta\theta, \bar{\theta} + \delta\bar{\theta}) = Y(x, \theta, \bar{\theta}) + -i\epsilon^\alpha[Q_\alpha, Y] + i\bar{\epsilon}^{\dot{\alpha}}[\bar{Q}_{\dot{\alpha}}, Y] + \dots \quad (3.34)$$

We see by comparing (3.33) and (3.34) that the operators Q , $\bar{Q}$ can be represented as differential operators

$$Q_\alpha = -i\partial_\alpha - \sigma_{\alpha\dot{\beta}}^\mu\bar{\theta}^{\dot{\beta}}\partial_\mu, \quad (3.35)$$

$$\bar{Q}_{\dot{\alpha}} = i\bar{\partial}_{\dot{\alpha}} + \theta^\beta\sigma_{\beta\dot{\alpha}}^\mu\partial_\mu. \quad (3.36)$$

One can check that indeed these operators are consistent with the supersymmetry algebra (3.5).

3.3.2 Supersymmetric action

We can now construct a Lagrangian that is invariant under super Poincaré transformations (up to a total derivative). And using the formalism of superspace and superfields it is a relatively trouble free task. This is because of the properties of the Grassmann variables we have that the integration of a superfield over superspace is automatically invariant under supersymmetry transformations. Under these transformations the variation of the field is [3]

$$\delta_{\epsilon\bar{\epsilon}}Y(x, \theta, \bar{\theta}) = (\epsilon^\alpha\partial_\alpha + \bar{\epsilon}^{\dot{\alpha}}\bar{\partial}_{\dot{\alpha}} + i\partial_\mu(\theta\sigma^\mu\bar{\epsilon} - \epsilon\sigma^\mu\bar{\theta}))Y. \quad (3.37)$$

The first two terms vanish after integration and we are only left with a total derivative. Hence the variation of the action is zero

$$\delta_{\epsilon\bar{\epsilon}}\mathcal{S} = \int d^4x d^2\theta d^2\bar{\theta} \delta_{\epsilon\bar{\epsilon}}Y(x, \theta, \bar{\theta}) = 0. \quad (3.38)$$

However, the general form of a superfield given by (3.30) does not transform in an irreducible representation of the group. Hence we need to put additional constraints on $Y(x, \theta, \bar{\theta})$ such that it remains a superfield and transforms in a nice way. Here are some of these fields

1. Chiral superfield Φ satisfying $\bar{\mathcal{D}}_{\dot{\alpha}}\Phi = 0$. Where $\bar{\mathcal{D}}_{\dot{\alpha}} = -\partial_{\dot{\alpha}} - i\theta^{\beta}\sigma_{\beta\dot{\alpha}}^{\mu}\partial_{\mu}$.
2. Anti-chiral superfield $\bar{\Phi}$ satisfying $\mathcal{D}_{\alpha}\bar{\Phi} = 0$. Where $\mathcal{D}_{\alpha} = \partial_{\alpha} + i\sigma_{\alpha\dot{\beta}}^{\mu}\bar{\theta}^{\dot{\beta}}\partial_{\mu}$.
3. Vector field V satisfying $V = V^{\dagger}$.

Then we can construct a Lagrangian using these definitions. For example the most general renormalizable theory that one can build from a set of (anti-) chiral superfields is [3]

$$\mathcal{L} = \int d^2\theta d^2\bar{\theta} \Phi^i \bar{\Phi}_i + \int d^2\theta W(\phi^i) + \int d^2\bar{\theta} \bar{W}(\bar{\Phi}_i) \quad (3.39)$$

with W being a **superpotential** with an expression

$$W(\Phi^i) = a_i \Phi^i + \frac{1}{2} m_{ij} \Phi^i \Phi^j + \frac{1}{3} g_{ijk} \Phi^i \Phi^j \Phi^k. \quad (3.40)$$

For vector superfield let us first consider the case where $\mathcal{N} = 1$. Imposing the condition $V = V^{\dagger}$ would give us eight bosonic and eight fermionic degrees of freedom. However a massless vector multiplet should have only two of each as we discussed previously. So we have a freedom to choose a gauge to reduce the number of degrees of freedom³. The gauge we choose is called the Wess-Zumino gauge and the explicit expression for the vector superfield is

$$V_{WZ} = \theta\sigma^{\mu}\bar{\theta}v_{\mu}(x) + i\theta\bar{\theta}\bar{\lambda}(x) - i\bar{\theta}\theta\lambda(x) + \frac{1}{2}\theta\bar{\theta}\bar{\theta}\bar{\theta}D(x), \quad (3.41)$$

where v_{μ} is a vector field, λ is a Weyl fermion (gaugino) and D is an auxiliary field that we can later integrate out. For an abelian gauge group $U(1)$ the role of the field strength in this theory is played by

$$W_{\alpha} = -\frac{1}{4}\bar{\mathcal{D}}\bar{\mathcal{D}}\mathcal{D}_{\alpha}V, \quad \bar{W}_{\dot{\alpha}} = \frac{1}{4}\mathcal{D}\mathcal{D}\bar{\mathcal{D}}_{\dot{\alpha}}V. \quad (3.42)$$

Since the algebra commutes with $\mathcal{D}$'s these are still superfields, in fact they are (anti-) chiral. For non-Abelian theories these are modified to

$$W_{\alpha} = -\frac{1}{4}\bar{\mathcal{D}}\bar{\mathcal{D}}(e^{-V}\mathcal{D}_{\alpha}e^V), \quad \bar{W}_{\dot{\alpha}} = \frac{1}{4}\mathcal{D}\mathcal{D}(e^V\bar{\mathcal{D}}_{\dot{\alpha}}e^{-V}). \quad (3.43)$$

Explicitly in $y^{\mu} = x^{\mu} + i\theta\sigma^{\mu}\bar{\theta}$ coordinates

$$W_{\alpha} = -i\lambda_{\alpha}(y) + \theta_{\alpha}D(y) + i(\sigma^{\mu\nu}\theta)_{\alpha}F_{\mu\nu} + \theta\bar{\theta}\sigma^{\mu}(\partial_{\mu}\bar{\lambda}_{\alpha}(y) - \frac{i}{2}[v_{\mu}, \bar{\lambda}_{\alpha}(y)]) \quad (3.44)$$

where $F_{\mu\nu} = \partial_{\mu}v_{\nu} - \partial_{\nu}v_{\mu} - \frac{i}{2}[v_{\nu}, v_{\mu}]$. Then the action of super Yang-Mills is

$$\mathcal{S} = \frac{1}{4g_{YM}^2} \int d^4x \left(\int d^2\theta \text{Tr}(W^{\alpha}W_{\alpha}) + \int d^2\bar{\theta} \text{Tr}(\bar{W}_{\dot{\alpha}}\bar{W}^{\dot{\alpha}}) \right) \quad (3.45)$$

This can be slightly simplified by introducing complex coupling $\tau = \frac{\vartheta}{2\pi} + i\frac{4\pi}{g_{YM}^2}$

$$\mathcal{S} = \frac{1}{8\pi^2} \int d^4x \text{Im Tr} \left(\tau \int d^2\theta \text{Tr} W^{\alpha}W_{\alpha} \right) \quad (3.46)$$

3.3.3 $\mathcal{N} = 4$ Super Yang-Mills theory in $d = 4$

In particular we will be interested in the extension of (3.46) to $\mathcal{N} = 4$ supersymmetry. In four spacetime dimensions this is the maximally extended supersymmetric field theory that includes particles with spin ≤ 1 and thus, does not include gravity. The $\mathcal{N} = 4$ super Yang-Mills (SYM) action can be derived in two ways. One by using $\mathcal{N} = 1$ superspace formalism. Second by a dimensional reduction of $\mathcal{N} = 1$ SYM theory in 10 dimensions. In the former case we require three

³Choosing a gauge reduces the degrees of freedom to four bosonic and four fermionic, then going on shell reduces the number to two bosonic and two fermionic.

chiral superfields Φ_i ($i = 1, 2, 3$) and one vector superfield V . Without going into too much details we can write down a particular action (see chapter 3.3 of [1] for more detailed explanation)

$$\begin{aligned} \mathcal{S} = \int d^4x \text{Tr} \left(\int d^2\theta d^2\bar{\theta} \Phi^{i\dagger} e^V \Phi_i e^{-V} + \frac{1}{8\pi^2} \text{Im} \left(\tau \int d^2\theta W^\alpha W_\alpha \right) \right. \\ \left. + \left(i g_{YM} \frac{\sqrt{2}}{3!} \int d^2\theta \epsilon_{ijk} \Phi^i [\Phi^j, \Phi^k] + h.c. \right) \right). \end{aligned} \quad (3.47)$$

Then writing this out in terms of component fields we get the action of $\mathcal{N} = 4$ super Yang-Mills in four spacetime dimensions with the Lagrangian

$$\begin{aligned} \mathcal{L} = \text{Tr} \left(-\frac{1}{2g_{YM}^2} F_{\mu\nu} F^{\mu\nu} + \frac{\not{D}}{16\pi^2} F_{\mu\nu} \tilde{F}^{\mu\nu} - i \bar{\lambda}^a \bar{\sigma}^\mu D_\mu \lambda_a - \sum_i D_\mu \phi^i D^\mu \phi_i \right. \\ \left. + g_{YM} \sum_{a,b,i} C_i^{ab} \lambda_a [\phi^i, \lambda_b] + g_{YM} \sum_{a,b,i} \bar{C}_{iab} \bar{\lambda}^a [\phi^i, \bar{\lambda}^b] + \frac{g_{YM}^2}{2} \sum_{i,j} [\psi^i, \phi^j]^2 \right) \end{aligned} \quad (3.48)$$

where $\tilde{F}_{\mu\nu} = \frac{1}{2} \epsilon_{\mu\nu\rho\sigma} F^{\rho\sigma}$, D_μ acts on the fields as $D_\mu \cdot = \partial_\mu \cdot + i[v_\mu, \cdot]$ and C_i^{ab} are the Clebsch-Gordon coefficients. This Lagrangian consists of a massless supersymmetry gauge multiplet that contains six real scalars ϕ_i ($i = 1, \dots, 6$), four Weyl fermions λ_α^a ($a = 1, \dots, 4$) and a gauge vector field v_μ . It is also invariant under supersymmetry transformations that act on these fields as [8]

$$\begin{aligned} \delta_\epsilon \phi^i &= [\epsilon_a^\alpha Q_\alpha^a, \phi^i] = \epsilon_a^\alpha C^{iab} \lambda_{\alpha b} \\ \delta_\epsilon \lambda_{\beta b} &= [\epsilon_a^\alpha Q_\alpha^a, \lambda_{\beta b}] = \frac{1}{2} (F_{\mu\nu} + \tilde{F}^{\mu\nu}) \epsilon_{\alpha b} (\sigma^{\mu\nu})^\alpha{}_\beta + [\phi^i, \phi^j] \epsilon_{\beta a} (C_{ij})^a{}_b \\ \delta_\epsilon \bar{\lambda}_\beta^b &= [\epsilon_a^\alpha Q_\alpha^a, \bar{\lambda}_\beta^b] = C_i^{ab} \epsilon_a^\alpha \bar{\sigma}_{\alpha\dot{\beta}}^\mu D_\mu \phi^i \\ \delta_\epsilon v_\mu &= [\epsilon_a^\alpha Q_\alpha^a, v_\mu] = \epsilon_a^\alpha (\sigma_\mu)_\alpha{}^\beta \bar{\lambda}_\beta^a \end{aligned} \quad (3.49)$$

where $(C_{ij})^a{}_b$ are related to bilinears in Clifford Dirac matrices of $SO(6)_R \sim SU(4)_R$. This theory is also invariant under the R-symmetry group $SU(4)_R$ where the vector field is in a singlet representation, the fermions form a fundamental representation and the scalars are in the 6-dimensional anti-symmetric representation of the group. An important property of this theory is that it is scale invariant since the fields are massless and the coupling constant is dimensionless.

Next we can look at the properties of a theory that is invariant under both supersymmetry and conformal transformations.

3.4 Superconformal symmetry

We saw that the $\mathcal{N} = 4$ SYM theory is invariant under super Poincaré group. Not only that the theory is also scale invariant due to dimensionless coupling g_{YM} . The Poincaré and conformal symmetries combine to form the group $SO(4, 2) \sim SU(2, 2)$ as we already discussed. Is there a larger group that also includes the supersymmetric generators Q and $\bar{Q}$? Yes, but for that we need to introduce additional fermionic generators to close the algebra. Let us review what symmetries we have for SYM [8]:

- Conformal symmetry $SO(2, 4) \sim SU(2, 2)$. Of which Poincaré is a subgroup generated by Lorentz transformations, translations. Together with scale and special conformal transformations generated by D and K^μ respectively.
- Poincaré supersymmetry. Generated by fermionic generators Q_α^I and $\bar{Q}_{I\dot{\alpha}}$. These are fermionic superpartners of the translations generator P^μ .
- R-symmetry $SU(4)_R \sim SO(6)_R$. Generated by T^A , $A = 1, \dots, 15$.

We include the conformal and supersymmetry into a one larger group **superconformal group** $SU(2, 2|4)$ (for $\mathcal{N} = 4$). But we need to introduce more fermionic generators. So the theory should also be invariant under

- Conformal supersymmetry. Generated by fermionic generators $S_{I\alpha}$ and $\bar{S}_{\dot{\alpha}}^I$ to ensure closure of the algebra. Similar to translation generator P^μ , these are the fermionic super partners of K^μ .

The superconformal algebra is defined by the following commutation relations (obviously including the commutators of (2.21) and (3.5)) [1]:

$$\begin{aligned}
[D, Q_\alpha^I] &= -\frac{i}{2} Q_\alpha^I, & [D, \bar{Q}_{\dot{\alpha}}] &= -\frac{i}{2} \bar{Q}_{\dot{\alpha}}, & [D, S_{I\alpha}] &= \frac{i}{2} S_{I\alpha}, \\
[D, \bar{S}_{\dot{\alpha}}^I] &= \frac{i}{2} \bar{S}_{\dot{\alpha}}^I, & [Q_\alpha^I, K^\mu] &= -i\sigma_{\alpha\dot{\alpha}}^\mu \bar{S}^{I\dot{\alpha}}, & [\bar{Q}_{I\dot{\alpha}}, K^\mu] &= i\epsilon_{\dot{\alpha}\beta}(\bar{\sigma}^\mu)^{\dot{\beta}\alpha} S_{I\alpha}, \\
[S_{I\alpha}, K^\mu] &= 0, & [\bar{S}_{\dot{\alpha}}^I, K^\mu] &= 0, & [S_{I\alpha}, P^\mu] &= i\sigma_{\alpha\dot{\alpha}}^\mu \epsilon^{\dot{\alpha}\beta} \bar{Q}_{I\dot{\beta}}, \\
[\bar{S}_{\dot{\alpha}}^I, P^\mu] &= -i\epsilon_{\dot{\alpha}\beta}(\bar{\sigma}^\mu)^{\dot{\beta}\alpha} Q_\alpha^I, & [S_{I\alpha}, J^{\mu\nu}] &= (\sigma^{\mu\nu})_\alpha{}^\beta S_{I\beta}, \\
[\bar{S}_{\dot{\alpha}}^I, J^{\mu\nu}] &= (\bar{\sigma}^{\mu\nu})^{\dot{\beta}}{}_{\dot{\alpha}} \bar{S}_{\dot{\beta}}^I, & \{S_{I\alpha}, \bar{S}_{\dot{\beta}}^J\} &= 2\delta_J^I (\sigma^\mu)_{\alpha\dot{\beta}} K_\mu, \\
\{Q_\alpha^I, \bar{S}_{\dot{\beta}}^J\} &= \{\bar{Q}_{I\dot{\alpha}}, S_{J\beta}\} = 0, & \{S_{I\alpha}, S_{J\beta}\} &= \{\bar{S}_{\dot{\alpha}}^I, \bar{S}_{\dot{\beta}}^J\} = 0, \\
\{Q_\alpha^I, S_{J\beta}\} &= 2\epsilon_{\alpha\beta} \delta_J^I D - i(\sigma^{\mu\nu})_\alpha{}^\gamma \epsilon_{\gamma\beta} J_{\mu\nu} \delta_J^I - 4i\epsilon_{\alpha\beta}(\delta_J^I T + B_J^I T^i), \\
\{\bar{Q}_{I\dot{\alpha}}, \bar{S}_{\dot{\beta}}^J\} &= 2\epsilon_{\dot{\alpha}\dot{\beta}} \delta_I^J D - i(\bar{\sigma}^{\mu\nu})^{\dot{\beta}}{}_{\dot{\alpha}} \epsilon^{\dot{\alpha}\gamma} J_{\mu\nu} \delta_I^J - 4i\epsilon_{\dot{\alpha}\dot{\beta}}(\delta_I^J T + B_I^J T^i).
\end{aligned} \tag{3.50}$$

Where B_I^{iJ} are defined by the commutation relations between generators of the R-symmetry and the supersymmetry charges which we did not include here.

Next we need to consider the representations of this algebra.

3.4.1 Representations of the superconformal algebra

Recall from before that in conformal field theory we call a conformal operator primary if it is annihilated by the generator K_μ . Consider an operator $\mathcal{O}$ with a definite scaling dimension $[D, \mathcal{O}] = -i\Delta\mathcal{O}$. Now looking at the commutation relations we can deduce that the fermionic operators S and $\bar{S}$ lower the dimension Δ by $\frac{1}{2}$. This can be seen from the Jacobi identity

$$\begin{aligned}
[D, [S_{I\alpha}, \mathcal{O}]] &= -[S_{I\alpha}, [\mathcal{O}, D]] - [\mathcal{O}, [D, S_{I\alpha}]] \\
&= -i(\Delta - \frac{1}{2})[S_{I\alpha}, \mathcal{O}].
\end{aligned} \tag{3.51}$$

We then define **superconformal primary operators** $\mathcal{O}$ to be the lowest dimension operators belonging to a superconformal multiplet of $\mathfrak{su}(2, 2|\mathcal{N})$ such that

$$[S_{I\alpha}^I, \mathcal{O}] = 0, \quad [\bar{S}_{I\dot{\alpha}}, \mathcal{O}] = 0. \tag{3.52}$$

In these cases we use the commutator if the operator is bosonic and anti-commutator if it is fermionic. To construct descendant operators we can act with other operators from the superconformal algebra (3.50). Then these descendants together with the primary operator transform in an irreducible representation of the superconformal algebra.

Similarly to (3.51) one can show that the fermionic superpartners of P_μ raises the dimension of an operator by $\frac{1}{2}$. Descendants of this type are called **superdescendants** and are defined by

$$\mathcal{O}' = [Q, \mathcal{O}]. \tag{3.53}$$

These operators are conformal primary operators since they are annihilated by K_μ

$$[K_\mu, [Q, \mathcal{O}]] = 0. \tag{3.54}$$

Each superdescendant operator defines a conformal multiplet called **Verma module** and these modules are related by a supersymmetry transformation [1].

Let us return to the theory of interest - $\mathcal{N} = 4$ super Yang-Mills. How can we realise these operators in terms of fields of our theory? In a gauge theory we only consider gauge invariant local operators $\mathcal{O}(x)$. And we know that all the fields: scalars ϕ^i , Weyl fermions λ^a and the field strength constructed from vector field v_μ all transform covariantly. Hence we can construct gauge

invariant operators by taking the trace. Important examples are **single-trace operators** obtained from the trace of the scalars

$$\mathcal{O}(x) = Str(\phi^{i_1}(x) \dots \phi^{i_n}(x)) = \text{Tr}(\phi^{(i_1}(x) \dots \phi^{i_n)}(x)) \quad (3.55)$$

The Str stands for **symmetrised trace**. In four dimensions the dimension of the fields ϕ^i is one. Hence the dimension of the operator (3.55) is $\Delta = n$, i.e. the number of scalar fields. Finally, how do we label these operators? We need to find a maximally commutative bosonic subalgebra of $\mathfrak{su}(2, 2|4)$. This is just the Lorentz algebra $\mathfrak{so}(1, 3)$ so spin j is a good label, then we have dilation subgroup $\mathfrak{so}(1, 1)$ with label Δ and we have the $\mathfrak{su}(4)_R$ R-symmetry subalgebra. These are labelled by something called Dynkin labels $[r_1, r_2, r_3]$ which determine the dimension of the representation [8].

Chapter 4

Anti-de Sitter spacetime

The theory of special and general relativity is formulated on Minkowski spacetime. This is a flat space and it is the simplest case one can consider. De-Sitter and anti-de Sitter spacetimes are closest relatives of Minkowski as they are the simplest non-flat spacetimes, since they have a constant curvature everywhere. There are a few ways to define anti-de Sitter spacetime, but I found the following way the most intuitive [17]. Consider a flat $d + 2$ -dimensional spacetime which has a metric that can be written as

$$ds^2 = -dX_0^2 + \sum_{i=1}^d dX_i^2 - dX_{d+1}^2. \quad (4.1)$$

We can embed a hypersurface in this spacetime defined by

$$X_0^2 - \sum_{i=1}^d X_i^2 - X_{d+1}^2 = -R^2 \quad (4.2)$$

where R is the radius of curvature of this surface. The result is the **anti-de Sitter** or **AdS** spacetime.

In this chapter we will briefly look at different ways of parametrizing this spacetime, see what symmetries it has and what is the conformal boundary.

4.1 Parametrization

There are different coordinate systems for AdS spacetime. Let's look at a few that are most widely used. The **global coordinates** of a $d + 1$ -dimensional AdS is defined by [17]

$$\begin{aligned} X^0 &= R \cosh \rho \cos \tau \\ X^i &= R \Omega_i \sinh \rho \\ X^{d+1} &= R \cosh \rho \sin \tau \end{aligned} \quad (4.3)$$

where $i \in \{1, \dots, d\}$, $\rho \in \mathbb{R}_+$, $\tau \in [0, 2\pi]$ and Ω_i is the usual parametrization of a $d - 1$ sphere. In these coordinates the metric takes the form

$$ds^2 = R^2 (-\cosh^2 \rho d\tau^2 + d\rho^2 + \sinh^2 \rho d\Omega_{d-1}^2). \quad (4.4)$$

With the metric $d\Omega_{d-1}^2$ of a unit sphere S^{d-1} . The metric is independent of τ , so ∂_τ can be taken to be a time coordinate. However, since this coordinate is periodic with a period of 2π the spacetime has closed timelike curves. One can deal with this by letting $\tau \in \mathbb{R}$ and "unwrapping" the circle, thus obtaining something called a **universal cover** of AdS [1]. When we study the boundary of AdS it will be useful to introduce a new parameter by defining $\tan \theta = \sinh \rho$ which changes the metric to

$$ds^2 = \frac{R^2}{\cos^2 \theta} (-d\tau^2 + d\theta^2 + \sin^2 \theta d\Omega_{d-1}^2). \quad (4.5)$$

Another useful parametrization is called the **Poincaré patch** with the following parametrization [17]

$$\begin{aligned} X^0 &= \frac{R^2}{2r} \left(1 + \frac{r^2}{R^4} (\vec{x}^2 - t^2 + R^2) \right) \\ X^i &= \frac{rx^i}{R} \end{aligned} \quad (4.6)$$

$$\begin{aligned} X^d &= \frac{R^2}{2r} \left(1 + \frac{r^2}{R^4} (\vec{x}^2 - t^2 - R^2) \right) \\ X^{d+1} &= \frac{rt}{R} \end{aligned} \quad (4.7)$$

where $i \in \{1, \dots, d-1\}$, $t \in \mathbb{R}$, $\vec{x} \in \mathbb{R}^{d-1}$ and $r \in \mathbb{R}_+$. In these coordinates the metric takes the form

$$ds^2 = \frac{R^2}{r^2} dr^2 + \frac{r^2}{R^2} (-dt^2 + d\vec{x}^2). \quad (4.8)$$

This parametrization only covers half of AdS since $z > 0$ and the boundary is now at $z = 0$ but we will talk more about that later. It is also useful to define $z = R^2/r$, in this case the metric is

$$ds^2 = \frac{R^2}{z^2} (dz^2 - dt^2 + d\vec{x}^2). \quad (4.9)$$

It is clear that at fixed value of z we get a flat Minkowski spacetime. Also an important property here is that this metric is invariant under a conformal transformation

$$(z, t, \vec{x}) \rightarrow \lambda(z, t, \vec{x}). \quad (4.10)$$

For AdS we can show that the Ricci scalar is [1]

$$R = -\frac{d(d+1)}{R^2} \quad (4.11)$$

and it satisfies the Einstein's vacuum field equations with a negative cosmological constant

$$\Lambda = -\frac{d(d-1)}{2R^2}. \quad (4.12)$$

4.2 Symmetries

The symmetries of a spacetime can be calculated by an infinitesimal shift in coordinates in the direction of some vector field k^μ . I.e. $x^\mu \rightarrow x^\mu + \epsilon k^\mu$ which changes the metric as

$$g_{\mu\nu}(x) \rightarrow \frac{\partial x'^\rho}{\partial x^\mu} \frac{\partial x'^\sigma}{\partial x^\nu} g_{\rho\sigma}(x'). \quad (4.13)$$

Expanding in orders of ϵ

$$\begin{aligned} \frac{\partial x'^\rho}{\partial x^\mu} \frac{\partial x'^\sigma}{\partial x^\nu} g_{\rho\sigma}(x') &= (\delta_\mu^\rho + \epsilon k^\rho{}_{,\mu})(\delta_\nu^\sigma + \epsilon k^\sigma{}_{,\nu})(g_{\rho\sigma}(x) + \epsilon k^\gamma g_{\rho\sigma,\gamma}(x) + \dots) \\ &= g_{\mu\nu}(x) + \epsilon(g_{\mu\sigma}(x)k^\sigma{}_{,\nu} + g_{\rho\nu}(x)k^\rho{}_{,\mu} + k^\gamma g_{\mu\nu,\gamma}(x)) + \dots \\ &= g_{\mu\nu}(x) + \epsilon \mathcal{L}_k g_{\mu\nu}(x) + \dots \end{aligned} \quad (4.14)$$

then demanding that the metric is left unchanged we get the **Killing equation**

$$\mathcal{L}_k g_{\mu\nu}(x) = 0. \quad (4.15)$$

This equation gives the isometries of spacetime (i.e. coordinate transformations that leave distances unchanged) [7]. For example d -dimensional Minkowski spacetime has d translations and $d(d-1)/2$ rotations and boosts. A total of $d(d+1)/2$ isometries. It turns out that is the maximum number of isometries a spacetime can have. These spacetimes are called **maximally symmetric** and have constant curvature everywhere. Note that the hypersurface defined by (4.2) is invariant under $SO(2, d)$ ¹ transformations acting on the coordinates. So a $d+1$ -dimensional AdS has $(d+1)(d+2)/2$ Killing generators. Hence AdS is a maximally symmetric spacetime which can also be described as a coset space $SO(d, 2)/SO(d, 1)$ [1].

¹This is the same symmetry group as a d -dimensional conformal group, which is an important fact for AdS/CFT correspondence.

4.3 Boundary of AdS

We can take the global AdS metric defined by (4.5) and multiply by a Weyl factor $\cos \theta / R^2$. This way we deform the spacetime, but still retain the causal structure. We can see that at the spacial infinity (boundary of AdS), when $\theta \rightarrow \pi/2$ we have a cylindrical conformal boundary $\mathbb{R} \times S^{d-1}$ with a metric

$$ds^2 = -d\tau^2 + d\Omega_{d-1}^2. \quad (4.16)$$

Note that this is an important result since it tells us that the boundary is actually a Lorentzian spacetime. If it were Euclidean then a quantum field theory on the boundary would have to be defined in a null spacetime. Which is one of the reasons we have AdS in the AdS/CFT correspondence.

Now consider the Poincaré AdS metric defined by (4.9). The boundary is located at $z = 0$ since this corresponds to $\theta = \pi/2$. We can also Multiply the metric by a Weyl factor z^2/R^2 . In this case the conformal boundary is a flat Minkowski spacetime

$$ds^2 = -dt^2 + d\vec{x}^2. \quad (4.17)$$

Notice that we get a different boundary then before. This is related to the fact that the Poincaré coordinates does not cover the whole manifold. In fact we can have infinitely many possibilities which are related to flat spacetime by a Weyl transformation. This can be done by approaching different points of the boundary at different rates [17].

Chapter 5

String theory

In 1960's physicists were trying to figure out a new model of strong interactions. Experimental data suggested that hadrons were not fundamental since their spin was proportional to mass squared of the particle. These are known as **Regge trajectories**. In 1968 Gabriele Veneziano proposed a model based on the S-matrix approach [37] which explained this behaviour. This marked the beginning of the string theory, since it led to the idea of hadrons behaving like strings. The first correct generalization of Gabriele's work was done by Nambu and Goto (independently) that described relativistic string [32]. However, string theory had many problems as a theory of strong interactions and the development of QCD led to its temporary demise. One of the problems the theory had was an appearance of spin-2 particle. It took scientists until mid-1970's to consider the string theory to be a theory of quantum gravity and the spin-2 particle to be identified with the graviton [31]. At the moment string theory is extremely popular in the physics community because it brings much promise not just in physics but also gives interesting insights in mathematics.

I start this chapter with a general approach to the subject by first introducing bosonic string theory. After that I give a very short description on the quantization of the string before discussing superstring theory and the low-energy effective action - supergravity. Finally, I end up by talking about D-branes which is very relevant for the next chapter.

String theory is a vast subject, so I will focus on giving a greatly simplified introduction, often skipping many steps along the way. However one should note that the subject gets very involved very quickly. I have mainly followed [36] for the first half of this chapter and chapter 4 of [1] for the second half.

5.1 Bosonic string theory

Until the development of string theory, we viewed point particles as the fundamental objects. In string theory these objects are one dimensional strings. I believe it would be wise to start by remembering a few things about the treatment of point particles in general relativity and draw analogies to it when we look at strings later on. The action of a massive relativistic particle is given by

$$\mathcal{S} = -m \int d\tau \sqrt{-\frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau} \eta_{\mu\nu}} \quad (5.1)$$

where the indices $\mu, \nu = 0, \dots, d-1$ and τ is a parameter that labels points of a particle on the worldline¹ [7]. We can interpret this action as simply being the proper time along a worldline. From (5.1) we could say that the action has d degrees of freedom. However we know that is not true since time is not really a dynamical degree of freedom. Instead a particle must move in time. This is actually just a redundancy in the description (a gauge symmetry) and actually the action is invariant under reparametrization $\tau' = \tau'(\tau)$. So one of the coordinates is a fake degree of freedom. The reason why we write the action like this is because the Poincaré symmetry is manifest.

A string on the other hand traces out a **worldsheet** on spacetime and it is a $1+1$ -dimensional sheet embedded in d -dimensional Minkowski space. So we will need another parameter to describe the shape of this surface. We choose the other parameter to be $\sigma \in [0, \sigma_0]$. For closed strings

¹Worldline is a path that a particle traces in spacetime. Naively these look like long strings in a d -dimensional Minkowski space.

σ_0 is taken to be 2π . So the embedding of the surface on the target space is given by $X^\mu(\tau, \sigma)$. And the two coordinates are commonly denoted $\sigma^\alpha = (\tau, \sigma)$ with $\alpha \in \{1, 2\}$. For convenience we define $f(\tau, \sigma) \equiv f(\sigma)$. How should the action of this theory look like? Firstly we would like it to be invariant under Poincaré. Secondly we want the action to be independent of our choice of parameters, i.e. it should be invariant under reparametrization. Looking back at the action for a particle we see that it is proportional to the length of the worldline. It turns out that the action for a string is proportional to the area of the worldsheet [36] and reads

$$\mathcal{S} = -T \int d^2\sigma \sqrt{-\det \gamma} \quad (5.2)$$

where

$$\gamma_{\alpha\beta} = \frac{\partial X^\mu}{\partial \sigma^\alpha} \frac{\partial X^\nu}{\partial \sigma^\beta} \eta_{\mu\nu} \quad (5.3)$$

is the induced metric on the worldsheet. Or for people familiar with differential geometry it is the flat metric pull-back on the target space. The parameter T is actually the tension of the string². The action (5.2) is called the **Nambu-Goto** action. This action has a square root in it and it makes quantization very difficult [36]. So we have an equivalent way of writing the action for a string:

$$\mathcal{S} = -\frac{1}{4\pi\alpha'} \int d^2\sigma \sqrt{-h} h^{\alpha\beta} \partial_\alpha X^\mu \partial_\beta X^\nu \eta_{\mu\nu} \quad (5.4)$$

where we have exchanged the square root for an additional auxiliary field $h_{\alpha\beta}$ which is the metric of the worldsheet. Using equations of motion for this field we would obtain the on-shell action that is just the Nambu-Goto action (5.2). This is known as the **Polyakov** action who was the first one to use it to quantize the string. The symmetries of this action are not just the familiar Poincaré and reparametrization symmetries, but also there is a new symmetry called the **Weyl invariance** [41] with the following transformations

$$X^\mu(\sigma) \rightarrow X^\mu(\sigma), \quad h_{\alpha\beta}(\sigma) \rightarrow \Omega^2(\sigma) h_{\alpha\beta}(\sigma) = e^{2\omega(\sigma)} h_{\alpha\beta}(\sigma). \quad (5.5)$$

Which is just the scale invariance which preserve angles we've seen in chapter 2. Using these symmetries we can choose a convenient gauge called the **conformal gauge**:

$$g_{\alpha\beta}(\sigma) = e^{2\omega(\sigma)} \eta_{\alpha\beta} \quad (5.6)$$

with $\eta_{\alpha\beta}$ being a 1+1-dimensional Minkowski metric. In this gauge the Polyakov action simplifies to [36]

$$\mathcal{S} = -\frac{1}{4\pi\alpha'} \int d^2\sigma \partial_\alpha X^\mu \partial^\alpha X^\nu \eta_{\mu\nu}. \quad (5.7)$$

Varying this action we get the equations of motion for $X^\mu(\sigma)$

$$\partial_\alpha \partial^\alpha X^\mu(\sigma) = \partial_+ \partial_- X^\mu(\sigma) = 0 \quad (5.8)$$

where we have introduced partial derivatives with respect to light cone coordinates $\sigma^\pm = \tau \pm \sigma$. These equations of motion are subject to boundary conditions and something called **Virasoro constraints** which are obtained as a consequence of the vanishing energy-momentum tensor of the worldsheet [1]. The boundary conditions simply arises from integration by parts when we vary the action. These of course should vanish. Explicitly we must satisfy the following condition

$$\partial_\sigma X^\mu \delta X_\mu|_0^{\sigma_0} = 0. \quad (5.9)$$

Considering for a moment open strings, there are two ways we can satisfy this condition:

1. Neumann boundary conditions

$$\partial_\sigma X^\mu(\sigma) = 0 \quad (5.10)$$

which is evaluated at the boundary $\sigma = 0$ and $\sigma = \sigma_0$ (for convenience we may take $\sigma_0 = \pi$).

²For historical reasons T is commonly expressed as $T = \frac{1}{2\pi\alpha'}$, where α' is the Regge slope that relates the spin of a particle with it's mass squared and is related to the string length by $\alpha' = l_s^2$.

2. Dirichlet boundary condition

$$\delta X^\mu(\sigma) = 0 \quad (5.11)$$

also evaluated at $\sigma = \{0, \pi\}$.

Intuitively, Neumann boundary condition allows the string to move freely (this condition must be imposed for $\mu = 0$), while the Dirichlet restricts the motion of the string in some directions. We can impose different boundary conditions for different coordinates for example if we have

$$\begin{aligned} \partial_\sigma X^a(\sigma) &= 0 \quad \text{for } a = 0, \dots, p \\ X^i(\sigma) &= \text{const.} \quad \text{for } i = p + 1, \dots, d - 1. \end{aligned} \quad (5.12)$$

Which is a mixture of both conditions, then the string end-points are constrained to live on on a $(p + 1)$ -dimensional hypersurface called **D-brane** (Dirichlet brane).

The Virasoro constraints arises when we calculate the energy-momentum tensor

$$T_{\alpha\beta} = \frac{4\pi\alpha'}{\sqrt{-h}} \frac{\delta\mathcal{S}}{\delta h^{\alpha\beta}}. \quad (5.13)$$

But this vanishes since we must satisfy the equations of motion for $h_{\alpha\beta}$ i.e. $\frac{\delta\mathcal{S}}{\delta h^{\alpha\beta}} = 0$. Explicitly in the light-cone coordinates these constraints are [1]

$$T_{++} = \partial_+ X^\mu \partial_+ X_\mu = 0, \quad T_{--} = \partial_- X^\mu \partial_- X_\mu = 0, \quad T_{+-} = T_{-+} = 0. \quad (5.14)$$

5.1.1 Classical solutions

The solutions of the equations of motion (5.8) can be decomposed into two modes: left moving modes $X_L^\mu(\sigma_+)$ and right moving modes $X_R^\mu(\sigma_-)$ which are functions of σ_+ and σ_- respectively. Then the general solutions takes the form [1]

$$X^\mu(\sigma) = X_L^\mu(\sigma_+) + X_R^\mu(\sigma_-). \quad (5.15)$$

These modes have Fourier expansions

$$\begin{aligned} X_L^\mu(\sigma_+) &= \frac{\tilde{X}_0^\mu}{2} + \frac{\alpha'}{2} \tilde{p}^\mu \sigma^+ + i\sqrt{\frac{\alpha'}{2}} \sum_{n \neq 0} \frac{\tilde{\alpha}_n^\mu}{n} e^{-in\sigma^+}, \\ X_R^\mu(\sigma_-) &= \frac{X_0^\mu}{2} + \frac{\alpha'}{2} p^\mu \sigma^- + i\sqrt{\frac{\alpha'}{2}} \sum_{n \neq 0} \frac{\alpha_n^\mu}{n} e^{-in\sigma^-}. \end{aligned} \quad (5.16)$$

Where the center of mass and the center of momentum is expressed by

$$X_C^\mu = \frac{\tilde{X}_0^\mu + X_0^\mu}{2}, \quad p_C^\mu = \frac{\tilde{p}^\mu + p^\mu}{2}. \quad (5.17)$$

Taking particular choices of these values we can satisfy the boundary conditions for the string. For a closed string, the solution has to be periodic with a period of σ_0 (which can be taken to be 2π) and the solutions must satisfy $p^\mu = \tilde{p}^\mu$. For open string we have a bit more freedom. We can have different boundary conditions for each end. For example if both ends satisfy Dirichlet boundary conditions then it is referred to as DD condition. If both are Neumann then it is NN (free moving string). We can have ND or DN as well.

The equations (5.16) can also be expanded in oscillator modes and Virasoro constraints put restrictions on the values of α_n^μ and $\tilde{\alpha}_n^\mu$.

5.1.2 String quantization

We mentioned earlier that this theory has a gauge symmetry. And quantizing a gauge theory gives rise to problems such as nonphysical states like we have in QED. Where we overcome these by fixing a gauge. Mainly by choosing to work in a particular gauge either use Gupta-Bleuler formalism in Lorentz gauge or only quantize the physical states of classical solutions in Coulomb gauge [25]. Similarly here we can use the diffeomorphism and Weyl symmetries to fix a gauge which lead us to Virasoro constraints [2].

Lets first start with defining the canonical momentum³

$$\Pi^\mu(\sigma) = \frac{\partial_\tau X^\mu(\sigma)}{2\pi\alpha'}. \quad (5.18)$$

Then we impose the canonical commutation relations

$$\begin{aligned} [X^\mu(\tau, \sigma), \Pi^\nu(\tau, \sigma')] &= i\eta^{\mu\nu} \delta(\sigma - \sigma'), \\ [X^\mu(\tau, \sigma), X^\nu(\tau, \sigma')] &= [\Pi^\mu(\tau, \sigma), \Pi^\nu(\tau, \sigma')] = 0. \end{aligned} \quad (5.19)$$

From this we can deduce that the Fourier modes of the expansion (5.16) has the following non-vanishing commutation relations [1]. For open strings satisfying NN boundary conditions we have

$$[X_0^\mu, p^\mu] = i\eta^{\mu\nu}, \quad [\alpha_n^\mu, \alpha_m^\nu] = n\eta^{\mu\nu} \delta_{n+m, 0}. \quad (5.20)$$

The first one is just the commutation relation for the operators of position and momentum of center of mass. The other one looks like the familiar creation and annihilation commutation relations, we just need to tweak it a little by defining

$$a_n^\mu = \frac{1}{\sqrt{n}} \alpha_n^\mu, \quad a_n^{\dagger\mu} = \frac{1}{\sqrt{n}} \alpha_{-n}^\mu \quad \forall n > 0. \quad (5.21)$$

Which yields $[a_n^\mu, a_m^{\dagger\nu}] = \delta_{m,n} \eta^{\mu\nu}$. These give rise to the Fock space of harmonic oscillators⁴. However, there is a problem with this space. The commutator $[a_n^0, a_m^{\dagger 0}] = -1$ gives rise to negative norm states. But as discussed we use the Virasoro constraints to decouple these states from the theory. So ignoring the unphysical solutions we can define a vacuum that satisfies

$$p^\mu |0, k\rangle = k^\mu |0, k\rangle, \quad a_n^i |0, k\rangle = 0. \quad (5.22)$$

Where $i = 1, \dots, d-2$ ⁵. Then we can build up the rest of the space by acting with creation operators. A general state then takes the form [1]

$$|N, k\rangle = \left(\prod_{i=1}^{d-2} \prod_{n=1}^{\infty} \frac{(a_n^{i\dagger})^{N_{in}}}{\sqrt{N_{in}!}} \right) |0, k\rangle \quad (5.23)$$

N_{in} is defined by

$$a_n^i a_n^{i\dagger} |N, k\rangle = N_{in} |N, k\rangle. \quad (5.24)$$

We can work out what is the mass of the state after doing a bit of work and using the ζ -function renormalization (which I will not include). This turns out to be

$$M^2 = \frac{1}{\alpha'} \left(N + \frac{2-d}{24} \right). \quad (5.25)$$

For the vacuum $N = 0$ this state has negative mass if $d > 2$, hence it is unstable. But this is solved in superstring theory. For the first excited state $N = 1$

$$M^2 = \frac{1}{\alpha'} \frac{d-26}{24}. \quad (5.26)$$

The state transforms under $SO(d-2)$ which implies that the state must be massless, hence the theory is only consistent in $d = 26$ dimensions [36].

For closed strings the only difference is that we would have two modes - left and right moving. We can define the other set of operators to be $\tilde{a}_n^i$ and $\tilde{a}_n^{i\dagger}$. Then a general closed string state can be expressed as [1]

$$|N, \tilde{N}, k\rangle = \left(\prod_{i=1}^{D-2} \prod_{n=1}^{\infty} \frac{(a_n^{i\dagger})^{N_{in}} (\tilde{a}_n^{i\dagger})^{\tilde{N}_{in}}}{\sqrt{N_{in}! \tilde{N}_{in}!}} \right) |0, 0, k\rangle. \quad (5.27)$$

Where the occupation numbers must satisfy $N = \tilde{N}$. Similarly here the mass of the first excited state is also tachyonic and for the first excited state we require $d = 26$ to get a $M^2 = 0$. These states are massless rank two tensors. The symmetric traceless part of this tensor is identified with the graviton. The scalar part is identified with the dilaton and the anti-symmetric part is the Kalb-Ramond field.

³The total momentum is then just the integral $p^\mu = \int_0^{\sigma_0} d\sigma \Pi^\mu(\sigma)$

⁴For a closed string there will be two spaces since we have two modes: left and right moving.

⁵The reason for this is because when quantizing we define light cone coordinates $X^\pm = X^0 \pm X^{d-2}$ to solve the Virasoro constraints and get rid of unphysical states. The dynamical degrees of freedom in terms of creation and annihilation operators are just a_m^i and $a_n^{i\dagger}$.

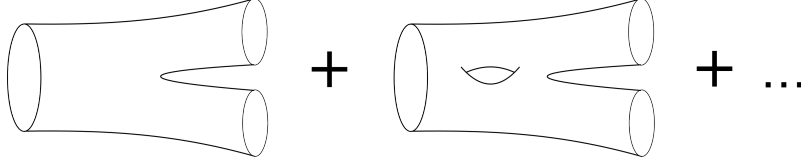


Figure 5.1: Lowest order string interaction expansion of a string splitting into two closed strings (from left to right).

5.1.3 Interactions

The discussion above was only for non-interacting strings with worldsheet topologies of a sheet or a cylinder. However in interacting theories the worldseets are more complicated. For example open strings can join and form a single string or the two endpoints of an open string can joint to make a closed string and so on. But it turns out that the story is actually similar to what we have in perturbation theory of QFT where we sum over different Feynmann diagrams with different numbers of interaction vertices. In string theory perturbative expansion we need to sum over different topologies of the worldsheet. For closed string the expansion is characterized by the number of "handles" or genus h which is related to Euler characteristic see figure 5.1. But to calculate these are extremely difficult so I will not talk about this in more detail since it is not very important for AdS/CFT. For a detailed explanation see chapter 6 of [36].

5.2 Superstring theory

The theory we considered so far has a few issues with it. One we already mentioned is that vacuum states have negative mass. The other issue we have not discussed is the fermionic degrees of freedom. So far there have been none. But fermions arises naturally if we introduce supersymmetry in the picture [2]. Supersymmetry not only gives us fermions, but also solves the tachyon problem. Here the spacetime coordinates (bosons⁶) $X^\mu(\sigma)$ are related by supersymmetry to their fermionic partners $\Psi^\mu(\sigma)$ which are two-component spinors

$$\Psi^\mu(\sigma) = \begin{pmatrix} \psi_-^\mu(\sigma) \\ \psi_+^\mu(\sigma) \end{pmatrix}. \quad (5.28)$$

In conformal gauge (5.6) the supersymmetric Polyakov action in d -dimensional flat spacetime $\eta_{\mu\nu}$ is given by [1]

$$\mathcal{S} = -\frac{1}{4\pi\alpha'} \int d^2\sigma \eta^{\alpha\beta} (\partial_\alpha X^\mu \partial_\beta X^\nu + i\bar{\Psi}^\mu \gamma_\alpha \partial_\beta \Psi^\nu) g_{\mu\nu} \quad (5.29)$$

with γ_α being the Dirac matrices of the world sheet. The action (5.29) is invariant under the following supersymmetry transformations

$$\delta_\epsilon X^\mu = \bar{\epsilon} \Psi^\mu, \quad \delta_\epsilon \Psi^\mu = \gamma_\alpha \partial_\alpha X^\mu \epsilon. \quad (5.30)$$

Let's just look at the fermionic part of this action which can be rewritten in the lightcone coordinates $\sigma^\pm$ as

$$\mathcal{S}_f = \frac{i}{2\pi\alpha'} \int d^2\sigma (\psi_-^\mu \partial_+ \psi_{\mu-} + \psi_+^\mu \partial_- \psi_{\mu+}). \quad (5.31)$$

As before the equations of motion describe left and right-moving modes

$$\partial_\pm \psi_\mp^\mu = 0 \quad (5.32)$$

and we get boundary terms from integration by parts which should vanish

$$(\psi_-^\mu \delta \psi_{\mu-} - \psi_+^\mu \delta \psi_{\mu+})|_0^{\sigma_0} = 0. \quad (5.33)$$

First consider open strings:

To satisfy the boundary conditions for open strings we can choose $\psi_+^\mu(\tau, 0) = \psi_-^\mu(\tau, 0)$ and then we are left with two choices

$$\psi_+^\mu(\tau, \sigma_0) = \psi_-^\mu(\tau, \sigma_0) \quad (5.34)$$

⁶If we look at the Polyakov action (5.4), from the perspective of the worldsheet it just looks like d scalars coupled to $2d$ gravity.

or

$$\psi_{+}^{\mu}(\tau, \sigma_0) = -\psi_{-}^{\mu}(\tau, \sigma_0). \quad (5.35)$$

These boundary conditions correspond to two different sectors of the theory. Solutions that satisfy the (5.34) condition belong to **Neveu-Schwarz** (NS) sector. The ones that satisfy (5.35) belong to **Ramond** (R) sector [41]. We can Fourier expand the solutions that satisfy these boundary conditions. For Ramond sector the expansion reads

$$\psi_{\pm}^{\mu}(\sigma) = \frac{1}{\sqrt{2}} \sum_{n \in \mathbb{Z}} d_n^{\mu} e^{-in\sigma_{\pm}} \quad (5.36)$$

and for Neveu-Schwarz sector we have

$$\psi_{\pm}^{\mu}(\sigma) = \frac{1}{\sqrt{2}} \sum_{n \in \mathbb{Z} - \frac{1}{2}} b_n^{\mu} e^{-in\sigma_{\pm}}. \quad (5.37)$$

The fermionic strings are then quantized by promoting the coefficients d_n^{μ} and b_n^{μ} to operators and imposing anti-commutation relations

$$\{d_n^{\mu}, d_m^{\nu}\} = \eta^{\mu\nu} \delta_{n,-m}, \quad \{b_n^{\mu}, b_m^{\nu}\} = \eta^{\mu\nu} \delta_{n,-m} \quad (5.38)$$

with the rest vanishing [1]. Again, one can see that the time component of these commutation relations gives rise to negative norm states. But again we can use (super-)Virasoro constraints to decouple those from the theory. Similarly as for the bosonic case we can define a vacuum state that is annihilated by the annihilation operator. Then we can build the rest of the representation space by acting on it with creation operators⁷. We can now calculate the mass spectrum of the states. The ground state of the NS sector is still tachyonic since $M^2 = -\frac{1}{2\alpha'}$. We can deal with this by consistently truncating the spectrum of the states based on the whether the state has an odd or an even number of creation operators applied on the vacuum. This prescription is called **Gliozzi, Schrek, Olive projection** (GSO). We only keep the odd or even number of operators applied to the vacuum and these are referred to as having positive or negative G-parity [41]. The first excited state transforms under the group $SO(d-2)$ as a vector, so again this implies that the state must be massless. Because the mass for this state is given by

$$M^2 = \frac{1}{\alpha'} \left(\frac{1}{2} - \frac{d-2}{16} \right) \quad (5.39)$$

we require that $d = 10$ for superstring theory. In the R sector the ground and excited states transform as massless spinors. In general, open strings are classified by their representation of the $SO(8)$ group.

Now consider closed strings:

States of closed strings are constructed from left and right moving modes. Each mode can have two different choices for satisfying the boundary conditions. Hence, we have four different sectors: R-R, NS-NS, R-NS and NS-R. The former two sectors are spacetime bosons and the latter two are spacetime fermions. The lowest closed string states are obtained from two open string states and depend on their GSO projection. The theories that are relevant for AdS/CFT are the **type IIA** and **type IIB**. The sectors contained in each of these theories are [1]

- Type IIA: (NS+, NS+), (R+, NS+), (NS+, R-), (R+, R-).
- Type IIB: (NS+, NS+), (R+, MS+), (NS+, R+), (R+, R+).

Where the $\pm$ denotes the G-parity or the fermion number from the GSO projection. In 10 dimensions the ground state of NS+ sector transforms in the fundamental representation of $SO(8)$ - $\mathbf{8}_V$. The R+ and R- ground states transform in spinorial representations with different chirality $\mathbf{8}$ and $\mathbf{8}'$. The NS- ground state transforms under the singlet representation and is tachyonic.

Using Young tableau we can decompose the closed string sectors into irreducible representations of $SO(8)$. These are summarized in table 5.1. Where $\mathbf{28}$ represents a two-form, $\mathbf{56}_t$ is a three-form, $\mathbf{35}_+$ a four-form. The $\mathbf{35}$ is a symmetric rank two tensor with vanishing trace. And the $\mathbf{56}$ and $\mathbf{56}'$ are vector spinors that we identify with gravitinos, the superpartners of a gravitons [8]. So the two theories consist of fields that transform in these representations. For example the type IIB string theory contains the following representations

$$\mathbf{1}^2 \oplus \mathbf{28}^2 \oplus \mathbf{35} \oplus \mathbf{35}_+ \oplus \mathbf{8}'^2 \oplus \mathbf{56}^2 \quad (5.40)$$

⁷Creation operators are d_n^{μ} and b_m^{μ} with $n, m < 0$

Sector	$SO(8)$ representations
NS+ $\otimes$ NS+	$\mathbf{8}_V \otimes \mathbf{8}_V = \mathbf{1} \oplus \mathbf{28} \oplus \mathbf{35}$
NS+ $\otimes$ R-	$\mathbf{8}_V \otimes \mathbf{8}' = \mathbf{8} \oplus \mathbf{56}'$
NS+ $\otimes$ R+	$\mathbf{8}_V \otimes \mathbf{8} = \mathbf{8} \oplus \mathbf{56}$
R+ $\otimes$ R-	$\mathbf{8} \otimes \mathbf{8}' = \mathbf{8}_V \oplus \mathbf{56}_t$
R+ $\otimes$ R+	$\mathbf{8} \otimes \mathbf{8} = \mathbf{1} \oplus \mathbf{28} \oplus \mathbf{35}_+$

Table 5.1: Irreducible representations of different sectors of superstring theory of closed strings [1].

5.2.1 Supergravity action

Lets look at low-energy action we can write down using these massless closed string states. This is the action of **supergravity**. One way of writing the action of supergravity is just considering the bosonic part of the full action⁸. Which reads [1]

$$\begin{aligned} \mathcal{S}_{IIB} = \frac{1}{2\tilde{\kappa}_{10}^2} & \left(\int d^{10}X \sqrt{-g} (e^{-2\phi} (R + 4\partial_\mu \phi \partial^\mu \phi - \frac{1}{2}|H_{(3)}|^2) \right. \\ & \left. - \frac{1}{2}|F_{(1)}|^2 - \frac{1}{2}|\tilde{F}_{(3)}|^2 - \frac{1}{4}|\tilde{F}_{(5)}|^2) - \frac{1}{2} \int C_{(4)} \wedge H_{(3)} \wedge F_{(3)} \right) \end{aligned} \quad (5.41)$$

where we have

$$2\tilde{\kappa}_{10}^2 = (2\pi)^7 \alpha', \quad (5.42)$$

$$\int d^{10}X \sqrt{-g} |F_{(p)}|^2 = \frac{1}{p!} \int d^{10}X \sqrt{-g} g_{\mu_1 \nu_1} \dots g_{\mu_p \nu_p} \tilde{F}^{\mu_1 \dots \mu_p} F^{\nu_1 \dots \nu_p} \quad (5.43)$$

and we have the following definitions for field strength tensors:

$$\begin{aligned} F_{(p)} &= dC_{(p-1)}, \quad H_{(3)} = dB_{(2)}, \quad \tilde{F}_{(3)} = F_{(3)} - C_{(0)}H_{(3)}, \\ \tilde{F}_{(5)} &= F_{(5)} - \frac{1}{2}C_{(2)} \wedge H_{(3)} + \frac{1}{2}B_{(2)} \wedge F_{(3)}. \end{aligned} \quad (5.44)$$

The field content is summarized in table 5.2.

Field	$SO(8)$ representations	
$g_{\mu\nu}$	$\mathbf{35}$	Graviton
$C_{(0)} + ie^{-\phi}$	$\mathbf{1}^2$	Axion-dilaton
$B_{(2)}, C_{(2)}$	$\mathbf{28}^2$	Two-form
$C_{(4)}$	$\mathbf{35}_+$	Four-form
$\Psi_{I\alpha}^\mu$	$\mathbf{56}'^2$	Majorana-Weyl gravitinos
$\lambda_{I\alpha}$	$\mathbf{8}'^2$	Majorana-Weyl dilatinos

Table 5.2: Type IIB supergravity fields and their representations

5.3 D-branes

Let us return to the discussion of boundary conditions for open strings. Recall that if the endpoint of a string satisfies Neumann boundary conditions in a particular direction, then it is free to move in that direction. If it satisfies Dirichlet, then it is fixed in that spacial direction. Thus, the endpoints of open strings are constrained to move on hypersurfaces called Dp -branes. Where p stands for the spacial dimensions of the hypersurface. Note that this brakes Lorentz invariance $SO(1, d-1) \rightarrow SO(1, p) \times SO(d-p-1)$.

How should we look at these objects? It turns out that D -branes are themselves dynamical objects. And we should be able to write down the action for them. Which is just the generalized Nambu-Goto action [36]

$$\mathcal{S} = -T_p \int d^{p+1}\xi \sqrt{-\det \gamma_{ab}} \quad (5.45)$$

⁸Another way to construct supergravity action is to consider local supersymmetry.

where T_p is the tension of the brane and γ again is the pull back

$$\gamma_{ab} = \frac{\partial X^\mu}{\partial \xi^a} \frac{\partial X^\nu}{\partial \xi^b} \eta_{\mu\nu} \quad (5.46)$$

and ξ^a ($a = 0, \dots, p$) are the coordinates of a $p + 1$ dimensional worldvolume that the D -brane sweeps in spacetime. This is the **Dirac action** which describes the transverse fluctuations of the brane that gives rise to massless scalar fields on it φ^I , $I = p + 1, \dots, D - 1$. But this is not the full story. Open strings are able to deform the D-brane which leads to gauge fields on the brane [24]. So the endpoints of a string are charged under these fields. However this is not present in (5.46). The more general form we can write down is the Dirac-Born-Infeld action⁹ (DBI) which is the bosonic part of a Dp -brane action:

$$\mathcal{S}_{DBI} = -\tau_p \int d^{p+1} \xi e^{-\phi} \sqrt{-\det(\gamma_{ab} + 2\pi\alpha' F_{ab})} \quad (5.47)$$

Where F_{ab} is the field strength of the $U(1)$ gauge field A_a that lives on the brane. We can consider constant dilaton field $e^\phi = g_s$, which can be identified with the string coupling constant. Then we see that it is related to the tension by $T_p = \frac{\tau_p}{g_s}$ ¹⁰. The above action includes the dynamics of the transverse fluctuations of the brane and the dynamics of the gauge field. Note that since $\mu, \nu = 0, \dots, d - 1$ it looks like we have d degrees of freedom. But we should only have $d - p - 1$ physical degrees of freedom that correspond to transverse fluctuations. However, we can use the reparametrization invariance of (5.47) to eliminate the unnecessary degrees of freedom which actually correspond to longitudinal fluctuations [36]. We can expand the DBI action in powers of α' ¹¹ to get

$$\mathcal{S}_{DBI} = -(2\pi\alpha')^2 \frac{\tau_p}{g_s} \int d^{p+1} \xi \left(\frac{1}{4} F^{ab} F_{ab} + \frac{1}{2} \partial_a \varphi^I \partial^a \varphi_I + \dots \right) \quad (5.48)$$

with $\varphi^I = \frac{X^I}{2\pi\alpha'}$. This is just the Maxwell action coupled to scalar fields. If we focus on the first part of (5.48) which is the Yang-Mills theory with $U(1)$ gauge group we can read off the coupling constant

$$g_{YM}^2 = \frac{g_s}{\tau_p (2\pi\alpha')} \quad (5.49)$$

5.3.1 Coincident D -branes

Let us expand on the previous ideas and consider N coincident D -branes. Then strings are able to stretch from one D -brane to another. This is characterized by **Chan-Paton** factors λ_{ij} which label strings stretching from i 'th brane to j 'th brane. The factors make up a Hermitian matrix of $U(N)$ Lie algebra, where N is the number of coincident D-branes [1]. The gauge field can then be expressed as $(A_a)^i_j$ which describes a $U(N)$ gauge symmetry. The scalars $(\varphi^I)^i_j$ then transform in the adjoint representation of $U(N)$. We can write down the non-Abelian action that describes the dynamics of N D-branes [36]

$$\mathcal{S} = -(2\pi\alpha')^2 \frac{\tau_p}{g_s} \int d^{p+1} \xi \text{Tr} \left(\frac{1}{4} F_{ab} F^{ab} + \frac{1}{2} \mathcal{D}_a \varphi^I \mathcal{D}^a \varphi_I - \frac{1}{4} \sum_{I \neq J} [\varphi^I, \varphi^J]^2 \right) \quad (5.50)$$

with covariant derivative

$$\mathcal{D}_a \varphi^I = \partial_a \varphi^I + i[A_a, \varphi^I] \quad (5.51)$$

and the field strength tensor

$$F_{ab} = \partial_a A_b - \partial_b A_a + i[A_a, A_b] \quad (5.52)$$

In Type II string theory not all Dp -branes are stable. We have that

- Type IIA has stable branes for even p 's.
- Type IIB has stable branes for odd p 's.

⁹The DBI action should also include the pullback of the bulk Kalb-Ramond field, but for simplicity we can assume it vanishes.

¹⁰ $\tau_p = (2\pi)^{-p} \alpha'^{-(p+1)/2}$

¹¹Using $\det(1 + M) = 1 - \frac{1}{2} \text{Tr}(M^2)$.

In flat spacetime Type IIB superstring theory is invariant under thirty-two supercharges in total. D -brane solutions are invariant under half of these supersymmetry generators. Theory with such number of generators is the maximally symmetric super Yang-Mills theory, where (5.50) is just the bosonic part of the action. The low-energy limit of $D3$ -branes has one vector field A_μ , six scalars φ^I together with four Weyl fermions. The theory that describes N of these branes is the $\mathcal{N} = 4$ super Yang-Mills theory with a gauge group $U(N)$ [1].

Chapter 6

AdS/CFT correspondence

Finally, we have worked through the main topics required to understand the first concrete example of the AdS/CFT correspondence that was given by Maldacena in his original paper [19]. The example we will work on is the duality between $\mathcal{N} = 4$ super-Yang-Mills in 3+1 dimensions with gauge group $SU(N)$ and type IIB string theory on $AdS_5 \times S^5$ with the following matching of the constants

$$g_{YM}^2 = 2\pi g_s, \quad 2g_{YM}^2 N = \frac{L^4}{\alpha'^2} \quad (6.1)$$

where L is the radius of curvature on string side, g_s is the string coupling constant, α' is related to the string length by $l_s^2 = \alpha'$, and g_{YM} is the coupling constant on the gauge side.

The statement above is referred to as the **strongest form** of the correspondence, i.e. valid for any values of the constants. However, calculations for this form are very difficult to perform. So we will only focus on the **weak form** of the duality by taking some limits that will simplify things a great deal. Firstly, we will work in the limit $g_s \ll 1$ since currently perturbation is the best way to understand string theory. On the gauge side this corresponds to $g_{YM} \ll 1$. Gauge theories simplify when taken large N limit ($N \rightarrow \infty$) as was noted by Gerard 't Hooft in 1974 [34]. This is referred to as the **'t Hooft limit** and actually AdS/CFT is a realization of his insight that by expanding a field theory in powers of $\frac{1}{N}$ it can be mapped to topological expansion of string worldsheet with a coupling $g_s \propto \frac{1}{N}$. A free parameter on the gauge side is the 't Hooft coupling $\lambda = g_{YM}^2 N$ which in the weak form of the correspondence is taken to be large and corresponds to strongly coupled field theories. This implies that $\frac{\sqrt{\alpha'}}{L} \rightarrow 0$ and the string length becomes very small which can be approximated as a point particle. The resulting theory is type IIB supergravity on $AdS_5 \times S^5$. Even though we take these limits in the following discussion the correspondence is believed to hold for any values of these constants.

In this chapter we motivate the correspondence by looking at two different faces of D-branes. Then we reinforce the arguments by performing some checks. Explicitly we check that the symmetries and representations match on both sides and we calculate correlation functions and compare them to those obtained in chapter 2. For the basic outline of this part of the dissertation I referred to chapter 5 of [1]. For the section on symmetry I mainly used [8]. For the discussion on representation mapping I followed the discussion of [9].

6.1 Different perspectives on D3-branes

The AdS/CFT correspondence can be motivated by looking at D3-branes in different ways depending on the value of $g_s N$ [19]. In the low energy regime these are commonly referred to as **closed** or **open** string perspectives.

- **Open string:** This is the view we have used so far. In this perspective, the D3-branes are viewed as hyperplanes where open strings end. This perspective is valid for small coupling $g_s N \ll 1$. Where N is the number of coincident branes or the number of the gauge group $U(N)$.
- **Closed string:** Here D3-branes are viewed as solutions of supergravity (superstring theory in the limit of low energy). They are viewed as massive objects that curve spacetime. This perspective is valid for $g_s N \gg 1$.

6.1.1 Weak coupling

First let's look at the open string perspective of type IIB superstring theory with N coincident $D3$ -branes embedded in flat ten-dimensional Minkowski spacetime. In perturbative regime the theory consists of open strings that corresponds to the excitations of $D3$ -branes, and closed strings that are identified with excitations of the whole ten-dimensional space. In this low energy limit we are effectively considering massless excitations. The total action can be written as the sum of actions for closed and open strings and their interaction [1]

$$\mathcal{S} = \mathcal{S}_{closed} + \mathcal{S}_{open} + \mathcal{S}_{int}. \quad (6.2)$$

Writing the metric as

$$g_{\mu\nu} = \eta_{\mu\nu} + \kappa h_{\mu\nu} \quad (6.3)$$

where $h_{\mu\nu}$ is a perturbation of the metric and $2\kappa^2 = (2\pi)^7 \alpha'^2 g_s^2$. Then the closed part of the action can be obtained by expanding the supergravity action. Schematically the action of the metric fluctuations to lowest order can be written as

$$\mathcal{S}_{closed} \sim -\frac{1}{2} \int d^{10}x \partial_\mu h \partial^\mu h + \mathcal{O}(\kappa). \quad (6.4)$$

The open string and interaction parts of the action (6.2) can be obtained by expanding the DBI action (5.47). The open part we already saw before

$$\mathcal{S}_{open} = -\frac{1}{2\pi g_s} \int d^{p+1}\xi \text{Tr} \left(\frac{1}{4} F_{ab} F^{ab} + \frac{1}{2} \mathcal{D}_a \varphi^I \mathcal{D}^a \varphi_I - \frac{1}{4} \sum_{I \neq J} [\varphi^I, \varphi^J]^2 + \mathcal{O}(\alpha') \right). \quad (6.5)$$

For the interaction part we also need to expand the dilaton field $e^{-\phi}$. Then to leading order in α' the interaction part is

$$\mathcal{S}_{int} = -\frac{1}{2\pi g_s} \int d^{p+1}\xi \text{Tr} \left(\frac{1}{4} F_{ab} F^{ab} \phi + \dots \right). \quad (6.6)$$

Important thing to note is that $\mathcal{S}_{int}$ ¹ is of order κ since we would need to rescale the dilaton field for canonical normalization [1]. Now let's take the limit $\alpha' \rightarrow 0$ ($\implies \kappa \rightarrow 0$)². Then we see that the interaction part of full action vanishes (since it is of order κ) and the theory decouples. Closed part of the action describes ten-dimensional supergravity. And as we mentioned before the free part of the action is just the bosonic part of $\mathcal{N} = 4$ super-Yang-Mills action.

6.1.2 Strong coupling

Now let's look at the closed string perspective ($g_s N \gg 1$). Here the $D3$ -branes are viewed as massive objects that curve spacetime. In type IIB theory the closed strings propagate in the curved geometry that is sourced by the branes. The metric that solves equations of motion of supergravity has the following form [1]

$$ds^2 = H(r)^{-\frac{1}{2}} \eta_{ab} dx^a dx^b + H(r)^{\frac{1}{2}} (dr^2 + r^2 d\Omega_5^2) \quad (6.7)$$

where η_{ab} $a = 0, \dots, 3$ is the metric along worldvolume of the $D3$ -brane and $d\Omega_5^2$ is the metric of a unit 5-sphere S^5 . Radial coordinate r is defined by $r^2 = \sum_{i=4}^9 x_i^2$ and

$$H(r) = 1 + \frac{L^4}{r^4}. \quad (6.8)$$

The solution also comes with a self-dual five-form. Using the arguments from string theory that the flux of this five-form on S^5 is quantized [9], we can show that

$$L^4 = 4\pi g_s N \alpha'^2. \quad (6.9)$$

¹The action implies in the lowest order non-trivial interaction a dilaton decays into two bosons.

²More precisely we should really take the limit while holding $\frac{r}{\alpha'}$ constant. Where r is related to the position of the branes. This is called the **Maldacena limit**.

There are two limiting cases we can take: $r \gg L$ or $r \ll L$. In the former case $H(r) \sim 1$ so the metric (6.7) simplifies to a ten-dimensional Minkowski metric. In the other case $H(r) \sim \frac{L^4}{r^4}$ and the metric is

$$ds^2 = \frac{L^2}{z^2}(\eta_{ab}dx^a dx^b + dz^2) + L^2 d\Omega_5^2 \quad (6.10)$$

with $z = \frac{L^2}{r}$. The region $r \ll L$ is referred to as the **near-horizon** region or **throat**. The first term in (6.10) is just AdS_5 so the spacetime is $AdS_5 \times S^5$. We can show that taking the low energy limit the two theories decouple as before. On one side we have type IIB supergravity in ten-dimensions and on the other we have type IIB supergravity on $AdS_5 \times S^5$.

6.2 Maldacena's argument

In the previous section we looked at two perspectives on the $D3$ -branes. In the low energy limit we saw that in both cases we get two decoupled effective theories. This is summarized in table 6.1. But this is just two different perspectives of the same physical theory. And since on both sides

Open string perspective $g_s N \ll 1$	Closed string perspective $g_s N \gg 1$
Type IIB supergravity in $\mathbb{R}^{9,1}$	Type IIB supergravity in $\mathbb{R}^{9,1}$
$\mathcal{N} = 4$ super-Yang-Mills in $\mathbb{R}^{3,1}$	Type IIB supergravity in $AdS_5 \times S^5$

Table 6.1: Decoupled theories in open and closed string perspectives in low energy limit.

we have a type IIB supergravity in flat ten-dimensional spacetime, Maldacena proposed that the two remaining theories are dual to each other. This should be true even at higher energies. More precisely, the $\mathcal{N} = 4$ super-Yang-Mills and type IIB string theory on $AdS_5 \times S^5$ are equivalent and describe the same physics from different points of view [9].

6.3 Checks of the correspondence

There is no rigorous proof of the correspondence and it will most likely remain this way for a while because we still lack the understanding of strongly coupled theories. However, it is widely believed that the conjecture holds regardless. Let us see what motivates us to believe that the correspondence should hold.

6.3.1 Symmetries

First, the most trivial thing we can ask is whether the symmetries of the two theories agree? We already discussed the symmetries of $\mathcal{N} = 4$ super-Yang-Mills. The theory is invariant under supergroup $PSU(2, 2|4)$. The bosonic subgroups of this supergroup are the conformal group $SO(4, 2) \sim SU(2, 2)$ and the $SO(6)_R \sim SU(4)_R$ R-symmetry group. The theory also has $\mathcal{N} = 4$ supersymmetry generated by superconformal and Poincaré supercharges S_α 's and Q_α 's which generate the fermionic subgroup of $PSU(2, 2|4)$.

The isometry group of AdS_5 space is $SO(4, 2)$ and for S^5 it is $SO(6)$. So we immediately see that the bosonic subgroups matches. The remaining fermionic symmetries also match because in type IIB string theory the $D3$ -brane solutions preserves 16 Poincaré supercharges (it is a 1/2 BPS solution that preserves half of the Poincaré supercharges). Also in the AdS limit there are additional 16 conformal supersymmetries that are broken by the geometry of $D3$ -branes [8]. Hence the symmetries of both theories agree as both sides are invariant under the full supergroup $PSU(2, 2|4)$.

6.3.2 Representation mapping

Since the symmetry groups of both theories match, we can talk about their representations. We should expect that the representations of the symmetry group should also coincide if the two theories are equivalent. Not only that there must be a map between the representations. This is commonly referred to as the **field-operator map** where the operators of super-Yang-Mills theory are identified with fields in type IIB supergravity on $AdS_5 \times S^5$ spacetime which transform in the

same representation of the symmetry group. We already discussed the representations of super-Yang-Mills in chapter 3. The important class of operators are the single-traced operators since these correspond to single-particle states in AdS [8]. To find the irreducible representations on the gravity side we perform something called the **Kaluza-Klein** compactification of $AdS_5 \times S^5$ onto S^5 . The fields $\varphi(\vec{x}, z, \Omega_5)$ living on the full space can be decomposed into spherical harmonics $Y_l(\Omega_5)$ living on S^5 and fields $\varphi_l(\vec{x}, z)$ living in AdS_5 spacetime. The decomposition reads

$$\varphi(\vec{x}, z, \Omega_5) = \sum_{l=0}^{\infty} \varphi_l(\vec{x}, z) Y_l(\Omega_5) \quad (6.11)$$

where $\vec{x}, z$ are the coordinates of AdS_5 and Ω_5 are the coordinates of S^5 [9]. The fields φ_l are Kaluza-Klein modes that have a dual operator in the gauge theory. For example, recall the expression for a single-trace operator

$$\mathcal{O}_{\Delta}(x) = \text{STr}(\phi^{i_1}(x) \dots \phi^{i_{\Delta}}(x)). \quad (6.12)$$

This operator is dual to a scalar field $s^l(\vec{x}, z)$ that can be constructed from the metric and the five-form of the supergravity Kaluza-Klein modes [1]. These fields satisfy

$$\square_{AdS_5} s^l(\vec{x}, z) = -\frac{1}{L^2} l(l-4) s^l(\vec{x}, z). \quad (6.13)$$

$\mathcal{O}_{\Delta}(x)$ and $s^l(\vec{x}, z)$ are in the same representation provided $l = \Delta$. We can insert the expansion (6.11) into the equations of motions to determine the relation between the mass and the scaling dimension. The relationship is summarized for various supergravity fields in table 6.2. Similarly

Field	Relation
Scalars	$m^2 L^2 = \Delta(\Delta - 4)$
Spin 1/2, 3/2	$ m L = \Delta - 2$
p-form	$m^2 L^2 = (\Delta - p)(\Delta + p - 4)$
Massive spin 2	$m^2 L^2 = \Delta(\Delta - 4)$
Massless spin 2	$m^2 L^2 = 0$
Rank s symmetric traceless tensor	$m^2 L^2 = (\Delta + s - 2)(\Delta - s - 2)$

Table 6.2: The relationship between mass and scaling dimensions of different field types [8]

the map exists not just for primary operator but also for their descendants.

Another important point to note is that the supergravity fields at the boundary can be interpreted as the source for the dual operators on the field theory side [1]. As an example consider again the action for a scalar field in AdS

$$\mathcal{S} = -\frac{C}{2} \int dz d^d x \sqrt{-g} (g^{mn} \partial_m \phi \partial_n \phi + m^2 \phi^2) \quad (6.14)$$

with the metric expressed in Poincaré coordinates

$$g_{mn} dx^m dx^n = \frac{L^2}{z^2} (dz^2 + \eta_{\mu\nu} dx^\mu dx^\nu). \quad (6.15)$$

Choosing to decompose the field into plane wave modes $\phi(\vec{x}, z) = e^{ip^\mu x_\mu} \phi_p(z)$ we get two independent solutions for $\phi_p(z)$ at the boundary $z \rightarrow 0$. Which are $\phi_p(z) \sim z^{\Delta_+}$ and $\phi_p(z) \sim z^{\Delta_-}$. With $\Delta_{\pm}$ being the roots of the relation between mass of the field and the conformal dimension

$$\Delta_{\pm} = \frac{d}{2} \pm \sqrt{\frac{d^2}{4} + m^2 L^2}. \quad (6.16)$$

The solution z^{Δ_+} is normalizable, i.e. the action evaluated on the solution is finite. And the other solution is non-normalizable. The full solution near the boundary can be approximated as

$$\phi(\vec{x}, z) \sim \phi_{(0)} z^{\Delta_-} + \phi_{(+)} z^{\Delta_+} + \dots \quad (6.17)$$

The normalizable mode $\phi_{(+)}$ is identified with the vacuum expectation value of the dual operator $\mathcal{O}_{\Delta}$ and the other mode $\phi_{(0)}$ is identified with the source of the dual operator [1].

6.3.3 Correlation functions

Having discussed the relationship between the fields and operators of both theories we can ask whether the correlation functions also agree on both sides. First let us recall how one can calculate correlations functions in quantum field theory for a field ϕ . A formal method in calculating correlators is to introduce an object called **generating functional**

$$Z[J] = \int \mathcal{D}\phi e^{iS[\phi] + i \int d^d x J(x)\phi(x)}. \quad (6.18)$$

Where we perturbed the action in the path integral by a source $J(x)$. We can functionally differentiate the generating functional with respect to source to obtain correlation functions [25]

$$\langle T\phi(x_1)\dots\phi(x_n) \rangle = \frac{1}{Z[0]} \frac{(-i)^n \delta^n Z[J]}{\delta J(x_1)\dots\delta J(x_n)} \Big|_{J=0}. \quad (6.19)$$

However in this case we would generate all correlation functions (Connected+disconnected+vacuum bubbles). So we define

$$\frac{Z[J]}{Z[0]} = e^{iW[J]} = \langle e^{i \int d^d x J(x)\phi(x)} \rangle \quad (6.20)$$

where the $W[J]$ generates only the connected Feynman diagrams

$$\langle T\phi(x_1)\dots\phi(x_n) \rangle_{connected} = \frac{(-i)^n \delta^n W[J]}{\delta J(x_1)\dots\delta J(x_n)} \Big|_{J=0}. \quad (6.21)$$

Recall from the previous section that the bulk field near the boundary $\phi_{(0)}$ has the interpretation of the source for a dual operator $\mathcal{O}$. So we can write the generating functional for the boundary quantum field theory as

$$\frac{Z_{CFT}[\phi_{(0)}]}{Z_{CFT}[0]} = e^{-W_{CFT}[\phi_{(0)}]} = \langle e^{\int d^d x \phi_{(0)}(x)\mathcal{O}(x)} \rangle_{CFT}. \quad (6.22)$$

Note that we defined (6.22) in Euclidean signature. The AdS/CFT correspondence precisely states that the generating functionals on both sides are equal

$$Z_{CFT}[\phi_{(0)}] = Z_{string}[\phi] \Big|_{z \rightarrow 0}. \quad (6.23)$$

The generating functional on the gravity side can be approximated as a saddle point of the superstring partition function [1] which is given by

$$Z_{string}[\phi_{(0)}] \sim e^{-S_{SUGRA}} \Big|_{z \rightarrow 0} \quad (6.24)$$

which implies that the generating functional of the connected correlation functions in four dimensions is identified with the supergravity action on AdS_5 where the fields are taken on the boundary $z \rightarrow 0$

$$W_{CFT}[\phi_{(0)}] = S_{SUGRA}[\phi] \Big|_{z \rightarrow 0}. \quad (6.25)$$

We can calculate the correlation functions of the operators $\mathcal{O}_i$ by

$$\langle \mathcal{O}_1(x_1)\mathcal{O}_2(x_2)\dots\mathcal{O}_n(x_n) \rangle = - \frac{\delta^n W[\phi_{(0)}^i]}{\delta \phi_{(0)}^1(x_1)\delta \phi_{(0)}^2(x_2)\dots\delta \phi_{(0)}^n(x_n)} \Big|_{\phi_{(0)}^i=0} \quad (6.26)$$

Which is equivalent to, as the correspondence states, to calculating tree level diagrams on the gravity side. There exists a set of rules to compute these diagrams, which are reminiscent of Feynman rules in QFT. The diagrams themselves are called the **Witten diagrams** and the rules to compute them can be summarized by the following points [1]

- The sources $\phi_{(0)}$ are represented by a circle (the boundary of AdS). And the AdS spacetime is the interior of that circle.
- **Boundary-to-boundary** propagator connects two points on the boundary of the circle.
- **Bulk-to-boundary** propagator connects points on the boundary of the circle with an interaction vertex in the bulk³.

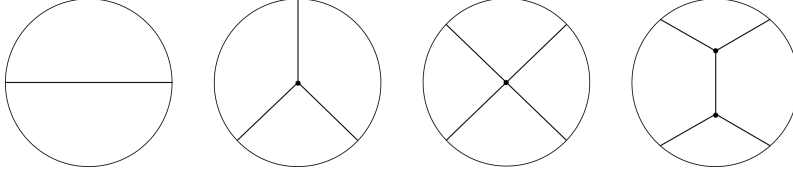


Figure 6.1: Examples of Witten diagrams. From left to right: two-point boundary-to boundary propagator; three-point bulk-to-boundary propagators; four-point bulk-to-boundary propagators; four-point bulk-to-boundary propagators and a bulk-to-bulk propagator.

- **Bulk-to-bulk** propagator connects two interaction vertices in the bulk.

Example diagrams are given in figure 6.1 The explicit expressions for the propagators of scalar field (dual to operator of dimension Δ) may be derived from operator $\square_g - m^2$ on AdS [8]. The bulk-to-bulk propagator G_Δ is defined by

$$(\square_g - m^2)G_\Delta(\vec{x}, z; \vec{y}, w) = \frac{\delta(z-w)\delta^d(\vec{x}-\vec{y})}{\sqrt{g}} \quad (6.27)$$

where $\vec{x}, \vec{y}$ denote the coordinates on the boundary and z, w denotes the coordinates of the bulk. The solution to (6.27) is rather messy and can be expressed in hypergeometric functions as

$$G_\Delta(\vec{x}, z; \vec{y}, w) = \frac{C_\Delta}{2^\Delta (2\Delta - d)} \xi^\Delta \cdot {}_2F_1\left(\frac{\Delta}{2}, \frac{\Delta+1}{2}; \Delta - \frac{d}{2} + 1, \xi^2\right) \quad (6.28)$$

where

$$C_\Delta = \frac{\Gamma(\Delta)}{\pi^{d/2} \Gamma(\Delta - \frac{d}{2})}, \quad \xi = \frac{2zw}{z^2 + w^2 + (\vec{x} + \vec{y})^2}. \quad (6.29)$$

The bulk-to-boundary propagator K_Δ is defined by taking the boundary limit of one of its coordinates

$$K(\vec{x}, z; \vec{y}) = \lim_{w \rightarrow 0} \frac{2\Delta - d}{w^\Delta} G_\Delta(\vec{x}, z; \vec{y}, w). \quad (6.30)$$

The explicit expression is

$$K(\vec{x}, z; \vec{y}) = C_\Delta \left(\frac{z}{z^2 + (\vec{x} - \vec{y})^2} \right)^\Delta. \quad (6.31)$$

Now the boundary-to-boundary propagator can be obtained from the boundary behaviour of the bulk-to-boundary propagator

$$B(\vec{x}; \vec{y}) = \lim_{z \rightarrow 0} z^{-\Delta} K_\Delta(\vec{x}, z; \vec{y}) \sim \frac{1}{(\vec{x} - \vec{y})^{2\Delta}}. \quad (6.32)$$

But the boundary-to-boundary propagator is just a two-point function of the operators in the field theory $\langle \mathcal{O}(\vec{x}) \mathcal{O}(\vec{y}) \rangle$. We already calculated this two-point function back in chapter 2 and of course this has the same form as what we had in (2.43). The two-point function can also be computed using (6.26) or equivalently using the gravity action $\mathcal{S}$

$$\langle \mathcal{O}(\vec{x}) \mathcal{O}(\vec{y}) \rangle = - \frac{\delta^2 \mathcal{S}|_{z \rightarrow 0}}{\delta \phi_{(0)}(\vec{x}) \delta \phi_{(0)}(\vec{y})} \Big|_{\phi_{(0)}=0}. \quad (6.33)$$

Since we do not care about interactions in this case we can write down the relevant part of $\mathcal{S}[\phi]$ in Euclidean signature as

$$\mathcal{S}[\phi] = \frac{C}{2} \int dz d^d x \sqrt{g} (g^{mn} \partial_m \phi \partial_n \phi + m^2 \phi^2). \quad (6.34)$$

We can integrate this action by parts and use the equations of motion to get the on-shell action which simplifies to

$$\mathcal{S}[\phi] = \frac{C}{2} \int dz d^d x \sqrt{g} (g^{zz} \phi(\vec{x}, z) \partial_z \phi(\vec{x}, z)) \Big|_{z=\epsilon}^{z=\infty}. \quad (6.35)$$

³The interaction terms are given by the supergravity action.

The integral evaluated at infinity vanishes but it diverges on $z \rightarrow 0$ hence the reason why we evaluate at $z = \epsilon$. The field solution to ϕ can be written in terms of the source as [1]

$$\phi(\vec{x}, z) = \int d^d y K_{\Delta}(\vec{x}, z; \vec{y}) \phi_{(0)}(\vec{y}) \quad (6.36)$$

then inserting this into (6.35) we get

$$\mathcal{S}[\phi] = -\frac{CL^{d-1}}{2\epsilon^{d-1}} \int d^d x d^d y d^d y' K_{\Delta}(\vec{x}, z; \vec{y}) \partial_z K_{\Delta}(\vec{x}, z; \vec{y}') \phi_{(0)}(\vec{y}) \phi_{(0)}(\vec{y}') \Big|_{z=\epsilon}. \quad (6.37)$$

Taking the derivative of the propagator, evaluating the result at $z \rightarrow 0$ and taking the double functional derivative as per (6.33) we get the expression for the two-point function

$$\langle \mathcal{O}(\vec{x}) \mathcal{O}(\vec{y}) \rangle = CL^{d-1} \frac{\Gamma(\Delta)}{\Gamma(\Delta - \frac{d}{2})} \frac{2\Delta - d}{\pi^{d/2} |\vec{x} - \vec{y}|^{2\Delta}}. \quad (6.38)$$

Which of course agrees with our previous results. Only here we used the supergravity action to compute the correlation function of a field theory.

We have only considered the weak form of the correspondence, however this can be extended for the the strongest form of the correspondence where we do not use the saddle point approximation of the partition function of string theory. However, we do not know the explicit form of Z_{string} . Presently it is not possible to give a rigorous proof of the AdS/CFT correspondence since we only understand quantum string theory using perturbation. Ragrdless of that, some very non trivial tests of the correspondence has been performed and the calculations on both sides always agrees [23].

Chapter 7

Conclusion

I have already mentioned in the introduction a few of the applications of the AdS/CFT correspondence. An excellent example is the study of confinement in QCD. Quarks behave as if they were connected by flux tubes and their energy scales with length. This flux can be described as a string in a dual theory, which gives familiar characteristics of confinement in certain geometries [5]. Another popular area of physics where the correspondence is realized is the study of strongly coupled systems in condensed matter theory. This is useful in two ways. The first is obviously that we can do simpler calculations in a dual theory, which has weak coupling. The other reason is that experiments of such systems are very accessible, so we can design experiments to test certain properties of the correspondence [14]. For me the most interesting development of the AdS/CFT correspondence is the study of the connection between the entanglement entropy and the entropy of a black hole, which is the generalization of Bekenstein-Hawking formula for the entropy of a black hole. This was done by Ryu and Takayanagi [29]. Their work gives a very explicit connection between quantum entanglement and the geometry of the bulk space in one higher dimension. Given more time I believe that this would be an interesting area to explore.

In this thesis I gave a basic introduction to AdS/CFT correspondence, which focuses on a particular example. Namely the one Maldacena proposed in his paper [19]. We focused on the weak form of the duality which states that $\mathcal{N} = 4$ super-Yang-Mills in 3+1 spacetime dimensions and gauge group $SU(N)$ is dual to type IIB supergravity on $AdS_5 \times S^5$. By dual I mean that they are equivalent and describe the same physics but from different perspectives.

I took a general approach to arrive at this statement that is characteristic to many reviews on AdS/CFT [8][9][24][14][27]. For the majority of the thesis I followed [1]. This was supplemented by various other reviews on more specific subjects such as [3] for chapter on supersymmetry and [36] for string theory.

I started with a chapter on conformal symmetry where I extended the Poincaré symmetry group to include conformal transformations and introduced a field theory that is invariant under these transformations. In the next chapter another non-trivial extension of Poincaré was introduced by considering graded Lie algebra and fermionic generators which form the supersymmetry algebra. Then I defined the superconformal group that includes both of these extensions. In this chapter I went into slightly more detail by also introducing superspace and superfield formalism and briefly explained how one arrives at the $\mathcal{N} = 4$ super-Yang-Mills Lagrangian which is invariant under the superconformal group. Moving on to the gravity side, I first gave a short introduction on anti-de Sitter spacetime and various parametrizations and its symmetries. The symmetries gave a first hint of the correspondence since both AdS_{d+1} spacetime and the d -dimensional conformal group is invariant under $SO(d, 2)$ transformations. The introduction to AdS was useful for the next chapter where the goal was to arrive at the type IIB superstring theory and the low energy solution - supergravity. I felt that it was necessary to give the reader a vague idea on where it comes from, which included a very short review of the development of string theory in general. Furthermore, I wanted to give reasoning for a few popular aspects of string theory such as why we need so many dimensions and the need for supersymmetry. After I arrived at superstring theory I introduced a low-energy effective action of supergravity and talked about D-branes. The last part was important since it was used to motivate the example of AdS/CFT correspondence given above in the next chapter. This was done by looking at D-branes from two different perspectives. Finally, I made a few easy checks of the correspondence such as making sure that the symmetries on both

sides agree, matching the representations (field-operator map), using the supergravity action to calculate the two point correlation function on field theory side and comparing to results obtained in chapter 2 on CFT's.

The duality is a major step forward in understanding quantum gravity and string theory. It is an important tool in studying quantum field theories in strong coupling regime where perturbation theory is no longer valid. The principles of this idea is applicable in many different fields such as condensed matter physics, nuclear physics or cosmology. Despite the fact that the duality remains a conjecture it is and increasingly popular field of study.

Appendix A

Spinor notation

Here I will give a short introduction on spinor notation that is used in supersymmetry. For more detailed introduction see [3]. Let us define some relationships between a few Lie groups that will help us deal with representations of the Lorentz group. First, the Lorentz group can be expressed as

$$SO(1, 3) \sim SU(2) \times SU(2)^*. \quad (\text{A.1})$$

Also Lorentz group can be expressed as a double cover of the group $SL(2, \mathbb{C})$, i.e.

$$SO(1, 3) \sim SL(2, \mathbb{C})/\mathbb{Z}_2. \quad (\text{A.2})$$

We can arrange the representation of the Lorentz group by the representation of the $SU(2)$ labelled by spins. But first let us define a spinor that carries a representation of $SL(2, \mathbb{C})$

$$\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}. \quad (\text{A.3})$$

With the following transformation property under $M \in SL(2, \mathbb{C})$

$$\psi_\alpha \longrightarrow M_\alpha{}^\beta \psi_\beta \quad (\text{A.4})$$

where $\alpha, \beta \in \{1, 2\}$. The complex conjugate representation is defined by

$$\bar{\psi}_{\dot{\alpha}} \longrightarrow M^*{}_{\dot{\alpha}}{}^{\dot{\beta}} \bar{\psi}_{\dot{\beta}} \quad (\text{A.5})$$

and $\dot{\alpha}, \dot{\beta} \in \{1, 2\}$. These are known as **Weyl spinors**. Note that $M \neq CM^*C^{-1}$ for some C , so the fundamental and the conjugate representations are not equivalent. ψ and $\bar{\psi}$ both transform in an irreducible representation of the Lorentz group and they can be labeled by their $SU(2)$ transformations due to the isomorphism (A.1)

$$\psi_\alpha = \begin{pmatrix} \frac{1}{2}, 0 \end{pmatrix}, \quad \text{Left-handed spinor.} \quad (\text{A.6})$$

$$\bar{\psi}^{\dot{\alpha}} = \begin{pmatrix} 0, \frac{1}{2} \end{pmatrix}, \quad \text{Right-handed spinor.} \quad (\text{A.7})$$

The invariant $SU(2)$ tensors are

$$\epsilon_{\alpha\beta} = -\epsilon^{\alpha\beta} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \quad (\text{A.8})$$

It is used to raise and lower indices of ψ

$$\psi^\alpha = \epsilon^{\alpha\beta} \psi_\beta, \quad \psi_\alpha = \epsilon_{\alpha\beta} \psi^\beta. \quad (\text{A.9})$$

The invariant tensor of the conjugate representation is

$$\epsilon_{\dot{\alpha}\dot{\beta}} = -\epsilon^{\dot{\alpha}\dot{\beta}} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad (\text{A.10})$$

which is used to lower and raise indices of $\bar{\psi}$

$$\bar{\psi}^{\dot{\alpha}} = \epsilon^{\dot{\alpha}\dot{\beta}} \bar{\psi}_{\dot{\beta}}, \quad \bar{\psi}_{\dot{\alpha}} = \epsilon_{\dot{\alpha}\dot{\beta}} \bar{\psi}^{\dot{\beta}}. \quad (\text{A.11})$$

The convention of contracting two indices are from upper left to lower left for undotted indices $\psi^\alpha \chi_\alpha \equiv \psi \chi$. For dotted it is the other way round $\bar{\psi}_{\dot{\alpha}} \bar{\chi}^{\dot{\alpha}} \equiv \bar{\psi} \bar{\chi}$. Spinors are Grassmann (anti-commuting) so we have the following relations (note the convention of contraction of indices!)

$$\psi \chi = \chi \psi, \quad \bar{\psi} \bar{\chi} = \bar{\chi} \bar{\psi}. \quad (\text{A.12})$$

We also have objects with mixed indices of $SO(3,1)$ and $SL(2, \mathbb{C})$

$$(\sigma^\mu)_{\alpha\dot{\alpha}}, \quad \sigma^\mu = (\sigma_0, \sigma_1, \sigma_2, \sigma_3) \quad (\text{A.13})$$

where the σ 's are the Pauli matrices. These transform as

$$(\sigma^\mu)_{\alpha\dot{\alpha}} \longrightarrow M_\alpha{}^\beta (\sigma^\nu)_{\beta\dot{\beta}} (\Lambda^{-1})^\mu{}_\nu M^*_{\dot{\alpha}}{}^{\dot{\beta}} \quad (\text{A.14})$$

where $\Lambda \in SO(3,1)$. This is used to define a product of a dotted and undotted spinor

$$\psi \sigma^\mu \bar{\chi} \equiv \psi^\alpha (\sigma^\mu)_{\alpha\dot{\alpha}} \bar{\chi}^{\dot{\alpha}}. \quad (\text{A.15})$$

For a finite Lorentz transformation we define the generators of the group as

$$(\sigma^{\mu\nu})_\alpha{}^\beta = \frac{i}{4} (\sigma^\mu \bar{\sigma}^\nu - \sigma^\nu \bar{\sigma}^\mu)_\alpha{}^\beta, \quad (\text{A.16})$$

$$(\bar{\sigma}^{\mu\nu})_{\dot{\alpha}}{}^{\dot{\beta}} = \frac{i}{4} (\bar{\sigma}^\mu \sigma^\nu - \bar{\sigma}^\nu \sigma^\mu)_{\dot{\alpha}}{}^{\dot{\beta}}. \quad (\text{A.17})$$

Where $(\bar{\sigma}^\mu)^{\alpha\dot{\alpha}} = \epsilon^{\alpha\beta} \epsilon^{\dot{\alpha}\dot{\beta}} (\sigma^\mu)_{\beta\dot{\beta}} = (\sigma_0, -\vec{\sigma})$. Then a finite Lorentz transformation is

$$\psi_\alpha \longrightarrow (e^{-\frac{i}{2} \omega_{\mu\nu} \sigma^{\mu\nu}})_\alpha{}^\beta \psi_\beta \quad (\text{A.18})$$

$$\bar{\psi}_{\dot{\alpha}} \longrightarrow (e^{-\frac{i}{2} \omega_{\mu\nu} \bar{\sigma}^{\mu\nu}})^{\dot{\alpha}}{}_{\dot{\beta}} \bar{\psi}^{\dot{\beta}} \quad (\text{A.19})$$

The **Dirac spinor** notation is also widely used so it is reasonable to mention that one can express a Dirac spinor as

$$\Psi = \begin{pmatrix} \psi_\alpha \\ \bar{\chi}^{\dot{\alpha}} \end{pmatrix} \quad (\text{A.20})$$

Which implies that Dirac spinors transforms in a reducible representation of the Lorentz group.

Appendix B

Grassmann variables

The simplest Grassmann number θ is defined such that

$$\theta^2 = 0. \quad (\text{B.1})$$

This stems from the fact that they are anti-commuting variables $\{\theta, \theta\} = 0$. A function of a Grassmann number has the following expansion

$$f(\theta) = \sum_{n=0}^{\infty} f_n \theta^n = f_0 + f_1 \theta + f_2 \theta^2 + \dots = f_0 + f_1 \theta. \quad (\text{B.2})$$

So it is linear. The derivatives with respect to θ are

$$\frac{\partial f}{\partial \theta} = f_1, \quad \frac{\partial^2 f}{\partial \theta^2} = 0. \quad (\text{B.3})$$

We define the integral with respect to θ such that it preserves the following property

$$\int d\theta \frac{\partial f}{\partial \theta} = 0 \quad (\text{B.4})$$

Then we define

$$\int \theta d\theta = 1 \quad (\text{B.5})$$

which implies that θ acts as a delta function $\delta(\theta) = \theta$. Then we have

$$\int f(\theta) d\theta = f_1 \quad (\text{B.6})$$

Hence we get that the integral of $f(\theta)$ is the same as the derivative

$$\int d\theta = \frac{\partial}{\partial \theta} \quad (\text{B.7})$$

This can be expanded for multiple Grassmann numbers. For a more general discussion see for example [4].

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Introduction to the AdS/CFT Correspondence

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Abstract

In this talk we give a brief introduction to the AdS/CFT correspondence, describe some tests, and mention some recent developments.

1 Introduction

The AdS/CFT correspondence is one of the most significant results that string theory has produced. It refers to the existence of amazing dualities between theories with gravity and theories without gravity, and is also sometimes referred to as the gauge theory-gravity correspondence. The prototype example of such a correspondence, as originally conjectured by Maldacena [1], is the exact equivalence between type IIB string theory compactified on $AdS_5 \times S^5$, and four-dimensional $N = 4$ supersymmetric Yang-Mills theory. The abbreviation AdS_5 refers to an anti-de Sitter space in five dimensions, S^5 refers to a five-dimensional sphere. Anti-de Sitter spaces are maximally symmetric solutions of the Einstein equations with a negative cosmological constant. The large symmetry group of 5d anti-de Sitter space matches precisely with the group of conformal symmetries of the $N = 4$ super Yang-Mills theory, which for a long time has been known to be conformally invariant. The term AdS/CFT correspondence has its origin in this particular example, CFT being an abbreviation for conformal field theory. Since then, many other examples of gauge theory/gravity dualities have been found, including ones where string theory is not compactified on an anti-de Sitter space and where the dual field theory is not conformal. Nevertheless, all these dualities are often still referred to as examples of the AdS/CFT correspondence. For more background information and more details, see the various reviews [2, 3, 4, 5, 6, 7, 8, 9, 10, 11].

At first sound, it is quite startling that a duality between a theory with gravity and a theory without gravity could exist. But what is a consistent theory of gravity anyway? String theory provides a consistent framework to compute finite quantum corrections to classical general relativity, but the full non-perturbative structure of string theory is not very well understood. The AdS/CFT correspondence relates a theory with gravity in d dimensions to a local field theory without gravity in $d - 1$ dimensions. If true, it implies that actual quantum degrees of freedom of gravity

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cannot resemble the degrees of freedom of a local field theory defined on the same space. That this has to be true follows among other things from the existence of black holes, as we briefly review in the next section. If the degrees of freedom in gravity would be local, one would imagine that one can have arbitrarily large volumes with fixed energy-density. However, we know that already in classical gravity this is not true: a fixed energy-density in a sufficiently large volume will collapse into a black hole.

In fact, hints for the existence of gauge theory/gravity duality have been around for quite some time, most notably in the case of three dimensions¹. Gravity in three dimensions can, modulo various subtleties, be described by a Chern–Simons theory [12, 13]. Chern–Simons theory is a topological field theory and when put on manifold with a boundary reduces to a $1 + 1$ dimensional field theory on the boundary [14, 15]. For compact gauge groups, a precise relation between Hilbert spaces and correlation functions can be established. For non-compact gauge groups such as the ones relevant for $2 + 1$ dimensional gravity, there still is an equivalence at the level of the actions, but the precise map between Hilbert spaces and correlations functions is not quite as well understood. Nevertheless, the duality between Chern–Simons theory and two-dimensional conformal field theory has many features in common with the AdS/CFT duality.

It is very hard to directly prove the equivalence between type IIB string theory on $AdS_5 \times S^5$, and four-dimensional $N = 4$ super Yang-Mills theory. For one, as alluded to above, we do not have a good definition of non-perturbative type IIB string theory. Even at string tree level, we do not (yet) know how to solve the theory completely. From this perspective, perhaps a more appropriate perspective is to view $N = 4$ super Yang-Mills theory as the definition of non-perturbative type IIB string theory on the $AdS_5 \times S^5$ background. This implies in particular that the non-perturbative quantum degrees of freedom of string theory do not resemble those it seems to contain at low energies.

A weaker form of the AdS/CFT correspondence is obtained by restricting to low-energies on the string theory side. At low-energies, type IIB string theory on $AdS_5 \times S^5$ reduces to type IIB supergravity on $AdS_5 \times S^5$. The corresponding limit on the gauge theory side is one where both N and $g_{YM}^2 N$ become large, where N is the rank of the $U(N)$ gauge group of the $N = 4$ supersymmetric gauge theory (not to be confused with the N appearing in $N = 4$), and g_{YM}^2 is the gauge coupling constant. The equivalence between type IIB supergravity on $AdS_5 \times S^5$ and $N = 4$ gauge theory in the large N , large $g_{YM}^2 N$ limit has been very well tested by now.

¹In three or less dimensions, gravity has no local propagating degrees of freedom, so perhaps it is not a good example; still, it will turn out that this case has many similarities with the story in dimensions above three.

2 Large N and holography

The AdS/CFT correspondence is related to two deep ideas in physics.

The first of these is the idea that large N gauge theory is equivalent to a string theory [16]. The perturbative expansion of a large N gauge theory in $1/N$ and $g_{YM}^2 N$ has the form

$$Z = \sum_{g \geq 0} N^{2-2g} f_g(\lambda) \quad (1)$$

where $\lambda \equiv g_{YM}^2 N$ is the so-called 't Hooft coupling. This is similar to the loop expansion in string theory

$$Z = \sum_{g \geq 0} g_s^{2g-2} Z_g, \quad (2)$$

with the string coupling g_s equal to $1/N$. Through some peculiar and not completely understood mechanism, Feynman diagrams of the gauge theory are turned into surfaces that represent interacting strings (but see [17]). Apparently, this is precisely what happens in the AdS/CFT correspondence.

The second is the idea of holography [18, 19]. This idea has its origin in the study of the thermodynamics of black holes. It was shown by Bekenstein and Hawking [20] that black holes can be viewed as thermodynamical systems with a temperature and an entropy. The temperature is directly related to the black body radiation emitted by the black hole, whereas the entropy is given by $S = A/4G$, with G the Newton constant and A the area of the horizon of the black hole. With these definitions, Einstein's equations of general relativity are consistent with the laws of thermodynamics. Since in statistical physics entropy is a measure for the number of degrees of freedom of a theory, it is rather surprising to see that the entropy of a black hole is proportional to the area of the horizon. If gravity would behave like a local field theory, one would have expected an entropy proportional to the volume. A consistent picture is reached if gravity in d dimensions is somehow equivalent to a local field theory in $d - 1$ dimensions. Both have an entropy proportional to the area in d dimensions, which is the same as the volume in $d - 1$ dimensions. The analogy of this situation to that of an hologram, which stores all information of a 3d image in a 2d picture, led to the name holography. The AdS/CFT correspondence is holographic, because it states that quantum gravity in five dimensions (forgetting the compact five sphere) is equivalent to a local field theory in four dimensions.

3 Anti-de Sitter space

To describe the correspondence in some more detail, we first need to describe the geometry of anti-de Sitter space in some more detail. Five-dimensional anti-de Sitter

space can be described as the five-dimensional manifold

$$-X_0^2 - X_1^2 + X_2^2 + X_3^2 + X_4^2 + X_5^2 = L^2$$

embedded in a six-dimensional space with metric

$$ds^2 = -dX_0^2 - dX_1^2 + dX_2^2 + dX_3^2 + dX_4^2 + dX_5^2.$$

It can be roughly thought of as a product of four-dimensional Minkowski space times an extra radial coordinate. The metric on Minkowski space is however multiplied by an exponential function of the radial coordinate, and anti-de Sitter space is therefore an example of a warped space: in a suitable local coordinate system,

$$ds^2 = L^2(dr^2 + e^{2r}(\eta_{\mu\nu}dx^\mu dx^\nu)). \quad (3)$$

The parameter L is just a scale factor. The limit where the radial coordinate goes to infinity and the exponential factor blows up is called the boundary of anti-de Sitter space. This boundary is the place where the dual field theory lives. One can indeed verify that string theory excitations in anti-de Sitter space extend all the way to the boundary [21]. In this way one obtains a map from string theory states to states in the field theory living on the boundary. We see that e^r sets the scale of the Minkowski part of the metric, and it turns out that e^r can quite literally also be viewed as a scale of the dual field theory (see e.g. [22]).

4 Correlation functions

The AdS/CFT correspondence in the form in which it was proposed in [1] did not yet provide a detailed map between AdS and CFT quantities. Such a map was given in [23, 24] and makes the correspondence much more explicit. To describe it, we first consider the AdS side, and in particular, we consider a free field with mass m propagating in anti-de Sitter space. The field equation

$$(\square + m^2)\phi = 0 \quad (4)$$

has two linearly independent solutions that behave respectively as

$$e^{-\Delta r}, \quad e^{(\Delta-4)r} \quad (5)$$

as $r \rightarrow \infty$, where

$$\Delta(\Delta - 4) = m^2. \quad (6)$$

Consider now a solution of the supergravity equations of motion with the boundary condition that the fields behave near $r = \infty$ as

$$\phi_i(r, x^\mu) \sim \phi_i^0(r, x^\mu)e^{(\Delta-4)r}. \quad (7)$$

Then the map between AdS and CFT quantities is given by

$$\exp(-\Gamma_{\text{sugra}}(\phi_i)) = \left\langle \exp \left(\int d^4x \phi_i^0 \mathcal{O}_i \right) \right\rangle \quad (8)$$

where the left-hand side is the supergravity action evaluated on the classical solution given by ϕ_i , and the right-hand side is a generating function for correlation functions in super Yang-Mills theory. Actually, we should really use the full string theory partition function subject to the relevant boundary conditions on the left-hand side, to which the supergravity approximation is only the saddle-point approximation. For many applications the above formula suffices though. We also see that there should be an operator $\mathcal{O}_i$ in Yang-Mills theory for every field ϕ_i in AdS. With some further work, one can show that this operator needs to have conformal dimension Δ_i . This yields a non-trivial prediction which we come back to in section 7.1. Finally, note that we restricted to scalar fields in (8). The full AdS/CFT correspondence should of course involve all the AdS degrees of freedom, not just the scalar ones.

5 Mapping between parameters

In order to illustrate the fact that the AdS/CFT duality is an example of a strong-weak coupling duality, we give the relations between the parameters of both theories.

String theory on $AdS_5 \times S^5$ has a dimensionless string coupling constant g_s , which measures the string interaction strength relevant for string splitting and joining, a dimensionful string length l_s , which sets the size of fluctuations of the string world-sheet, and another dimensionful parameter L , the curvature radius of AdS_5 and S^5 that appears in (3).

Four-dimensional $N = 4$ super Yang-Mills theory with gauge group $U(N)$ has, besides the rank N of the gauge group, a dimensionless coupling constant g_{YM}^2 .

The identification of the parameters reads

$$g_s = g_{YM}^2, \quad (L/l_s)^4 = 4\pi g_{YM}^2 N = 4\pi\lambda. \quad (9)$$

By comparing the expansions (1) and (2), we now see that the AdS/CFT correspondence is indeed an example of a weak/strong coupling duality. Depending on the choice of parameters, either AdS or the CFT provides a weakly coupled description of the system, but never both at the same time. Gauge theory is a good description for small $g_{YM}^2 N$ and small g_{YM}^2 , whereas string theory is good for large $g_{YM}^2 N$ and small g_{YM}^2 . Therefore, the AdS/CFT correspondence can be applied in two directions. We can use string theory to learn about gauge theory, and we can use gauge theory to learn about string theory.

6 Derivation of the AdS/CFT correspondence

The derivation of the AdS/CFT correspondence given in [1] crucially involves the notion of D-branes. D-branes are certain extended objects in string theory, that were introduced by Polchinski in [25]. They are labeled by the number of dimensions of the object, so that a D0 brane is like a particle, a D1 brane is like a string, a D2 brane is like a membrane, etc. There are two ways to think about D-branes. On the one hand, they are solitonic solutions of the equations of motion of low-energy closed string theory. On the other hand, they are objects in open string theory with the property that open strings can end on them. Open strings have a finite tension, and their center of mass cannot be taken arbitrarily far away from the D-brane. As a consequence, the degrees of freedom of the open string can effectively only propagate in a direction parallel to the brane: one says that they are confined to the brane, or that they live on the brane. The open string spectrum can be reproduced directly from the soliton in the closed string description via a collective coordinate quantization.

As a very crude analogy, one can think about two ways to describe a monopole. On the one hand, one can think of it as the 't Hooft-Polyakov monopole, in which case it is an extended soliton solution of the Yang-Mills-Higgs equations of motion. On the other hand, one can view a monopole as a point particle, on which magnetic field lines can end. Both descriptions have their advantages, as do the open and closed string descriptions of D-branes.

To derive the AdS/CFT correspondence, one starts with a stack of D3-branes. This has a description both in terms of open and closed strings. Next, one takes a suitable low-energy limit of the system, which involves taking $l_s \rightarrow 0$. The open string description reduces to $N = 4$ super Yang-Mills theory, whereas the closed string description reduces to string theory on $AdS_5 \times S^5$. Thus, the AdS/CFT duality arises as a consequence of the duality between open and closed strings. This duality is most easily visualized by thinking of a cylinder. It can be viewed either as the world-sheet of a closed string moving along an interval, and equivalently as an open string moving along a circle.

7 Tests of the AdS/CFT correspondence

7.1 spectrum of operators

In order for (8) to hold, there should for every gauge-invariant operator in the gauge theory exist a corresponding closed string field on $AdS_5 \times S^5$, whose mass is related to the scaling dimension of the operator according to (6).

It is difficult to test this statement in full generality, because we don't know the spectrum of super Yang-Mills theory at strong coupling, nor do we know the string spectrum at strong coupling. However, we do know the spectrum of a subset of the

operators of super Yang-Mills theory at strong coupling, namely the so-called BPS operators.

BPS operators $\mathcal{O}$ have the property that the corresponding state $|\psi\rangle = \mathcal{O}|0\rangle$ is annihilated by some hermitian supersymmetry generator Q , which satisfies $Q^2 = \Delta + J$, with J some $U(1)$ generator². If a state $|\psi\rangle$ is annihilated by Q , it obeys

$$0 = |Q|\psi\rangle|^2 = (\Delta + J)|\psi\rangle|^2 \quad (10)$$

and therefore $\Delta = -J$. The eigenvalues of J are quantized, and therefore as long as there is no phase transition at finite coupling constant the weak and strong coupling values of Δ have to be identical.

In ten-dimensional string theory, there are massless and massive fields. The dimensions of the operators corresponding to massive fields scale as $(L/l_s) \sim (g_{YM}^2 N)^{1/4}$, and vary continuously with the Yang-Mills coupling constant. Such operators can therefore not be BPS, and it is difficult to compare them to operators in the gauge theory.

Massless fields, on the other hand, have scaling dimensions that are independent of $g_{YM}^2 N$, and these should therefore correspond to BPS operators. It has been verified [23] that there is indeed a precise match between massless fields in string theory, and BPS operators in the gauge theory. A first necessary condition for such a matching to be possible is that the global symmetries match. The space $AdS_5 \times S^5$ has isometry group $SO(2,4) \times SO(6)$, whereas $N = 4$ super Yang-Mills theory is invariant under the conformal group $SO(2,4)$ and the R-symmetry group $SO(6)$. The bosonic global symmetries therefore indeed agree. In addition, there are fermionic symmetries that also agree, and together with the bosonic symmetries these form the supergroup $PSU(2,2|4)$. One can verify directly that the massless fields in string theory fall in the same multiplets of $PSU(2,2|4)$ as the BPS operators of super Yang-Mills theory.

7.2 correlation functions

Correlation functions have been considered in great detail, see e.g. [11]. The results are quite encouraging, although one has to face the same problem as when comparing the spectrum of operators, namely one can only compare quantities that can be computed in the field theory at strong coupling. This restricts the set of correlation functions that can be compared to the ones that satisfy non-renormalization theorems in the field theory, so that the strong coupling answer is a straightforward extrapolation of the weak coupling answer. Such correlation functions have been compared to the string theory answer, so far always with success. In addition, the AdS/CFT

²The reason that Δ appears is that the Hamiltonian of a conformal field theory has a continuous spectrum when quantized on R^4 , but on a cylinder $S^3 \times R$ it has a discrete spectrum, with eigenvalues given by Δ .

correspondence has suggested new non-renormalization theorems, which remain to be proven in field theory, for extremal correlation functions of the form

$$\langle \mathcal{O}_\Delta(x) \mathcal{O}_{\Delta_1}(x_1) \dots \mathcal{O}_{\Delta_n}(x_n) \rangle \quad (11)$$

where $\Delta = \sum_i \Delta_i$.

7.3 Wilson loops

Another interesting quantity to compare is the vacuum expectation value of a Wilson loop,

$$\langle \text{Tr}(\text{Pexp} \oint_C A_\mu dx^\mu) \rangle. \quad (12)$$

A Wilson loop in field theory can be expanded in terms of local operators, and using (8) one could in principle use this to determine which computation in supergravity (or string theory) one would have to do to compute this same expectation value. However, it turns out that the string theory calculation has a direct geometric interpretation [26, 27]. A Wilson loop is described by a string world-sheet in AdS, which extends all the way to the boundary of AdS, and approaches there the curve C . A classically stable string world-sheet is a minimal area surface, so that the computation of the Wilson loop vacuum expectation value reduces to a minimal area problem. For a rectangular Wilson loop, one can solve the minimal area problem and in particular find the quark-quark energy as a function of their separation r ,

$$E = -\frac{4\pi^2(2g_{YM}^2 N)^{1/2}}{\Gamma(\frac{1}{4})^4 r}. \quad (13)$$

Interestingly, the 't Hooft coupling $g_{YM}^2 N$ appears with a fractional power, and therefore this result cannot remain valid at weak coupling, where the answer will be qualitatively different. The result in (13) is an example of a non-trivial prediction of the AdS/CFT correspondence.

It is amusing to notice that the fundamental string in AdS is the same as the QCD string of Yang-Mills theory. In some sense we have come a full circle towards the original motivation for string theory, namely as an effective theory for the strong interactions.

7.4 finite temperature

So far the field theory, $N = 4$ super Yang-Mills theory, was a theory at zero temperature, and it was obtained in the derivation of the AdS/CFT correspondence as the low energy limit of the degrees of freedom associated to a stack of D3-branes. It turns out that this setup can be generalized to non-zero temperature by employing so-called near extremal D3-branes. In particular, we can compute the free energy

as a function of temperature, using either the field theory or the dual gravitational description. The results one obtains are

$$F_{\text{sugra}} = -\frac{\pi^2}{8}N^2VT^4, \quad F_{\text{SYM}} = -\frac{\pi^2}{6}N^2VT^4. \quad (14)$$

There is no disagreement, since the first result is valid at strong coupling in the gauge theory, the second at weak coupling. As the coupling varies, one expects a smooth transition between these two results (for a discussion, see e.g. [10]), but the possibility of a phase transition at a finite value of the coupling has not been ruled out completely.

More generally, we can directly consider a finite temperature version of the AdS/CFT correspondence, by replacing the boundary geometry R^4 in (3) by $S^3 \times S^1$, where the S^1 represents periodic imaginary time so that the system is indeed at finite temperature [28]. The five-dimensional geometry that solves the string equations of motion and has this boundary is no longer anti-de Sitter space, but a space with a different metric. Actually, there are two different five manifolds with conformal boundary $S^3 \times S^1$. One is a finite temperature version of Euclidean AdS, the other describes a Schwarzschild black hole in AdS.

The finite temperature version of AdS dominates the supergravity path integral at low temperatures. In this geometry, the expectation value of the Wilson loop vanishes, and the center of the gauge group is unbroken, as in a confining phase.

The second solution has a nonzero expectation value for the Wilson loop, and the center is broken, as in a deconfining phase.

Altogether this provides a tantalizing geometric picture of the confinement/deconfinement transition: it is mapped to topology changing process in gravity!

7.5 glue balls and the string tension

So far, successful tests of AdS/CFT relied heavily on supersymmetry in order to compare answers at weak and strong coupling. Of course, for more realistic applications, it is desirable to consider cases with less or no supersymmetry. One way to break supersymmetry is to take supersymmetric Yang-Mills at finite temperature, and to take the $T \rightarrow \infty$ limit. Naively, the field theory reduces to some non-supersymmetric extension of three-dimensional QCD. To test this, one can try to compute quantities in this version of AdS/CFT and compare those to lattice results in three-dimensional QCD. In particular, one can compute glueball masses using field equations in AdS, and string tensions using the Wilson loop.

There is a considerable disagreement with lattice results. The string theory in question has many additional states with a mass of the order of $T \sim \Lambda_{\text{QCD}}$, which is also the mass scale of the glueballs. At present, no controlled way to remove the cutoff and reach a weak coupling limit has been found.

The string that appears in the AdS/CFT correspondence yields results that differ from the strong coupling lattice results, and this in turn differs from the continuum lattice results. For a detailed overview, see [29]. The latter two are separated by a phase transition, but whether there are any such phase transitions in AdS/CFT remains an open problem.

8 Other solutions

Besides the prototype example of the AdS/CFT correspondence, namely the duality between string theory on $AdS_5 \times S^5$ and $N = 4$ supersymmetric Yang-Mills theory, many other AdS/CFT dualities have been found. These involve p -dimensional AdS_p spaces for various values of p , and are dual to suitable conformal field theories. For instance, type IIB string theory compactified on $AdS_3 \times S^3 \times T^4$ is dual to a certain $1 + 1$ dimensional conformal field theory [1]. This CFT is relatively well-understood, and it is this CFT that featured in the first counting of microstates of a black hole by Strominger and Vafa [30].

AdS/CFT dualities with less supersymmetry can be found by dividing by a discrete group. For example, if G is a discrete subgroup of $SO(6)$, we can consider $AdS_5 \times S^5/G$, which is dual to a gauge theory with less supersymmetry. This gauge theory is typically of ‘quiver type,’ ‘moose type,’ or ‘deconstruction type’ [31].

9 Interpretation of the extra dimension

Other examples of the AdS/CFT duality no longer involve an anti-de Sitter space. For instance, one can deform the gauge theory by a relevant operator, and in some cases one can solve explicitly for the corresponding supergravity solutions. These gravity solutions are then dual to renormalization group flows that flow between a UV and an IR fixed point. The UV fixed point is the original undeformed gauge theory, the IR fixed point is the low energy limit of the deformed CFT. For such flows one can prove a c -theorem: c , the central charge, is a quantity that appears in the conformal anomaly of the gauge theory, and one can prove that it always decreases along renormalization group flow trajectories. In the gravitational description, c is directly related to the metric of the supergravity solution.

We have actually been somewhat careless in setting up the correspondence as in eqn (8). To do things properly, one should choose a cutoff $r = r_0$, evaluate the supergravity quantities on the truncated space where $r \leq r_0$, and finally take the cutoff r_0 to infinity. In this process, the supergravity action diverges. It can be rendered finite by subtracting local counterterms, exactly as in the standard renormalization of field theories. For a review, see [32]. This illustrates once more the statement that the fifth dimension can be viewed as an energy scale.

In fact, one of the questions that the AdS/CFT immediately raises and perhaps we should have considered right at the beginning, is that of the interpretation of the extra fifth dimension in the field theory. As suggested above, it is closely related to the energy scale in the dual field theory. From the 5d gravitational point of view, low-energy processes in field theory stay close to the boundary of AdS, whereas high-energy processes penetrate deeper in the interior [22]. One can even show that the invariance under 5d general coordinate transformations implies the Callan-Symanzik renormalization group equations in the field theory [33, 34]. Thus, from the 5d point of view, the renormalization group is on an equal footing with 4d Poincaré invariance.

10 AdS/CFT with a cutoff

What happens if we do not remove the cutoff r_0 , but instead keep it finite? On AdS, many fields have nonnormalizable modes that correspond to parameters and coupling constants in the dual field theory. Once we have a finite cutoff, all these fields become normalizable, and they therefore correspond to dynamical degrees of freedom in the dual description. In particular, the dual theory will no longer be a field theory, but it will become a field theory coupled to dynamical gravity. Whether one uses this description, or the higher dimensional one that involves truncated AdS, depends on the choices of parameters. Normally the most weakly coupled one will be the most useful.

This version of the AdS/CFT correspondence is very relevant for brane world scenarios and generic string compactifications. The latter typically contain branes, and whether one uses the actual brane description or the dual supergravity description depends on the number of branes, their coupling constants, and the curvature of the ambient geometry. Unfortunately we have no time to venture further in this interesting direction.

11 High energy scattering/deep inelastic scattering

At first sight, the AdS/CFT correspondence, or any duality between string theory and gauge theory, seems at odds with the known fact that the scattering of glueballs at high energies is hard, whereas string scattering at high energies is soft, due to their extended nature. The resolution sits in the fact that AdS is a warped space. When an object moves away from the boundary of AdS, its size is exponentially reduced. Very high energy processes in the gauge theory are described by strings which propagate a long distance from the boundary of AdS before they interact. By that time, the size of the strings has been exponentially reduced, and this compensates for the softness

of string interactions to make it into a hard process in the gauge theory [35]. Besides such effects, which are due to the geometric warping of AdS, other gauge theory processes crucially involve strong gravity physics like black hole formation [36]. It is also possible to study the physics of deep inelastic scattering and the parton model from the AdS/CFT point of view [37].

12 Towards a QCD string?

A more involved version of the AdS/CFT correspondence is the one given in [38]. It was discovered by studying branes stuck in singularities in string theory. The gauge theory that appears is an $N = 1$ theory in four dimensions with gauge group $SU(N) \times SU(N + M)$. There are two chiral superfields A_i in the $(N, \overline{N + M})$ representation of the gauge group, and two chiral superfields B_i in the $(\overline{N}, N + M)$ representation. In addition, there is a nontrivial superpotential of the form $W \sim \epsilon^{ij} \epsilon^{kl} \text{tr}(A_i B_k A_j B_l)$.

This field theory has a remarkable property: it has running gauge couplings, but does not become free at either low or high energies. The gauge coupling becomes strong either way. Strongly coupled $N = 1$ theories in four dimensions often admit a dual weakly coupled description, a duality known as Seiberg duality [39]. The same is true here: both at low energies and at high energies there exist dual descriptions. However, these dual descriptions have the same problem: they are not weakly coupled at either low or high energies. Again, they admit suitable dual descriptions. The full picture that emerges is that of an infinite “cascade” of gauge theories, that continues indefinitely at high energies, with an ever increasing rank of the gauge group, but terminates at low energies once e.g. the rank of one of the gauge groups becomes one. At that point, the gauge theory becomes confining. Strictly speaking we need an infinite amount of fine tuning of irrelevant operators to obtain this infinite cascade, but quite remarkable, the dual description of this gauge theory quite naturally sees the same cascade. The ranks N and M of the gauge group become non-trivial functions of the radial coordinate of the dual AdS-like geometry. This also confirms once more the interpretation of the extra fifth dimension in the AdS/CFT correspondence as an energy scale in the field theory.

The AdS-like geometry that is dual to this infinite cascade has several nice features. String theory on this background exhibits (i) confinement, (ii) glueballs and baryons with a mass scale that emerges through dimensional transmutation, exactly as in the gauge theory, (iii) gluino condensates that break the $\mathbf{Z}_{2M}$ chiral symmetry to $\mathbf{Z}_2$, and (iv) domain walls separating different vacua.

The gauge theory at low energies reduces to a pure $N = 1$ supersymmetric Yang-Mills theory. Does the dual geometry therefore provide a dual string theory for pure supersymmetric Yang-Mills theory, the long sought for QCD string? Not really, because it has new degrees of freedom beyond those of the field theory that appear at

Λ_{QCD} , as we already mentioned in the section 7.5. This is a generic problem in trying to find weakly coupled string theory descriptions of gauge theories. To decouple the additional degrees of freedom, we need to make the curvature of the AdS-like geometry large, while keeping the string coupling g_s small. String theory in a strongly curved background is described by a strongly coupled $1 + 1$ dimensional field theory. The structure of the sigma models relevant for the AdS/CFT correspondence is not very well understood, but there has been progress in this direction recently (see [40] and references therein), and the prospect of finding a string theory dual of QCD remains an exciting possibility.

13 Other string effects in gauge theories: large quantum numbers and pp-waves

Instead of trying to find a precise string theory dual description of pure $N = 1$ supersymmetric Yang-Mills theory, it is also interesting to look for more qualitative stringy behavior in gauge theories.

One place to find such behavior is to look at states with a large scaling dimension proportional to N , the rank of the gauge group. Many gauge theories have baryons with such scaling dimensions, and it turns out that they are not described by strings but by branes in the dual geometrical description [41]. Thus, it is also possible to discover branes in gauge theory.

A related example is to consider operators with a large spin s , like for example $\text{tr}(\Phi D_{\mu_1} \dots D_{\mu_s} \Phi)$, where Φ is some field that transforms in the adjoint representation of $U(N)$. Such operators correspond to folded rotating closed strings in the dual geometrical description. One can compute the scaling dimension of these operators both in the field theory and in the dual geometrical description. This confirms the equivalence between the two, as one finds in both cases that it behaves like $s + \log s$ [42].

A more complete way to recover string theory from a gauge theory has been described in [43]. The idea is to take a particular scaling limit of the AdS/CFT correspondence. This scaling limit, when applied to the AdS geometry, yields a different geometry known as a “pp-wave”. In fact, many geometries admit scaling limits in which they reduce to pp-waves, as originally shown by Penrose [44]. String theory on the pp-wave, in the absence of string interactions, can be exactly solved, and in particular the free string spectrum can be obtained.

On the field theory side, the same limit can be taken. In this limit only a subset of the operators of the full $N = 4$ super Yang-Mills theory survive, namely those for which the scaling dimension Δ and a certain global $U(1)$ quantum number J have the property that $\Delta + J$ scales as $N^{1/2}$, while $\Delta - J$ is kept finite, as one takes $N \rightarrow \infty$.

Interestingly, this set of operators is in one-to-one correspondence with the set of

free string states. This is the first example where a complete string spectrum has been obtained from a gauge theory, albeit in a special scaling limit.

The ground state of the string theory (in light-cone quantization) is described in the gauge theory by the operator $\text{tr}(Z^J)$, where Z is a complex scalar field in the adjoint representation of the gauge group with charge $J = +1$ under the distinguished global $U(1)$ symmetry. The simplest excited states of the string are operators of the form $\sum_i a_i \text{tr}(Z^i \Phi Z^{J-i})$ where the a_i are phases. The string appears from this point of view as composed of “string bits.” The string bits are the operators Z , and the string is composed of a string of J bits. Exciting the string amounts to introducing impurities like Φ that are distributed with phases (i.e. a discrete momentum) along the chain of Z ’s.

A similar discretized picture of string theory can be obtained in several other gauge theories as well, see e.g. [45]. It is also presently being investigated whether one can correctly recover string interactions, or even the full string field theory, from the gauge theory [46]. The emerging picture is that there is a complete agreement, for infinitesimal ’t Hooft coupling. However, the only quantities for which the Penrose limit exists are correlation functions in the field theory where one set of operators is taken to $t = -\infty$, the ‘in’ states, while the other set is taken to $t = +\infty$, the ‘out’ states. Such correlation functions map to string theory S-matrix elements between corresponding string theory in and out states.

14 Outlook

The AdS/CFT correspondence is still a very lively research area, with many open problems. Clearly, it would be great if one could identify the dual string theories directly for more realistic gauge theories, like $N = 1$ QCD, and other asymptotically free gauge theories. Also, one would like to find holographic dual descriptions of flat space, and of time-dependent backgrounds such as our universe. In addition, it would be very helpful if one could solve string theory on AdS at tree level, in particular at strong coupling, which should correspond to free super Yang-Mills theory at large N .

But one of the most important, difficult and unsolved problems in the AdS/CFT correspondence is to reconstruct 5d local gravitational physics directly from the dual 4d field theory point of view. In particular, we would like to know in what way the local gravitational description breaks down. Does such a breakdown occur in a local way at the Planck length, or in a non-local way at much larger length scales? The AdS/CFT correspondence seems to prefer the second answer, which is also the answer that may provide a resolution to the black hole information paradox. This paradox is based on the fact that semiclassically, everything that falls into a black hole is converted into purely thermal radiation, with no memory of the object that fell in except for its mass and perhaps a few other quantum numbers. Such a process

contradicts the usual rules of quantum mechanics, and we can either give up on quantum mechanics or give up on the semiclassical approximation to quantum gravity; AdS/CFT prefers the latter.

Acknowledgements

I would like to thank the organizers of SUSY'02 for the invitation to present this lecture, and the stichting FOM for partial support.

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AdS/CFT Correspondence and Differential Geometry

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1. Introduction: The AdS/CFT correspondence
2. Conformal Anomaly
3. AdS/CFT for field theories with $\mathcal{N} = 1$ Supersymmetry
4. Example: Sasaki-Einstein manifolds

AdS/CFT Correspondence

(Maldacena 1997, AdS: Anti de Sitter space, CFT: conformal field theory)

Witten; Gubser, Klebanov, Polyakov

- Duality Quantum Field Theory $\Leftrightarrow$ Gravity Theory
- Arises from String Theory in a particular low-energy limit
- Duality: Quantum field theory at strong coupling
 $\Leftrightarrow$ Gravity theory at weak coupling

Conformal field theory in four dimensions

$\Leftrightarrow$ Supergravity Theory on $AdS_5 \times S^5$

Anti-de Sitter space

Anti de Sitter space: Einstein space with constant negative curvature has a **boundary** which is the upper half of the Einstein static universe (locally this may be conformally mapped to four-dimensional Minkowski space)

Isometry group of AdS_5 : $SO(4, 2)$

AdS/CFT:

relates conformal field theory at the boundary of AdS_5
to gravity theory on $AdS_5 \times S^5$

Isometry group of S^5 : $SO(6)$ ($\sim SU(4)$)

- **Anti-de Sitter space:** Einstein space with constant negative curvature

AdS space has a boundary

$$\text{Metric: } ds^2 = e^{2r/L} \eta_{\mu\nu} dx^\mu dx^\nu + dr^2$$

- Isometry group of **$(d + 1)$ -dimensional AdS space** coincides with **conformal group in d dimensions** ($SO(d, 2)$).
- AdS/CFT correspondence provides **dictionary** between field theory operators and supergravity fields

$$\mathcal{O}_\Delta \leftrightarrow \phi_m \ , \quad \Delta = \frac{d}{2} + \sqrt{\frac{d^2}{4} + L^2 m^2}$$

- Items in the same dictionary entry have the same quantum numbers under superconformal symmetry $SU(2, 2|4)$.

Field theory side of AdS/CFT correspondence

Consider (3+1)-dimensional Minkowski space

Quantum field theory at the boundary of Anti-de Sitter space:

$\mathcal{N} = 4$ supersymmetric $SU(N)$ gauge theory ($N \rightarrow \infty$)

Fields transform in irreps of $SU(2, 2|4)$, superconformal group

Bosonic subgroup: $SO(4, 2) \times SU(4)_R$

- 1 vector field A_μ
- 4 complex Weyl fermions $\lambda_{\alpha A}$ ($\bar{4}$ of $SU(4)_R$)
- 6 real scalars ϕ_i (6 of $SU(4)_R$)

(All fields in adjoint representation of gauge group)

$\beta \equiv 0$, theory conformal

(9 + 1)-dimensional supergravity: equations of motion allow for
D3 brane solutions

(3 + 1)-dimensional (flat) hypersurfaces with invariance group

$$\mathbb{R}^{3,1} \times SO(3, 1) \times SO(6)$$

Inserting corresponding ansatz into the equation of motion gives

$$ds^2 = H(y)^{-1/2} \eta_{\mu\nu} dx^\mu dx^\nu + H(y)^{1/2} dy^2$$

H harmonic with respect to y

Boundary condition: $\lim_{y \rightarrow \infty} H = 1$

$$\Rightarrow \begin{aligned} H(y) &= 1 + \frac{L^4}{y^4} \\ L^4 &= 4\pi g_s N \alpha'^2 \end{aligned}$$

In addition: self-dual five-form F_5^+

Maldacena limit

For $|y| < L$: Perform coordinate transformation $u = L^2/y$

Asymptotically for u large:

$$ds^2 = L^2 \left[\frac{1}{u^2} \eta_{\mu\nu} dx^\mu dx^\nu + \frac{du^2}{u^2} + d\Omega_5^2 \right]$$

Metric of $AdS_5 \times S^5$

Limit:

$N \rightarrow \infty$ while keeping $g_s N$ large and fixed

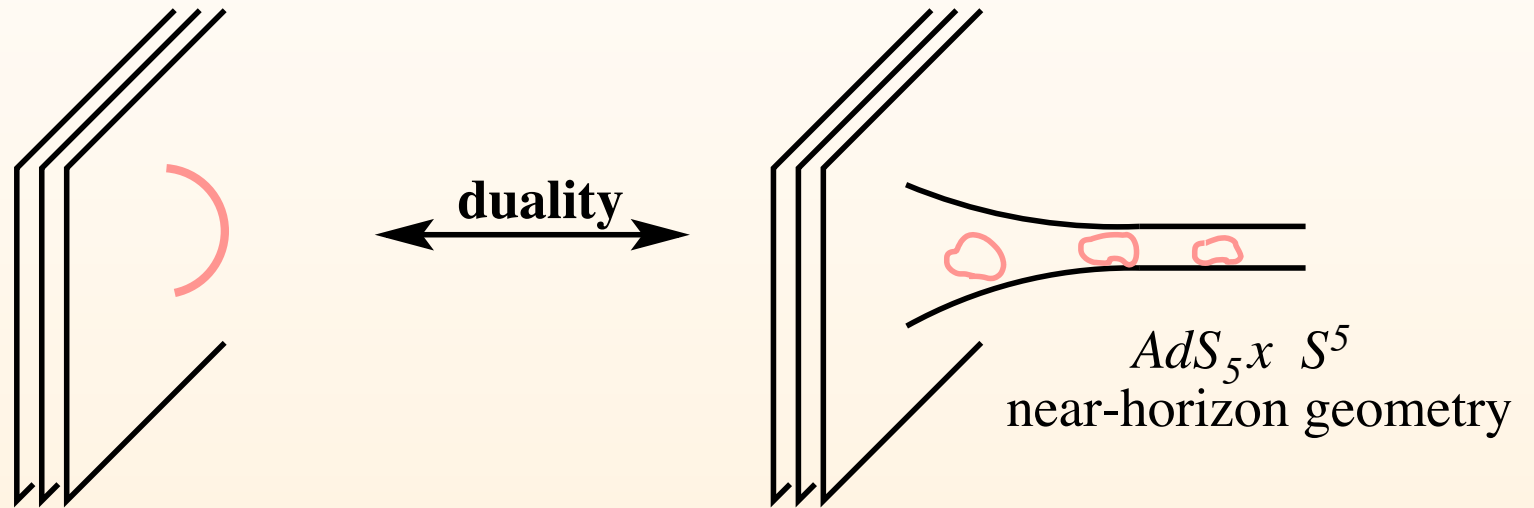
$(l_s \rightarrow 0)$

Isometries $SO(4, 2) \times SO(6)$ of $AdS_5 \times S^5$ coincide with

global symmetries of $\mathcal{N} = 4$ Super Yang-Mills theory

String theory origin of AdS/CFT correspondence

D3 branes in 10d



↓ Low-energy limit

$\mathcal{N} = 4$ SUSY $SU(N)$ gauge
theory in four dimensions
($N \rightarrow \infty$)

IIB Supergravity on $AdS_5 \times S^5$

Conformal anomaly in field theory

Classical action functional $S_{Matter} = \int d^4x \sqrt{-g} \mathcal{L}_M$

Consider variation of the metric $g_{\mu\nu} \rightarrow g_{\mu\nu} + \delta g_{\mu\nu}$

$$\delta S_M = \frac{1}{2} \int d^m x \sqrt{-g} T^{\mu\nu} \delta g_{\mu\nu}$$

$T_{\mu\nu}$ **energy-momentum tensor**, $T_{\mu\nu} = T_{\nu\mu}$, $\nabla^\mu T_{\mu\nu} = 0$

In conformally covariant theories: $T_\mu{}^\mu = 0$

Quantised theory: Generating functional

$$Z[g] \equiv e^{-W[g]} = \int \mathcal{D}\phi_M \exp \left[- \int d^4x \sqrt{-g} \mathcal{L}_m \right]$$

$$\delta W[g] = \int d^4x \langle T^{\mu\nu} \rangle \delta g_{\mu\nu}$$

Consider $\delta g_{\mu\nu} = -2\sigma(x)g_{\mu\nu}$, **Weyl variation:** **Generically** $\langle T_\mu{}^\mu \rangle \neq 0!$

Conformal anomaly

In (3+1) dimensions

$$\langle T_\mu^\mu \rangle = \frac{c}{16\pi^2} C^{\mu\nu\sigma\rho} C_{\mu\nu\sigma\rho} - \frac{a}{16\pi^2} \frac{1}{4} \varepsilon_{\alpha\beta\gamma\delta} \varepsilon_{\mu\nu\rho\sigma} R^{\alpha\beta\mu\nu} R^{\gamma\delta\rho\sigma}$$

C Weyl tensor, $\frac{1}{4} \varepsilon_{\alpha\beta\gamma\delta} \varepsilon_{\mu\nu\rho\sigma} R^{\alpha\beta\mu\nu} R^{\gamma\delta\rho\sigma}$ Euler density

Coefficients c, a depend on $\mathcal{L}_M$

Many explicit calculation methods, for instance heat kernel

$\mathcal{N} = 4$ supersymmetric theory: $c = a = \frac{1}{4}(N^2 - 1)$

$$\langle T_\mu^\mu \rangle = \frac{N^2 - 1}{8\pi^2} \left(R^{\mu\nu} R_{\mu\nu} - \frac{1}{3} R^2 \right)$$

Henningson+Skenderis '98, Theisen et al '99

- Calculation of conformal anomaly using Anti-de Sitter space
- Powerful test of AdS/CFT correspondence
- Write metric of Einstein space in Fefferman-Graham form
(requires equations of motion)

$$ds^2 = L^2 \left(\frac{d\rho^2}{4\rho^2} + \frac{1}{\rho} g_{\mu\nu}(x, \rho) dx^\mu dx^\nu \right)$$
$$g_{\mu\nu}(x, \rho) = \bar{g}_{\mu\nu}(x) + \rho g^{(2)}_{\mu\nu}(x) + \rho^2 g^{(4)}_{\mu\nu}(x) + \rho^2 \ln \rho h^{(4)}_{\mu\nu}(x) + \dots$$

- Insert Fefferman-Graham metric into five-dimensional action

$$S = -\frac{1}{16\pi G_5} \int d^5 z \sqrt{|g|} \left(R + \frac{12}{L^2} \right) ,$$
$$S_\varepsilon = -\frac{1}{16\pi G_5} \int d^4 x \int_{\rho=\varepsilon} \frac{d\rho}{\rho} \left(a^{(0)}(x) + a^{(2)}(x)\rho + a^{(4)}(x)\rho^2 + \dots \right)$$

- Action divergent as $\varepsilon \rightarrow 0$
- Regularisation: Minimal Subtraction of counterterm

- Weyl transformation gives conformal anomaly:

$$\begin{aligned}\langle T_{\mu}^{\mu}(x) \rangle &= - \lim_{\varepsilon \rightarrow 0} \frac{1}{\sqrt{|g|}} \frac{\delta}{\delta \sigma} [S_{\varepsilon}[\bar{g}] - S_{ct}[\bar{g}]] \\ &= \frac{N^2}{32\pi^2} \left(R^{\mu\nu} R_{\mu\nu} - \frac{1}{3} R^2 \right)\end{aligned}$$

- Coincides with $\mathcal{N} = 4$ field theory result
- Important:** Coefficient determined by volume of internal space:

$$a = \frac{\pi^3}{4} N^2 \text{Vol}(S^5) \quad (N \gg 1)$$

Field-theory coefficients a, c are related to volume of internal manifold (S^5 for $\mathcal{N} = 4$ supersymmetry)

Ultimate goal: To find gravity dual of the field theories in the Standard Model of elementary particle physics

First step: Consider more involved internal spaces

Example: Instead of D3 branes in flat space, consider D3 branes at the tip of a six-dimensional toric non-compact Calabi-Yau cone

Field theory: has $\mathcal{N} = 1$ supersymmetry, ie. $U(1)_R$ R symmetry
(instead of the $SU(4)_R$ of $\mathcal{N} = 4$ theory)

Quiver gauge theory: Product gauge group $SU(N) \times SU(M) \times SU(P) \times \dots$

Matter fields in bifundamental representations of the gauge group

Conformal anomaly coefficient of these field theories can be determined by a maximization principle

In general for $\mathcal{N} = 1$ theories:

$$a = \frac{3}{32} \left(3 \sum_i R_i^3 - \sum_i R_i \right)$$

R_i charges of the different fields under $U(1)_R$ symmetry

If other $U(1)$ symmetries are present (for instance flavour symmetries), it is difficult in general to identify the correct R charges.

Result (Intriligator, Wecht 2004): The correct R charges maximise a !

Local maximum of this function determines R symmetry of theory at its superconformal point.

Critical value agrees with central charge of superconformal theory.

Metric

$$ds^2 = \frac{L^2}{r^2} (\eta_{\mu\nu} dx^\mu dx^\nu + dr^2 + r^2 ds^2(Y))$$

with

$$ds^2(X) = dr^2 + r^2 ds^2(Y)$$

(X, ω) Kähler cone of complex dimension n ($n = 3$)

$$X = \mathbb{R}^+ \times Y, \quad r > 0$$

X Kähler and Ricci flat $\Leftrightarrow Y = X|_{r=1}$ **Sasaki-Einstein manifold**

$$\mathcal{L}_{r\partial/\partial r}\omega = 2\omega \quad \rightarrow \quad \omega \text{ exact: } \omega = -\frac{1}{2}d(r^2\eta), \quad \eta \text{ global one-form on } Y$$

- Kähler cone X has a covariantly constant complex structure tensor $\mathcal{I}$
- **Reeb vector** $K \equiv \mathcal{I}(r \frac{\partial}{\partial r})$
- **Constant norm Killing vector field**
- Reeb vector dual to $r^2 \eta \rightarrow \eta = \mathcal{I}(\frac{dr}{r})$
- Reeb vector generates the AdS/CFT dual of $U(1)_R$ symmetry
- Sasaki-Einstein manifold $U(1)$ bundle over Kähler-Einstein manifold, $U(1)$ generated by Reeb vector

Martelli, Sparks, Yau 2006

Variational problem on space of **toric** Sasakian metrics

toric cone X

real torus T^n acts on X preserving the Kähler form –

supersymmetric three cycles

Einstein-Hilbert action on toric Sasaki Y reduces to volume function $vol(Y)$

Kähler form: $\omega = \sum_{i=1}^3 dy_i \wedge d\phi_i$

Symplectic coordinates (y_i, ϕ_i) ,

ϕ_i angular coordinates along the orbit of the torus action

For general toric Sasaki manifold define vector $K' = \sum_{i=1}^3 b_i \frac{\partial}{\partial \phi_i}$

$\Rightarrow vol[Y] = vol[Y](b_i)$

Geometrical equivalent of a maximization

Reeb vector selecting Sasaki-Einstein manifold corresponds to those b_i which minimise volume of Y

Volume minimization $\Rightarrow$ Gravity dual of a maximization

Volume calculable even for Sasaki-Einstein manifolds for which metric is not known (toric data)

Example: Conifold

Base of cone: $Y = T^{(1,1)}$, $T^{(1,1)} = (SU(2) \times SU(2))/U(1)$

Symmetry $SU(2) \times SU(2) \times U(1)$, topology $S^2 \times S^3$

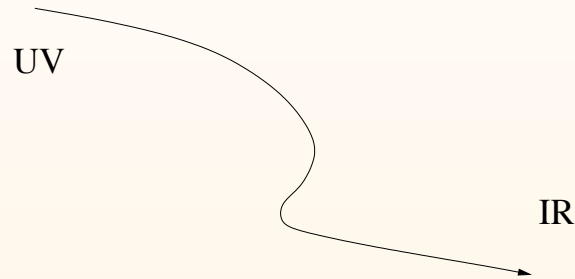
Dual field theory has gauge group $SU(N) \times SU(N)$

$$Vol(T^{(1,1)}) = \frac{16}{27}\pi^3$$

- can be calculated using volume minimisation as described
- gives correct result for anomaly coefficient in dual field theory

There exists an infinite family of Sasaki-Einstein metrics $Y^{p,q}$

C-Theorem (Zamolodchikov 1986) in $d = 2$: $\dot{C}(g^i) \leq 0$, $C = C(g^i(\mu))$



So far no field-theory proof in $d = 4$ exists

There is a version of the C theorem in non-conformal generalisations of AdS/CFT

Metric: $ds^2 = e^{2A(r)} \eta_{\mu\nu} dx^\mu dx^\nu + dr^2$

C-Function:

$$C(r) = \frac{c}{A'(r)^3}$$

Outlook

To investigate **non-conformal examples** of gauge theory/gravity duality
with methods of differential geometry

Conclusions

- AdS/CFT provides a powerful relation between gauge theory and gravity.
- It originates from string theory.
- Calculation of conformal anomaly provides powerful check.
- Generalisations to less symmetric field theories are possible.
- Further generalisations will provide
 - new insights into the structure of string theory
 - new non-perturbative tools to describe field theories.

AdS/CFT and applications

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Preparatory Minicourse

Holography@25 School, ICTP-SAIFR

June 2023

Plan:

- Lecture 1. Elements of General Relativity and AdS space
- Lecture 2. Elements of string theory and D-branes
- Lecture 3. Black holes in supergravity vs. D-branes
- Lecture 4. AdS/CFT and gauge/gravity duality in Euclidean and Lorentzian signatures
- Lecture 5. Holographic renormalization and holographic RG flow
- Lecture 6. Finite temperature and $\mathcal{N} = 4$ SYM plasma
- Lecture 7. Solitons and probes in AdS/CFT
- Lecture 8. The pp wave correspondence and spin chains
- Lecture 9. Applications to condensed matter: AdS/CMT
- Lecture 10. Applications to QCD

Lecture 1

Elements of General Relativity and AdS space

- **Special relativity:** speed of light = const. in all inertial reference frames, $c = 1 \Rightarrow$

$$ds^2 = -dt^2 + d\vec{x}^2 = \eta_{ij} dx^i dx^j$$

is invariant $\rightarrow$ invariant distance. SR: Physics is Lorentz invariant, i.e. invariant under

$$x'^i = \Lambda^i_j x^j; \quad \Lambda^i_j \in SO(1, 3)$$

- **General relativity:** General spacetime: curved. Distance between points is

$$ds^2 = g_{ij}(x) dx^i dx^j$$

Here $g_{ij}(x)$ = arbitrary functions: the metric.

- E.g. 2-sphere in 3d Euclidean space

$$\begin{aligned} ds^2 &= dx_1^2 + dx_2^2 + dx_3^2; \quad x_1^2 + x_2^2 + x_3^2 = R^2 \Rightarrow \\ ds^2 &= dx_1^2 \left(1 + \frac{x_1^2}{R^2 - x_1^2 - x_2^2}\right) + dx_2^2 \left(1 + \frac{x_2^2}{R^2 - x_1^2 - x_2^2}\right) + 2dx_1 dx_2 \frac{x_1 x_2}{R^2 - x_1^2 - x_2^2} \\ &\equiv g_{ij} dx^i dx^j (= d\theta^2 + \sin^2 \theta d\phi^2) \end{aligned}$$

- But for arbitrary symmetric metric $g_{ij}(x)$, we cannot embed in flat space: $\exists \frac{d(d+1)}{2}$ functions $g_{ij}(x)$ - d functions $x'_i(x_j)$ and moreover: signature of embedding space is not fixed.

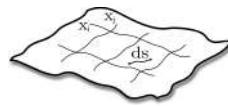
- E.g. 2d surfaces can be embedded in 3d with Euclidean OR Minkowski signature. So: general space is *intrinsically curved*.

- Curved space: triangle made by geodesics has angles $\alpha + \beta + \gamma \neq \pi$. E.G. sphere $\alpha + \beta + \gamma > \pi$: positive curvature $R > 0$.

- But $\exists$ also spaces of negative curvature, for which $\alpha + \beta + \gamma < \pi$, e.g. Lobachevski space (or "Euclidean AdS_2),

$$ds^2 = dx^2 + dy^2 - dz^2; \quad x^2 + y^2 - z^2 = -R^2$$

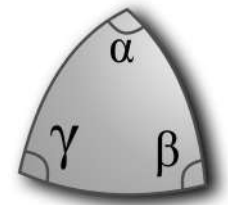
but $\det g_{ij} > 0 \Rightarrow$ space has (intrinsic) Euclidean signature.



a)



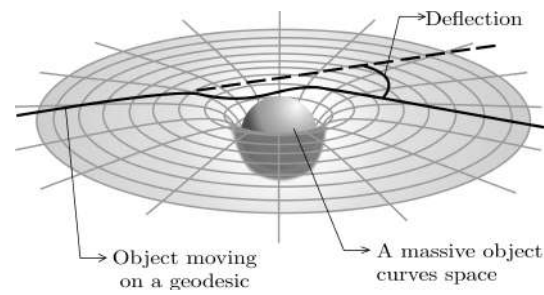
b)



c)



d)



Einstein's theory of general relativity:

- A1: Gravity is geometry: matter follows geodesic in curved space, and to us it appears as gravity.
- A2: Matter sources gravity: matter curves space $\Rightarrow$ Princ.:
- 1. Physics is invariant under general coordinate transformations:

$$x'_i = x'_i(x_j) \Rightarrow ds^2 = g_{ij}(x)dx^i dx^j = g'_{ij}(x')dx'^i dx'^j$$

- 2. Equivalence principle: there is no difference between acceleration and gravity

$$m_i = m_g, \text{ where } \vec{F} = m_i \vec{a}(\text{Newton}) \quad \vec{F}_g = m_g \vec{g}(\text{gravity})$$

- Dynamics of gravity: Einstein's eqs.

- Before that: define kinematics. $g_{\mu\nu}$ can be put locally to zero by coordinate transformations

$$g'_{\mu\nu}(x') = \frac{\partial x^\rho}{\partial x'^\mu} \frac{\partial x^\sigma}{\partial x'^\nu} g_{\rho\sigma}(x)$$

→ not a good measure of gravity. What else?

- Define tensors: A^μ transforms like dx^μ , B_μ transforms like $\partial/\partial x^\mu$, mixed transform as the product.
- Define: inverse metric $g^{\mu\nu} = g_{\mu\nu}^{-1}$, and then Christoffel symbol:

$$\Gamma^\mu_{\nu\rho} = \frac{1}{2} g^{\mu\sigma} (\partial_\rho g_{\nu\sigma} + \partial_\nu g_{\sigma\rho} - \partial_\sigma g_{\nu\rho}) ,$$

and Riemann tensor

$$R^\mu_{\nu\rho\sigma}(\Gamma) = \partial_\rho \Gamma^\mu_{\nu\sigma} - \partial_\sigma \Gamma^\mu_{\nu\rho} + \Gamma^\mu_{\lambda\rho} \Gamma^\lambda_{\nu\sigma} - \Gamma^\mu_{\lambda\sigma} \Gamma^\lambda_{\nu\rho}$$

- $\Gamma^\mu_{\nu\rho} \sim$ gauge field of gravity. $R^\mu_{\nu\rho\sigma} \sim$ field strength. Indeed, analogous to field strength of $SO(d-1, 1)$ gauge group,

$$F^A_{\mu\nu} = \partial_\mu A^A_\nu - \partial_\nu A^A_\mu + f^A_{BC} (A^B_\mu A^C_\nu - A^B_\nu A^C_\mu),$$

- Moreover, covariant derivative of tensor is

$$D_\mu T_\nu^\rho \equiv \partial_\mu T_\nu^\rho + \Gamma^\rho_{\sigma\mu} T_\nu^\sigma - \Gamma^\sigma_{\mu\nu} T_\sigma^\rho ,$$

similar to $D_\mu \phi^a = \partial_\mu \phi^a + (A^a_b)_\mu \phi^b$, $A = (a_b)$. Note $D_\rho g_{\mu\nu} = 0$.

- Then $R^\mu{}_{\nu\rho\sigma} \rightarrow$ tensor, as are $R_{\mu\nu} = R^\lambda{}_{\mu\lambda\nu}$, $R = R_{\mu\nu} g^{\mu\nu}$. R is coordinate invariant $\rightarrow$ true measure of curvature of space.
- The simplest choice for action for gravity is correct (compatible with experiment)

$$S_{gravity} = \frac{1}{16\pi G} \int d^d x \sqrt{-\det(g_{\mu\nu})} R$$

$\Rightarrow$ Einstein's equation

$$8\pi G \left[\frac{\delta S_{gravity}}{\sqrt{-g} \delta g^{\mu\nu}} + \frac{\delta S_{matter}}{\sqrt{-g} \delta g^{\mu\nu}} \right] = 0 \Rightarrow R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi G T_{\mu\nu}$$

- Indeed,

$$\delta S_{gravity} = \frac{1}{16\pi G_N} \int d^d x \sqrt{-g} \delta g^{\mu\nu} \left[R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R \right] .$$

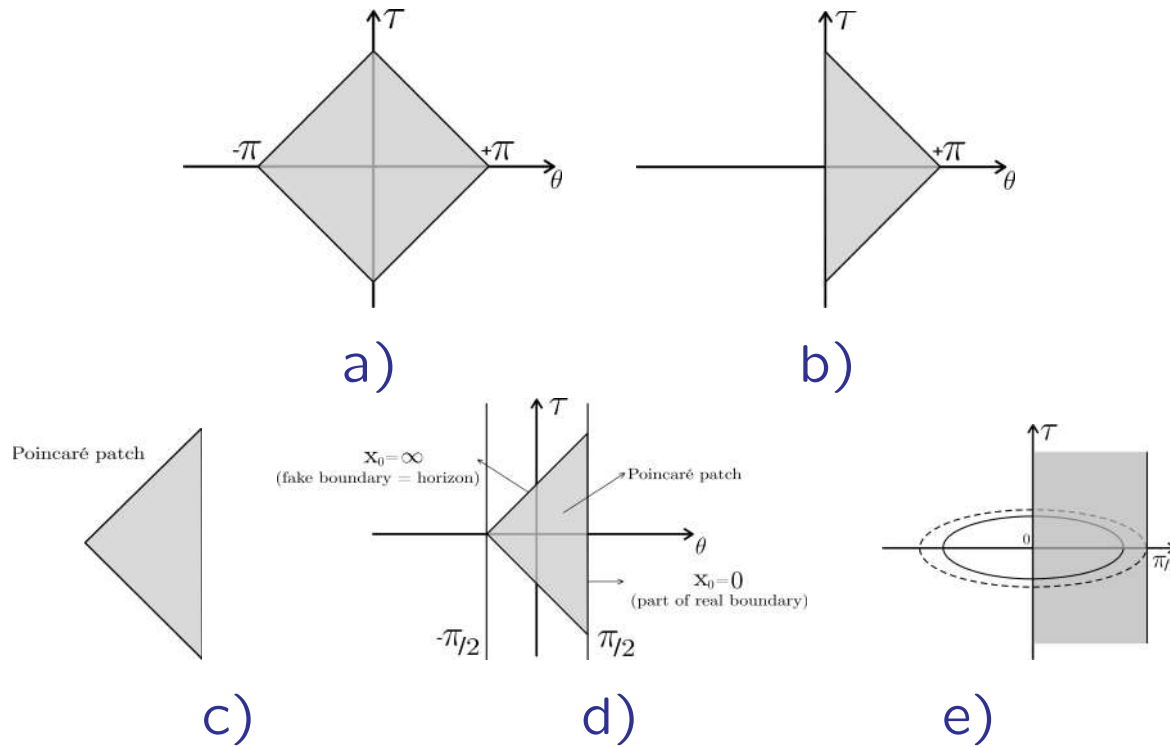
$$T_{\mu\nu} = -\frac{2}{\sqrt{-g}} \frac{\delta S_{matter}}{\delta g^{\mu\nu}} .$$

Global structure: Penrose diagrams

- To understand topological & causal structure: Penrose diagrams.
- For light propagation, $ds^2 = 0 \Rightarrow$ conformal factor is irrelevant.
- Make coordinate transformation that bring ∞ to finite distance, drop conformal factors. E.g. 2d Minkowski,

$$\begin{aligned} ds^2 &= -dt^2 + dx^2; \quad u_{\pm} = t \pm x \Rightarrow ds^2 = -du_+ du_- \\ u_{\pm} &= \tan \tilde{u}_{\pm}, \quad \tilde{u}_{\pm} = \frac{\tau \pm \theta}{2} \Rightarrow \\ ds^2 &= \left[\frac{1}{4 \cos^2 \tilde{u}_+ \cos^2 \tilde{u}_-} \right] (-d\tau^2 + d\theta^2) \end{aligned}$$

Here $|\tilde{u}_{\pm}| \leq \pi/2 \Rightarrow |\tau \pm \theta| \leq \pi \Rightarrow$ diamond Penrose diagram



Penrose diagrams. a) Penrose diagram of 2 dimensional Minkowski space. b) Penrose diagram of 3 dimensional Minkowski space. c) Penrose diagram of the Poincaré patch of Anti-de Sitter space. d) Penrose diagram of global AdS_2 (2 dimensional Anti-de Sitter), with the Poincaré patch emphasized; $x_0 = 0$ is part of the boundary, but $x_0 = \infty$ is a fake boundary (horizon). e) Penrose diagram of global AdS_d for $d \geq 2$. It is half the Penrose diagram of AdS_2 rotated around the $\theta = 0$ axis.

Anti-de Sitter space

- We saw examples of 2d (curved) surfaces of Euclidean signature (usual):

- 2d sphere, embedded in 3d Euclidean space:

$$\begin{aligned} ds_3^2 &= dx_1^2 + dx_2^2 + dx_3^2; & x_1^2 + x_2^2 + x_3^2 &= R^2 \Rightarrow \\ ds_2^2 &= g_{ij}dx^i dx^j; & \det(g_{ij}) &> 0 \end{aligned}$$

is explicitly $SO(3)$ invariant by construction, and $\mathcal{R} > 0$.

- 2d Lobachevski space, embedded in 3d Minkowski space:

$$\begin{aligned} ds_3^2 &= dx_1^2 + dx_2^2 - dx_3^2; & x_1^2 + x_2^2 - x_3^2 &= -R^2 \Rightarrow \\ ds_2^2 &= g_{ij}dx^i dx^j; & \det(g_{ij}) &> 0 \end{aligned}$$

- Is explicitly $SO(2, 1)$ invariant by construction, and $\mathcal{R} < 0$.
- Generalize to Lorentzian signature. $\mathcal{R} > 0$ case (generalization of the sphere) = de Sitter space. $\mathcal{R} < 0$ case (generalization of Lobachevski) = Anti de Sitter. So, sometimes: sphere = "Euclidean de Sitter" and Lobachevski = "Euclidean Anti de Sitter"

- Thus, d -dimensional de Sitter space:

$$ds^2 = -dx_0^2 + \sum_{i=1}^{d-1} dx_i^2 + dx_{d+1}^2; \quad -x_0^2 + \sum_{i=1}^{d-1} x_i^2 + x_{d+1}^2 = R^2$$

is explicitly invariant under $SO(d, 1)$ by construction and $\mathcal{R} > 0$.

- d -dimensional Anti de Sitter space:

$$ds^2 = -dx_0^2 + \sum_{i=1}^{d-1} dx_i^2 - dx_{d+1}^2; \quad -x_0^2 + \sum_{i=1}^{d-1} x_i^2 - x_{d+1}^2 = -R^2$$

is explicitly invariant under $SO(d-1, 2)$ by construction and $\mathcal{R} < 0$.

- **Metrics:** Poincare coordinates $(t, x_i \in R, x_0 \in R_+)$

$$ds^2 = \frac{R^2}{x_0^2} \left(-dt^2 + \sum_{i=1}^{d-2} dx_i^2 + dx_0^2 \right)$$

- Up to conformal factor, same as flat space $\Rightarrow$ Penrose diagram is the same. For $d > 2$ however, we use radial coordinate $\rho > 0$ instead of spatial coordinate $x \in R \Rightarrow$ obtain half of diamond = triangle.

- We can make explicit also the exponential "warp factor"

$$ds^2 = e^{2y} \left(-dt^2 + \sum_{i=1}^{d-2} dx_i^2 \right) + dy^2 \quad (x_0 = e^{-y})$$

- Even though r, x_i, x_0 are ∞ in extent, space is not complete: Infinity at $y = \infty$ is reached in finite time by a null ray:

$$ds^2 = 0 \Rightarrow dt^2 = e^{-2y} dy^2 \Rightarrow t = \int^{\infty} e^{-y} dy < \infty$$

- $\Rightarrow \exists$ other coordinates covering whole space: global coordinates:

$$\begin{aligned} \text{AdS :} \quad ds_d^2 &= R^2 (-\cosh^2 \rho d\tau^2 + d\rho^2 + \sinh^2 \rho d\vec{\Omega}_{d-2}^2) \\ \text{sphere :} \quad ds_d^2 &= R^2 (\cos^2 \rho dw^2 + d\rho^2 + \sin^2 \rho d\vec{\Omega}_{d-2}^2) \end{aligned}$$

- Coordinate transf.: global $\leftrightarrow$ embedding:

$$X_0 = R \cosh \rho \cos \tau; \quad X_i = R \sinh \rho \Omega_i; \quad X_{d+1} = R \cosh \rho \sin \tau$$

- Finally, coordinate transf. $\tan \theta = \sinh \rho \Rightarrow$

$$ds_d^2 = \frac{R^2}{\cos^2 \theta} (-d\tau^2 + d\theta^2 + \sin^2 \theta d\vec{\Omega}_{d-2}^2)$$

- Here $0 \leq \theta \leq \pi/2$, $\tau \in R \Rightarrow$ infinite cylinder. Poincare patch: figure of revolution obtained by rotating triangle around a side, situated along the axis of the cylinder:

.

- Boundary of cylinder still reached by light ray in finite time (and reflected back).

- AdS is somewhat like a finite box, with a boundary.

- d -dimensional boundary of AdS_{d+1} space: In Poincare coordinates, at $x_0 = \epsilon$ (and fixed) is Minkowski $_d$,

$$ds^2 = \frac{R^2}{\epsilon^2} [-dt^2 + \sum_{i=1}^{d-1} dx_i^2]$$

- In global coordinates, at $\theta = \pi/2 - \epsilon$ is $S^3 \times R$ cylinder

$$ds^2 = \frac{R^2}{\epsilon^2} [-d\tau^2 + d\vec{\Omega}_{d-2}^2]$$

- But the 2 are related, in the Euclideanized version, by a conformal transformation:

$$ds^2 = d\rho^2 = \rho^2 d\Omega_{d-1}^2 = e^{2\tau} [d\tau^2 + d\Omega_{d-1}^2]; \quad \rho = e^\tau$$

- So conformal symmetry of boundary = invariance symmetry of AdS_{d+1} .

- Perhaps physics in AdS_{d+1} space is **holographic**: Physics inside AdS = physics at its boundary.

- Reason why possible: Boundary of space is reached in finite time.

- Anti-de Sitter space is a maximally symmetric space: constant negative curvature $\mathcal{R} < 0$.
- Solution of Einstein's eq. with a cosmological constant, $T_{\mu\nu} = \Lambda g_{\mu\nu}$ and $\Lambda < 0$, i.e.

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 8\pi GT_{\mu\nu}$$

- Observation: Light takes an ∞ time to reach the middle of AdS space, $\rho = 0$ or $x_0 = \infty$:

$$ds^2 = 0 \Rightarrow t = \int_0 \frac{dx_0}{x_0} \sim -\ln x_0|_{x_0 \sim 0} \rightarrow \infty$$

- $\Rightarrow$ In order for AdS to become truly like a box of finite size, we must cut out a tube in the middle.

Vielbein-spin connection formulation of GR: 1st vs. 2nd order

- Any space is locally flat: tangent space: Lorentz invariance that is local (at any point).
- **Vielbein** e_μ^a : "square root" of metric, making local Lorentz invariance manifest:

$$g_{\mu\nu}(x) = e_\mu^a(x)e_\nu^b(x)\eta_{ab}$$

$$\rightarrow e_\mu^a \rightarrow \Lambda^a_b e_\mu^b.$$

- Covariant derivative acting on tensors (bosons): with $\Gamma^\mu_{\nu\rho}$

$$D_\mu T_\nu^\rho \equiv \partial_\mu T_\nu^\rho + \Gamma^\rho_{\mu\sigma} T_\nu^\sigma - \Gamma^\sigma_{\mu\nu} T_\sigma^\rho$$

- Covariant derivative acting on spinors (fermions): with **spin connection** ω_μ^{ab} , multiplying the generator of the Lorentz group in the spinor representation, $\frac{1}{4}\Gamma_{ab}$,

$$D_\mu \Psi = \partial_\mu \Psi + \frac{1}{4} \omega_\mu^{ab} \Gamma_{ab} \Psi$$

- Second order formulation: $\omega_\mu^{ab} = \omega_\mu^{ab}(e)$ satisfies "vielbein postulate", or "no torsion constraint" ($T_{\mu\nu}^a = \text{torsion}$),

$$T_{[\mu\nu]}^a = D_{[\mu} e_{\nu]}^a = \partial_{[\mu} e_{\nu]}^a + \omega_{[\mu}^{ab} e_{\nu]}^b = 0$$

(if there are no fundamental fermions; if there are, there are extra terms). Equivalently,

$$D_\mu e_\nu^a \equiv \partial_\mu e_\nu^a + \omega_\mu^{ab} e_\nu^b - \Gamma^\rho_{\mu\nu} e_\rho^a = 0$$

- The solution is

$$\omega_\mu^{ab}(e) = \frac{1}{2} e^{a\nu} (\partial_\mu e_\nu^b - \partial_\nu e_\mu^b) - \frac{1}{2} e^{b\nu} (\partial_\mu e_\nu^a - \partial_\nu e_\mu^a) - \frac{1}{2} e^{a\rho} e^{b\sigma} (\partial_\rho e_{c\sigma} - \partial_\sigma e_{c\rho}) e_\mu^c.$$

- Define the field strength of ω_μ^{ab} ($=SO(1, d-1)$ gauge field)

$$R_{\mu\nu}^{ab}(\omega) = \partial_\mu \omega_\nu^{ab} - \partial_\nu \omega_\mu^{ab} + \omega_\mu^{ab} \omega_\nu^{bc} - \omega_\nu^{ab} \omega_\mu^{bc}.$$

- Then we have

$$R_{\rho\sigma}^{ab}(\omega(e)) = e_{\mu}^a e^{-1,\nu b} R^{\mu}_{\nu\rho\sigma}(\Gamma(e)) , \quad R = R_{\mu\nu}^{ab} e_a^{-1\mu} e_b^{-1\nu}$$

so that the Einstein-Hilbert action *in second order formulation* ($\omega = \omega(e)$) is

$$S_{EH} = \frac{1}{16\pi G_N} \int d^d x (\det e) R_{\mu\nu}^{ab}(\omega(e)) e_a^{-1,\mu} e_b^{-1,\nu}.$$

- But then: first order formulation: $\omega_{\mu}^{ab} =$ independent variable in the same action, rewritten as

$$\begin{aligned} S_{EH} &= \frac{1}{16\pi G_N} \frac{1}{4} \int d^4 x \epsilon^{\mu\nu\rho\sigma} \epsilon_{abcd} R_{\mu\nu}^{ab}(\omega) e_{\rho}^c e_{\sigma}^d \\ &= \frac{1}{16\pi G_N} \int \epsilon_{abcd} R^{ab}(\omega) \wedge e^c \wedge e^d \end{aligned}$$

- Then the ω_{μ}^{ab} equation of motion is

$$T_{[\mu\nu]}^a \equiv 2D_{[\mu} e_{\nu]}^a = 0$$

so solving it, we are back at the second order formulation.

Black holes

- The Schwarzschild solution = most general solution of vacuum ($T_{\mu\nu} = 0$) Einstein equation, with spherical symmetry:

$$ds^2 = - \left(1 - \frac{2MG_N}{r} \right) dt^2 + \frac{dr^2}{1 - \frac{2MG_N}{r}} + R^2 d\Omega_2^2.$$

- In the Newtonian limit, with $g_{\mu\nu} = \eta_{\mu\nu} + \kappa h_{\mu\nu}$,

$$\kappa_N h_{00} = \kappa_N h_{ii} = -2U_N = + \frac{2MG_N}{r},$$

so we are back to the Newtonian potential, satisfying

$$\Delta U_{\text{Newton}} = 4\pi G_N M \delta^3(x) \Rightarrow \Delta \kappa_N h_{00} = -8\pi G_N M \delta^3(x).$$

- Solution *apparently* singular at $r = r_H = 2MG_N$, so we cannot reach the source at $r = 0$?
- If the solution is valid all the way to r_H , we have a (Schwarzschild) black hole.
- $r = r_H$ is the *event horizon*: light, at $ds^2 = 0$, gives

$$dt = \frac{dr}{1 - \frac{2MG_N}{r}}, \quad r \rightarrow r_H : dt \simeq 2MG_N \frac{dr}{r - 2MG_N} \Rightarrow t \simeq 2MG_N \ln(r - 2MG_N) \rightarrow \infty.$$

- But at the horizon, $R \sim 1/r_H^2 = 1/(2MG_N)^2 = \text{finite!} \Rightarrow$ not singular. Need better coordinates: Kruskal.

- First, tortoise coordinates,

$$\frac{dr}{1 - \frac{2MG_N}{r}} = dr_* \Rightarrow r_* = r + 2MG_N \ln \left(\frac{r}{2MG_N} - 1 \right) ,$$

then Eddington-Finkelstein ones,

$$u = t - r_*; \quad v = t + r_* ,$$

and finally Kruskal coordinates,

$$\bar{u} = -4MG_N e^{-\frac{u}{4MG_N}}; \quad \bar{v} = +4MG_N e^{\frac{v}{4MG_N}} ,$$

so that the metric is (region $r \geq 2MG_N$ becomes $-\infty < r_* < +\infty$, thus $-\infty < \bar{u} \leq 0, 0 \leq \bar{v} < +\infty$)

$$ds^2 = -\frac{2MG_N}{r} e^{-\frac{r}{2MG_N}} d\bar{u}d\bar{v} + r^2 d\Omega_2^2 ,$$

- Then we have

$$-\frac{\bar{u}\bar{v}}{(4MG_N)^2} = e^{\frac{v-u}{4MG_N}} = e^{\frac{r^*}{2MG_N}} = e^{\frac{r}{2MG_N}} \left(\frac{r}{2MG_N} - 1 \right)$$

so the $r = 0$ singularity corresponds to $\bar{u}\bar{v} = (4MG_N)^2$, while $r = 2MG_N$ is $\bar{u} = 0$ or $\bar{v} = 0$. Define $\bar{t}, \bar{r}$ as $\bar{u} = \bar{t} - \bar{r}$, $\bar{v} = \bar{t} + \bar{r} \rightarrow$ Kruskal diagram. $r = 0$ is then $\bar{t}^2 - \bar{r}^2 = (4MG_N)^2$.

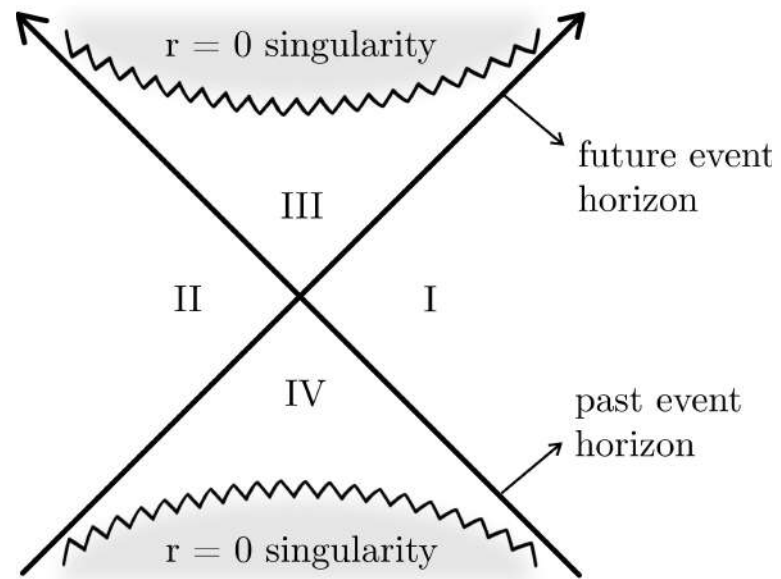
- Penrose diagram: drop $d\Omega_2^2$ and conformal factor in Kruskal metric. Then: subset of flat space, restricted by $\bar{t}^2 - \bar{r}^2 = (4MG_N)^2$. The usual transformation is

$$\bar{u} = 4MG_N \tan \tilde{u}_+; \quad \bar{v} = 4MG_N \tan \tilde{u}_-; \quad \tilde{u}_\pm = \frac{\tau \pm \theta}{2},$$

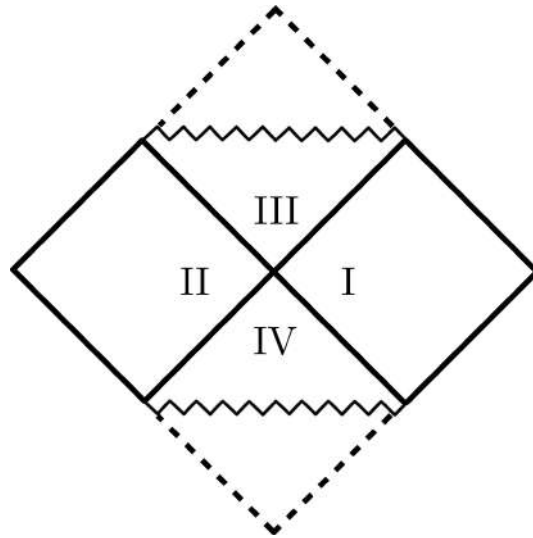
and the $r = 0$ singularity is then

$$1 = \tan \frac{\tau + \theta}{2} \tan \frac{\tau - \theta}{2} = \frac{\sin^2(\tau/2) - \sin^2(\theta/2)}{1 - \sin^2(\tau/2) - \sin^2(\theta/2)},$$

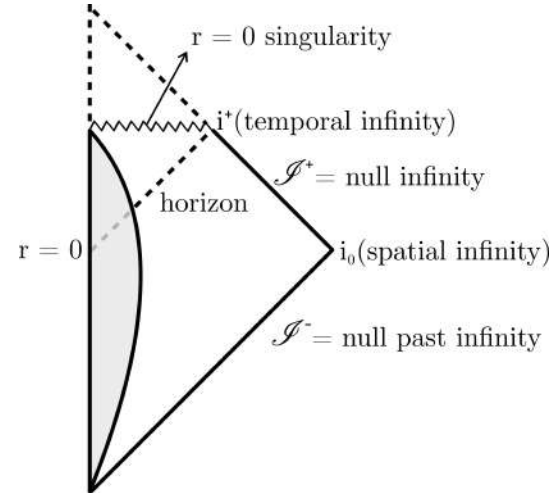
leading to $\sin^2(\tau/2) = 1/2$, thus $\tau = \pm\pi/2$ ($\tau_{\max}$, for $\theta = 0$, is π).



Kruskal diagram of the Schwarzschild black hole.



a)



b)

a) Penrose diagram of the eternal Schwarzschild black hole (time independent solution). The dotted line gives the completion to the Penrose diagram of flat 2 dimensional (Minkowski) space. b) Penrose diagram of a physical black hole, obtained from a collapsing star (the curved line). The dotted line gives the completion to the Penrose diagram of flat $d > 2$ dimensional (Minkowski) space.

Lecture 2

Elements of string theory and D-branes

Wordline particle action: analogy

- Action on particle worldline = proper time: in terms of $X^\mu(\tau)$:

$$S = -m \int d\tau \sqrt{-\dot{X}^\mu \dot{X}^\nu \eta_{\mu\nu}}.$$

- In the nonrelativistic limit: OK:

$$S = -mc^2 \int dt \sqrt{1 - \frac{v^2}{c^2}} \simeq \int dt \left[-mc^2 + \frac{mv^2}{2} \right].$$

- Action is reparametrization invariant, $\tau' = \tau'(\tau)$, $dx^\mu/d\tau = (dx^\mu/d\tau')d\tau'/d\tau$; $X'^\mu(\tau'(\tau)) = X^\mu(\tau)$.
- Equations of motion and boundary conditions from δS :

$$\delta S = +m \int d\tau \frac{d}{d\tau} \left[-\frac{\eta_{\mu\nu} \dot{X}^\mu}{\sqrt{-\dot{X}^\rho \dot{X}_\rho}} \right] \delta X^\nu + \delta X^\mu m \frac{dX_\mu}{d\tau} \Big|_{\tau_i}^{\tau_f} \Rightarrow \frac{dp_\mu}{d\tau} = 0$$

- Couple to gravity: nontrivial: $\eta_{\mu\nu} \rightarrow g_{\mu\nu}$, geodesic equation.
- Couple to background charge: add worldline term $= \int A_\mu j^\mu$,

$$\int d\tau A_\mu(X^\rho(\tau)) \left(q \frac{dX^\mu}{d\tau} \right) = \int d^4x A_\mu(X^\rho(\tau)) q \frac{dX^\mu}{d\tau} \delta^3(X^\rho(\tau)) \equiv \int d^4x A_\mu(X^\rho(\tau)) j^\mu(X^\rho(\tau)).$$

- First order action: introduce auxiliary field = independent world-line metric $\gamma_{\tau\tau}$, or einbein (vielbein) $e(\tau) = \sqrt{-\gamma_{\tau\tau}(\tau)}$. Write action for massive scalars X^μ in 1d on worldline, coupled to gravity (GR). Use $\sqrt{-\det \gamma} \times \gamma^{\tau\tau} = e^{-1}(\tau)$ and $\sqrt{-\det \gamma} = e(\tau)$.

$$S_p = \frac{1}{2} \int d\tau \left(e^{-1}(\tau) \frac{dX^\mu}{d\tau} \frac{dX^\nu}{d\tau} \eta_{\mu\nu} - em^2 \right) ,$$

- Write the equation of motion for $e(\tau)$, solve it, and replace it:

$$-\frac{1}{e^2} \dot{X}^2 - m^2 = 0 \Rightarrow e^2(\tau) = -\frac{\dot{X}^\mu \dot{X}_\mu}{m^2} \Rightarrow$$

$$S_p = \frac{1}{2} \int d\tau \left[\frac{m}{\sqrt{-\dot{X}^2}} \dot{X}^2 - \frac{\sqrt{-\dot{X}^2}}{m} m^2 \right] = -m \int d\tau \sqrt{-\dot{X}^\mu \dot{X}_\mu} \equiv S_1 .$$

- Take $m \rightarrow 0$ limit, then fix a gauge for reparametrization invariance ($e(\tau) \rightarrow e'(\tau')$), $e(\tau) = 1$:

$$S_{m=0,e=1} = \frac{1}{2} \int d\tau \frac{dX^\mu}{d\tau} \frac{dX^\nu}{d\tau} \eta_{\mu\nu} .$$

- Equation of motion ($X^\mu(\tau)$) and constraint (for $e(\tau) = 1$: previous eq. for $e(\tau)$):

$$\frac{d}{d\tau} \left(\frac{dX^\mu}{d\tau} \right) = 0 , \quad -\frac{ds^2}{d\tau^2} = \frac{dX^\mu}{d\tau} \frac{dX^\nu}{d\tau} \eta_{\mu\nu} \equiv T = 0 .$$

Strings

- Nambu-Goto action for bosonic string = area of "worldsheet" spanned by string $\times$ string tension. Generalization of particle action: area of worldsheet. $X^\mu(\sigma, \tau)$ = coordinates in spacetime. $\xi^a = (\sigma, \tau)$ = intrinsic coordinates on worldsheet.

$$S_{NG} = -\frac{1}{2\pi\alpha'} \int d\sigma d\tau \sqrt{\det(h_{ab})}$$

where h_{ab} = metric induced on worldsheet (pullback)

$$\begin{aligned} ds_{ind}^2 &= dx^\mu dx^\nu g_{\mu\nu}(X) = d\xi^\mu d\xi^\nu h_{ab}(\xi) \Rightarrow \\ h_{ab}(\sigma, \tau) &= \partial_a X^\mu \partial_b X^\nu g_{\mu\nu}(X) \end{aligned}$$

- Is worldsheet diffeomorphism (gen. coord., or reparametrization) invariant.

- First order form: again introduce auxiliary field = independent worldsheet metric.

- $\Rightarrow$ Polyakov action. In flat spacetime,

$$S_P[X, \gamma] = -\frac{1}{4\pi\alpha'} \int d\sigma d\tau \sqrt{-\gamma} \gamma^{ab} \partial_a X^\mu \partial_b X^\nu \eta_{\mu\nu}$$

- Symmetries:

- Spacetime Poincare invariance

- Worldsheet diffeomorphism invariance: $X'^\mu(\sigma', \tau') = X^\mu(\sigma, \tau)$

- Worldsheet Weyl invariance: $\gamma'_{ab} = e^{2\omega(\sigma, \tau)} \gamma_{ab}$

- Use them to fix conformal (unit) gauge: $\gamma_{\alpha\beta} = \eta_{\alpha\beta}$.

- Action becomes

$$S = -\frac{T}{2} \int d^2\sigma \eta^{\alpha\beta} \partial_\alpha X^\mu \partial_\beta X^\nu \eta_{\mu\nu}$$

→ action for free massless scalars in 2d: **conformally invariant** (conf. inv. = residual gauge invariance: dependence on $\sigma + \tau$ only), with equations of motion

$$\square X^\mu = \left(\frac{\partial^2}{\partial \sigma^2} - \frac{\partial^2}{\partial \tau^2} \right) X^\mu = 0 \Rightarrow X^\mu(\sigma, \tau) = X_R^\mu(\sigma - \tau) + X_L^\mu(\sigma + \tau)$$

- Boundary term: gives string types:

$$-\frac{1}{2\pi\alpha'} \int d\tau \sqrt{-\gamma} \delta X^\mu \partial_\sigma X_\mu \Big|_{\sigma=0}^{\sigma=l} = 0 \Rightarrow$$

- Closed strings (periodic): $X^\mu(\tau, l) = X^\mu(\tau, 0)$; $\gamma_{ab}(\tau, l) = \gamma_{ab}(\tau, 0)$.

- Neumann open strings (free endpoints, $v = c$): $\partial^\sigma X^\mu(\tau, 0) = \partial^\sigma X^\mu(\tau, l)$.

- Dirichlet open strings (fixed endpoints): $\delta X^\mu(\tau, 0) = \delta X^\mu(\tau, l) = 0$.

- (Virasoro) Constraints: equations of motion of γ_{ab} (fixed to unit) $= T_{ab}$

$$T_{ab} = -\frac{1}{4\pi} \frac{1}{\sqrt{-\gamma}} \frac{\delta S_P}{\delta \gamma^{ab}} \Big|_{\gamma_{ab}=\eta_{ab}} = \frac{1}{\alpha'} \left(\partial_a X^\mu \partial_b X_\mu - \frac{1}{2} \eta_{ab} \partial_c X^\mu \partial^c X_\mu \right) \Rightarrow$$

$$\alpha' T_{01} = \alpha' T_{10} = \dot{X} \cdot X' , \quad \alpha' T_{00} = \alpha' T_{11} = \frac{1}{2} (\dot{X}^2 + X'^2).$$

- Closed strings: expand $X_R^\mu(\tau - \sigma)$ and $X_L^\mu(\tau + \sigma)$ in Fourier modes $\alpha_n^\mu, \tilde{\alpha}_n^\mu$,

$$X^\mu(\sigma, \tau) = x^\mu + \alpha' p^\mu \tau + i \frac{\sqrt{2\alpha'}}{2} \sum_{n \neq 0} \frac{1}{n} \left[\alpha_n^\mu e^{-in(\tau-\sigma)} + \tilde{\alpha}_n^\mu e^{-in(\tau+\sigma)} \right].$$

- Neumann open strings: identify $\alpha_n^\mu = \tilde{\alpha}_n^\mu$.
- Fourier modes $L_m, \bar{L}_m$ of constraints T_{--}, T_{++} are $L_m, \bar{L}_m$, for closed strings

$$L_m = \frac{1}{2} \sum_{n=-\infty}^{+\infty} \alpha_{m-n}^\mu \alpha_n^\mu , \quad \bar{L}_m = \frac{1}{2} \sum_{n=-\infty}^{+\infty} \tilde{\alpha}_{m-n}^\mu \tilde{\alpha}_n^\mu.$$

and $H = L_0 + \bar{L}_0 = 0$ (closed) or $H = L_0$ (open) give (classically)

$$M_{\text{closed}}^2 \equiv -p_\mu p^\mu = \frac{2}{\alpha'} \sum_{n \geq 1} (\alpha_{-n}^\mu \alpha_n^\mu + \tilde{\alpha}_{-n}^\mu \tilde{\alpha}_n^\mu) , \quad M_{\text{open}}^2 \equiv -p_\mu p^\mu = -\frac{\alpha_0^2}{2\alpha'} = \frac{1}{\alpha'} \sum_{n \geq 1} \alpha_{-n}^\mu \alpha_n^\mu$$

- What does the string action represent? Particle action: is first quantized: Need to also define vertices and propagators. String action: defines the propagator; vertex is unique!!

- **Quantization:** $\alpha_{-n}^\mu, \tilde{\alpha}_{-n}^\mu$: creation operators. More precisely, $\alpha_m^\mu = \sqrt{m} a_m^\mu$, $\alpha_{-m}^\mu = \sqrt{m} a_m^{\dagger\mu}$ for $m > 0$.

- But $\exists$ gauge inv.: easiest in light-cone gauge. $X^\pm$ auxiliary, X^i physical. Then $H = p^-$ and the open string mass spectrum is

$$M^2 \equiv 2p^+p^- - p^ip^i = \frac{1}{\alpha'}(N - a), \quad N = \sum_{n \geq 1} \alpha_{-n}^i \alpha_n^i = \sum_{n \geq 1} n a_n^{\dagger i} a_n^i,$$

where

$$a = - \sum_{i=1}^{D-2} \sum_{n \geq 1} \frac{n}{2} = - \frac{D-2}{2} \sum_{n \geq 1} n = \frac{D-2}{24} = 1 \Rightarrow D = 26.$$

- Bosonic closed string spectrum is similar, but with N and $\bar{N}$,

$$\sim a_{n_1}^{i_1} \dots a_{n_k}^{i_k} \tilde{a}_{\bar{m}_1}^{\tilde{i}_1} \dots \tilde{a}_{\bar{m}_j}^{\tilde{i}_j} |0\rangle,$$

with the constraint $P = L_0 - \bar{L}_0 = 0$, so $N = \bar{N}$. Spectrum $\rightarrow$ different fields $\Rightarrow$ String theory = field theory of infinite number of different kinds of fields.

- Massless fields: $A_{\mu\nu} = \alpha_{-1}^\mu \tilde{\alpha}_{-1}^\nu |0\rangle = \{A_{((\mu\nu))} = g_{\mu\nu}, A^{[\mu\nu]} = B^{\mu\nu}, \phi = A^{\mu\mu}\} \rightarrow$ spacetime fields.

- These massless fields create a spacetime background for the string

$$S = -\frac{1}{4\pi\alpha'} \int d^2\sigma [\sqrt{h} h^{\alpha\beta} \partial_\alpha X^\mu \partial_\beta X^\nu g_{\mu\nu}(X^\rho) + \epsilon^{\alpha\beta} \partial_\alpha X^\mu \partial_\beta X^\nu B_{\mu\nu}(X^\rho) - \alpha' \sqrt{h} \mathcal{R}^{(2)} \Phi(X^\rho)]$$

- But bosonic string has tachyonic vacuum: $M^2(|0\rangle) < 0$. Need to get rid of it:

- **Superstring**: Supersymmetric string. In **Green-Schwarz formulation**, spacetime susy + κ symmetry. (Fix a gauge for κ symmetry $\Rightarrow$ worldsheet susy). Introduce θ^A = spacetime spinors and worldsheet scalars. Replace $\partial_a X^\mu$ with spacetime susy invariant

$$\Pi_a^\mu = \partial_a X^\mu - i \bar{\theta}^A \Gamma^\mu \partial_a \theta^A$$

invariant under

$$\delta X^\mu = -\bar{\epsilon}^A \Gamma^\mu \partial_a \theta^A, \quad \delta \theta^A = \epsilon^A$$

$$S = -\frac{1}{4\pi\alpha'} \int d\tau d\sigma \sqrt{-\gamma} \gamma^{ab} \Pi_a^\mu \Pi_b^\nu g_{\mu\nu} + \int d\tau d\sigma \epsilon^{ab} \Pi_a^M \Pi_b^N B_{MN}$$

- flat space:

$$B \equiv \epsilon^{ab} \Pi_a^M \Pi_b^N B_{MN} = -idX^\mu \wedge (\bar{\theta}^1 \Gamma_\mu d\theta^2 - \bar{\theta}^2 \Gamma_\mu d\theta^1) + \bar{\theta}^1 \Gamma^\mu d\theta^1 \wedge \bar{\theta}^2 \Gamma_\mu d\theta^2$$

- Kappa symmetry,

$$\delta_\kappa \theta^A = -2\Gamma_\mu \Pi_a^\mu \kappa^{Aa}, \quad \delta_\kappa X^\mu = -\bar{\theta}^A \Gamma^\mu \delta \theta^A, \quad \dots$$

is fixed by the condition (together with lightcone gauge for bosons)

$$\Gamma^+ \theta^1 = \Gamma^+ \theta^2 = 0, \quad \Gamma^\pm = (\Gamma^0 \pm \Gamma^9)/\sqrt{2}$$

and $\theta^{A\alpha}$ are regrouped as 2-comp. Majorana worldsheet spinors S^m , m spinor of $SO(8)$,

$$S_{lc} = -\frac{1}{4\pi\alpha'} \int d^2\sigma \left[\partial_a X^i \partial^a X^i + 2\alpha' \bar{S}^m \gamma^a \partial_a S^m \right].$$

- Supersymmetry means tachyons (and other states) are out of the spectrum. Vacuum: massless states $A^{\mu\nu} = \{g^{\mu\nu}, B^{\mu\nu}, \phi\} + \text{others}$.
- Strings $\rightarrow$ couple to $B_{\mu\nu}$.

- Gauge fixed Green-Schwarz action = gauge fixed action with manifest worldsheet susy:
- **Spinning string**: Neveu-Schwarz-Ramond (NSR) action, with ψ^μ = worldsheet spinors, spacetime vectors:

$$S = -\frac{1}{4\pi\alpha'} \int d^2\sigma \left[\partial_a X^\mu \partial^a X_\mu + \bar{\psi}^\mu \gamma^a \partial_a \psi_\mu \right] ,$$

with worldsheet susy:

$$\delta X^\mu = \bar{\epsilon} \psi^\mu , \quad \delta \psi^\mu = \gamma^a \partial_a X^\mu \epsilon$$

- Fermionic boundary term (for open string)

$$\psi_+ \delta \psi_+ - \psi_- \delta \psi_- \Big|_0^\pi ,$$

means we can impose at 0 $\psi_+^\mu(0, \tau) = \psi_-^\mu(0, \tau)$, but then at π we have 2 possibilities,

$$\psi_+(\pi, \tau) = \pm \psi_-^\mu(\pi, \tau),$$

giving the Ramond (+) $\Rightarrow$ spacetime fermions, or Neveu-Schwarz (-) $\Rightarrow$ spacetime bosons sectors. Closed strings: indep. for L or R $\Rightarrow$ NS-NS and R-R (bosons) or NS-R and R-NS (fermions).

- Chirality of θ^A in closed string GS theory ($N = 2$): same $\Rightarrow$ **type IIB string theory**; opposite $\Rightarrow$ **type IIA string theory**. Open strings ($N = 1$): single θ ; can couple to $SO(32)$ Yang-Mills fields (non-anomalous theory): **type I string theory**. Other $N = 1$ theories: **heterotic** (left movers bosonic, right movers supersymmetric): $SO(32)$ or $E_8 \times E_8$.

Conformal invariance in 2 dimensions

- Conformal invariance: symmetry of QFT in flat space, under coordinate transformation that generalizes scale transf., of the type

$$x'^{\mu} = \alpha x^{\mu} \Rightarrow ds^2 = d\vec{x}'^2 = \alpha^2 d\vec{x}^2 \rightarrow$$

$$x_{\mu} \rightarrow x'_{\mu}(x) \text{ s.t. } ds^2 = dx'_{\mu} dx'_{\mu} = [\Omega(x)]^{-2} dx_{\mu} dx_{\mu}$$

- Obs: transf. on **flat space**. Transf. are a **subclass** of general coordinate transf. For strings in unit gauge, $\sigma^+ \rightarrow \tilde{\sigma}^+ = f(\sigma^+)$, $\sigma^- \rightarrow \tilde{\sigma}^- = g(\sigma^-)$ (residual gauge inv.).

- In Minkowski $(d - 1, 1)$ spacetime, **for** $d > 2$, the conformal group is $SO(d, 2)$.
- **d=2**: special. Invariance group is not finite dimensional (like $SO(d, 2)$), but infinite dimensional: Virasoro algebra

$$\begin{aligned}
 ds^2 &= dzd\bar{z} \Rightarrow z' = f(z) \\
 ds'^2 &= dz'd\bar{z}' = \frac{\partial z'}{\partial z} \frac{\partial \bar{z}'}{\partial \bar{z}} dzd\bar{z} = \Omega^{-2}(z, \bar{z}) dzd\bar{z}
 \end{aligned}$$

- So any **holomorphic** transformation is conformal. Generators $\{L_m\}$ (qu. version of string constraint modes): Virasoro algebra:

$$[L_m, L_n] = (m - n)L_{m+n} + \frac{c}{12}(m^3 - m)\delta_{m,-n}$$

- GR tensors: covariant tensor $T_{i_1 \dots i_n}$ transforms as

$$T_{i_1 \dots i_n}(z_1, z_2) = T'_{j_1 \dots j_n}(z'_1, z'_2) \frac{\partial z'^{j_1}}{\partial z^{i_1}} \dots \frac{\partial z'^{j_n}}{\partial z^{i_n}}.$$

- Primary fields or tensor operators of CFT defined by analogy: " $T_{z \dots z \bar{z} \dots \bar{z}}$ " is a primary field of dimensions $(h, \bar{h})$ (even if $h, \bar{h} \notin \mathbb{Z}$) if

$$T_{z \dots z \bar{z} \dots \bar{z}} = T'_{z \dots z \bar{z} \dots \bar{z}} \left(\frac{dz'}{dz} \right)^h \left(\frac{d\bar{z}'}{d\bar{z}} \right)^{\bar{h}},$$

- Operator product expansion (OPE) (valid in any QFT, but there only asymptotically):

$$\mathcal{O}_i(x_i)\mathcal{O}_j(x_j) = \sum_k c^k_{ij}(x_i - x_j)\mathcal{O}_k(x_j).$$

and if operators $\mathcal{O}_i$ have dimensions Δ_i ,

$$\langle \mathcal{O}_i(x_i)\mathcal{O}_j(x_j)\dots \rangle = \sum_k \frac{c^k_{ij}}{|x_i - x_j|^{\Delta_i + \Delta_j - \Delta_k}} \left\langle \mathcal{O}_k\left(\frac{x_i + x_j}{2}\right) \dots \right\rangle ,$$

whereas with the energy-momentum tensor $T(z)$,

$$T(z)\mathcal{O}(0,0) = \dots + \frac{h}{z^2}\mathcal{O}(0,0) + \frac{1}{z}\partial\mathcal{O}(0,0) + \dots ,$$

and for a primary field

$$T(z)\phi_i^{(h_i, \tilde{h}_i)} = \frac{h_i}{z^2}\phi_i^{(h_i, \tilde{h}_i)} + \frac{1}{z}\partial\phi_i^{(h_i, \tilde{h}_i)} + \text{nonsingular}.$$

- Example: free scalars (Polyakov string action in unit gauge)

$$S_E = \frac{1}{4\pi\alpha'} \int d^2\sigma [\partial_1 X^\mu \partial_1 X_\mu + \partial_2 X^\mu \partial_2 X_\mu] = \frac{1}{2\pi\alpha'} \int d^2z \partial X^\mu \bar{\partial} X_\mu.$$

- **T-duality of closed and open strings:** symmetry of string perturbation theory on compact spaces.
- For a string winding m times around X^{25} , bound. cond.

$$X^{25}(\tau, \sigma + 2\pi) = X^{25}(\tau, \sigma) + 2\pi\alpha'w.$$

- The classical solution is

$$X^{25}(\tau, \sigma) = X_L + X_R = x_0 + \alpha'p\tau + \alpha'w\sigma + i\sqrt{\frac{\alpha'}{2}} \sum_{n \neq 0} \frac{e^{-in\tau}}{n} (\alpha_n e^{in\sigma} + \tilde{\alpha}_n e^{-in\sigma}),$$

where $p = n/R$ and $w = mR/\alpha'$. The constraint is now $L_0 - \tilde{L}_0 = \alpha'pw + N^\perp - \tilde{N}^\perp$ and gives the spectrum

$$\begin{aligned} M_{\text{compact}}^2 &= p^2 + w^2 + \frac{2}{\alpha'}(N^\perp + \tilde{N}^\perp - 2) \\ &= \left(\frac{n}{R}\right)^2 + \left(\frac{mR}{\alpha'}\right)^2 + \frac{2}{\alpha'}(N^\perp + \tilde{N}^\perp - 2). \end{aligned}$$

- We observe the T-duality symmetry of the spectrum

$$M^2(R; n, m) = M^2(\tilde{R}; m, n).$$

extended to

$$x_0 \leftrightarrow q_0; \quad p \leftrightarrow w; \quad \alpha_n \leftrightarrow -\alpha_n; \quad \tilde{\alpha}_n \leftrightarrow \tilde{\alpha}_n,$$

- This T-duality exchanges then:

$$X^{25}(\tau, \sigma)X_L(\tau+\sigma)+XR(\tau-\sigma) \leftrightarrow X'^{25}(\tau, \sigma) = X_L(\tau+\sigma)-X_R(\tau-\sigma)$$

- **T-duality of open strings:** Do the same exchange for the open string solution. Obtain

$$X'^{25}(\tau, \sigma) = X_L^{25}(\tau + \sigma) - X_R^{25}(\tau - \sigma) = q_0^{25} + \sqrt{2\alpha'}\alpha_0^{25}\sigma + \sqrt{2\alpha'} \sum_{n \neq 0} \frac{\alpha_n^{25}}{n} e^{-in\tau} \sin n\sigma ,$$

$$\alpha_0^{25} = \frac{1}{\sqrt{2\alpha'}} \frac{x_2^{25} - x_1^{25}}{\pi}.$$

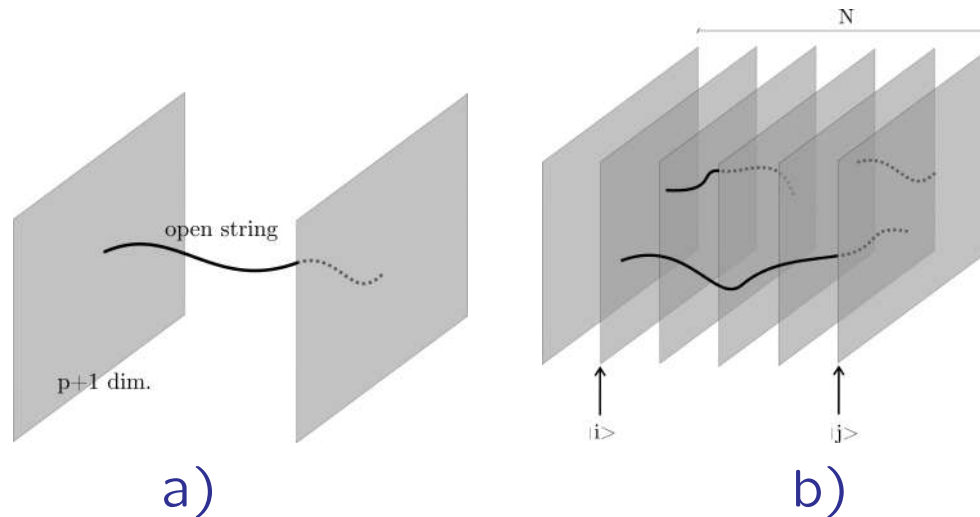
- But then the boundary condition changes from Neumann to Dirichlet and vice versa,

$$\partial_\alpha X^{25} = \epsilon_{\alpha\beta} \partial_\beta X'^{25}.$$

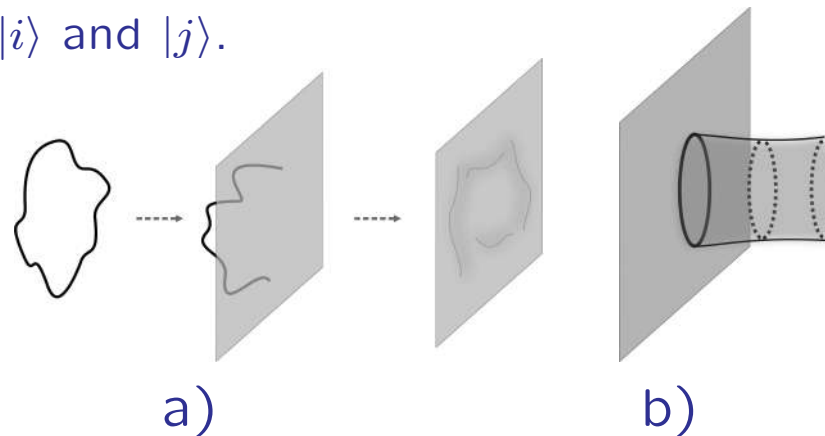
- **Reminder:** Vary Polyakov action $\Rightarrow$ equations of motion, and boundary term

$$\delta S_{P,bd.} = -\frac{1}{2\pi\alpha'} \int d\tau \sqrt{-\gamma} \delta X^\mu \partial^\sigma X_\mu \Big|_{\sigma=0}^{\sigma=\pi} \Rightarrow$$

- Neumann boundary condition: $\partial^\sigma X_\mu = 0|_{\sigma=0,\pi} \Rightarrow$ endpoints of string move at the speed of light: usual.
- Dirichlet boundary condition: $\delta X^\mu = 0|_{\sigma=0,\pi} \Rightarrow X^\mu = \text{constant}$ at $\sigma = 0, \pi$. $\rightarrow$ endpoints fixed.
- We can have Neumann for $p + 1$ coordinates and Dirichlet for $D - p - 1 \Rightarrow$ "Dp-brane".
- Spacetime fields can excite coordinates X^μ transverse to the Dp-brane (Dirichlet directions) $\rightarrow$ fluctuations $\Rightarrow$ this is Dp-brane is a dynamical object.



a) Open string between two D- p -branes ($p + 1$ dimensional "walls"). b) The endpoints of the open string are labelled by the D-brane they end on (out of N D-branes), here $|i\rangle$ and $|j\rangle$.



a) Closed string colliding with a D-brane, exciting an open string mode and making it vibrate b) String worldsheet corresponding to it, with a closed string tube coming from infinity and ending on the D-brane as an open string boundary. Allows us to calculate the D-brane action and couplings.

- Compute charges and tensions of Dp-branes and compare with supergravity p-brane solutions (Polchinski, 1995) $\Rightarrow$ Dp-brane = extremal p-brane solution of supergravity.

- Open strings have "Chan-Paton factors" at endpoints $\rightarrow$ indices $\Rightarrow$ open string. $\lambda_{ij}^a |i\rangle \otimes |j\rangle \Rightarrow$ massless open string state is $A_\mu^a = \alpha_{-1}^\mu \lambda_{ij}^a |i\rangle \otimes |j\rangle =$ vector in $U(N)$ gauge group for N D-branes.

- Action for a single D-brane is

$$S_p = T_p \int d^{p+1} \xi e^{-\phi} \sqrt{-\det(h_{ij} + \alpha' (F_{ij} + B_{ij}))} + \text{fermi} + \text{WZ}$$

- Static gauge: $X^i = \xi^i, i = 0, \dots, p$ and $g_{\mu\nu} = \eta_{\mu\nu} \Rightarrow$

$$\begin{aligned} h_{ij} &= \partial_i X^\mu \partial_j X^\nu g_{\mu\nu} = \eta_{ij} + \partial_i X^m \partial_j X_m \\ B_{ij} &= \partial_u X^\mu \partial_j X^\nu B_{\mu\nu} \end{aligned}$$

- WZ term: $\int_{M_p} e^{\wedge F/2\pi} \wedge \sum_n A_n$, e.g. a term on D5 in type IIB is

$$\frac{1}{2\pi} \int_{M_6} d^6 x \epsilon^{\mu_1 \dots \mu_6} A_{\mu_1} F_{\mu_2 \dots \mu_6}^+$$

- Then, for $p = 3$ and a single brane

$$S_2 = \text{const.} + \int d^3x \left(-\frac{F_{ij}^2}{4} - \frac{1}{2} \partial_i X^m \partial^i X_m + \text{fermi} \right)$$

- In fact, the action: " $\mathcal{N} = 4$ supersymmetric Yang-Mills" for N D3-branes.

- Fields: $\{A_i^a, X^a[IJ], \psi_\alpha^{aI}\}$, $a \in SU(N)$, $I \in SU(4)$, $[IJ] \rightarrow$ anti-symmetric of $SU(4)$: **6** representation. ($m = 1, \dots, 6$: transverse to D3).

- Action

$$S_{\mathcal{N}=4SYM} = -2 \int d^4x \text{tr} \left[-\frac{1}{4} F_{\mu\nu}^2 - \frac{1}{2} \bar{\Psi}_I \not{D} \Psi^I - \frac{1}{2} D_\mu X_{IJ} D^\mu X^{IJ} \right. \\ \left. + ig \bar{\Psi}^I [X_{IJ}, \Psi^J] - g^2 [X_{IJ}, X_{KL}] [X^{IJ}, X^{KL}] \right]$$

- **Observation:** Bosonic Nambu-Goto version $\rightarrow$ also volume spanned by worldvolume:

$$S_p = T_p \int d^{p+1}\xi \sqrt{-\det(h_{ab})}$$

$$h_{ab} = \partial_a \xi^\mu \partial_b \xi^\nu g_{\mu\nu}$$

- In fact, strings massless fields form **spacetime supergravity multiplet**.
- Supergravity has extremal p-branes solution $\Rightarrow$ p-branes are string theory nonperturbative objects: D-branes.
- Super p-brane: generalization

$$S_p = T_p \int d^{p+1}\xi \left(-\frac{1}{2} \sqrt{-\gamma} \gamma^{ij} \Pi_i^A \Pi_j^B \eta_{MN} e^{\frac{a(p)\phi}{p+1}} \right. \\ \left. + \frac{p-1}{2} \sqrt{-\gamma} - \frac{1}{(p+1)!} \epsilon^{i_1 \dots i_{p+1}} \Pi_{i_1}^{A_1} \dots \Pi_{i_{p+1}}^{A_{p+1}} A_{A_1 \dots A_{p+1}} \right)$$

- Bosonic: $\Pi_i^A \rightarrow \Pi_i^a = \partial_i X^\mu E_\mu^a$.

Lecture 3

Black holes in supergravity vs. D-branes

Supersymmetry

- Bose-fermi symmetry. e.g. 2d: 1 Majorana spinor Ψ + 1 real scalar ϕ . **On-shell supersymmetry:** 1 bose degree of freedom, 1 fermi d.o.f.

$$S = -\frac{1}{2} \int d^2x [(\partial_\mu \phi)^2 + \bar{\Psi} \not{\partial} \Psi]$$

- Dimensions: $[\phi] = 0, [\Psi] = 1/2$. Fermi-bose $\Rightarrow$ start as

$$\begin{aligned} \delta \phi &= \bar{\epsilon} \Psi \Rightarrow [\epsilon] = -1/2 \Rightarrow \\ \delta \Psi &= \not{\partial} \phi \epsilon \end{aligned}$$

- Action is on-shell invariant.
- **Off-shell supersymmetry:** Ψ has 2 d.o.f. $\Rightarrow$ need to add 1 auxiliary field

$$\begin{aligned} S &= -\frac{1}{2} \int d^2x [(\partial_\mu \phi)^2 + \bar{\Psi} \not{\partial} \Psi - F^2] \\ \delta F &= \bar{\epsilon} \not{\partial} \Psi; \quad \delta \Psi = \not{\partial} \phi \epsilon + F \epsilon; \quad \delta \phi = \bar{\epsilon} \Psi \end{aligned}$$

- However, off-shell susy means that *the algebra of susy is satisfied off-shell (without the use of the eqs. of motion)*.
- The most general N -extended superalgebra in 4d, with central charges, is

$$\{Q_\alpha^i, Q_\beta^j\} = 2(C\gamma^\mu)_{\alpha\beta}P_\mu\delta^{ij} + C_{\alpha\beta}U^{ij} + (C\gamma_5)_{\alpha\beta}V^{ij} ,$$

and must be satisfied on all fields. In 2d, for the WZ model above,

$$\{Q_\alpha^i, Q_\beta^j\} = 2(C\gamma^\mu)_{\alpha\beta}P_\mu\delta^{ij} \Rightarrow [\delta_{\epsilon_1}, \delta_{\epsilon_2}] = 2\bar{\epsilon}_2\gamma^\mu\epsilon_1\partial_\mu.$$

- Representing the algebra with central charges and massive states using the Wigner method, we find

$$\begin{aligned} a_\alpha &= \frac{1}{\sqrt{2}}[Q_\alpha^1 + \epsilon_{\alpha\dot{\beta}}\bar{Q}_{2\dot{\beta}}] & a_\alpha^\dagger &= \frac{1}{\sqrt{2}}[\bar{Q}_{1\dot{\alpha}} + \epsilon_{\alpha\beta}Q_\beta^2] \\ b_\alpha &= \frac{1}{\sqrt{2}}[Q_\alpha^1 - \epsilon_{\alpha\dot{\beta}}\bar{Q}_{2\dot{\beta}}] & a_\alpha^\dagger &= \frac{1}{\sqrt{2}}[\bar{Q}_{1\dot{\alpha}} - \epsilon_{\alpha\beta}Q_\beta^2] , \end{aligned}$$

so we obtain the algebra

$$\{a_\alpha, a_\beta^\dagger\} = 2(M - Z)\delta_{\alpha\beta}; \quad \{b_\alpha, b_\beta^\dagger\} = 2(M + Z)\delta_{\alpha\beta} \Rightarrow M \geq |Z|.$$

and the rest zero, giving the **BPS bound**.

- The interacting $\mathcal{N} = 1$ chiral (WZ) model in 4d is

$$S = \int d^4x \left[\phi^* \square \phi - i(\partial_\mu \bar{\psi})(\sigma^\mu)^T \psi - m \bar{\psi} \psi - 2\text{Re}[g\phi \bar{\psi} \psi] - |\lambda + m\phi + g\phi^2|^2 \right] ,$$

where the auxiliary field was solved as $F^* = -(\lambda + m\phi + g\phi^2)$.

- The $\mathcal{N} = 1$ vector multiplet (vector+spinor) in 4d (off-shell) is

$$S_{\mathcal{N}=1 \text{ SYM}} = (-2) \int d^4x \text{Tr} \left[-\frac{1}{4} F_{\mu\nu}^2 - \frac{1}{2} \bar{\lambda} \not{D} \lambda + \frac{D^2}{2} \right] ,$$

invariant under

$$\delta A_\mu^a = \bar{\epsilon} \gamma_\mu \lambda^a , \quad \delta \lambda^a = \left[-\frac{1}{2} \gamma^{\mu\nu} F_{\mu\nu}^a + i \gamma_5 D^a \right] \epsilon , \quad \delta D^a = i \bar{\epsilon} \gamma_5 \not{D} \lambda^a .$$

- When we couple to WZ multiplets, we obtain the D-term (auxiliary field) $D^a = \phi^{\dagger i} (T^a)_{ij} \phi^j$, and the scalar potential is

$$V = \sum_i |F_i|^2 + g^2 D^a D^a .$$

- $\mathcal{N} = 4$ SYM is obtained as $\mathcal{N} = 1$ SYM in 10d reduced to 4d,

$$\begin{aligned} S_{10d, \mathcal{N}=1 \text{ SYM}} &= (-2) \int d^{10}x \text{Tr} \left[-\frac{1}{4} F^{MN} F_{MN} - \frac{1}{2} \bar{\lambda} \Gamma^M D_M \lambda \right] \Rightarrow \\ S_{4d, \mathcal{N}=4 \text{ SYM}} &= (-2) \int d^4x \text{Tr} \left[-\frac{1}{4} F_{\mu\nu}^2 - \frac{1}{2} \bar{\psi}_i \not{D} \psi^i - \frac{1}{2} D_\mu \phi_{ij} D^\mu \phi^{ij} \right. \\ &\quad \left. - g \bar{\psi}^i [\phi_{ij}, \psi^j] - \frac{g^2}{4} [\phi_{ij}, \phi_{kl}] [\phi^{ij}, \phi^{kl}] \right] \end{aligned}$$

Supergravity

- Supergravity = supersymmetric theory of gravity, OR: theory of local supersymmetry.

- Local supersymmetry $\Rightarrow \epsilon^\alpha(x) \Rightarrow \exists$ "gauge field of supersymmetry", " $A_\mu^\alpha(x)$ " $\rightarrow$ gravitino $\Psi_{\mu\alpha}(x)$: supersymmetric partner of $e_\mu^a(x)$.

- $\mathcal{N} = 1$ supergravity in 4d: $\{e_\mu^a, \Psi_{\mu\alpha}\}$. Supersymmetry laws:

$$\begin{aligned}\delta e_\mu^a &= \frac{\kappa_N}{2} \bar{\epsilon} \gamma^a \Psi_\mu \\ \delta \Psi_\mu &= \frac{1}{\kappa_N} D_\mu \epsilon; \quad D_\mu \epsilon = \partial_\mu \epsilon + \frac{1}{4} \omega_\mu^{ab} \gamma_{ab} \epsilon\end{aligned}$$

- Action:

$$\begin{aligned}S &= S_{E-H}(\omega, e) + S_{RS}(\Psi_\mu) \\ &= \frac{1}{16\pi G} \int d^d x (\det e) R_{\mu\nu}^{ab}(\omega) e_a^{-1\mu} e_b^{-1\nu} - \frac{1}{2} \int d^d x (\det e) \bar{\Psi}_\mu \gamma^{\mu\nu\rho} D_\nu \Psi_\rho\end{aligned}$$

- Second order formalism: $e_\mu^a, \psi_{\mu\alpha}$ indep., ω_μ^{ab} dependent. However, $\exists$ dynamical fermions, so $\omega_\mu^{ab} = \omega_\mu^{ab}(e) + \psi\psi$ terms, obtained by varying action with respect to ω_μ^{ab} (as in first order formalism) $\Rightarrow \omega_\mu^{ab}(e, \psi)$.

- First order formalism: $e_\mu^a, \psi_{\mu\alpha}, \omega_\mu^{ab}$ independent.

- 1.5 order formalism (best): Use 2nd order formalism, but in $S(e, \psi, \omega(e, \psi))$, we don't vary $\omega(e, \psi)$, since it is multiplied by $\delta S / \delta \omega = 0$ (in second order formalism).

- In 4d, maximal susy (for multiplets of spins ≤ 2) is $\mathcal{N} = 8$. It has graviton e_μ^a , 8 gravitini $\psi_{\mu\alpha}^i$, 28 vectors A_μ^{IJ} , 56 fermions χ_{ijk}^α and 35 scalars forming a matrix ν or, in terms of $\mathcal{N} = 1$ multiplets, 1 supergravity, 7 gravitino, 21 vectors and 35 chiral (WZ) multiplets.

- It is the dimensional reduction of an $\mathcal{N} = 1$ supergravity multiplet in 11 dimensions, with graviton e_μ^a , gravitino $\psi_{\mu\alpha}$ and 3-index antisymmetric tensor $A_{\mu\nu\rho}$.

- Higher ($\mathcal{N} = 1$) supersymmetry in $d > 4 \Rightarrow$ matter fields in the same multiplet: vectors A_μ , fermions Ψ_α , also p-form antisymmetric fields $A_{\mu_1 \dots \mu_n}$

$$S = \int d^d x (\det e) F_{\mu_1, \dots, \mu_{n+1}}^2$$

$$F_{\mu_1 \dots \mu_{n+1}} = \partial_{[\mu_1} A_{\mu_2 \dots \mu_{n+1}]}$$

- Generalized Maxwell invariance

$$\delta A_{\mu_1 \dots \mu_n} = \partial_{[\mu_1} \Lambda_{\mu_2 \dots \mu_n]}$$

- **Black holes and p-branes:** Most general solution with spherical symmetry of Einstein's equations in vacuum ($T_{\mu\nu} = 0$): $R_{\mu\nu} - 1/2 g_{\mu\nu} R = 0$ is the Schwarzschild solution. In 4d,

$$ds^2 = - \left(1 - \frac{2mG}{r} \right) dt^2 + \frac{dr^2}{1 - \frac{2mG}{r}} + R^2 d\Omega_2^2$$

- Is solution of a point particle of mass M , or of a spherical mass distribution of radius R at $r > R$. If it's valid down to $r = r_H \equiv 2MG$: **black hole**. $r = r_H$: event horizon.

- Can add charge Q . BPS bound $|Q| \leq M$. For $|Q| = M$: extremal black hole ($r_+ = r_-$).

- Solution with charge: modify the Newtonian potential defining solution,

$$U_N(r) = -\frac{MG_N}{r} + \frac{Q^2 G_N}{4\pi\epsilon_0^2 4r^2},$$

where $ds^2 = -(1 + 2U_N(r))dt^2 + dr^2/(1 + 2U_N(r)) + r^2 d\Omega_2^2$.

- In D dimensions, the Schwarzschild solution is

$$ds^2 = -\left(1 - \frac{2C^{(D)}G_N^{(D)}M}{r^{D-3}}\right)dt^2 + \frac{dr^2}{1 - \frac{2C^{(D)}G_N^{(D)}M}{r^{D-3}}} + r^2 d\Omega_{D-2}^2.$$

- The Schwarzschild p -brane solution in D dimension is obtained by trivial extension of T^p ,

$$ds^2 = -\left(1 - \frac{2C^{(D-p)}G_N^{(D-p)}M}{r^{D-3-p}}\right)dt^2 + d\vec{x}_p^2 + \frac{dr^2}{1 - \frac{2C^{(D-p)}G_N^{(D-p)}M}{r^{D-3-p}}} + r^2 d\Omega_{D-2-p}^2.$$

- In 4d Einstein-Maxwell, the charged black hole is rewritten as

$$ds^2 = -\Delta dt^2 + \frac{dr^2}{\Delta} + r^2 d\Omega_2^2, \quad \Delta = \left(1 - \frac{r_+}{r}\right) \left(1 - \frac{r_-}{r}\right)$$

- For the extremal case, $r_+ = r_-$, and after $r = M + \bar{r}$, we obtain

$$ds^2 = -H(\bar{r})^{-2} dt^2 + H(\bar{r})^2 (d\bar{r}^2 + \bar{r}^2 d\Omega_2^2),$$

where $H = 1 + M/\bar{r}$ is harmonic,

$$\Delta_{(3)} H = -4\pi M \delta^3(r).$$

- The AdS-Reissner-Nordstrom solution (charged BH in AdS) in 4d is

$$ds^2 = -\Delta dt^2 + \frac{dr^2}{\Delta} + r^2 d\Omega_2^2; \quad \Delta \equiv 1 - \frac{2MG_N}{r} + \frac{\tilde{Q}^2 G_N}{r^2} - \frac{8\pi G_N \Lambda r^2}{3}.$$

- In D dimensions, the extremal Reissner-Nordstrom (charged with respect to A_μ) black hole is rewritten by $r^{D-3} = \bar{r}^{D-3} + r_H^{D-3}$ as

$$ds^2 = -f(\bar{r})^{-2} dt^2 + f(\bar{r})^{\frac{2}{D-3}} (d\bar{r}^2 + \bar{r}^2 d\Omega_{D-2}^2) ,$$

- in terms of the harmonic function

$$f(\bar{r}) = 1 + \left(\frac{r_H}{\bar{r}} \right)^{D-3} .$$

- In supergravity however, we can add charge Q_p associated with an $A_{\mu_1 \dots \mu_{p+1}}$, with source term in the action $Q_p \int d^{p+1} \xi A_{01 \dots p+1} = \int d^D x j^{\mu_1 \dots \mu_{p+1}} A_{\mu_1 \dots \mu_{p+1}}$, giving

$$A_{01 \dots p} = -\frac{C_p Q_p}{r^{D-p-3}} .$$

- The source term can be rewritten as (on the worldvolume)

$$-\frac{1}{(p+1)!} T_P \int d^{p+1} \xi \epsilon^{i_1 \dots i_{p+1}} \partial_{i_1} X^{M_1} \dots \partial_{i_{p+1}} X^{M_{p+1}} A_{M_1 \dots M_{p+1}} ,$$

but there is actually also a coupling to the dilaton and the metric, like for the string case.

- Extremal solutions $M = |Q_p|$ play a fundamental role: extended in $p+1$ dimensions: p -branes. Important nonperturbative objects.

- Extremal p-brane solutions of supergravity (with $M = |Q_p|$), with action

$$S_D = \frac{1}{2k^2} \int d^D x \sqrt{-g} \left(R - \frac{1}{2}(\partial\phi)^2 - \frac{1}{2(d+1)!} e^{-a(d)\phi} F_{d+1}^2 \right)$$

(here ϕ is a scalar = "dilaton"), are of type

$$\begin{aligned} ds_{Einstein}^2 &= e^{-\frac{\phi}{2}} ds_{string}^2; \quad H_p = 1 + \frac{\alpha_p Q_p}{|\vec{x}_\perp|^{7-p}} \\ ds_{string}^2 &= H_p^{-1/2} (-dt^2 + d\vec{x}_p^2) + H_p^{1/2} d\vec{x}_{9-p}^2 \\ e^{-4\phi} &= H_p^{\frac{p-3}{4}} \\ A_{01\dots p} &= -\frac{1}{2}(H_p^{-1} - 1) \end{aligned}$$

with source term $\int A_{(p+1)} = \int d^d x j^{\mu_1 \dots \mu_{p+1}} A_{\mu_1 \dots \mu_{p+1}}$ added to S_D .

- Spans a $p+1$ -dimensional "worldvolume".
- Play a special role when supergravity is embedded in string theory: are equal to D-branes.

Conformal field theory in d=4

- For $d \neq 2$, infinitesimal transf. is

$$\begin{aligned}x'_\mu &= x_\mu + v_\mu(x) \Rightarrow \\v_\mu(x) &= a_\mu + \omega_{\mu\nu}x_\nu + \lambda x_\mu + b_\mu x^2 - 2x_\mu b \cdot x\end{aligned}$$

- Generators: $(a_\mu, \omega_{\mu\nu}) \rightarrow (P_\mu, J_{\mu\nu})$: Poincare ($\Omega(x) = 1$). $b_\mu \rightarrow K_\mu$: special conformal transformation, $\lambda \rightarrow D$: dilatation.

- Form group: $SO(d, 2)$ in Minkowski $(d-1, 1)$.

- Obs: All conformal transf. obtained from rotations, translations and inversions:

$$I : x'_\mu = \frac{x_\mu}{x^2} \Rightarrow \Omega(x) = x^2$$

- So we only need to check invariance under inversions for a Poincare invariant theory.

- Scaling dimension: eigenvalue $-i\Delta$ of scaling operator D

$$\phi(x) \rightarrow \phi'(x) = \lambda^\Delta \phi(\lambda x)$$

- From $SO(d, 2)$ conformal algebra,

$$[D, P_\mu] = -iP_\mu \Rightarrow D(P_\mu \phi) = -i(\Delta + 1)(P_\mu \phi)$$

$$[D, K_\mu] = +iK_\mu \Rightarrow D(K_\mu \phi) = -i(\Delta - 1)(K_\mu \phi)$$

- Thus $K_\mu \sim a$ (annihilation) and $P_\mu \sim a^\dagger$ (creation). Generate representation of conformal group. Operator of lowest dimension in it = **primary operator** ϕ_0 ($\sim |0\rangle$), s.t. $K_\mu \phi_0 = 0$.

- The orthogonal matrix representing inversion is

$$R_{\mu\nu}(x) \equiv I_{\mu\nu}(x) = \delta_{\mu\nu} - 2 \frac{x^\mu x^\nu}{x^2} ,$$

and transforms under conformal transf. as

$$I_{\mu\nu}(x' - y') = R_{\mu\rho} R_{\nu\sigma} I_{\rho\sigma}(x - y); \quad (x' - y')^2 = \frac{(x - y)^2}{\Omega(x)\Omega(y)}.$$

- 2-point correlators for scalar operators $\mathcal{O}_i$ and conserved currents J_μ^a are

$$\begin{aligned} \langle \mathcal{O}_i(x) \mathcal{O}_j(y) \rangle &= \frac{C \delta_{ij}}{|x - y|^{2\Delta_i}} \\ \langle J_\mu^a(x) J_\nu^b(y) \rangle &= C \frac{\delta^{ab} I_{\mu\nu}(x - y)}{|x - y|^{2(d-1)}} , \end{aligned}$$

while 3-point correlators of scalar operators are

$$\langle \mathcal{O}_i(x) \mathcal{O}_j(y) \mathcal{O}_k(z) \rangle = \frac{C_{ijk}}{|x - y|^{\Delta_i + \Delta_j - \Delta_k} |y - z|^{\Delta_j + \Delta_k - \Delta_i} |z - x|^{\Delta_k + \Delta_i - \Delta_j}}.$$

- $d = 4 \rightarrow \mathcal{N} = 4$ Super Yang-Mills = representation of conformal group $\{A_\mu^a, \Psi_\alpha^{aI}, X_{[IJ]}^a\}$.

- In $\mathcal{N} = 4$ SUM, beta function = 0 $\Rightarrow$ scale and conformal invariant. But $\Delta = \Delta_0 + \mathcal{O}(g)$ in general. No *infinities*, but $\exists$ finite renormalizations.

- So classically: $[A_\mu^a] = 1, [\Psi_\alpha^{aI}] = 3/2, [X_{[IJ]}^a] = 1$, but composite gauge invariant ops., e.g. $\text{tr } F_{\mu\nu}^2$, have $\Delta(g)$.

- $\mathcal{N} = 4$ susy invariance of SYM:

$$\delta A_\mu^a = \bar{\epsilon}_I \gamma_\mu \Psi^{aI}$$

$$\delta X_a^{[IJ]} = \frac{i}{2} \bar{\epsilon}^{[I} \Psi^{J]a}$$

$$\delta \Psi^{aI} = -\frac{\gamma^{\mu\nu}}{2} F_{\mu\nu}^a \epsilon^I + 2i \gamma^\mu D_\mu X^{a,[IJ]} \epsilon_J - 2g f^a_{bc} (X^b X^c)^{[IJ]} \epsilon_J$$

AdS/CFT in original formulation (Maldacena, 1997)

- String theory in $AdS_5 \times S^5 = \mathcal{N} = 4$ SYM with $SU(N)$ gauge group (low energy theory on N D3-branes), living at the boundary of $AdS_5 \times S^5$, involving a certain limit.

- Heuristical derivation:

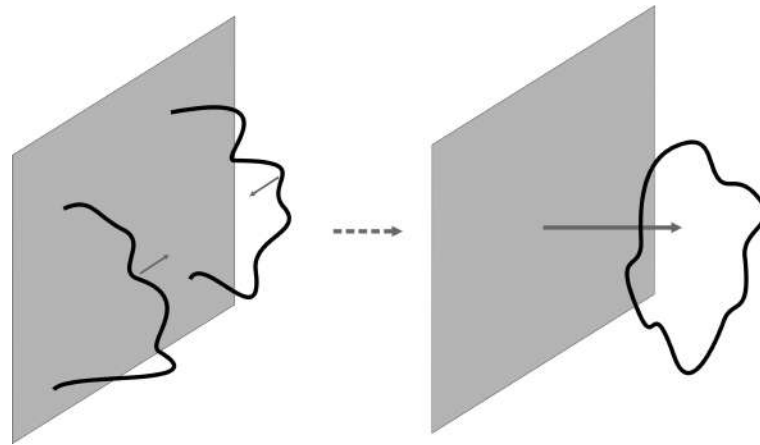
- D-branes = extremal p-branes $\Rightarrow$ curve space. Solution:

$$ds^2 = H^{-1/2}(r) d\vec{x}_{||}^2 + H^{1/2}(r) (dr^2 + r^2 d\Omega_5^2)$$

$$F_5 = (1 + *) dt \wedge dx_1 \wedge dx_2 \wedge dx_3 \wedge (dH^{-1})$$

$$H(r) = 1 + \frac{R^4}{r^4}; \quad R = 4\pi g_s N \alpha'^2; \quad Q = g_s N$$

- Add a $\delta M \rightarrow$ near extremal: $M = Q + \delta M \Rightarrow$ horizon $\Rightarrow$ emits Hawking radiation: 2 open strings on D3 collide and form a closed string that peels off and goes into the bulk.



Two open strings living on a D-brane collide and form a closed string, that can then peel off and go away from the brane.

● **P.O.V. nr. 1** D3-branes = endpoints of strings. String theory gives:

-open strings on D3. Low energy ($\alpha' \rightarrow 0$) $\Rightarrow \mathcal{N} = 4$ SYM

-closed strings in bulk (all spacetime): supergravity + massive modes of string. Low energy: supergravity only.

-interactions, giving e.g. Hawking radiation as above.

$$S = S_{bulk} + S_{brane} + S_{interactions}$$

● Low energy limit, $\alpha' \rightarrow 0$, $\Rightarrow S_{bulk} \rightarrow S_{supergravity}$, $S_{brane} \rightarrow S_{\mathcal{N}=4SYM}$, $S_{int} \propto \kappa_{Newton} \sim g_s \alpha'^2 \rightarrow 0$. Moreover, since Newton $\kappa_N \rightarrow 0$, $\Rightarrow$ free gravity. Thus:

● free gravity in bulk

● 4d $\mathcal{N} = 4$ SYM on D3's.

● Obs: $\partial(AdS_5 \times S^5) = R^{3,1}$ or $S^3 \times R$ (4 dimensional!): S^5 shrinks to zero size at boundary.

- **P.O.V. nr. 2** D3-branes replaced by p-branes (supergravity solutions).

- Geometry has two asymptotic regions: $r \rightarrow 0$: $AdS_5 \times S^5$ and $r \rightarrow \infty$: Minkowski₁₀. Infinitely long throat:

- Energy at point r is

$$E_r \sim \frac{d}{d\tau} = \frac{1}{\sqrt{-g_{00}}} \frac{d}{dt} \sim \frac{1}{\sqrt{-g_{00}}} E_\infty \Rightarrow E_\infty = H^{-1/4} E_r \sim r E_r$$

- Then at $r \rightarrow 0$, for fixed E_r (energy of the throat) $E_\infty \rightarrow 0 \Rightarrow$ low energy excitations.

- At $r \rightarrow \infty$, long distance $\delta r \rightarrow \infty \Leftrightarrow E \rightarrow 0$, effective gravity coupling $GE^{D-2} \rightarrow 0 \Rightarrow$ free gravity $\rightarrow$ in the bulk.

- Compare POV 1 with POV 2. Same free gravity in the bulk $\Rightarrow$ Identify the others $\Rightarrow$
- 4d $\mathcal{N} = 4$ SYM with $SU(N)$ on D3 = gravity at $r \rightarrow 0$ in D-brane background, for $\alpha' \rightarrow 0$.
- Background for $r \rightarrow 0$, with $r/R \equiv R/x_0$.

$$ds^2 = R^2 \frac{-dt^2 + d\vec{x}_3^2 + dx_0^2}{x_0^2} + R^2 d\Omega_5^2 : AdS_5 \times S^5$$

Lecture 4

AdS/CFT and gauge/gravity duality in
Euclidean and Lorentzian signatures

- **Last time:** 4d $\mathcal{N} = 4$ SYM with $SU(N)$ on D3 = gravity at $r \rightarrow 0$ in D-brane background, for $\alpha' \rightarrow 0$.

- Background for $r \rightarrow 0$, with $r/R \equiv R/x_0$.

$$ds^2 = R^2 \frac{-dt^2 + d\vec{x}_3^2 + dx_0^2}{x_0^2} + R^2 d\Omega_5^2 : AdS_5 \times S^5$$

- **Now:** Define limit further: $r \rightarrow 0 \Rightarrow M \sim R^4/r^4 \propto \alpha'^2/r^4$.

- $E_r \sqrt{\alpha'}$ fixed and E_∞ fixed $\Rightarrow$

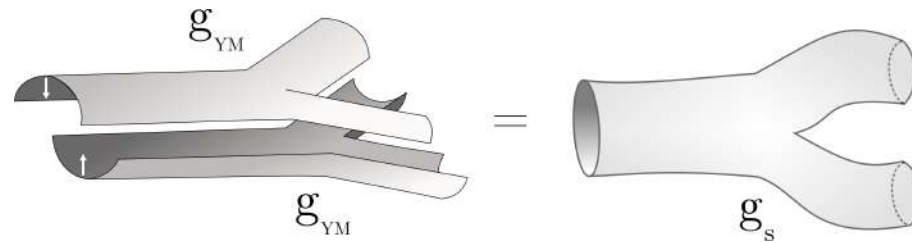
$$\frac{E_\infty}{E_r \sqrt{\alpha'}} = \frac{r}{\alpha'} \equiv U = \text{fixed (energy scale)}$$

- Then, metric is

$$ds^2 = \alpha' \left[\frac{U^2}{\sqrt{4\pi g_s N}} (-dt^2 + d\vec{x}_3^2) + \sqrt{4\pi g_s N} \left(\frac{dU^2}{U^2} + d\Omega_5^2 \right) \right]$$

- And ds^2/α' finite, but in order to have small string corrections we need also $g_s \rightarrow 0$ (quantum string corrections) and

- $R_{AdS}^2 = \sqrt{4\pi g_s N}$ = fixed and large (small α' corrections).

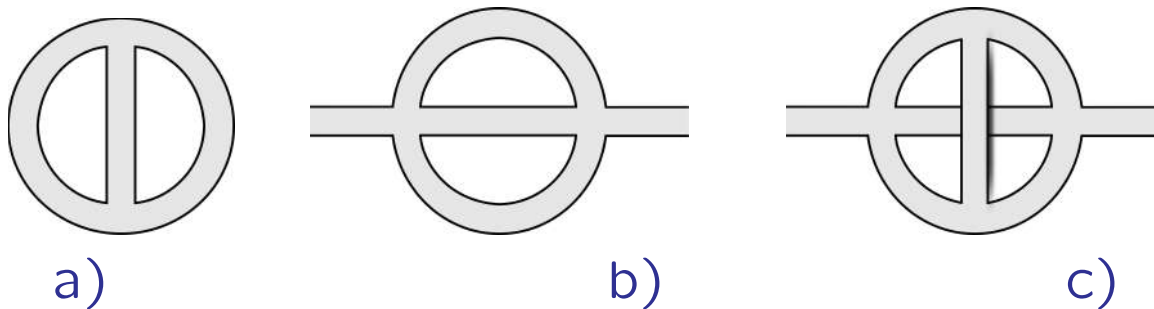


Two open string splitting interactions can be glued on the edges to give a closed string interaction ("pair of pants"), therefore $g_{YM}^2 = g_s$.

• Large N limit: 't Hooft: for gauge theories with only adjoints, we have an *effective, or 't Hooft coupling* $\lambda = g_{YM}^2 N$, besides $1/N$. The dependence of amplitudes is

$$\mathcal{A} \sim (g^2 N)^L N^{1-2h},$$

where $L = \text{loop nr.}$, $h = \text{handle nr.}$, $\chi = 2 - 2h - l$ ($l = \text{nr. of quark, external lines}$) is the Euler characteristic of a surface. Planar limit $\equiv h = 0$



a) Planar 2-loop diagram with 2 3-point vertices b) Planar 2-loop diagram with 2 4-point vertices c) Nonplanar 3-loop diagram.

- (open string)² $\sim$ closed string $\Rightarrow g_s = g_{YM}^2$.
- So, limit: $g_{YM}^2 \rightarrow 0, N \rightarrow \infty$, but 't Hooft coupling $\lambda = g_{YM}^2 N$ large and fixed.
- Opposite of perturbation theory ($\lambda \ll 1$) $\Rightarrow$ duality:
 Perturbation theory in string theory $\Rightarrow$ nonperturbative in SYM
 and vice versa: Hard to test, but useful $\rightarrow$ calculate nonperturbative effects.
- 3 possible versions of AdS/CFT:
 - Weakest: only at $g_s \rightarrow 0$ and $g_s N$ large $\rightarrow$ string theory $\simeq$ supergravity. α' and g_s corrections might disagree.
 - Stronger: valid at any finite $g_s N$, but only at $g_s \rightarrow 0, N \rightarrow \infty$, i.e. $\alpha'/R^2 = 1/\sqrt{g_s N}$ corrections agree, but not g_s corrections.
 - Strongest: believed to be correct: valid at any g_s and N (or g_s and α').

- **Defining map:** (Witten, 1998)

- Gauge invariant **operator** $\mathcal{O}$ of $\mathcal{N} = 4$ SYM, with conformal dimension Δ and representation I_n of $SO(6) = SU(4) \leftrightarrow$ **field in** AdS_5 , of mass m and representation I_n of $SO(6)$ = symmetry of S^5

- Reduce 10d fields on S^5 :

$$\phi(x, y) = \sum_n \sum_{I_n} \phi_{(n)}^{I_n}(x) Y_{(n)}^{I_n}(y)$$

- Then $\phi_{(n)}^{I_n} \leftrightarrow \mathcal{O}_{(n)}^{I_n}$, with

$$\Delta = \frac{d}{2} + \sqrt{\frac{d^2}{4} + m^2 R^2}$$

- But Δ doesn't receive quantum corrections ($\Delta(\lambda \rightarrow \infty) = \Delta(\lambda = 0)$) only for chiral primary operators = primary operators preserving some susy: $[Q_{comb}] \mathcal{O}_{ch.pr} = 0$

● **"Experimental" evidence:** towers of multiplets of operators in $\mathcal{N} = 4$ SYM $\leftrightarrow$ towers of KK fields on $AdS_5 \times S^5$.

● 6 families of chiral primary scalar representations:

$Q^2 \mathcal{O}_n = \text{Tr}(\phi^{I_1} \dots \phi^{I_n}) \leftrightarrow$ scalars with $m^2 R^2 = n(n-4), n \geq 2$;

$\mathcal{O}_n = \text{Tr}(\epsilon^{\alpha\beta} \lambda_{\alpha A} \lambda_{\beta B} \phi^{I_1} \dots \phi^{I_n}) \leftrightarrow m^2 R^2 = (n+3)(n-1), n \geq 0$;

$Q^2 \bar{Q}^2 \mathcal{O}_n = \text{Tr}(\epsilon^{\alpha\beta} \epsilon^{\bar{\alpha}\bar{\beta}} \lambda_{\alpha A_1} \lambda_{\beta A_2} \tilde{\lambda}_{\dot{\alpha}}^{B_1} \tilde{\lambda}_{\dot{\beta}}^{B_2} \phi^{I_1} \dots \phi^{I_n}) \leftrightarrow m^2 R^2 = (n+6)(n+2), n \geq 0$;

$Q^4 \mathcal{O}_n = \text{Tr}(F_{\mu\nu} F^{\mu\nu} \phi^{I_1} \dots \phi^{I_n}) \leftrightarrow m^2 R^2 = n(n+4), n \geq 0$;

$Q^4 \bar{Q}^2 \mathcal{O}_n = \text{Tr}(\epsilon^{\alpha\beta} \lambda_{\alpha A} \lambda_{\beta B} F_{\mu\nu}^2 \phi^{I_1} \dots \phi^{I_n}) \leftrightarrow m^2 R^2 = (n+3)(n+7), n \geq 0$;

$Q^4 \bar{Q}^4 \mathcal{O}_n = \text{Tr}(F_{\mu\nu}^4 \phi^{I_1} \dots \phi^{I_n}) \leftrightarrow m^2 R^2 = (n+4)(n+8), n \geq 0$.

- **Global AdS/CFT**. Metric of global $AdS_5 \times S^5$ and boundary: cylinder:

$$ds^2 = \frac{R^2}{\cos^2 \theta} (d\tau^2 + d\theta^2 + \sin^2 \theta d\Omega_3^2) + R^2 d\Omega_5^2 \rightarrow ds^2 = \frac{R^2}{\epsilon^2} (d\tau^2 + d\Omega_3^2).$$

- Metric in Poincaré coords, and boundary: plane:

$$ds^2 = R^2 \frac{d\vec{x}^2 + dx_0^2 + x_0^2 d\Omega_5^2}{x_0^2} \rightarrow ds^2 = \frac{R^2}{\epsilon^2} d\vec{x}^2.$$

- Boundary $\mathbb{R}_t \times S^3$ (cylinder) and $\mathbb{R}^4$ (plane) are related by conf. transf. (irrelevant for CFT):

$$ds^2 = d\vec{x}^2 = dx^2 + x^2 d\Omega_3^2 = x^2 ((d \ln x)^2 + d\Omega_3^2) = x^2 (d\tau^2 + d\Omega_3^2).$$

- CFT: operator-state correspondence. In 2d, $z = e^{-w}$ maps cylinder (in w : $w \sim w + 2\pi$) to plane (in z) and incoming states (at $w = -i\infty$) with operators on the plane.

- e.g. Closed string. Taylor exp. of operator on plane $\Rightarrow$ state-operator map:

$$\alpha_{-m}^\mu = \sqrt{\frac{2}{\alpha'}} \frac{i}{(m-1)!} \partial^m X^\mu(0) \Rightarrow \alpha_{-m}^\mu |0,0\rangle \leftrightarrow \sqrt{\frac{2}{\alpha'}} \frac{i}{(m-1)!} \partial^m X^\mu(0).$$

- Hamiltonian on $\mathbb{R}_\tau$ (cylinder KK reduced on circle) is the same as the "dilatation operator" on plane, generator of $\vec{r} \rightarrow \lambda \vec{r}$, since ($d\vec{r}^2 = dr^2 + r^2 d\theta^2$)

$$H_{\mathbb{R}_\tau} = i\partial_\tau = ir\partial_r = D_{\mathbb{R}^2} .$$

- In 4d, similar. Only, conformal invariance requires $\int R\phi^2$ term in action: on plane, =0; on cylinder: mass term $-\int \phi^2/2$.

- Scalars Z in $\mathbb{R}^4$: Taylor expansion for this op. are

$$z_{\alpha_1 \dots \alpha_m}^{(m)} \sim (\partial_{\alpha_1} \dots \partial_{\alpha_m}) Z ,$$

and correspond to KK states on cylinder, but const. term has energy $E = 1$ due to mass term (is a harmonic oscillator).

- Again, QM Hamiltonian for KK states on $\mathbb{R}_\tau$ (cylinder reduced on S^3), same as dilatation operator on $\mathbb{R}^4$, for $\vec{r} \rightarrow \lambda \vec{r}$,

$$H_{\mathbb{R}_\tau} = i\partial_\tau = ir\partial_r = D_{\text{plane}} .$$

Witten construction

- Near boundary $x_0 = 0$ in Poincaré coords., $\square\phi = 0$ has solutions $\phi \rightarrow \phi_0$ and $\phi \rightarrow x_0^d \phi_0$, and $(\square - m^2)\phi = 0$ (massive) has solutions $\phi \rightarrow x_0^{d-\Delta} \phi_0$ and $\phi \rightarrow x_0^\Delta \phi_0$ ($\Delta = \text{dim. of dual op.}$).
- Then $\phi_0 =$ source for dual operator $\mathcal{O}$.
- Observables for $\mathcal{O} \leftrightarrow \phi$: generating functional for $\mathcal{O}$:

$$Z_{\text{boundary}} = Z_{\mathcal{O},CFT}[\phi_0] = \int \mathcal{D}[\text{SYM fields}] e^{-S_{N=4\text{ SYM}} + \int d^4x \mathcal{O}(x) \phi_0(x)}$$

- Fundamental idea: $Z_{\text{boundary}} = Z_{\text{bulk}} = Z_{\text{string}}[\phi_0]$, where $\phi_0 =$ boundary sources. But for $\alpha' \rightarrow 0, g_s \rightarrow 0, R^4/\alpha'^2 \gg 1 \rightarrow \text{string} \simeq$ classical supergravity, and $Z_{\text{string}}[\phi_0] = e^{-S_{\text{sugra}}[\phi[\phi_0]]}$.

$$\Rightarrow Z_{\mathcal{O},CFT}[\phi_0] = e^{-S_{\text{sugra}}[\phi[\phi_0]]}$$

- But in CFT, correlators are obtained by derivation:

$$\begin{aligned} \langle \mathcal{O}(x_1) \dots \mathcal{O}(x_n) \rangle &= \frac{\delta^n}{\delta \phi_0(x_1) \dots \delta \phi_0(x_n)} Z_{\mathcal{O}}[\phi_0] \big|_{\phi_0=0} \\ &= \frac{\delta^n}{\delta \phi_0(x_1) \dots \delta \phi_0(x_n)} e^{-S_{\text{sugra}}[\phi[\phi_0]]} \big|_{\phi_0=0} \end{aligned}$$

- Define "bulk to boundary propagator" K_B , a propagator with the free leg on the boundary,

$$" \square_{\vec{x}, x_0} " K_B(\vec{x}, x_0; \vec{x}') = \delta^4(\vec{x} - \vec{x}') ,$$

such that the field in the bulk is written as a convolution of K_B with ϕ_0 ,

$$\phi(\vec{x}, x_0) = \int d^4 \vec{x}' K_B(\vec{x}, x_0; \vec{x}') \phi_0(\vec{x}') ,$$

and replaced in the sugra action allows us to calculate correlators.

• In our case,

$$K_{B,\Delta}(\vec{x}, x_0; \vec{x}') = C_d \left[\frac{x_0}{x_0^2 + (\vec{x} - \vec{x}')^2} \right]^\Delta \cdot \xrightarrow{x_0 \rightarrow 0} x_0^{d-\Delta} \delta(\vec{x} - \vec{x}') , \quad x_0 \frac{\partial}{\partial x_0} K_B \Big|_{x_0 \rightarrow 0} \stackrel{\Delta=d}{=} \frac{dx_0^d}{|\vec{x} - \vec{x}'|^{2d}}$$

so • Example: **2 point function of scalars**

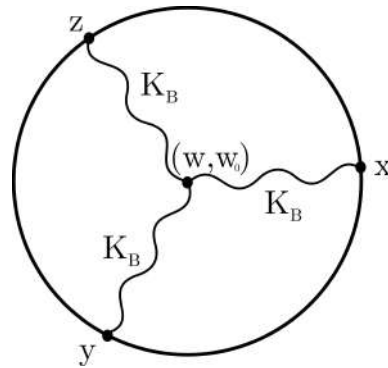
$$\begin{aligned} \langle \mathcal{O}(x_1) \mathcal{O}(x_2) \rangle &= - \frac{\delta^2 S_{sugra}[\phi[\phi_0]]}{\delta \phi_0(x_1) \delta \phi_0(x_2)} \Big|_{\phi_0=0} \\ S_{sugra}[\phi] &= \frac{1}{2} \int_{boundary} d^4 x \sqrt{h} (\phi \vec{n} \cdot \vec{\nabla} \phi) = \frac{Cd}{2} \int d^d \vec{x} \int d^d \vec{x}' \frac{\phi_0(\vec{x}) \phi_0(\vec{x}')}{|\vec{x} - \vec{x}'|^{2d}} \Rightarrow \\ \langle \mathcal{O}(x_1) \mathcal{O}(x_2) \rangle &= - \frac{Cd/2}{|\vec{x} - \vec{x}'|^{2d}}. \quad \text{OK!} \end{aligned}$$

- Another way to think about it, which can be generalized:

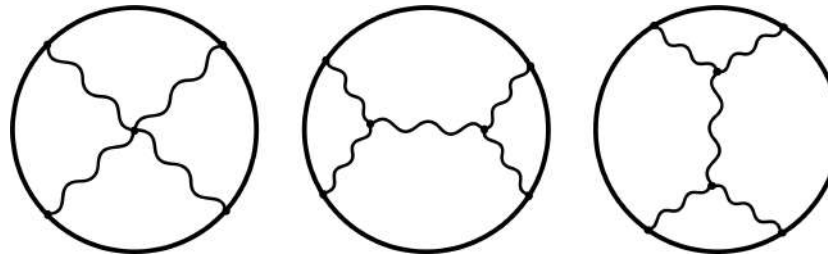
$$S_{sugra}[\phi[\phi_0]] = \int d^5x \sqrt{g} \int d^4\vec{x}' \int d^4\vec{y}' \partial_\mu K_B(\vec{x}, x_0; \vec{x}') \phi_0(\vec{x}') \\ \times \partial^\mu K_B(\vec{x}, x_0; \vec{y}') \phi_0(\vec{y}') + \mathcal{O}(\phi_0^3)$$

$$\Rightarrow \langle \mathcal{O}(\vec{y}) \mathcal{O}(\vec{z}) \rangle = \int d^5x \sqrt{g} \partial_{\mu \vec{x}, x_0} K_B(\vec{x}, x_0; \vec{x}') \partial_{\vec{x}, x_0}^\mu K_B(\vec{x}, x_0; \vec{y}')$$

- Generalize: Boundary Feynman diagrams (Witten) for $\langle \mathcal{O}(x_1) \dots \mathcal{O}(x_n) \rangle$, e.g.



a)



b)

a) Tree level "Witten diagram" for the 3-point function in AdS space. b) Tree level Witten diagrams for the 4-point function in AdS space.

• Example: 3-point function R-current anomaly. In general, for 3-point functions, we obtain

$$\langle \mathcal{O}(x_1) \mathcal{O}(x_2) \mathcal{O}(x_3) \rangle = -\lambda \int \frac{d^d z dz_0}{z_0^{d+1}} K_{B, \Delta_1}(z_0, \vec{z}; \vec{x}_1) K_{B, \Delta_2}(z_0, \vec{z}; \vec{x}_2) K_{B, \Delta_3}(z_0, \vec{z}; \vec{x}_3).$$

• For the anomaly, we calculate

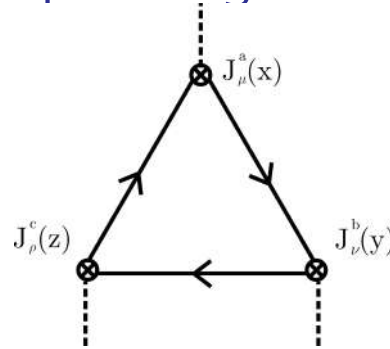
$$\langle J^{ia}(x_1) J^{jb}(x_2) J^{kc}(x_3) \rangle_{\text{CFT, d}_{\text{abc}} \text{ part}} = - \left. \frac{\delta^3 S_{\text{CS, sugra}}^{3\text{-pnt vertex}}[A_\mu^a[a_l^d]]}{\delta a_i^a(x_1) \delta a_j^b(x_2) \delta a_k^c(x_3)} \right|_{a=0},$$

using $S_{\text{CS}}(A) = \frac{N^2}{18\pi^2} \text{Tr} \int_{B_5 = \partial M_6} \epsilon^{\mu\nu\rho\sigma\tau} (A_\mu (\partial_\nu A_\rho) \partial_\sigma A_\tau + A^4 \text{ terms} + A^5 \text{ terms}),$

and find equality with the CFT result

$$\frac{\partial}{\partial z^k} \langle J_i^a(x) J_j^b(y) J_k^c(z) \rangle_{\text{CFT, d}_{\text{abc}}} = - \frac{(N^2 - 1) i d_{abc}}{48\pi^2} \epsilon^{ijkl} \frac{\partial}{\partial x_k} \frac{\partial}{\partial y_l} \delta(x - y) \delta(y - z),$$

coming from the one-loop triangle anomaly (which is one-loop exact!),



Triangle diagram contributing to the $\langle J_i^a(x) J_j^b(y) J_k^c(z) \rangle$ correlator. Chiral fermions run in the loop.

- **(Euclidean)** Bulk to bulk propagator satisfies

$$(\square_x - m^2)G(x, y) = \frac{1}{\sqrt{g_y}} \delta^{d+1}(x - y) = -\delta^d(\vec{x} - \vec{y}) \delta(x_0 - y_0) \frac{y_0^{d+1}}{R^{d+1}},$$

and is found to be

$$G(x, y) = (x_0 y_0)^{d/2} \int \frac{d^d k}{(2\pi)^d} e^{i\vec{k} \cdot (\vec{x} - \vec{y})} I_\nu(k x_0^<) K_\nu(k x_0^>),$$

composed, as usual, of the two independent solutions to the homogenous eq. $(\square - m^2)\Phi = 0$,

$$\Phi_{1,\vec{k}} \propto e^{i\vec{k} \cdot \vec{x}} x_0^{d/2} K_\nu(k x_0) \phi_0(\vec{k}), \quad \Phi_{2,\vec{k}} \propto e^{i\vec{k} \cdot \vec{x}} x_0^{d/2} I_\nu(k x_0) \phi_0(\vec{k}),$$

where $\Phi_{1,\vec{k}}$ is regular everywhere (also in the center $x_0 = +\infty$), and at the boundary ($x_0 = 0$) is a combination of the non-normalizable mode $x_0^{\Delta-}$ and the normalizable mode $x_0^{\Delta+}$ (thus the non-normalizable mode is leading), while $\Phi_{2,\vec{k}}$ is the normalizable mode $x_0^{\Delta-}$ at the boundary, but blows up in the center (so is not a physical mode). Therefore

$$\begin{aligned} \Phi_{1,2} &\sim x_0^{\Delta_\pm}, \quad \Delta_\pm = \frac{d}{2} \pm \sqrt{\frac{d^2}{4} + m^2 R^2} \Rightarrow \\ \phi(\vec{x}, x_0) &= \int d^d y K_B(\vec{x}, x_0; \vec{y}) \phi_0(\vec{y}) \sim x_0^{d-\Delta} \phi_0(\vec{x}) = x_0^{\Delta-} \phi_0(\vec{x}). \end{aligned}$$

- **Poincaré, Lorentzian signature, modes:** Then $k^2 = -m^2 < 0$ on-shell, so solutions

$$\Phi^\pm \propto e^{ik \cdot x} (x_0)^{d/2} J_{\pm\nu}(|k|x_0) ,$$

where $\nu = \sqrt{d^2/4 + m^2 R^2}$. Thus near boundary $x_0 = 0$,

$$\Phi^- \sim (x_0)^{\Delta_-} , \quad \Phi^+ \sim (x_0)^{\Delta_+} ,$$

so Φ^- is non-normalizable and Φ^+ is normalizable (like in Euclidean case), but now **both** are regular in the center! (unlike Euclidean case). Thus in the Lorentzian case we must understand Φ^+ (normalizable and regular).

- **Natural Lorentzian map:** $\Phi^- \rightarrow$ sources, $\Phi^+ \rightarrow$ states, in boundary CFT.

- Wick rotating partition function, Witten map is now ($|s\rangle =$ state)

$$Z_{\text{sugra}}[\phi_0] = e^{iS_{\text{sugra}}[\phi(\phi_0)]} = Z_{\text{CFT}}[\phi_0] = \langle s | e^{i \int_{\partial M} \phi_0 \mathcal{O}} | s \rangle .$$

- But state $|s\rangle$ mapped to normalizable mode ϕ_n , so

$$\begin{aligned}\phi(x_0, \vec{x}) &= \phi_n(x_0, \vec{x}) + \int d^d y K_B(x_0, \vec{x}; \vec{y}) \phi_0(\vec{y}) \\ &= \phi_n(x_0, \vec{x}) + c \int d^d y \frac{x_0^d}{(x_0^2 + (\vec{x} - \vec{y})^2)^d} \phi_0(\vec{y}).\end{aligned}$$

- Substituting in sugra action, we get

$$\langle \tilde{\phi}_n | \mathcal{O}(\vec{x}) | \tilde{\phi}_n \rangle_{\phi_0} = \frac{\delta}{\delta \phi_0(\vec{x})} S_{\text{sugra}}[\phi(\phi_0)] = d\tilde{\phi}_n(\vec{x}) + cd \int d^d x' \frac{\phi_0(\vec{x}')}{|\vec{x} - \vec{x}'|^{2d}}.$$

- Then *non-normalizable modes are mapped to sources and normalizable modes to VEVs (or states)*,

$$\phi \sim \alpha_i(x_0)^{d-\Delta} + \beta_i(x_0)^\Delta,$$

implying

$$\begin{aligned}H &= H_{CFT} + \alpha_i \mathcal{O}_i, \\ \langle \beta_i | \mathcal{O} | \beta_i \rangle &= \beta_i + (\alpha_i \text{ piece}).\end{aligned}$$

• **Solutions and propagators in global Lorentzian space.** The metric is

$$ds^2 = \frac{R^2}{\cos^2 \rho} (-dt^2 + d\rho^2 + \sin^2 \rho d\Omega_{d-1}^2).$$

• Only one solution regular in the center of AdS,

$$\psi_1 = e^{-i\omega t} Y_{l,\{m\}}(\Omega) (\cos \rho)^{\Delta_+} (\sin \rho)^l {}_2F_1 \left(\frac{\Delta_+ + l + \omega}{2}, \frac{\Delta_+ + l - \omega}{2}; l + \frac{d}{2}; \sin^2 \rho \right).$$

• At boundary $x_0 = \cos \rho = 0$, two possible behaviours (solutions)

$$\begin{aligned} \Phi^+ &= e^{-i\omega t} Y_{l,\{m\}}(\Omega) (\cos \rho)^{\Delta_+} (\sin \rho)^l {}_2F_1 \left(\frac{\Delta_+ + l + \omega}{2}, \frac{\Delta_+ + l - \omega}{2}; \Delta_+ + 1 - \frac{d}{2}; \cos^2 \rho \right) \\ \Phi^- &= e^{-i\omega t} Y_{l,\{m\}}(\Omega) (\cos \rho)^{\Delta_-} (\sin \rho)^l {}_2F_1 \left(\frac{\Delta_- + l + \omega}{2}, \frac{\Delta_- + l - \omega}{2}; \Delta_- + 1 - \frac{d}{2}; \cos^2 \rho \right), \end{aligned}$$

where Φ^+ is normalizable, Φ^- is non-normalizable, and in general $\psi_1 \sim C^+ \Phi^+ + C^- \Phi^-$. But if

$$\omega_{nl} = \pm(\Delta_+ + l + 2n),$$

$C^- = 0$, so ψ_1 is normalizable (and regular).

- So unlike in Poincaré coords., *the general solution is non-normalizable, but particular discrete frequencies, it is normalizable.*
- Non-normalizable modes: sources for CFT operators, but at special AdS frequencies, normalizable modes, corresponding to states of CFT on the cylinder, with energies $\omega_{nl} = \Delta + 2n + l$.
- We can compute global bulk to bulk propagator,

$$iG(x, y) = \frac{C_B}{(\cosh^2 \frac{s}{R})^{\frac{\Delta_+}{2}}} {}_2F_1 \left(\frac{\Delta_+}{2}, \frac{\Delta_+ + 1}{2}; \nu + 1; \frac{1}{\cosh^2 \frac{s}{R}} - i\epsilon \right),$$

and take limits to obtain the bulk to boundary, and boundary to boundary (thus, CFT) propagators,

$$K_B(b, x) = C_B \left[\frac{\cos \rho'}{\cos(t - t') - \sin \rho' \Omega \cdot \Omega' + i\epsilon} \right]^{\Delta_+}$$

$$G_\partial(b, b') \propto \frac{1}{[(\cos(t - t') - \Omega \cdot \Omega')^2 + i\epsilon]^{\frac{\Delta_+}{2}}}.$$

- However, the Poincaré boundary to boundary propagator is different: $x_{12}^2 = |x_1 - x_2|^2$ in global coordinates is different than the denominator of the above,

$$x_{12}^2 = \frac{2(\cos(t_1 - t_2) - \Omega \cdot \Omega')}{(\cos \tau_1 - \Omega_1^d)(\cos \tau_2 - \Omega_2^d)},$$

so in each coordinate set we must define things from the start!

Gauge/gravity duality

- **Generalize:** other max. susy CFT cases: $AdS_7 \times S^4$, $AdS_4 \times S^7$
 $\rightarrow$ "gravity dual"
- Conformal invariance $\leftrightarrow$ AdS space. But we can obtain less susy by taking $AdS \times X$, e.g. by dividing by a finite group S^k/Γ .
- We can also break conformal invariance $\rightarrow$ modify AdS space.
- Theories with **mass gap**: AdS space like finite quantum mechanical box: must cut out a thin cylinder from the middle of the AdS cylinder.
- We have an UV-IR correspondence:
 $E \sim U = r/\alpha' \Rightarrow$ IR in CFT $= r \rightarrow 0$ (UV) in AdS. Cut out around $r = r_{min}$.
- Motion in $U = r/\alpha' \rightarrow$ Renormalization group flow in QFT.

Minimal ingredients to simulate QCD:

- A large N quantum gauge theory ($N \rightarrow \infty$ for small g_s corrections)
- Boundary at infinity identified with flat space of QCD, but better: field theory at energy scale U corresponds to flat space at position r in the gravity dual.
- Thus $d + 1$ dimensional gravity dual corresponds to d -dim. field theory plus its energy scale U .
- Since motion in U is RG flow, mass gap corresponds to minimum r of gravity dual.
- Gauge group appears in gravity dual only through N .

Map field theory/gravity dual

- Global symmetries in Mink_d field th. $\leftrightarrow$ gauge symmetries in $d+1$ -dim. gravity dual. $\rightarrow$ global symmetries of compact space X_m . J_μ^a couple to A_μ^a .
- P_μ Noether current: $T_{\mu\nu} \leftrightarrow$ (couples to) $g_{\mu\nu}$. So d -dim. transl. inv. $\leftrightarrow$ diffeomorphism invariance in $d+1$ dimensions.
- Open/closed coupling: $g_s = g_{YM}^2/(4\pi)$.
- Gauge invariant operators $\leftrightarrow$ (sourced by) gravity dual fields in $d+1$ dimensions:
 - Supergravity fields in $d+1$ dim. (reduced on X_m) $\leftrightarrow$ SYM operators (made of adjoints) ("glueballs").
 - For quarks (fundamentals of gauge group and of some global symmetry G), introduce SYM fields for the group G in the gravity dual, coupling to G -charged, pion-like operators (made of quarks), so "SYM $\leftrightarrow$ pion fields".

- Thus: supergravity modes $\leftrightarrow$ glueballs, SYM fields $\leftrightarrow$ mesons.
- Mass spectrum of tower of glueballs = mass spectrum for wave eq. of sugra mode in gravity dual. Similar for mesons.
- Baryons: more than two fields, e.g. $B^{IJK} = \epsilon_{ijk} q^{Ii} q^{Jj} q^{Kk}$. In field theory: solitonic. $\rightarrow$ e.g. topological solitons in Skyrme model. In gravity dual: solitons: branes wrapped on cycles.
- Wave functions of states in field theory, $e^{ik \cdot x}$, correspond to gravity dual wave functions $\Phi(x, U, X_m) = e^{ik \cdot x} \Psi(U, X_m)$.

General properties for gravity duals for QCD-like, or SQCD-like theories:

- At high energy: conformal (all mass scales irrelevant). Thus, for $U \rightarrow \infty$, $AdS_5 \times X_5$, or maybe with subleading corrections to metric.
- At low energy, mass gap, so gravity dual must terminate at some $U_{\min}$, such that "warp factor" U^2 in front of $d\vec{x}^2$ remains finite.
- For fundamental quarks, open string modes on some brane must be introduced. Couple to meson-like operators. Alternative: free probe branes, probing physics at various energy scales.
- If QCD-like theory has global symm. (like flavor, or R, symm.), gravity dual, so X_m , must have this.

Lecture 5

Holographic renormalization and holographic RG flow

Holographic renormalization

- There are infinities in the AdS (or gravity dual) calculation of the on-shell sugra action: must renormalize them.
 - In part., volume near bound. is ∞ : $\int dx_0 \sqrt{g} \propto \int dx_0 / (x_0)^{d+1} |_{z_0 \rightarrow 0} \rightarrow \infty$.
 - Must add counterterms to on-shell sugra action: $S_{\text{ren}} = S_{\text{on-shell, sugra}} + S_{\text{ct.}}$, and take derivatives of S_{ren} with respect to boundary fields.
- We will obtain for the exact one-point functions *for nonzero source*

$$\begin{aligned}\langle \mathcal{O}(x) \rangle_{\phi_{(0)}} &= -\frac{1}{\sqrt{g_{(0)}}} \frac{\delta S_{\text{ren}}}{\delta \phi_{(0)}(x)} \sim \phi_{(2\Delta-d)}(x) \\ \langle J_i(x) \rangle_{A_{(0)i}} &= -\frac{1}{\sqrt{g_{(0)}}} \frac{\delta S_{\text{ren}}}{\delta A_{(0)i}(x)} \sim A_{(n)i}(x) \\ \langle T_{ij}(x) \rangle_{g_{(0)ij}} &= -\frac{1}{\sqrt{g_{(0)}}} \frac{\delta S_{\text{ren}}}{\delta g_{(0)ij}} \sim g_{(d)ij}(x).\end{aligned}$$

- $\phi_{(2\Delta-d)}$ = coefficient in expansion near boundary of *the exact solution* (not determined by the near-boundary expansion of the *equations of motion*), and similar for $A_{(n)i}$ and $g_{(d)ij}$.

- From the exact one-point function *with source*, so the exact solutions $\phi_{(2\Delta-d)}, A_{(n)i}, g_{(d)ij}$, we can calculate the n-point functions via derivatives,

$$\langle \mathcal{O}(x_1) \dots \mathcal{O}(x_n) \rangle \sim (-1)^{n-1} \frac{\delta^{n-1} \phi_{(2\Delta-d)}(x_1)}{\delta \phi_{(0)}(x_2) \dots \delta \phi_{(0)}(x_n)} \Big|_{\phi_{(0)}=0}.$$

- Also diff. and conformal Ward identities are obtained as

$$\nabla^i \langle T_{ij} \rangle_{g_{(0)ij}} = 0; \quad \langle T^i_i \rangle_{g_{(0)ij}} = \mathcal{A}.$$

- **Asymptotic expansion.** Define asymptotically AdS spacetimes by near boundary ($z = 0$) expansion. Metric can be put into

$$ds^2 = \frac{1}{z^2} (dz^2 + g_{ij} dx^i dx^j),$$

- where $g_{ij}(\vec{x}, z)$ solves Einstein's equation and admits Taylor expansion (is smooth),

$$g_{ij}(\vec{x}, z) = g_{(0)ij}(\vec{x}) + z g_{(1)ij}(\vec{x}) + z^2 g_{(2)ij} + \dots$$

- Einstein's equations fix all $g_{(n)}(\vec{x})$ with $n > 0$ in terms of $g_{(0)}(\vec{x})$. (in part., in pure gravity all odd powers vanish up to z^d). If d is even, we can also have a logarithmic term at order z^d , so

$$g_{ij}(\vec{x}, z) = g_{(0)ij}(\vec{x}) + z^2 g_{(2)ij}(\vec{x}) + \dots + z^d (g_{(d)ij}(\vec{x}) + h_{(d)ij}(\vec{x}) \log z^2) + \dots$$

- Solutions for $g_{(n)}$ in terms of $g_{(0)}$ is *algebraic*. In the above, $g_{(d)}$ is determined by $g_{(0)}$, but $h_{(d)}$ equals the variation of the conformal anomaly with respect to the metric.

- A general field $\Phi(\vec{x}, z)$ has the near boundary expansion

$$\Phi(\vec{x}, z) = z^m (\Phi_{(0)}(\vec{x}) + z^2 \Phi_{(2)}(\vec{x}) + \dots + z^{2n} (\Phi_{(2n)}(\vec{x}) + \log z^2 \tilde{\Phi}_{(2n)}(\vec{x})) + \dots$$

- The field equation for Φ (second order in derivatives) has solutions near the boundary z^m and z^{m+2n} , and their coefficients, $\Phi_{(0)}$ and $\Phi_{(2n)}$, correspond to the source for the dual operator, and to $\langle \mathcal{O} \rangle$ (VEV), respectively (this is true for the exact one-point function).

- **Regularization and counterterms:** Regularize the boundary: at $z = \epsilon$ instead of $z = 0$. Then regularized action is

$$S_{\text{reg}}[\Phi_{(0)}, \epsilon] = \int_{(z=\epsilon)} d^d x \sqrt{g_{(0)}} [\epsilon^{-2\nu} s_{(0)}[\Phi_{(0)}] + \epsilon^{-2\nu+2} s_{(2)}[\Phi_{(0)}] + \dots - \log \epsilon s_{(2\nu)}[\Phi_{(0)}] + \text{finite}].$$

- Leading divergent term: $\sim \int dz/z^{d+1} \phi^2 \sim \epsilon^{d-\Delta}$; must be cancelled by counterterm. Minimal subtraction scheme:

$$S_{\text{ct}}[\Phi(\vec{x}, \epsilon); \epsilon] = -\text{div.terms in } S_{\text{reg}}[\Phi_{(0)}(\Phi(\vec{x}, \epsilon)); \epsilon].$$

- Then the *subtracted action*, varied in order to obtain correlation functions, is

$$S_{\text{sub}}[\Phi(\vec{x}, \epsilon); \epsilon] = S_{\text{reg}}[\Phi_{(0)}; \epsilon] + S_{\text{ct}}[\Phi(\vec{x}, \epsilon); \epsilon],$$

and has ϵ finite (must be kept so; put $\epsilon \rightarrow 0$ only at the end of the calculation). However, the renormalized on-shell action is its $\epsilon \rightarrow 0$ limit,

$$S_{\text{ren}}[\Phi_{(0)}] = \lim_{\epsilon \rightarrow 0} S_{\text{sub}}[\Phi(\vec{x}, \epsilon); \epsilon].$$

- The one-point function, from which we can calculate the other correlation functions, is roughly

$$\langle \mathcal{O}(\vec{x}) \rangle_{\Phi_{(0)}} = " \frac{1}{\sqrt{g_{(0)}}} \frac{\delta S_{\text{ren}}}{\delta \Phi_{(0)}(\vec{x})} ",$$

but really keep ϵ finite and put it to zero at the end, so really, $\Phi(\vec{x}, \epsilon) = \epsilon^m \Phi_{(0)} + \dots$ and $\gamma_{ij} = g_{(0)ij}/\epsilon^2 + \dots$. Then

$$\langle \mathcal{O}(\vec{x}) \rangle_{\Phi_{(0)}} = \lim_{\epsilon \rightarrow 0} \frac{1}{\epsilon^{d-m}} \frac{1}{\sqrt{\gamma}} \frac{\delta S_{\text{sub}}}{\delta \Phi(\vec{x}, \epsilon)}.$$

- The result is proportional to the linearly independent coefficient $\Phi_{(2n)}(\vec{x})$, but there could also be a local function of the source $\Phi_{(0)}$ that leads to contact terms in the higher n -point functions, and is scheme dependent, so

$$\langle \mathcal{O}(\vec{x}) \rangle_{\Phi_{(0)}} \sim \Phi_{(2n)}(\vec{x}) + F(\Phi_{(0)}).$$

- RG transformations in field theory arise from bulk diffeomorphisms that induce Weyl transformations on the boundary, so

$$x^i = \mu x^{i'}; \quad z = \mu z'.$$

Example: massive scalar in AdS

- Writing first $\Phi(\vec{x}, z) = z^{d-\Delta} \phi(\vec{x}, z)$, $\phi(\vec{x}, z)$ is finite at the boundary, and has a Taylor expansion in even powers of z ,

$$\phi(\vec{x}, z) = \phi_{(0)} + z^2 \phi_{(2)} + z^4 \phi_{(4)} + \dots$$

- Substituting in the KG eq. expanded in z , we find first $m^2 R^2 = \Delta(\Delta - d)$, then

$$\phi_{(2)}(\vec{x}) = \frac{1}{2(2\Delta - d - 2)} \partial_i \partial_i \phi_{(0)}$$

$$\phi_{(4)}(\vec{x}) = \frac{1}{4(2\Delta - d - 4)} \partial_i \partial_i \phi_{(2)}, \dots, \phi_{(2n)} = \frac{1}{2n(2\Delta - d - 2n)} \partial_i \partial_i \phi_{(2n-2)},$$

and so on, and the series ends when $2\Delta - d - 2n = 0$, where we need to introduce a $z^\delta \log z^2$ term in Φ , so

$$\phi(\vec{x}, z) = \phi_{(0)} + z^2 \phi_{(2)} + \dots + z^{2\Delta-d} (\phi_{(2\Delta-d)} + (\log z^2) \tilde{\phi}_{(2\Delta-d)}) + \dots$$

- From the equations of motion, expanded in z around the boundary, we find

$$\tilde{\phi}_{(2\Delta-d)} = -\frac{1}{2^{2\Delta-d} \Gamma(\Delta - \frac{d}{2}) (\Delta - \frac{d-2}{2})} (\partial_i \partial_i)^{\Delta - \frac{d}{2}} \phi_{(0)},$$

on the other hand $\phi_{(2\Delta-d)}$ is not fixed by them.

•**Regularization and counterterms:** Regularized kinetic on-shell action for Φ is

$$\begin{aligned}
S_{\text{reg.}} &= \frac{1}{2} \int_{z \geq \epsilon} d^{d+1}x \sqrt{g_{d+1}} (g^{\mu\nu} \partial_\mu \Phi \partial_\nu \Phi + m^2 \Phi^2) \\
&= \frac{1}{2} \int_{z \geq \epsilon} d^{d+1}x \sqrt{g_{d+1}} \Phi (-\square_{g_{\mu\nu}} + m^2) \Phi - \frac{1}{2} \int_{z=\epsilon} d^d x \sqrt{g_{d+1}} g^{zz} \Phi \partial_z \Phi , \\
&= - \int_{z=\epsilon} d^d x \epsilon^{-2\Delta+d} \left(\frac{1}{2} (d - \Delta) \phi(\vec{x}, \epsilon)^2 + \frac{1}{2} \epsilon \phi(\vec{x}, \epsilon) \partial_\epsilon \phi(\vec{x}, \epsilon) \right) ,
\end{aligned}$$

where we integrated by parts, used the equations of motion, and expressed Φ in terms of ϕ . This is of the general form, with

$$\begin{aligned}
s_{(0)} &= -\frac{1}{2} (d - \Delta) \phi_{(0)}^2 \\
s_{(2)} &= -(d - \Delta + 1) \phi_{(0)} \phi_{(2)} = -\frac{d - \Delta + 1}{2(2\Delta - d - 2)} \phi_{(0)} \partial_i \partial_i \phi_{(0)}, \quad \dots , \\
s_{(2\Delta-d)} &= d \phi_{(0)} \tilde{\phi}_{(2\Delta-d)} = -\frac{d}{2^{2\Delta-d} \Gamma\left(\Delta - \frac{d}{2}\right) \left(\Delta - \frac{d-2}{2}\right)} \phi_{(0)} (\partial_i \partial_i)^{\Delta - \frac{d}{2}} \phi_{(0)}
\end{aligned}$$

- For the counterterm action, cancel the divergences, but with $\phi_{(2k)}$ re-expressed in terms of $\Phi(\vec{x}, \epsilon)$. Inverting $\Phi(\vec{x}, \epsilon)$ to second order in ϵ^2 , we obtain

$$\begin{aligned}\phi_{(0)} &= \epsilon^{-(d-\Delta)} \left(\Phi(\vec{x}, \epsilon) - \frac{1}{2(2\Delta - d - 2)} \square_\gamma \Phi(\vec{x}, \epsilon) \right) \\ \phi_{(2)} &= \epsilon^{-(d-\Delta)-2} \frac{1}{2(2\Delta - d - 2)} \square_\gamma \Phi(\vec{x}, \epsilon) ,\end{aligned}$$

where $\square_\gamma$ is the Laplacean of $\gamma_{ij} = \delta_{ij}/\epsilon^2$ (induced metric at $z = \epsilon$). Then the counterterm action is

$$S_{\text{ct.}} = \int_{\text{boundary}} d^d x \sqrt{\gamma} \left(\frac{d-\Delta}{2} \Phi^2 + \frac{1}{2(2\Delta - d - 2)} \Phi \square_\gamma \Phi \right) + \mathcal{O}(\square_\gamma^2) ,$$

- For $\Delta = d/2 + k$, the coefficient of $\Phi \square_\gamma \Phi$ has a $\log \epsilon$ (for $k = 1, -\frac{1}{2} \log \epsilon$).

• Then for $\Delta = d/2 + 1$, we obtain

$$\begin{aligned}
\delta S_{\text{sub.}} &= -\frac{1}{2}\delta \int_{z=\epsilon} d^d x \sqrt{g} g^{zz} \Phi \partial_z \Phi + \delta \int_{z=\epsilon} d^d x \sqrt{\gamma} \left(\frac{d-\Delta}{2} \Phi^2 - \frac{1}{2} \log z \Phi \square_\gamma \Phi \right) + .. \\
&= \int_{z=\epsilon} d^d x \sqrt{\gamma} \delta \Phi (-\epsilon \partial_\epsilon \Phi + (d-\Delta) \Phi - \log \epsilon \square_\gamma \Phi) \Rightarrow \\
\frac{1}{\sqrt{\gamma}} \frac{\delta S_{\text{sub.}}}{\delta \Phi} &= -\epsilon \partial_\epsilon \Phi + (d-\Delta) \Phi - \log \epsilon \square_\gamma \Phi ,
\end{aligned}$$

so that

$$\begin{aligned}
\frac{1}{\sqrt{\gamma}} \frac{\delta S_{\text{sub.}}}{\delta \Phi} &= -\epsilon \partial_\epsilon \Phi + (d-\Delta) \Phi - \log \epsilon \square_\gamma \Phi \\
\langle \mathcal{O} \rangle_{\phi_{(0)}} &= \lim_{\epsilon \rightarrow 0} \left(\frac{1}{\epsilon^\Delta} \frac{1}{\sqrt{\gamma}} \frac{\delta S_{\text{sub.}}}{\delta \Phi} \right) = -2(\phi_{(2)} + \tilde{\phi}_{(2)}) ,
\end{aligned}$$

which is of the general form $\langle \mathcal{O} \rangle_{\phi_{(0)}} \sim \phi_{(2\Delta-d)} + F(\phi_{(0)})$, since for $\Delta = d/2 + 1$, $\phi_{(2\Delta-d)} = \phi_{(2)}$, and $\tilde{\phi}_{(2)} = \tilde{\phi}_{(2\Delta-d)} = F(\phi_{(0)})$. For general Δ , one finds (one can show)

$$\langle \mathcal{O} \rangle_{\phi_{(0)}} = -(2\Delta - d) \phi_{(2\Delta-d)} + F(\phi_{(0)}) .$$

- This proves that indeed, $\phi_{(2\Delta-d)}$ (the coefficient of x^Δ) gives the (deformation of the) operator VEV, while we had that $\phi_{(0)}$ (the coefficient of $z^{d-\Delta}$) gave the operator deformation of the theory (source).
- To calculate the 2-point function from the 1-point function, we need $\phi_{(2\Delta-d)}$ and F as a function of $\phi_{(0)}$, which only is true for the exact solution.
- For example, for $d = 4$ and $\Delta = d/2 + 1 = 3$, the regular solution of the KG equation in momentum space (regular also at the center) is

$$\Phi = z^2 K_1(z) ,$$

expanded near the boundary $z = 0$ as

$$\Phi(k, z) = \frac{1}{k} z \left[1 + k^2 z^2 \left(\frac{1}{4}(2\gamma - 1) - \frac{1}{2} \log 2 + \frac{1}{2} \log(kz) \right) \right] + \dots ,$$

so we have

$$\begin{aligned} \tilde{\phi}_{(2)}(k) &= \frac{k^2}{4} \phi_{(0)}(k) , \quad \phi_{(2)}(k) = \phi_{(0)}(k) k^2 \left[\frac{1}{4}(2\gamma - 1) + \frac{1}{2} \log \frac{k}{2} \right] \Rightarrow \\ \langle \mathcal{O} \rangle_{\phi_{(0)}} &= -2(\phi_{(2)} + \tilde{\phi}_{(2)}) = -2\phi_{(0)}(k) \left[k^2 \left(\frac{1}{4}(2\gamma - 1) - \frac{1}{2} \log 2 + \frac{1}{4} \right) + \frac{k^2}{4} \log k^2 \right] . \end{aligned}$$

•Then

$$\langle \mathcal{O}(k) \mathcal{O}(-k) \rangle = -\frac{\delta\phi_{(2)}(k)}{\delta\phi_{(0)}(-k)} = \frac{k^2}{2} \log k^2 + \text{contact terms},$$

which Fourier transforms to the x space into

$$\langle \mathcal{O}(x) \mathcal{O}(0) \rangle = \frac{4}{\pi^4} \mathcal{R} \frac{1}{x^6},$$

where $\mathcal{R}1/x^6$ equals $1/x^6$ away from $x = 0$. For $\Delta = d/2 + k$, one obtains

$$\langle \mathcal{O}(x) \mathcal{O}(0) \rangle = (2\Delta - d) \frac{\Gamma(\Delta)}{\pi^{d/2} \Gamma(\Delta - d/2)} \mathcal{R} \frac{1}{x^{2\Delta}}.$$

•Note that naive Witten prescription calculation differs by $(2\Delta - d)/\Delta$.

- RG transformations, $\vec{x} = \vec{x}'\mu, z = z'\mu$, onto scalar $\Phi(\vec{x}, z)$, imply

$$\begin{aligned}\phi'_{(2k)}(\vec{x}') &= \mu^{d-\Delta+2k} \phi_{(2k)}(\vec{x}'\mu), \quad 2k < 2\Delta - d \\ \tilde{\phi}'_{(2\Delta-d)}(\vec{x}') &= \mu^\Delta \tilde{\phi}_{(2\Delta-d)}(\vec{x}'\mu) \\ \phi'_{(2\Delta-d)}(\vec{x}') &= \mu^\Delta [\phi_{(2\Delta-d)}(\vec{x}'\mu) + \log \mu^2 \tilde{\phi}_{(2\Delta-d)}(\vec{x}'\mu)] ,\end{aligned}$$

leading to

$$\begin{aligned}\mu \frac{\partial}{\partial \mu} \phi_{(0)}(\vec{x}\mu) &= (\Delta - d) \phi_{(0)}(\vec{x}\mu) , \\ \langle \mathcal{O}(\vec{x}') \rangle'_{\phi_{(0)}} &= \mu^\Delta \left(\langle \mathcal{O}(\vec{x}'\mu) \rangle_{\phi_{(0)}} - (2\Delta - d) \log \mu^2 \tilde{\phi}_{(2\Delta-d)}(\vec{x}'\mu) \right) ,\end{aligned}$$

consistent with $\phi_{(0)}$ a source for an operator of dimension Δ , and $\tilde{\phi}_{(2\Delta-d)}$ giving the conformal anomaly.

- We already saw that $\phi_{(2\Delta-d)}$ was a deformation of the VEV of the theory, and the above RG flow is also consistent with that.

Holographic RG flow

- Motion in radial coordinate of the sugra solution, starting and ending at solutions with AdS symmetry $\leftrightarrow$ RG flow between fixed points of the field theory.
- RG flow initiated by relevant deformation of CFT: must deform basis theory by some operator.
- **Example:** $\mathcal{N} = 1$ susy deformation of $\mathcal{N} = 4$ SYM. The superpotential of $\mathcal{N} = 4$ SYM,

$$W = \text{Tr}(\Phi_3[\Phi_1, \Phi_2]) ,$$

is deformed by a supersymmetric mass deformation

$$\delta W = \frac{m}{2} \text{Tr}(\Phi_3^2) ,$$

- **Obs:** Superpotential is in *superspace*: Superfields $\Phi(x, \theta)$, where θ is a fermionic (anticommuting) variable. For a 4d chiral superfield,

$$\Phi(x, \theta) = \Phi(y, \theta) = \phi(y) + \sqrt{2}\theta\psi(y) + \theta\theta F(y) , \quad y^\mu = x^\mu + i\theta\sigma^\mu\bar{\theta}$$

- In 4d, $\exists$ 2 anomaly coefficients characterizing fixed points, central charges c and a . $g_{\mu\nu} \leftrightarrow T_{\mu\nu}$, $A_\mu \leftrightarrow J_\mu$. Consider J_μ the R-current ($SU(4) = SO(6)$ global symmetry, acting on the fermions).
- Then, anomaly in VEV of $T^\mu{}_\mu$ (classically = 0 by conformal invariance) and VEV of $\partial^\mu J_\mu$,

$$\begin{aligned}\langle T^\mu{}_\mu \rangle_{g_{\mu\nu}, A_\mu} &= \frac{c}{16\pi^2} C_{\mu\nu\rho\sigma}^2 - \frac{a}{16\pi^2} \tilde{R}_{\mu\nu\rho\sigma} \tilde{R}^{\mu\nu\rho\sigma} + \frac{c}{6\pi^2} F_{\mu\nu}^2 \\ \langle \partial_\mu \sqrt{g} J^\mu \rangle_{g_{\mu\nu}, A_\mu} &= -\frac{a-c}{24\pi^2} R_{\mu\nu\rho\sigma} \tilde{R}^{\mu\nu\rho\sigma} + \frac{5a-3c}{9\pi^2} F_{\mu\nu} \tilde{F}^{\mu\nu}.\end{aligned}$$

where $\tilde{F}^{\mu\nu} = \frac{1}{2} \epsilon^{\mu\nu\rho\sigma} F_{\rho\sigma}$, $\tilde{R}^{\mu\nu}{}_{\rho\sigma} = \frac{1}{2} \epsilon^{\mu\nu\lambda\tau} R_{\lambda\tau\rho\sigma}$, and $C_{\mu\nu\rho\sigma}$ is the (conformal invariant) Weyl tensor,

$$C_{\mu\nu\rho\sigma} = R_{\mu\nu\rho\sigma} - \frac{2}{d-2} (g_{\mu[\rho} R_{\sigma]\nu} - g_{\nu[\rho} R_{\sigma]\mu}) + \frac{2}{(d-1)(d-2)} R g_{\mu[\rho} g_{\sigma]\nu}.$$

In 4d we have the topological density E_4 and the conformally invariant I_4 ,

$$\begin{aligned}\tilde{R}_{\mu\nu\rho\sigma} \tilde{R}^{\mu\nu\rho\sigma} &= R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} - 4 R_{\mu\nu} R^{\mu\nu} + R^2 = E_4 \\ C_{\mu\nu\rho\sigma} C^{\mu\nu\rho\sigma} &= R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} - 2 R_{\mu\nu} R^{\mu\nu} + \frac{R^2}{3} = I_4\end{aligned}$$

- Central charge c counts perturbative massless degrees of freedom in CFT, up to normalization. Here, normalization chosen such that $c = \frac{1}{4}(N_c^2 - 1)$ for $\mathcal{N} = 4$ SYM.

- Anomaly contributions in $\langle \partial^\mu J_\mu \rangle$: $a - c$ from $\partial^\mu J_\mu - T_{\mu\nu} - T_{\mu\nu}$ triangle, prop. to $\sum_\chi R(\chi)$, and $5a - 3c$ from $\partial^\mu J_\mu - J_\mu - J_\mu$, prop. to $\sum_\chi R(\chi)^3$:

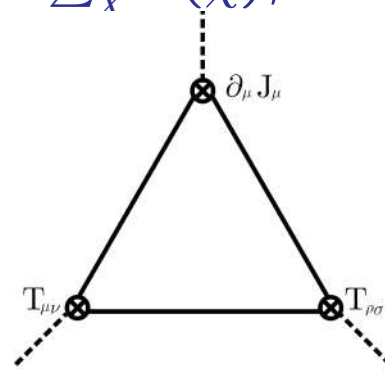


diagram for a-c

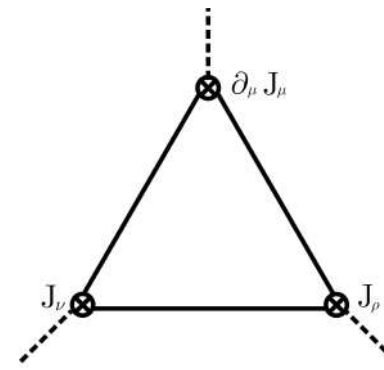


diagram for 5a-3c

a) Anomalous diagram contributing to $a - c$ b) Anomalous diagram contributing to $5a - 3c$.

- For $\mathcal{N} = 1$ RG flow on $\mathcal{N} = 4$ SYM, UV: $\sum_\chi R(\chi) = 0, \sum_\chi R(\chi)^3$, IR: $\sum_\chi R(\chi) = 0, \sum_\chi R(\chi)^3 = \frac{3}{4}(N_c^2 - 1)$, so

$$\begin{aligned}
 a_{UV} - c_{UV} &= 0; \quad 5a_{UV} - 3c_{UV} \propto \frac{8}{9}(N_c^2 - 1), \\
 a_{IR} - c_{IR} &= 0; \quad 5a_{IR} - 3c_{IR} \propto \frac{3}{4}(N_c^2 - 1) \Rightarrow \\
 \frac{a_{IR}}{a_{UV}} &= \frac{c_{IR}}{c_{UV}} = \frac{27}{32}.
 \end{aligned}$$

- **c-theorem**: (2d: Zamolodchikov) For an RG flow between 2 fixed points, $\exists$ *monotonically decreasing* function along RG flow, with value c_{UV} in the UV and c_{IR} in the IR, called c-function. In 2d, $\hat{c}$ appears in the trace anomaly (in conformal invariance),

$$\langle T^\mu{}_\mu \rangle = -\frac{c}{12}R.$$

- In 4d, similar statement (Komargodski and Schwimmer, after conjecture by Cardy): *a*-theorem: for the *a* charge. Will be proven *constructively* via AdS/CFT.

- Cardy's statement applies *in general dimension* (thus including the c-theorem and the a-theorem) to the coefficient of $E_d = \tilde{R}_{\mu\nu\rho\sigma}\tilde{R}^{\mu\nu\rho\sigma}$, in

$$\langle T^\mu{}_\mu \rangle = -2(-)^{d/2}AE_d + \dots$$

- Central charges in $\mathcal{N} = 4$ SYM: from holographic Weyl anomaly.

● **Holographic Weyl anomaly and central charge:** The variation of the on-shell sugra action under a Weyl transformation $\delta g_{(0)} = 2\sigma g_{(0)}$; $\delta\epsilon = 2\delta\sigma\epsilon$ gives the one-point function of $T^\mu{}_\mu$ ($\langle T_{\mu\nu} \rangle = \lim_{\epsilon \rightarrow 0} \frac{1}{\epsilon^{\Delta_T}} \frac{1}{\sqrt{-\gamma}} \frac{\delta S_{\text{sub}}}{\delta g^{\mu\nu}}$, for $\Delta_T = d$). But

$$S_{\text{reg}} = (16\pi G_N^{(d+1)})^{-1} \int d^d x \sqrt{g_{(0)}} [\dots + (-\log \epsilon) s_{(d)}] + S_{\text{finite}} ,$$

and the finite part is cancelled by the counterterm, so we obtain

$$\delta S_{\text{finite}} = - \int d^d x \sqrt{g_{(0)}} \delta\sigma \mathcal{A} \quad \dots \Rightarrow \mathcal{A} = \frac{1}{16\pi G_N^{(d+1)}} (-2s_{(d)}) .$$

● A holographic calculation leads to $a = c$, so

$$\mathcal{A} = -\frac{a}{16\pi^2} (E_4 + I_4) ,$$

and ($G_{N,5} = G_N^{(10)}/R^5\Omega_5$, and $\Omega_5 = \pi^3$)

$$a = c = \frac{\pi^2 R^3}{l_{P,5}^3} = \frac{\pi R^3}{8G_N^{(5)}} .$$

- **Holographic RG flow and c -function: kink ansatz.** Holographic RG flow ansatz must be kink-type:

$$ds^2 = e^{2A(r)}(-dt^2 + d\vec{x}_{d-1}^2) + dr^2 = e^{2A(r)}\eta_{\mu\nu}dx^\mu dx^\nu + dr^2 ,$$

and $\phi_i = \phi_i(r)$. AdS: $A(r) = r/R, \phi_i = 0$. Thus, at endpoints: $A_1(r) \simeq r/R_1(UV), A_2(r) \simeq r/R_2(IR), \phi_i \simeq 0$, with $R_2 < R_1$.

- Consider perfect fluid $T_{\mu\nu} = \text{diag}(\rho, p_1, \dots, p_{d-1})$, satisfying weakest energy condition (satisfied by all QFTs), $\rho + p_i \geq 0$, then Einstein equations for the above ansatz, with

$$R_{\mu\nu} = e^{2A(r)}[A'' + d(A')^2]\eta_{\mu\nu}; \quad R_{rr} = -d[A'' + (A')^2] ,$$

one finds the condition $A'' \leq 0$. This means that we have the monotonically non-increasing function

$$C(r) = a(r) = \frac{a_0}{(A')^{d-1}}.$$

- With the proper normalization, we find the c -function (or a -function) in d dimensions, which gives c and a at the endpoints (fixed points),

$$C(r) = a(r) = \frac{\pi^{d/2}}{\Gamma(d/2)(l_P^{(d+1)} A'(r))^{d-1}} .$$

- Note this also gives the value of the holographic central charge in d dimensions!
- **Supersymmetric flow** For the $\mathcal{N} = 1$ mass def. of $\mathcal{N} = 4$ SYM, in $\mathcal{N} = 8$ sugra: interpolate between the $\mathcal{N} = 8$ susy AdS_5 and another $\mathcal{N} = 2$ AdS_5 (1/4 susy). Thus susy kink $\Rightarrow \delta_{\text{SUSY}} \psi = 0$.
- Supergravity scalar potential V in terms of superpotential W is

$$V = \frac{9}{8} \sum_j \left| \frac{\partial W}{\partial \phi_j} \right|^2 - 3l_P^2 |W|^2.$$

- Then from gravitino variation $\delta \psi_\mu^a = 0$ and spin 1/2 variation $\delta \chi^{abc} = 0$, we find

$$A' = -l_P^2 W; \quad \frac{d\phi_i}{dr} = \frac{3}{2} l_P^2 \frac{\partial W}{\partial \phi_i},$$

- so the c -function or a -function in this case is

$$C(r) = a(r) = \frac{\pi^2}{(l_P^{(5)})^9 W^3} ,$$

- For the flow of interest, we indeed find $a(0)/a(\infty) = 27/32!$
- Extension and application: radial time evolution (holographic cosmology).** Consider the Domain Wall/Cosmology correspondence = double Wick rotation redefining radial r as time t , with background taken together as (note: z is not the previous one!!)

$$ds^2 = \eta dz^2 + a^2(z) d\vec{x}^2 , \quad \Phi = \varphi(z)$$

where $\eta = \pm 1$, solutions to the action ($\kappa^2 = 8\pi G_N$)

$$S = \frac{\eta}{2\kappa^2} \int d^4x \sqrt{|g|} \left[-R + (\partial\Phi)^2 + 2\kappa^2 V(\Phi) \right]$$

- Cosmologies are solutions to the equations

$$\frac{\dot{a}}{a} = -\frac{1}{2}W , \quad \dot{\varphi} = W_{,\varphi} , \quad 2\eta\kappa^2 V = (W_{,\varphi})^2 - \frac{3}{2}W^2 ,$$

where W is a "fake superpotential" ($V(W)$ the same as in susy).

•Asympt. AdS: $a(z) \sim e^z$, $\varphi \sim 0$ at $z \rightarrow \infty$. Formalism includes also sympt. power law (non-conformal AdS/CFT), $a(z) \sim (z/z_0)^n$, $\varphi \sim \sqrt{2n} \log(z/z_0)$, $z_0 = n - 1$. These domain walls become cosmologies for $\eta = -1$.

•Asympt. AdS in Fefferman-Graham coords. vs. ADM parametrization:

$$ds^2 = \frac{1}{z^2} \left[dz^2 + (g_{(0)ij} + \dots + z^d g_{(d)ij} + \dots) \right]$$

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu = \hat{\gamma}_{ij} d\hat{x}^i d\hat{x}^j + 2N^i d\hat{x}^i dr + (N^2 + N_i N^i) dr^2$$

match in the gauge $N = 1$, $N_i = 0$, for $z = e^{-r}$, so $\hat{\gamma}_{ij} = g_{ij} = \frac{1}{z^2} (g_{(0)ij}(\vec{x}) + \dots) = e^{-2r} (g_{(0)ij}(\vec{x}) + \dots)$.

•The usual canonical momenta in ADM formalism (with action $-\frac{1}{2\kappa^2} \int d^{d+1}x \sqrt{\hat{\gamma}} N (\hat{R} + \hat{K} - \hat{K}_{\mu\nu} \hat{K}^{\mu\nu} - 2\kappa^2 \mathcal{L}_m)$),

$$\pi^{\mu\nu} \equiv \frac{\delta L}{\delta \dot{\gamma}_{\mu\nu}} = -\frac{1}{2\kappa^2} (\hat{K} \hat{\gamma}^{\mu\nu} - \hat{K}^{\mu\nu}) , \quad \pi^I \equiv \frac{\delta L}{\delta \dot{\Phi}_I}$$

become equal to the momenta obtained from the variation of the on-shell action with respect to the boundary variables,

$$\pi^{\mu\nu}(r_1, \vec{x}) = \frac{\delta S_{\text{on-shell}}}{\delta \hat{\gamma}_{\mu\nu}(r_1, \vec{x})} , \quad \pi^I(r_1, \vec{x}) = \frac{\delta S_{\text{on-shell}}}{\delta \Phi_I} , \quad r_1 \rightarrow \infty.$$

•But since

$$\langle T_{ij}(\vec{x}) \rangle = \frac{1}{\sqrt{g_{(0)}(\vec{x})}} \frac{\delta S_{\text{on-shell}}(g_{(0)}, \dots)}{\delta g_{(0)}^{ij}(\vec{x})}, \quad \langle \mathcal{O}_I \rangle = \frac{\delta S_{\text{on-shell}}}{\delta \Phi^I},$$

we obtain

$$\begin{aligned} \langle T_{ij}(\vec{x}) \rangle &= \left(-\frac{2}{\sqrt{g}} \pi_{ij} \right)_{(d)} = -\frac{1}{8\pi G_N} (\hat{K}_{ij} - \hat{K} \hat{\gamma}_{ij})_{(d)} \simeq -\frac{d}{16\pi G_N} g_{(d)ij}(\vec{x}) \\ \langle \mathcal{O}(\vec{x}) \rangle &= \frac{1}{\sqrt{\hat{\gamma}}} \pi^I_{(\Delta_I)}, \end{aligned}$$

where the subscript means keep the part with the corresponding engineering dimension (or dilatation eigenvalue), d for T or Δ for $\mathcal{O}$; or the given term in the near boundary expansion in z (terms of less dimension are divergent, and are removed by renormalization). ($\delta_r \simeq \delta_D$, and $\delta_D \pi_{(n)ij} = -n \pi_{(n)ij}$).

•Then, according to the general theory, 2-point function from 1-point function with nonzero source,

$$\langle T_{ij}(x) T_{kl}(y) \rangle = \frac{1}{\sqrt{g_{(0)ij}}} \frac{\delta}{\delta g_{(0)kl}(y)} \langle T_{ij}(x) \rangle = \frac{1}{\sqrt{g_{(0)ij}}} \frac{\delta}{\delta g_{(0)kl}(y)} \left(-\frac{2}{\sqrt{g}} \pi_{ij}(x) \right)_{(d)}.$$

•The r.h.s. can be calculated in gravity, and is related to the 2-point functions of gauge invariant fluctuation modes γ_{ij} and ζ , linear combinations of the h_{ij} and ϕ fluctuation modes of the cosmology.

Lecture 6

Finite temperature and $\mathcal{N} = 4$ SYM plasma

• **Finite temperature in field theory:** QM transition amplitude

$$\langle q', t' | q, t \rangle = \langle q' | e^{-i\hat{H}(t'-t)} | q \rangle = \sum_n \psi_n(q') \psi_n^*(q) e^{-iE_n(t'-t)},$$

with $t \rightarrow -it_E, t' - t \rightarrow -i\beta, iS \rightarrow -S_E$, gives

$$\langle q', \beta | q, 0 \rangle = \sum_n \psi_n(q') \psi_n^*(q) e^{-\beta E_n}.$$

• In the case $q' = q$ and integrating over q , we get

$$\int dq \langle q, \beta | q, 0 \rangle = \int dq \sum_n |\psi_n(q)|^2 e^{-\beta E_n} = \text{Tr}\{e^{-\beta \hat{H}}\} = Z[\beta].$$

• But the transition amplitude is a path integral,

$$\langle q', t' | q, t \rangle = \int \mathcal{D}q(t) e^{iS[q(t)]},$$

so Wick rotating it to periodic euclidean time, we obtain

$$Z_E[\beta] = \int_{\phi(\vec{x}, t_E + \beta) = \phi(\vec{x}, t_E)} \mathcal{D}\phi e^{-S_E[\phi]} = \text{Tr}(e^{-\beta \hat{H}})$$

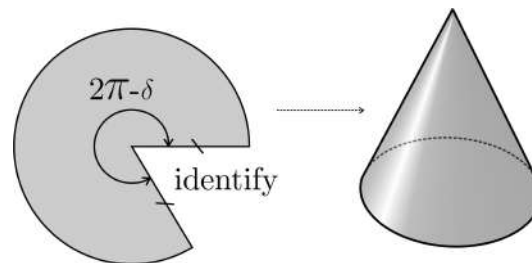
- Euclidean (Wick-rotated) Schwarzschild black hole

$$ds^2 = + \left(1 - \frac{2MG_N}{r} \right) d\tau^2 + \frac{dr^2}{1 - \frac{2MG_N}{r}} + r^2 d\Omega_2^2$$

- Does not make sense for $r < 2MG_N$ (unlike Minkowski signature) since signature of space changes! But near $r = 2MG_N$, $r - 2MG_N = \tilde{r} = \rho^2 \Rightarrow$

$$ds^2 \simeq 8MG_N \left[d\rho^2 + \frac{\rho^2 d\tau^2}{(4MG_N)^2} \right] + (2MG_N)^2 d\Omega_2^2$$

- This is of the type of a cone, $ds^2 = d\rho^2 + \rho^2 d\theta^2$ in general, and only if $\theta \sim \theta + 2\pi$ is a plane.



A flat cone is obtained by cutting out an angle from flat space, so that $\theta \in [0, 2\pi - \Delta]$ and identifying the cut.

- To avoid conical singularity at $r - 2MG_N = \rho^2 = 0$, we need $\tau/(4MG_N)$ to have period 2π , so $\exists$ temperature

$$T_{BH} = \frac{1}{\beta_\tau} = \frac{1}{8\pi MG_N}$$

- But Schwarzschild black hole is thermodynamically unstable, since the specific heat

$$C = \frac{\partial M}{\partial T} = -\frac{1}{8\pi T^2 G_N} < 0$$

- So we can't interpret the black hole as putting the QFT at finite temperature. In AdS space however, this can be done, as we will see.

• **Spin structure in black hole background.** At $r \rightarrow \infty$, the Euclidean BH solution is

$$ds^2 \simeq d\tau^2 + d\vec{x}^2 ,$$

where $\tau \sim \tau + 8\pi MG_N$, so $\mathbb{R}^3 \times S^1_\tau$: KK vacuum.

• Expand in Fourier modes. But fermions can acquire a phase $e^{i\alpha}$ around a circle, here S^1_τ at infinity,

$$\psi \rightarrow e^{i\alpha} \psi ,$$

known as *spin structures*. $\alpha = 0, \pi$ always OK, others depending on $\mathcal{L}$.

• At horizon $r = 2MG_N$, metric is $\simeq \mathbb{R}^2 \times S^2$, where Ω_2 is the sphere at ∞ . But $\mathbb{R}^2 \times S^2$ is simply connected, i.e., $\nexists$ nontrivial cycles: any loop can be smoothly shrunk to 0. Thus no non-trivial fermion phases possible around any loop, so $\exists$ unique spin structure!

• But what is it at infinity? $\tau \rightarrow \tau + \beta$ at infinity is $\theta \rightarrow \theta + 2\pi$ near horizon, i.e., rotation in 2d plane, under which a fermion gets a minus sign. Thus unique spin structure is antiperiodic around circle at infinity.

- **KK masses and susy breaking:** If fermions are antiperiodic at infinity, they must depend on the Euclidean time θ , $\psi = \psi(\theta)$. Then at infinity

$$\not\partial\psi = 0 \Rightarrow \not\partial^2\psi = \square_{4d}\psi = 0.$$

- But under dimensional reduction $\square_{4d} = \square_{3d} + \partial^2/\partial\theta^2$, so

$$0 = \square_{4d}\psi = \left(\square_{3d} + \frac{\partial^2}{\partial\theta^2}\right)\psi = (\square_{3d} - m^2)\psi,$$

so fermions become massive in the presence of the black hole (from the p.o.v. of the reduced 3d theory).

- Bosons can be periodic at infinity in $\theta \sim \theta + 2\pi$, so, e.g. for a scalar under KK reduction

$$0 = \square_{4d}\phi = \left(\square_{3d} + \frac{\partial^2}{\partial\theta^2}\right)\phi = \square_{3d}\phi,$$

so they remain massless.

- But for susy we need $m_{\text{scalar}} = m_{\text{fermion}}$, so susy is broken by the black hole.

- In fact, one can prove that finite temperature always breaks susy in QFT (regardless of the existence of AdS/CFT). So **we have a way of breaking the unrealistic $\mathcal{N} = 4$ susy by finite T !**

The AdS black hole and Witten's finite temperature prescription

Witten, 1998

- Witten: to put AdS/CFT at finite T , put a black hole in AdS_5 . Black hole in *global* AdS_{n+1} :

$$ds^2 = - \left(\frac{r^2}{R^2} + 1 - \frac{w_n M}{r^{n-2}} \right) dt^2 + \frac{dr^2}{\frac{r^2}{R^2} + 1 - \frac{w_n M}{r^{n-2}}} + r^2 d\Omega_{n-1}^2 .$$

- At $M = 0$, we obtain AdS_5 in global coord. ($w_n = \frac{8\pi G_N^{(n+1)}}{(n-2)\Omega_{n-1}}$.)
- Follow Schw. case, first: (outer) horizon r_+ is largest sol. of

$$\frac{r^2}{R^2} + 1 - \frac{w_n M}{r^{n-2}} = 0.$$

- Euclidean sol. near outer horizon ($\delta r = r - r_+$) is

$$ds^2 \simeq \left(\frac{2r_+}{R^2} + \frac{(n-2)w_n M}{r_+^{n-1}} \right) \delta r dt^2 + \frac{(d\delta r)^2}{\delta r \left(\frac{2r_+}{R^2} + \frac{(n-2)w_n M}{r_+^{n-1}} \right)} + r_+^2 d\Omega_2^2.$$

- Metric is free of conical singularities if the period in t is

$$\beta = \frac{4\pi}{\frac{2r_+}{R^2} + \frac{(n-2)w_n M}{r_+^{n-1}}} = \frac{4\pi}{\frac{nr_+}{R^2} + \frac{(n-2)}{r_+}} .$$

Then the temperature of the AdS_5 black hole is

$$T = \frac{nr_+^2 + (n-2)R^2}{4\pi R^2 r_+}$$

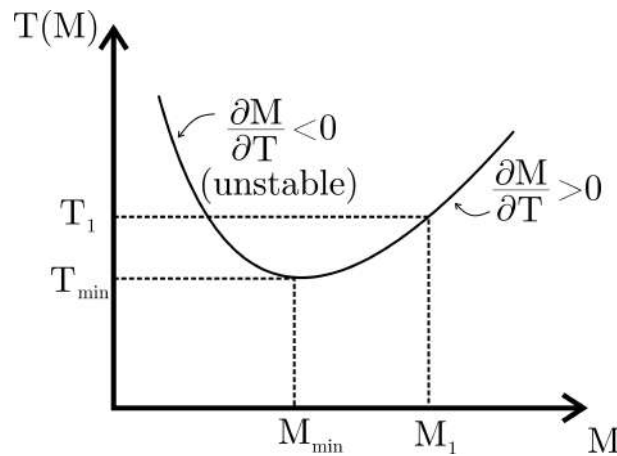
• From $T = T(r_+)$ and $r_+ = r_+(M)$, we get $T = T(M)$. From d/dM on the horizon eq., we find

$$\frac{dr_+}{dM} \left[\frac{nr_+^{n-1}}{R^2} + (n-2)r_+^{n-3} \right] = w_n ,$$

so $dr_+/dM > 0$. Therefore the min. of $T(M)$ is when $dT/dr_+ = 0$, giving

$$r_+ = R\sqrt{\frac{n-2}{n}} \Rightarrow T_{\min} = \frac{nr_+}{2\pi R} = \frac{\sqrt{n(n-2)}}{2\pi R}.$$

- Low M branch ($M < M(T_{\min})$) has $C = \partial M / \partial T < 0$, so is thermodynamically unstable (is a small enough perturbation of the flat space: Schwarzschild; black hole small w.r.t. AdS radius)
- High M branch ($M > M(T_{\min})$) has $C = \partial M / \partial T > 0$, so thermodynamically stable.



$T(M)$ for the AdS black hole. The lower M branch is unstable, having $\partial M/\partial T < 0$. The higher M branch has $\partial M/\partial T > 0$, and above T_1 it is stable.

• Need to check also that free energy, $F_{\text{BH}} < F_{\text{AdS}}$. Free energy is def. by $Z = e^{-F}$, but in gravitational theory

$$Z_{\text{grav}} = e^{-S},$$

where S =Euclidean gravity action. Then

$$S(\text{Euclidean action}) = \frac{F}{T},$$

so the comparison we need to do (and can prove) is

$$F_{\text{BH}} - F_{\text{AdS}} = T(S_{\text{BH}} - S_{\text{AdS}}) \Rightarrow T > T_1 \equiv \frac{n-1}{2\pi R} > T_{\text{min}}.$$

- One more problem: Metric at $r \rightarrow \infty$ is

$$ds^2 \simeq \left(\frac{r}{R} dt \right)^2 + \left(\frac{R}{r} dr \right)^2 + r^2 d\Omega_{n-1}^2 ,$$

and, while the transverse S^{n-1} has radius $r \rightarrow \infty$, only the *relative* distances matter in CFT at infinity, so r scales out, and boundary is $S^{n-1} \times S^1$, instead of $\mathbb{R}^{n-1} \times S^1$ (flat space with periodic Euclidean time). Then we must have

$$\frac{\frac{r}{R}}{\frac{1}{T}} = R \cdot T \rightarrow \infty \Rightarrow T \rightarrow \infty \Rightarrow M \rightarrow \infty ,$$

and we need to rescale coords. to get finite results. We find $M \propto r^n$ and $r^2 dt^2$ must be finite, so

$$r = \left(\frac{w_n M}{R^{n-2}} \right)^{1/n} \rho; \quad t = \left(\frac{w_n M}{R^{n-2}} \right)^{-1/n} \tau ,$$

and $M \rightarrow \infty$. Then $(dx_i = (w_n M / R^{n-2})^2 d\Omega_i)$

$$ds^2 = \left(\frac{\rho^2}{R^2} - \frac{R^{n-2}}{\rho^{n-2}} \right) d\tau^2 + \frac{d\rho^2}{\frac{\rho^2}{R^2} - \frac{R^{n-2}}{\rho^{n-2}}} + \rho^2 \sum_{i=1}^{n-1} dx_i^2 ,$$

- The temperature is found from the period of τ , now

$$\beta_1 = \frac{4\pi R}{n} \Rightarrow T = \frac{R}{\beta_1} = \frac{n}{4\pi}.$$

- For $n = 4$ dim., the metric becomes

$$ds^2 = \frac{\rho^2}{R^2} \left[\left(1 - \frac{R^4}{\rho^4} \right) d\tau^2 + R^2 d\vec{x}^2 \right] + R^2 \frac{d\rho^2}{\rho^2 \left(1 - \frac{R^4}{\rho^4} \right)},$$

and after $\frac{\rho}{R} = \frac{U}{U_0}$; $\tau = t \frac{U_0}{R}$; $\vec{x} = \vec{y} \frac{U_0}{R^2}$, and adding back in $R^2 d\Omega_5^2$, we find

$$ds^2 = \frac{U^2}{R^2} \left[-f(U) dt^2 + d\vec{y}^2 \right] + R^2 \frac{dU^2}{U^2 f(U)} + R^2 d\Omega_5^2$$

$$f(U) \equiv 1 - \frac{U_0^4}{U^4}.$$

- This is the nonextremal $AdS_5 \times S^4$ in Poincaré (!) coords. But that was near-horizon of D3-brane, so this can be obtained also as near-horizon of near-extremal D3-brane.

- Moreover, after $U/R = R/z$ and $U_0/R = R/z_0$ as usual, we find finite T version of Poincaré-AdS

$$ds^2 = \frac{R^2}{z^2} \left[-f(z) dt^2 + d\vec{y}^2 + \frac{dz^2}{f(z)} \right] + R^2 d\Omega_5^2$$

$$f(z) = 1 - \frac{z^4}{z_0^4},$$

with temperature $T = 1/(\pi z_0)$ (consistent with previous, $\beta_\tau = \pi R \Rightarrow \beta_t = \pi z_0$)

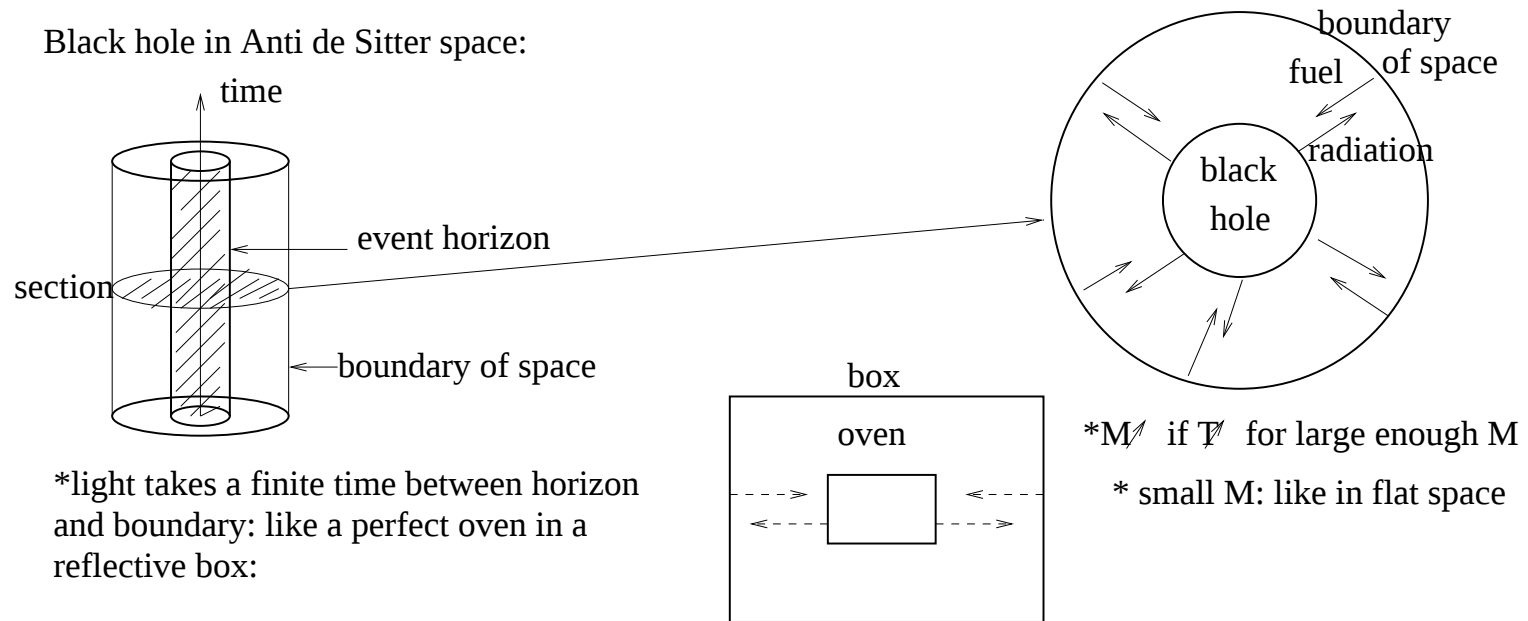
- So putting a large M black hole in AdS space $\leftrightarrow$ putting the boundary field theory at finite temperature.
- Interpretation: It takes radiation a finite time to get to ∞ and back $\Rightarrow$ radiation of black hole gets back in: stability (unlike flat space, where the time is infinite, so only BH is at finite T).

- In AdS-BH, susy is broken. Fermions antiperiodic around Euclidean time (which is an S^1), so we KK reduce $\mathcal{N} = 4$ SYM to 3d.
- Fermions are massive, gauge fields are massless (protected by gauge inv.), but scalars, classically massless, become massive through a quantum fermion loop. Then we have **3d pure glue theory** (A_μ^a)!
- Can we understand **mass gap in pure glue theory from AdS/CFT**? $\rightarrow$ spontaneous appearance of mass for quantum physical states in QFT $\rightarrow$ of classical physical states in AdS.
- Scalar field sol. to $\square\phi = 0$ can be put in factorized form

$$\phi(\rho, \vec{x}, \tau) = f(\rho)e^{i\vec{k}\cdot\vec{x}}.$$

- Horizon: bd. cond. that sol. is smooth, $df/d\rho = 0$ (horizon = like origin of plane in cyl. coords.). At $\rho \rightarrow \infty$, impose normalizability (state = physical), so $f \sim 1/\rho^4$. The 2 conditions give a quantization condition on parameters, $\vec{k}^2 = m^2 \Rightarrow$ discrete spectrum.
- Effective QM box between $r_{\text{horizon}} = r_{\text{min}}$ and $r = \infty$ (light takes a finite time) $\Rightarrow$ discrete modes $m_n \leftrightarrow$ masses of nonperturbative objects = glueballs in QFT. **Simplest model with a mass gap!**

- Black hole horizon at finite $r \Rightarrow$ cuts out a small cylinder from the middle of AdS $\rightarrow$ like the case of mass gap described before:



$\rightarrow$ Black hole can be in equilibrium with Anti de Sitter
 $\rightarrow$ fixed temperature inside Anti de Sitter
 Black hole in AdS $\Leftrightarrow$ fixed, finite temperature in flat 4 dimensional world (Witten, 1998)

$\mathcal{N} = 4$ SYM plasma from AdS/CFT

- Without dim. red. to 3d, $\mathcal{N} = 4$ SYM at finite T : very different from QCD. Nevertheless, find that various results in $\mathcal{N} = 4$ SYM at finite T are similar to QCD at finite T : universality?
- Brookhaven's RHIC and LHC's (CERN) ALICE, one obtains sQGP (strongly-coupled quark-gluon plasma). Even though dynamical and spatially bounded, we can use the previous methods to good approx. But they have also finite density ρ , finite chem. pot. μ , and magnetic B fields important, so need to describe.

Bulk properties: Entropy of 5d BH: Bekenstein-Hawking:

$$S = \frac{A}{4G_N},$$

and should equal QFT's entropy. But area of horizon, $A = \frac{R^3}{z_0^3} \int dy_1 dy_2 dy_3$ is ∞ , Therefore the entropy density is

$$s = \frac{S}{\int dy_1 dy_2 dy_3} = \frac{R^3}{4G_{N,5} z_0^3}.$$

• But $2\kappa_N^2 = 16\pi G_{N,10} = (2\pi)^7 g_s^2 \alpha'^4$ and in $AdS_5 \times S^5$, $R^4 = \alpha'^2 g_{YM}^2 N = \alpha'^2 (4\pi g_s) N$.

- Then reducing on an S^5 of radius R , with $\Omega_5 = \pi^3$ gives

$$G_{N,10} = \frac{\pi^4}{2N^2} R^8 \Rightarrow G_{N,5} = \frac{G_{N,10}}{\Omega_5 R^5} = \frac{\pi}{2N^2} R^3 \Rightarrow s_{\lambda=\infty} = \frac{\pi^2}{2} N^2 T^3.$$

- This is entropy density at ∞ coupling. From $\sigma = \partial P / \partial T$ and $\epsilon = -P + Ts$, we find

$$P_{\lambda=\infty} = \frac{\pi^2}{8} N^2 T^4, \quad \epsilon_{\lambda=\infty} = \frac{3\pi^2}{8} N^2 T^4,$$

at infinite coupling. But at weak coupling, one free bosonic d.o.f has $s = 2\pi^2 T/45$, and one free fermionic d.o.f. has $7/8$ of that. The for $\mathcal{N} = 4$ SYM (8 bosonic d.o.f and 8 fermionic d.o.f., all in adjoint of $SU(N)$), we have

$$s_{\lambda=0} = \left(8 + 8 \frac{7}{8}\right) (N^2 - 1) \frac{2\pi^2 T^3}{45} \simeq \frac{2\pi^2}{3} N^2 T^3,$$

so we obtain the ratios (for pressure, we use the same thermod. relations)

$$\frac{s_{\lambda=\infty}}{s_{\lambda=0}} = \frac{3}{4}, \quad \frac{P_{\lambda=\infty}}{P_{\lambda=0}} = \frac{\epsilon_{\lambda=\infty}}{\epsilon_{\lambda=0}} = \frac{3}{4}.$$

- "Experimentally" (in lattice QCD), we find 80%, close to the above 75%.

- **Energy loss: drag on heavy quarks** When a fast heavy quark (would-be "jet") passes through sQGP plasma, loses energy at high rate: "jet quenching": observed experimentally at RHIC and ALICE.

- AdS/CFT: heavy quark is long string stretching between the boundary at ∞ and interior of AdS. Moving heavy quark on straight path: string moving at ∞ on straight path. The other end: asymptotically to the horizon. Force against momentum loss keeps it at constant velocity.

- Ansatz: Boundary endpoint: $z = 0$ and $y(t, z) = vt + h(z)$. Static gauge $\sigma = z, \tau = t$, so NG action in AdS-BH is:

$$S = -\frac{R^2}{2\pi\alpha'} \left(\int dt \right) \int \frac{dz}{z^2} \sqrt{\frac{f(z) - v^2 + f(z)^2 h'^2(z)}{f(z)}}.$$

- $S = S[h'(z)] \Rightarrow$ canonical mom. is conserved:

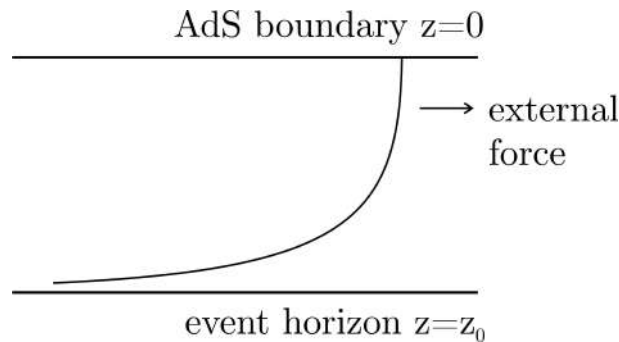
$$P_z^1 = \frac{\delta S}{\delta h'(z)} = \frac{\delta S}{\delta y^{1'}(z)} = -\frac{R^2}{2\pi\alpha'} \frac{1}{z^2} \frac{f^{3/2}(z) h'(z)}{\sqrt{f(z) - v^2 + f(z)^2 h'^2(z)}}.$$

- Solve for $h'(z)$. Then denom. and num. =0 at same time, so

$$h'^2(z) = \left(\frac{2\pi\alpha' P_z^1}{R^2} \right)^2 \frac{z^4}{f(z)^2} \frac{f(z) - v^2}{f(z) - \left(\frac{2\pi\alpha' P_z^1}{R^2} \right)^2 z^4} \Rightarrow P_z^1 = \pm \frac{R^2}{2\pi\alpha' z_0^2} \gamma v.$$

- Finally, integrating $h'(z)$, we find

$$h(z) = -\frac{vz_0}{2} \left[\operatorname{arctanh} \left(\frac{z}{z_0} \right) - \arctan \left(\frac{z}{z_0} \right) \right].$$



A quark being pulled by an external force at the boundary, and a string trailing behind it, hanging down from the boundary.

- Then momentum loss in the plasma is

$$\frac{dp}{dt} = -P_z^1 = -\frac{R^2}{2\pi\alpha'z_0^2}\gamma v.$$

- In $\mathcal{N} = 4$ SYM variables, $R^2/\alpha' = \sqrt{\lambda}$ and $T = 1/(\pi z_0)$, so ($p = M\gamma v$ is the heavy quark momentum, η_D is the drag coefficient)

$$\frac{dp}{dt} = -\frac{\pi}{2}\sqrt{\lambda}T^2\gamma v = -\frac{\pi}{2M}\sqrt{\lambda}T^2p \equiv -\eta_D p.$$

- Reasonable comparison to QCD, if $g_{YM}^2 \rightarrow g_{QCD}^2(\mu)$.

- **Adding finite chemical potential** $\mu \neq 0$. $\mathcal{N} = 4$ SYM charge = R-charge (for $SU(4) = SO(6)$ global symm.). Gravity side: sourced by A_μ^a .

- Coupling $\int d^d x J^\mu a_\mu$, a_μ = boundary value of A_μ . Then charge density J^0 couples to a_0 . Thus boundary condition for nonzero charge is

$$A = A_0(z)dt + \dots, \quad z \rightarrow 0$$

- But source coupling must be $q\mu$, so $A_0(z=0) = a_0 = \mu$, chemical potential of R-charge. So boundary cond. is $A \rightarrow \mu dt$ as $z \rightarrow 0$.

- Solution with nonzero gauge field at finite temperature: Reissner-Nordstrom-AdS. In Poincaré coords. we just change $f(z)$ to charged expression, and add the gauge field,

$$\begin{aligned} ds^2 &= \frac{R^2}{z^2} \left(-f(z)dt^2 + d\vec{x}^2 + \frac{dz^2}{f(z)} \right) \\ f(z) &= 1 - (1 + Kz_+^2\mu^2) \left(\frac{z}{z_+} \right)^d + Kz_+^2\mu^2 \left(\frac{z}{z_+} \right)^{2(d-1)} \\ K &\equiv \frac{(d-2)\kappa_{N,d+1}^2}{(d-1)g^2R^2} \\ A_0 &= \mu \left[1 - \left(\frac{z}{z_+} \right)^{d-2} \right]. \end{aligned}$$

- We cannot drop constant part of A_0 , since we need $A = 0$ at horizon $z = z_+$ (nonsingular A at horizon).

- Temperature of BH solution: like before, resulting in

$$T = \frac{1}{4\pi z_+} (d - K(d-2)z_+^2\mu^2) .$$

- Thermodynamic potential at constant μ (as opposed to constant charge density ρ), the grand-canonical potential $\Omega(\mu, V, T) = U - TS - \mu N$, is found from the same on-shell sugra action (always the thermod. pot. equals $S_{\text{on-shell}}$),

$$Z_{\text{CFT}} = e^{-\beta\Omega} = Z_{\text{sugra}} = e^{-S_{\text{sugra}}} \Rightarrow \Omega = TS_{\text{sugra}} .$$

- One finds

$$\Omega = -\frac{R^{d-1}}{2\kappa_N^2 z_+^d} (1 + Kz_+^2\mu^2) V_{d-1} .$$

- The charge density is a one-point function,

$$\rho = \langle J^0 \rangle = \left. \frac{\delta S_{\text{sugra}}}{\delta a_0} \right|_{a_0=0} .$$

- Alternative: keep fixed charge density ρ , so thermod. pot. is $F = \Omega + \mu Q$. Thus add a term linear in $\mu = a_0$ to $S_{\text{on-shell}}$: boundary term (h_{ab} = boundary metric)

$$+ \frac{1}{g^2} \int_{z \rightarrow 0} d^d x \sqrt{-h} n^a F_{ab} A^b ,$$

- So we keep $n^a F_{ab}$ fixed instead of A_a on boundary, hence the conjugate of $a_0 = \mu$, i.e., ρ .

- **Adding magnetic field** $B \neq 0$. Magnetic field in gauge theory is magnetic field in the bulk: Magnetic field: *gauge* the $U(1) \subset SU(4)$ symm. by coupling the current to a gauge field $\in J^\mu a_\mu$ (without kinetic term for a_μ : external magnetic field): same as $\int J^\mu a_\mu$ coupling to bulk.

- AdS_5 more complicated ($F = B dx^1 \wedge dx^2$ breaks isotropy), so show AdS_4 : generalize previous sol. (for $d = 3$) to $A = A_0(z)dt + B(z)xdy$. Then replace $f(z)$ with electric-magnetic duality inv.

$$\begin{aligned} f(z) &= 1 - [1 + K(z_+^2 \mu^2 + z_+^4 B^2)] \left(\frac{z}{z_+}\right)^3 + K(z_+^2 \mu^2 + z_+^4 B^2) \left(\frac{z}{z_+}\right)^4 \\ &= 1 - [1 + h^2 + q^2] \left(\frac{z}{z_+}\right)^3 + (h^2 + q^2) \left(\frac{z}{z_+}\right)^4, \end{aligned}$$

and add gauge field

$$A = \mu \left[1 - \frac{z}{z_+}\right] dt + Bxdy \Rightarrow F = \frac{1}{z_+^2 \sqrt{K}} [h dx \wedge dy + q dt \wedge dz].$$

- Both $\vec{E}$ and $\vec{B}$ finite at boundary, but $\vec{B}$ =external magnetic field, and $\vec{E}$ =source for charge density.

- Temperature and thermodynamical pot. (grand-canonical) are

$$\begin{aligned} T &= \frac{1}{4\pi} [3 - K(z_+^2 \mu^2 + z_+^4 B^2)] \\ \Omega &= -\frac{R^2}{2\kappa_N^2 z_+^3} [1 + K(z_+^2 \mu^2 - 3z_+^4 B^2)] V_2. \end{aligned}$$

Lecture 7

Solitons and probes in AdS/CFT

Instantons vs. D-instantons

- Instantons in G gauge theories: BPST $SU(2)$ instantons, embedded in G . YM action

$$\frac{1}{4g^2} \int d^4x (F_{\mu\nu}^a)^2 = \int d^4x \left[\frac{1}{4g^2} F_{\mu\nu}^a * F^{a\mu\nu} + \frac{1}{8g^2} (F_{\mu\nu}^a - *F_{\mu\nu}^a)^2 \right],$$

- is minimized on self-dual configurations, $F_{\mu\nu}^a = *F_{\mu\nu}^a$, which are real only in Euclidean space: BPS bound. But

$$n = \frac{g^2}{16\pi^2} \int d^4x \text{Tr} [F_{\mu\nu} * F^{\mu\nu}]$$

is a topological invariant = instanton nr. or Pontryagin index. [It's topological since $\text{Tr} [F_{\mu\nu} * F^{\mu\nu}] = 4 \text{Tr} [F \wedge F]$ and $\text{Tr} [F \wedge F] = d\mathcal{L}_{\text{CS}} = d \left[A dA + \frac{2}{3} A \wedge A \wedge A \right]$.]

- BPST instanton solution (quantum properties: 't Hooft) is

$$A_\mu^a = \frac{2}{g} \frac{\eta_{\mu\nu}^a (x - x_i)_\nu}{(x - x_i)^2 + \rho^2},$$

where η_{ij}^a is 't Hooft symbol, $\eta_{ij}^a = \epsilon^{aij}$, $\eta_{i4}^a = \delta_i^a$, $\eta_{4i}^a = -\delta_i^a$.
Then $-\frac{1}{2} \text{Tr} [F_{\mu\nu} F^{\mu\nu}] = \frac{48}{g^2} \frac{\rho^4}{[(x - x_i)^2 + \rho^2]^4} \Rightarrow S_{\text{inst}} = \frac{8\pi^2}{g^2}$.

- Euclidean instanton action gives transition probabilities, between static configurations at $x_4 = -\infty$ and $x_4 = +\infty$, = configs. of different winding numbers. Probability $\simeq e^{-S_{\text{inst}}}$.

• D-instantons in string theory: D(-1)-branes: Neumann bd. cd. in all directions. But D p -brane = source for A_{p+1} , so D-instanton = source for IIB axion scalar a . In flat space, $e^\phi = H_{-1}$ and $a - a_\infty = H_{-1}^{-1} - 1 \propto e^{-\phi} - 1/g_s$.

• D-instantons in $AdS_5 \times S^5$, near the boundary of AdS_5 ($x_0 = 0$) is found to be

$$e^\phi = g_s + \frac{24\pi}{N^2} \frac{x_0^4 \tilde{x}_0^4}{[\tilde{x}_0^2 + |\vec{x} - \vec{x}_a|^2]^4} + \dots$$

$$a = a_\infty + e^{-\phi} - \frac{1}{g_s}.$$

• The variation of the on-shell dilaton action gives

$$\delta S = -\frac{1}{2\kappa_5^2} \int d^4x \frac{R^3}{z^3} \delta\phi \partial_z \phi|_{z=0}.$$

• For the dilaton profile of the D-instanton, with $R^3 = \kappa_5^2 N^2 / (4\pi^2)$,

$$\begin{aligned} \frac{\delta S}{\delta\phi_0(\vec{x})} &= -\frac{48}{4\pi g_s} \frac{\tilde{z}^4}{[\tilde{z}^2 + |\vec{x} - \vec{x}_a|^2]^4} \\ &= \frac{1}{2g_{YM}^2} \langle \text{Tr} [F_{\mu\nu}^2(\vec{x})] \rangle, \end{aligned}$$

so with $4\pi g_s = g_{YM}^2$, we find the instanton bgr., with $\rho = \tilde{z}$,

$$-\frac{1}{2g_{YM}^2} \langle \text{Tr} [F_{\mu\nu}^2(\vec{x})] \rangle = \frac{48}{g_{YM}^2} \frac{\tilde{z}^4}{[\tilde{z}^2 + |\vec{x} - \vec{x}_a|^2]^4}.$$

Baryons as solitons via AdS/CFT

- Mesons in $SU(N_c)$ gauge theory with fundamental quarks: $M^{IJ} = \bar{q}_i^I q^{Ji}$ (group inv. δ_i^j). Baryons: with group invariant $\epsilon_{i_1 \dots i_N}$, in QCD ($N_c = 3$), ϵ_{ijk} :

$$B^{I_1 \dots I_N} = \epsilon_{i_1 \dots i_N} q^{I_1 i_1} \dots q^{I_N i_N}; \quad \text{QCD: } B^{IJK} = \epsilon_{ijk} q^{Ii} q^{Jj} q^{Kk}$$

- In $SU(N_c)$ gauge theory without quarks (like $\mathcal{N} = 4$ SYM), define *baryon vertex*, connection N external (heavy, non-dynamical) quarks, formally $\epsilon_{i_1 \dots i_N}$ above. The baryon vertex has (solitonic) energy, even in the presence of external quarks only.

- **Baryons as solitons in Skyrme model:** Low energy QCD: theory of pions $\vec{\pi}$ (Goldstone bosons for $SU(2)_A$ breaking); together with σ ("Higgs" for breaking), element in $SO(4) \simeq SU(2)_L \times SU(2)_R$ global symm. group:

$$U = \exp \left[\frac{i}{f_\pi} (\sigma + \vec{\pi} \cdot \vec{\tau}) \right] .$$

- Low energy QCD action in terms of $L_\mu = U^{-1} \partial_\mu U$:

$$\mathcal{L} = \mathcal{L}_{\text{kin}} + \mathcal{L}_{\text{int}} , \quad \mathcal{L}_{\text{kin}} = \frac{f_\pi^2}{4} \text{Tr} [L_\mu L^\mu] , \quad \text{e.g. of } \mathcal{L}_{\text{int}} = \frac{\epsilon^2}{4} \text{Tr} ([L_\mu, L_\nu]^2)$$

- Conserved topological current and charge:

$$B^\mu = \frac{1}{24\pi^2} \epsilon^{\mu\nu\rho\sigma} \text{Tr} [L_\nu L_\rho L_\sigma] , \quad B = \frac{1}{24\pi^2} \int d^3x \epsilon^{ijk} \text{Tr} [L_i L_j L_k].$$

- For small fields $\vec{\pi}$ we obtain an explicit map between $SU(2) \simeq SO(3)$ of group (index a) and $SO(3)$ of spatial rotations (index i), so B counts wrappings of the former on the latter:

$$B \simeq \frac{1}{12\pi^2 f_\pi^3} \epsilon^{ijk} \epsilon_{abc} \partial_i \pi^a \partial_j \pi^b \partial_k \pi^c + \dots ,$$

- Configuration with $B \neq 0$ = soliton, identified with baryon: hedgehog configuration,

$$U = \exp [iF(r)\vec{n} \cdot \vec{\tau}] , \quad n \equiv \frac{\vec{r}}{r}$$

- **Baryon as wrapped branes in AdS/CFT:** Strings ending on D-branes with $|i\bar{j}\rangle$ in $N \otimes \bar{N}$. External fundamental quark: long and massive: one end on a separated D-brane. In AdS/CFT: one end at infinity, the other in AdS.

- Baryon vertex: in AdS, place where N fundamental strings end $\Rightarrow$ must be a D-brane. In fact: D5-brane at a point in AdS_5 and wrapping S^5 . Indeed, it *needs* exactly N strings to end on it. But in AdS_5 , $\int_{S^5} d^5x \epsilon^{\mu_1 \dots \mu_5} \frac{F_{\mu_1 \dots \mu_5}^+}{2\pi} = N$, so from WZ term on D5-brane,

$$\frac{1}{2\pi} \int_{S^5 \times \mathbb{R}} d^6x \epsilon^{\mu_1 \dots \mu_6} A_{\mu_1} F_{\mu_2 \dots \mu_6}^+ = N \int_{\mathbb{R}_\tau} dx^\mu A_\mu ,$$

N units of A_μ charge on S^5 , that *need* to be absorbed by N strings. Also, vertex energy $\propto \frac{1}{g_s} \sim N$. OK!

Wilson loops in QCD and AdS/CFT

- QCD: dynamical quarks, but we can also consider external (very heavy) quarks. Yet only in gauge invariant combinations: N_c quarks + baryon vertex (before), or quark-antiquark (since QCD is *confining*). Very heavy quarks $\Rightarrow$ fixed: define contour. Observable: $q\bar{q}$ potential, $V_{q\bar{q}}(L)$.

- Define *Wilson line* = path ordered exponential in a gauge theory,

$$\Phi(y, x; P) = P \exp \left\{ i \int_x^y A_\mu(\xi) d\xi^\mu \right\} \equiv \lim_{n \rightarrow \infty} \prod_n e^{i A_\mu(\xi_n^\mu - \xi_{n-1}^\mu)} .$$

- Under an (Abelian or non-Abelian) gauge transf. with $\Omega = e^{i\chi(x)}$, it transforms as

$$\Phi(y, x; P) \rightarrow e^{i\chi(y)} \Phi(y, x; P) e^{-i\chi(x)} .$$

- It provides parallel transport along the curve, since, for a charged scalar field,

$$\phi(x) \rightarrow e^{i\chi(x)} \phi(x) \Rightarrow e^{i\chi(y)} (\Phi(y, x; P) \phi(x)) .$$

- For a closed path ($y = x$) AND taking the trace, the *Wilson loop* is gauge invariant and indep. on x , only on the curve C ,

$$W(C) = \text{Tr} \Phi(x, x; C) ,$$

due to cyclicity under the trace.

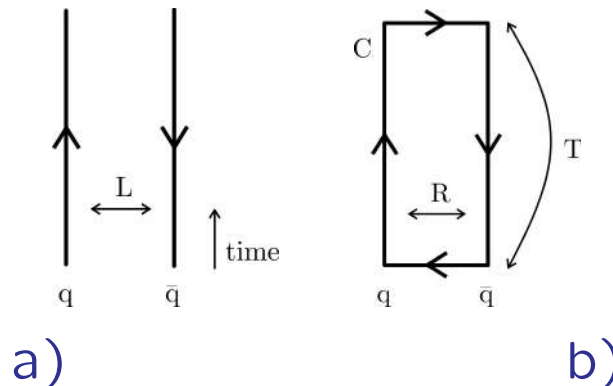
- In Abelian case, for $x = y$, we can put Φ in explicitly gauge inv. form,

$$\Phi_C = e^{i \int_{C=\partial\Sigma} A_\mu d\xi^\mu} = e^{i \int_\Sigma F_{\mu\nu} d\sigma^{\mu\nu}},$$

and in the non-Abelian case only $W[C]$ can be put, to first nontrivial order,

$$\Phi_{\square_{\mu\nu}} = e^{ia^2 F_{\mu\nu}} + O(a^4) \Rightarrow W_{\square_{\mu\nu}} = \frac{1}{N} \text{Tr} \{ \Phi_{\square_{\mu\nu}} \} = 1 - \frac{a^4}{2N} \text{Tr} \{ F_{\mu\nu} F_{\mu\nu} \} + O(a^6).$$

- Define the Wilson loop for the calc. of $q\bar{q}$ potential, a very long rectangle in the time direction (and short in the spatial one).



a) Heavy quark and antiquark staying at a fixed distance L . b) Wilson loop contour C for the calculation of the quark-antiquark potential.

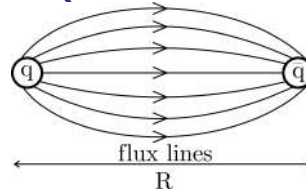
- Then from the VEV of the Wilson loop, as $T \rightarrow \infty$, extract $q\bar{q}$ potential,

$$\langle W(C) \rangle_0 \propto e^{-V_{q\bar{q}}(R)T}.$$

- Confining theory: constant force $\rightarrow$ linear potential,

$$V_{q\bar{q}}(R) \sim \sigma R,$$

σ = QCD string tension. QCD string = flux tube of constant cross section.



Between a quark and an antiquark in QCD, flux lines are confined: they live in a flux tube.

- For a conformal (scale inv.) theory, like QED, Coulomb potential,

$$V_{q\bar{q}}(R) \sim \frac{\alpha}{R},$$

- Then in a confining theory like QCD, *area law*,

$$\langle W(C) \rangle_0 \propto e^{-\sigma T \cdot R} = e^{-\sigma A(C)},$$

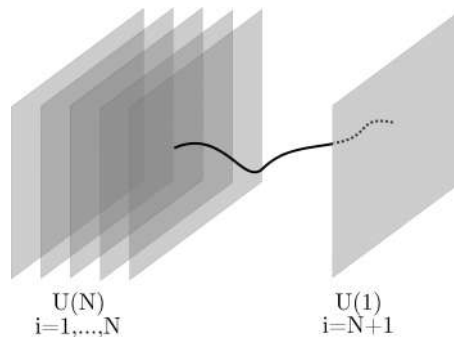
while in a conformal theory like QED, *conf. inv. result*, e.g.

$$\langle W(C) \rangle_0 \propto e^{-\alpha \frac{T}{R}}.$$

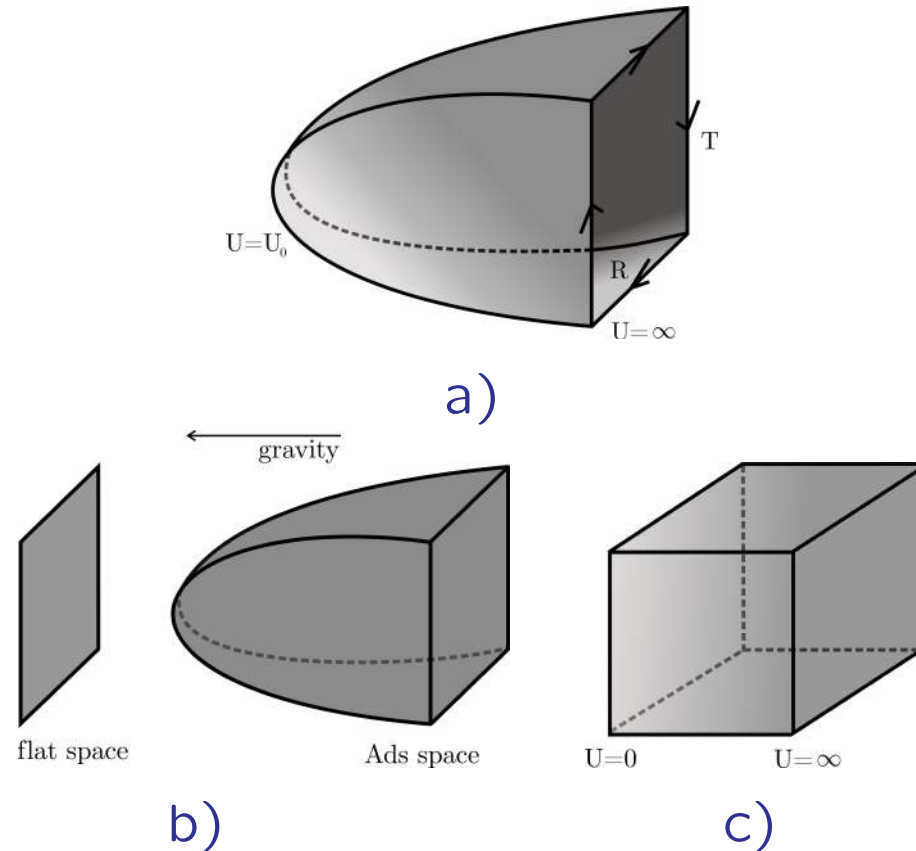
- Separate one D-brane to obtain $U(N)$ fundamental quarks, with Chan-Paton state $|N+1\rangle \otimes |i\rangle$.
- AdS/CFT: Mass $M = \frac{r}{2\pi\alpha'} = \frac{U}{2\pi}$. One end is at infinity (separated D-brane), one in AdS (the rest of the D-branes). Infinite mass: $U \rightarrow \infty$. From the point of view of $U(N+1)$ gauge theory, string is a "W boson" (vector field made massive by Higgsing to $U(N) \times U(1)$ via (bi-)fundamental scalar).
- For Wilson loop then, put Wilson contour C at infinity (boundary condition). String stretches inside AdS and forms a smooth surface. Indeed, there is a gravitational potential. Qualitatively, compare with Newtonian approx. to see that there is a potential leading to minimum U :

$$ds^2 = \alpha' \frac{U^2}{R^2/\alpha'} (-dt^2 + d\vec{x}^2) + \dots \leftrightarrow ds^2 = (1 + 2V)(-dt^2 + \dots).$$

So string at $U = \infty$ drops down to $U = U_0$, where it is held back by its tension.



One D-brane separated from the rest (N) D-branes acts as a probe on which the Wilson loop is located.



a) The Wilson loop contour C is located at $U = \infty$ and the string worldsheet ends on it and stretches down to $U = U_0$. b) In flat space, the string worldsheet would form a flat surface ending on C , but in AdS space 5 dimensional gravity pulls the string inside AdS. c) The free "W bosons" are strings that would stretch in all of the AdS space, from $U = \infty$ to $U = 0$, straight down, forming an area proportional to the perimeter of the contour C .

- But strings situated also on S^5 , with coordinates $\theta^I \leftrightarrow X^I$, scalars of $\mathcal{N} = 4$ SYM, transf. under $SO(6)$ R-symmetry. Then, string worldsheet is source for *susy generalized Wilson loop*,

$$W[C] = \frac{1}{N} \text{Tr} \, P \exp \left[\oint \left(i A_\mu \dot{x}^\mu + \theta^I X^I(x^\mu) \sqrt{\dot{x}^2} \right) d\tau \right] .$$

$x^\mu(\tau)$: loop, θ^I : on unit S^5 . We consider only $\theta^I = \text{const.}$: rectangular Wilson loop is 1/2 susy. (invariant under susy transf.). It is always *locally* susy,

$$\delta_{\text{susy}} W[C](x) \propto \left(i \delta A_\mu \dot{x}^\mu + \theta^I \delta X^I(x^\mu) \sqrt{\dot{x}^2} \right) = 0 ,$$

but *globally* susy only if variations at each point commute.

- Then the Maldacena prescription in sugra limit ($g_s \rightarrow 0$, $g_s N$ fixed and large) is derived

$$\langle W[C] \rangle = Z_{\text{string}}[C] = e^{-S_{\text{string}}[C]} ,$$

but the naive result is infinite, since U goes from ∞ to U_0 . But: must subtract the "free W boson" (no $\mathcal{N} = 4$ SYM interactions) mass term, from $U = \infty$ straight down (parallel to C) to $U = 0$. Then true prescription is

$$\langle W[C] \rangle = e^{-(S_\phi - l\phi)} .$$

• **Calculating the $q\bar{q}$ potential.** Contour C with q at $x = -L/2$ and $\bar{q}$ at $x = +L/2$, and (since $T \rightarrow \infty$) *approximate* worldsheet as time-translation inv. Then, using Euclidean $AdS_5 \times S^5$ with Euclidean NG string action, and static gauge $\sigma = x, \tau = t$, then single variable is $U(x)$, and the action becomes ($\tilde{R}^2 = R^2/\alpha'$)

$$S_{\text{string}} = \frac{1}{2\pi} T \int dx \sqrt{(\partial_x U)^2 + \frac{U^4}{\tilde{R}^4}}.$$

• Then implicit solution for $x = x(y, U_0), y = U/U_0$, and the corresponding $L/2 = x(\infty, U_0)$ is

$$x = \frac{\tilde{R}^2}{U_0} \int_1^{U/U_0} \frac{dy}{y^2 \sqrt{y^4 - 1}}, \quad \frac{L}{2} = \frac{\tilde{R}^2}{U_0} \int_1^\infty \frac{dy}{y^2 \sqrt{y^4 - 1}} = \frac{\tilde{R}^2}{U_0} \frac{\sqrt{2}\pi^{3/2}}{\Gamma(1/4)^2}.$$

• Finally we find

$$\begin{aligned} TV_{q\bar{q}}(L) &= S_\phi - l\phi = T \frac{2U_0}{2\pi} \left[\int_1^\infty dy \left(\frac{y^2}{\sqrt{y^4 - 1}} - 1 \right) - 1 \right] \\ &= -T \cdot \frac{4\pi^2}{\Gamma(1/4)^4} \frac{\sqrt{2g_{YM}^2 N}}{L}. \end{aligned}$$

so a nonperturbative result $\propto \sqrt{g_{YM}^2 N}$.

• Nonsusy (regular) Wilson loop (Alday+Maldacena, 2007): same prescription, except Neumann bd. cond. on S^5 , instead of Dirichlet.

Brane probes in AdS/CFT: We have seen that some solitonic branes, like D5 wrapped on S^5 , correspond to solitons in QFT.

- But a single moving brane can probe the QFT. If motion in U : can probe different energies (in non-conformal gauge/gravity duality). Brane excitations (fluctuations): meson spectra: massless scalar = pion, massive vectors: vector mesons, etc.
- Also baryons can be understood, but as solitonic solutions on the probe brane.
- Then motion in U : (Hamiltonian motion on the) RG flow, for the various hadrons.

"Hard-wall" model for QCD and Polchinski-Strassler scenario for scattering

- QCD in the UV: approx. conformal $\Rightarrow$ gravity dual should be approx. $AdS_5 \times X^5$ at large r (fifth dimension). Modified in some way at small r . Simplest: hard cut-off at $r = r_{\min}$ (space terminates).

$$\begin{aligned} ds^2 &= \frac{r^2}{R^2} d\vec{x}^2 + R^2 \frac{dr^2}{r^2} + R^2 ds_X^2 \\ &= e^{-2y/R} d\vec{x}^2 + dy^2 + R^2 ds_X^2. \end{aligned}$$

- Momenta $p_i = -i\partial_i$ in QCD are related to 10d momenta $\tilde{p}_i$ by

$$\tilde{p}_\mu = \frac{R}{r} p_\mu.$$

- A characteristic mom. scale in 10d is $\tilde{p} \sim 1/R$, so QCD mom. $p \sim r/R^2$. But the characteristic QCD momentum scale is Λ_{QCD} , so

$$r_{\min} = R^2 \Lambda_{\text{QCD}}.$$

- Scattering in this "hard-wall" model: AdS fields (states): glueballs or mesons/baryons, coupling to glueball operators in QFT.

- Wavefunction for glueball $e^{ik \cdot x}$ mapped to gravity state wavefunction in $AdS_5 \times X^5$,

$$\Phi = e^{ik \cdot x} \times \Psi(\rho, \vec{\Omega}_{X_5}).$$

- Assume gravitational states scatter locally as in flat space, ansatz for scattering (Polchinski+Strassler, 2001):

$$\mathcal{A}_{\text{QCD}}(p_i) = \int dr d^5 \Omega_{X_5} \sqrt{-g} \mathcal{A}_{\text{string}}(\tilde{p}_i) \prod_i \Psi_i(r, \vec{\Omega}_{X_5}).$$

- Define a "QCD string scale" $\hat{\alpha}'$ in hard-wall model,

$$\hat{\alpha}' = (g_{YM}^2 N)^{-1/2} \Lambda^{-2}, \text{ such that } \sqrt{\alpha'} \tilde{p} = \sqrt{\hat{\alpha}'} p \left(\frac{r_{\min}}{r} \right) \leq \sqrt{\hat{\alpha}'} p.$$

- Also, since $M_{\text{Pl}}^8 \sim 1/(g_s^2 \alpha'^4)$, we define a "QCD Planck scale",

$$\hat{M}_{\text{Pl}} = g_s^{-1/4} \hat{\alpha}'^{-1/2} = g_{YM}^{-1/2} \Lambda (g_{YM}^2 N)^{1/4} = N^{1/4} \Lambda.$$

- **Regge behaviour:** When $s \gg -t > 0$, gauge theories expected to have *Regge behaviour*,

$$\mathcal{A}(s, t) \simeq \beta(t) s^{\alpha(t)}, \quad \alpha(t) = \alpha_0 + \frac{\hat{\alpha}'}{2} t.$$

- But string $2 \rightarrow 2$ flat space amplitude (Virasoro-Shapiro),

$$\mathcal{A}_{\text{string}} = g_s^2 \alpha'^3 \left[\prod_{x=s,t,u} \frac{\Gamma(-\alpha' \tilde{x}/4)}{\Gamma(1 + \alpha' \tilde{x}/4)} \right] K(\sqrt{\alpha'} \tilde{p})$$

becomes in the Regge limit $\alpha' s \gg 1, \alpha' |t|$ fixed, of the same Regge form,

$$\mathcal{A}_{\text{string}}(s, t) \simeq g_s^2 \alpha'^3 [\text{polariz. tensors}] (\alpha' s)^{\alpha' t/2 + 2} \frac{\Gamma(-\alpha' t/4)}{\Gamma(1 + \alpha' t/4)}.$$

- But, doing the PS integral in the "hard-wall" of the Regge limit, the integral is dominated by lowest $r = r_{\min}$, so by the integrand there = flat space amplitude. So: QCD has Regge behaviour:

$$\mathcal{A}_{\text{QCD}}(p) \sim \beta(t) (\hat{\alpha}' s)^{2 + \hat{\alpha}' t/2}.$$

Gravitational shockwave scattering as a model of QCD high energy scattering

- High energy scattering in gravity: particles become grav. shockwaves of "parallel plane" (pp) type. In flat space: Aichelburg-Sexl:

$$ds^2 = 2dx^+ dx^- + H(x^+, x^i)(dx^+)^2 + \sum_i dx_i^2 ,$$

which is an *exact* solution to Einstein's equations with a massless pointlike source, reducing to

$$R_{++} = -\frac{1}{2}\partial_i^2 H(x^+, x^i) = 8\pi G T_{++} = p\delta^{d-2}(x^i)\delta(x^+).$$

Then the solution is

$$H(x^+, x^i) = \delta(x^+)\Phi(x^i) , \quad \partial_i^2 \Phi(x^i) = -16\pi G_{N,d} p \delta^{d-2}(x^i) ,$$

leading in flat space to

$$\begin{aligned} \Phi &= -4G_{N,4} \ln \rho^2 , \quad (d=4) \\ \Phi &= \frac{16\pi G_{N,d}}{\Omega_{d-3}(d-4)} \frac{p}{\rho^{d-4}} , \quad (d>4). \end{aligned}$$

- Note that for pp wave solutions, $\nexists$ α' corrections: all R^2 corr. vanish on-shell, so: Exact string solutions!
- Then for scattering with $G_N s \sim 1$, but $G_N s < 1$, we have one particle an A-S wave, the other moving in it. For $G_N s > 1$, both particles are A-S waves. *Before* collision,

$$ds^2 = 2dx^+ dx^- + dx_i^2 + (dx^+)^2 \Phi_1(x^i) + (dx^-)^2 \Phi_2(x^i).$$

- Then (Penrose, at $b = 0$, Eardley+Giddings at $0 < b \leq b_{\max}$), a "marginally trapped surface" forms at collision point $x^+ = x^- = 0$ (Schwarzschild BH: $r_H = 2MG_N$ is a marginally trapped surface), so (GR theorem): BH must form in the future of the collision. Then, BH formation in collision, with $\sigma_{\text{BH}} \geq \pi b_{\max}^2$.
- In hard-wall model (cut-off $AdS_5 \times X^5$: curved space): same mechanism. Then, use PS formula to related to QCD. But: need gravity amplitude corresponding to $b_{\max}$. This is obtained in the *black disk eikonal approx.*, $S = e^{i\delta}$, with $\text{Re}[\delta(b, s)] = 0$, $\text{Im}[\delta(b, s)] = 0$ for $b > b_{\max}$, $\text{Im}[\delta(b, s)] = \infty$ for $b \leq b_{\max}$. Then the amplitude is

$$\begin{aligned} \frac{1}{s} \mathcal{A}(s, t) &= -i \int d^2 b e^{i\vec{q} \cdot \vec{b}} (e^{i\delta} - 1) = i \int_0^{b_{\max}(s)} b db \int_0^{2\pi} d\theta e^{iqb \cos \theta} \\ &= 2\pi i \frac{b_{\max}(s)}{\sqrt{t}} J_1 \left(\sqrt{t} b_{\max}(s) \right). \end{aligned}$$

- Energy regime in gravity dual ("hard-wall") of QCD: $\Lambda_{\text{QCD}} < (\hat{\alpha}_s)^{-1/2} = (g_{YM}^2 N)^{1/2} \Lambda_{\text{QCD}} < \hat{M}_P = N^{1/4} \Lambda_{\text{QCD}}$. For $\sqrt{s} > \hat{M}_P$, in dual we create small black holes. But for $E > E_R = M_P^8 R^7$ in gravity dual, so $\sqrt{s} > \hat{E}_R = N^2 \Lambda_{\text{QCD}}$ in QCD, BH of size larger than R_{AdS} .

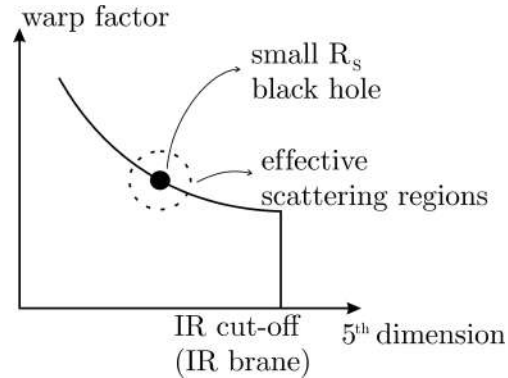
- Yet $\exists$ higher energy scale in QCD, depending on the details of the gravity dual: $E_F \leftrightarrow \hat{E}_F$, such that BH is effectively on the IR cut-off r_{min} itself. Then, in QCD, *Froissart unitarity bound*,

$$\sigma_{\text{tot}} \leq C \ln^2 \frac{s}{s_0}; \quad C \leq \frac{\pi}{m_\pi^2},$$

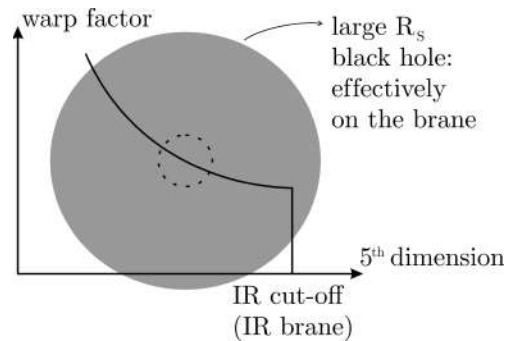
where m_π is lowest energy state in theory.

- Describe this via collision of 2 (A-S-type) gravitational shock-waves on the IR cut-off = IR brane, in a symmetrical situation (IR brane position acts as a pion field, see before),

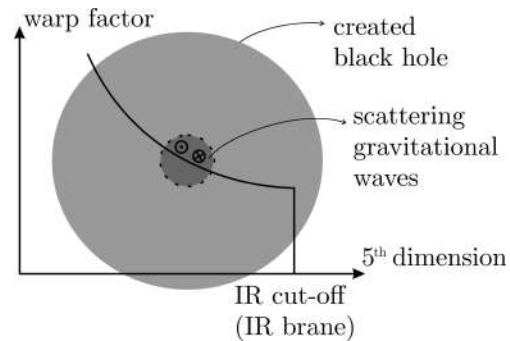
$$ds^2 = e^{\frac{2|y|}{R}} d\vec{x}^2 + dy^2 + R^2 ds_X^2.$$



a)



b)



c)

a) At small enough energies, the created black hole is small, and fluctuates (is created at a random point) inside a small region of effective scattering. b) At large enough energies, the created black hole is so large, that it is effectively fixed (has small fluctuations) and it looks like it sits on the IR brane. c) At these large energies, the process is effectively classical: two shockwaves going in opposite directions scatter creating a black hole larger than the scattering region.

- The shockwave profile is found to be (M_1 is the first KK mode when reducing 5d theory onto IR brane, $R_s = G_{N,4}\sqrt{s}$)

$$\Phi(r, y) = \frac{4G_{N,d+1}p}{(2\pi)^{\frac{d-4}{2}}} \frac{e^{-\frac{d|y|}{2R}}}{r^{\frac{d-4}{2}}} \int_0^\infty dq q^{\frac{d-4}{2}} J_{\frac{d-4}{2}}(qr) \frac{I_{d/2}\left(e^{-\frac{|y|}{R}}Rq\right)}{I_{d/2-1}(Rq)}$$

$$\Phi(r, y=0) \simeq R_s \sqrt{\frac{2\pi R}{r}} C_1 e^{-M_1 r}, \quad M_1 = \frac{j_{1,1}}{R}.$$

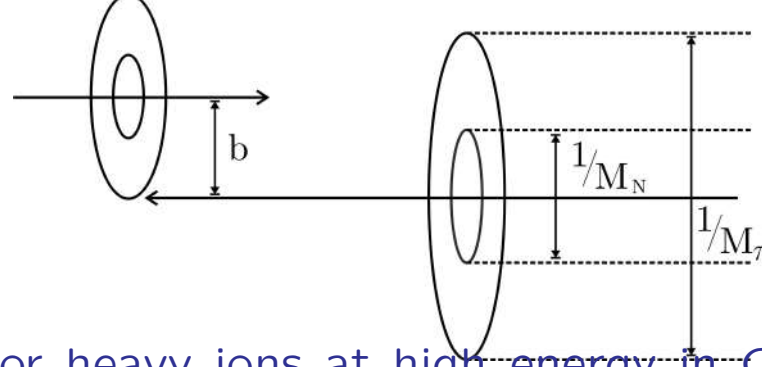
- $\exists$ Exact analysis, but simple arg.: $\Phi \sim \sqrt{s} e^{-M_1 r}$, so for 2 A-S waves at impact parameter b , emitted energy $\propto \sqrt{s} e^{-M_1 r}$. Minimum energy = $\hat{M}_P$, reached at $b_{\max}$, so

$$b_{\max} \sim \frac{1}{M_1} \ln \frac{s}{\hat{M}_{\text{PI}}} \Rightarrow \sigma_{\text{tot}} = \pi b_{\max}^2 \sim \frac{\pi}{M_1^2} \ln^2 \frac{s}{\hat{M}_{\text{PI}}}.$$

- Description matches 1952 Heisenberg model for nucleon-nucleon high energy collisions. $\Phi \rightarrow$ pion wavefunction (indeed, IR brane position = pion field). But, Heisenberg: pion field overlap $\sim e^{-\mu_\pi b}$, so emitted energy $\sim \sqrt{s} e^{-m_\pi b}$. $b_{\max}$: when emitted energy = average per pion emitted energy $\langle E_0 \rangle$, which is $\simeq$ constant only for DBI action (action of IR brane), whereas for canonical scalar with polynomial V , $\langle E_0 \rangle \propto \sqrt{s}$. Then

$$\sqrt{s} e^{-m_\pi b_{\max}} = \langle E_0 \rangle \Rightarrow b_{\max} = \frac{1}{m_\pi} \ln \frac{\sqrt{s}}{\langle E_0 \rangle} \Rightarrow$$

$$\sigma_{\text{tot}} = \pi b_{\max}^2 = \frac{\pi}{m_\pi^2} \ln^2 \frac{\sqrt{s}}{\langle E_0 \rangle}.$$



Scattering of nuclei or heavy ions at high energy in QCD described by the Heisenberg model.

- Lower bound on entropy formed in collisions:

$$A_{\text{ev.hor.}} \geq A_{\text{marg.trap.}} \Rightarrow S_{\text{emitted}} = S_{\text{BH}} = \frac{A_{\text{ev.hor.}}}{4G_{N,5}} \geq \frac{A_{\text{marg.trap.}}}{4G_{N,5}}.$$

- Gubser et al.: shockwave with $T_{++} = E\delta(x^+)\delta(z-R)\delta^{d-2}(x^i)$,

$$ds^2 = \frac{R^2}{z^2} (2dx^+dx^- + (dx^1)^2 + (dx^2)^2 + dz^2) + \frac{R}{z} \Phi(x^1, x^2, z) \delta(x^+) (dx^+)^2 \Rightarrow$$

$$\Phi(x^1, x^2, z) = \frac{2G_{N,5}E}{R} \frac{1 + 8q(1+q) - 4\sqrt{q(1+q)}(1+2q)}{\sqrt{q(1+q)}}, \quad q \equiv \frac{(x^1)^2 + (x^2)^2 + (z-R)^2}{4zR}.$$

- Treating $\delta g_{ij} = R/z\Phi(x^1, x^2, z)\delta(x^+)$ as a perturbation,

$$\langle T_{ij}(\vec{x}) \rangle = \frac{R^3}{4\pi G_{N,5}} \lim_{z \rightarrow 0} \frac{1}{z^4} \delta g_{ij} = \frac{2R^4 E}{\pi [R^2 + (x^1)^2 + (x^2)^2]^3} \delta(x^+).$$

Lecture 8

The pp wave correspondence and spin chains

The Penrose limit and pp waves

- PP waves: Linearized solution is exact! Only nontrivial Ricci is $R_{++} = -\frac{1}{2}\partial_i^2 H(x^+, x^i)$. For shockwaves, $T_{++} \propto \delta(x^+)$, so $H(x^+, x^i) = \delta(x^+)\Phi(x^i)$, but here: waves not localized in x^+ .
- In supergravity: 11d sugra, pp wave solutions with

$$F_4 = dx^+ \wedge \phi : F_{(4)+\mu_1\mu_2\mu_3} = \phi_{(3)\mu_1\mu_2\mu_3}$$

$$d\phi = 0, \quad d*\phi = 0, \quad -\partial_i^2 H = |\phi|^2.$$

- For $H = \sum_{ij} A_{ij}x^i x^j$, $-2\text{Tr} A = |\phi|^2$, we have solutions with 1/2 susy, but there is a unique sol. with ALL susy,

$$H = \sum_{i,j} A_{ij}x^i x^j = - \sum_{i=1,2,3} \frac{\mu^2}{9} x_i^2 - \sum_{i=4}^9 \frac{\mu^2}{36} x_i^2$$

$$\phi = \mu dx^1 \wedge dx^2 \wedge dx^3.$$

- In 10d IIB sugra, pp wave solutions with

$$F_5 = dx^+ \wedge (\omega + *\omega) : F_{+\mu_1\dots\mu_4} = \omega_{\mu_1\dots\mu_4}; \quad F_{+\mu_5\dots\mu_8} = \omega_{\mu_5\dots\mu_8}$$

$$H = \sum_{ij} A_{ij}x^i x^j; \quad \phi = \phi_0, \quad d\omega = 0, \quad d*\omega = 0, \quad \partial_i^2 H = -|\omega|^2.$$

- Again, sols. have 1/2 susy, but $\exists!$ sol. with full susy,

$$H = -\frac{\mu^2}{64} \sum_i x_i^2; \quad \omega = \frac{\mu}{2} dx^1 \wedge dx^2 \wedge dx^3 \wedge dx^4.$$

• **Penrose limit:** Penrose theorem: near a null geodesic in any metric, the spacetime becomes a pp wave. Null geodesic defined by $V = Y^i = 0, U = \tau$, we can always put the metric in form (Penrose):

$$ds^2 = dV \left(dU + \alpha dV + \sum_i \beta_i dY^i \right) + \sum_{ij} C_{ij} dY^i dY^j ,$$

where U, V are lightcone coords., and take the limit

$$U = u; \quad V = \frac{v}{R^2}; \quad Y^i = \frac{y^i}{R}; \quad R \rightarrow \infty ,$$

to obtain a pp wave in u, v, y^i , but in Rosen coordinates,

$$ds^2 = 2dudv + g_{ij}(u)dy^i dy^j , \quad g_{ij}(u) = C_{ij}(U = u, V = 0, Y^i = 0).$$

• To go to the standard Brinkmann coordinates form, write $g_{ij}(u) = e_i^a(u)e_j^b(u)\delta_{ab}$, then

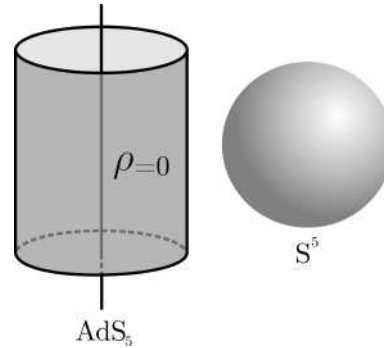
$$u = x^+ , \quad v = x^- + \frac{1}{2}\ddot{e}_{ai}e_b^i x^a x^b , \quad y^i = e_a^i x^a ,$$

then obtain $A_{ab} = \ddot{e}_{ai}e_b^i$. Interpretation of Penrose limit: boost along direction x , while taking the overall scale of metric to infinity:

$$\begin{aligned} t' &= \cosh \beta \ t + \sinh \beta \ x; & x' &= \sinh \beta \ t + \cosh \beta \ x \Rightarrow \\ x' - t' &= e^{-\beta}(x - t); & x' + t' &= e^{\beta}(x + t) , \end{aligned}$$

then scale all coords. by $1/R$ and identify $e^{\beta} = R \rightarrow \infty$.

- Penrose limit of $AdS_5 \times S^5$: boost along an equator of S^5 defined by $\theta = 0$ and stay at center of AdS_5 at $\rho = 0$ (is a null geodesic).



Null geodesic in $AdS_5 \times S_5$ for the Penrose limit giving the maximally supersymmetric wave. It is in the center of AdS_5 , at $\rho = 0$, and on an equator of S_5 , at $\theta = 0$.

$$\begin{aligned}
 ds^2 &= R^2 \left(-\cosh^2 \rho \, d\tau^2 + d\rho^2 + \sinh^2 \rho \, d\Omega_3^2 \right) + R^2 \left(\cos^2 \theta \, d\psi^2 + d\theta^2 + \sin^2 \theta \, d\Omega_3'^2 \right) \\
 &\simeq R^2 \left[- (1 + \rho^2) \, d\tau^2 + d\rho^2 + \rho^2 d\Omega_3^2 \right] + R^2 \left[(1 - \theta^2) \, d\psi^2 + d\theta^2 + \theta^2 d\Omega_3'^2 \right].
 \end{aligned}$$

- Then define null coords. $\tilde{x}^\pm = (\tau \pm \psi)/\sqrt{2}$, and rescale to obtain the pp wave,

$$\begin{aligned}
 \tilde{x}^+ &= x^+; \quad \tilde{x}^- = \frac{x^-}{R^2}; \quad \rho = \frac{r}{R}; \quad \theta = \frac{y}{R} \Rightarrow \\
 ds^2 &= -2dx^+ dx^- - \mu^2 (\vec{r}^2 + \vec{y}^2) (dx^+)^2 + d\vec{y}^2 + d\vec{r}^2.
 \end{aligned}$$

Penrose limit of AdS/CFT: large R-charge Berenstein, Maldacena, Nastase, 2002

- $E = i\partial_\tau$ in global AdS_5 , and $J = -i\partial_\psi$ (for rot. $X^5 \leftrightarrow X^6$).
- But $E \leftrightarrow \Delta$ and $J \leftrightarrow U(1) \subset SU(4) = SO(6)$ R-charge rotating $X^5 \leftrightarrow X^6$.
- Penrose limit

$$p^- = -p_+ = i\partial_{x^+} = i\partial_{\tilde{x}^+} = \frac{i}{\sqrt{2}}(\partial_\tau + \partial_\psi) = \frac{1}{\sqrt{2}}(\Delta - J)$$

$$p^+ = -p_- = i\partial_{x^-} = i\frac{\partial_{\tilde{x}^-}}{R^2} = \frac{i}{\sqrt{2}R^2}(\partial_\tau - \partial_\psi) = \frac{\Delta + J}{\sqrt{2}R^2}.$$

- Rescale p^- by $\mu\sqrt{2}$ and p^+ by $1/\mu\sqrt{2}$:

$$\frac{p^-}{\mu} = \Delta - J; \quad 2\mu p^+ = \frac{\Delta + J}{R^2}.$$

- For string theory on pp wave, p^+, p^- finite, so as $R \rightarrow \infty$, keep $\Delta - J$ and $(\Delta + J)/R^2$ fixed, so $\Delta \simeq J \sim R^2 \rightarrow \infty$. Thus *Penrose limit is large R-charge limit in AdS/CFT!*

- In $\mathcal{N} = 4$ SYM, $\frac{R^2}{\alpha'} = \sqrt{4\pi g_s N} = \sqrt{g_{YM}^2 N}$, so for g_s fixed, we have $J \sim R^2 \sim \sqrt{N}$, so

$$\frac{J}{\sqrt{N}} = \text{fixed} \quad \text{and} \quad \frac{g_{YM}^2 N}{J^2} = \text{fixed}.$$

String quantization and Hamiltonian on pp wave

- Polyakov action on pp wave ($x^i = (\vec{r}, \vec{y})$)

$$S = -\frac{1}{2\pi\alpha'} \int_0^l d\sigma \int d\tau \frac{1}{2} \sqrt{-\gamma} \gamma^{ab} [-2\partial_a x^+ \partial_b x^- - \mu^2 x_i^2 \partial_a x^+ \partial_b x^+ + \partial_a x^i \partial_b x^i].$$

- In conf. gauge, $\sqrt{-\gamma} \gamma^{ab} = \eta^{ab}$, light-cone gauge $x^+(\sigma, \tau) = \tau$ (rescale τ by $\alpha' p^+$), and then $l = 2\pi\alpha' p^+$,

$$S = -\frac{1}{2\pi\alpha'} \int d\tau \int_0^{2\pi\alpha' p^+} d\sigma \left[\frac{1}{2} (-(\dot{x}^i)^2 + (x'^i)^2) + \frac{\mu^2}{2} x_i^2 \right].$$

- The equations of motion and solutions are

$$(-\partial_\tau^2 + \partial_\sigma^2) x^i - \mu^2 x^i = 0. \Rightarrow x^i \propto e^{-i\omega_n \tau + i k_n \sigma}, \quad \omega_n^2 = k_n^2 + \mu^2.$$

- In flat space $\mu = 0$, $\omega_n = k_n = n$, but now we rescaled by $\alpha' p^+$, so

$$\omega_n = \sqrt{\mu^2 + \frac{n^2}{(\alpha' p^+)^2}}.$$

- Light-cone Hamiltonian $H_{\text{l.c.}} = p^-$ has no 0-modes p^i (x^i massive), so

$$H = \sum_{n \in \mathbb{Z}} N_n \omega_n, \quad N_n = \sum_i a_n^{i\dagger} a_n^i + \sum_\alpha b_n^{\alpha\dagger} b_n^\alpha.$$

- As for flat space, transl.inv.: $P = \sum_n n N_n = 0$. In $\mathcal{N} = 4$ SYM, $E/\mu = \Delta - J, 2\mu p^+ \simeq 2J/R^2$, so

$$(\Delta - J)_n = \frac{\omega_n}{\mu} = \sqrt{1 + \frac{g_{YM}^2 N n^2}{J^2}}.$$

String states from $\mathcal{N} = 4$ SYM; BMN operators

- Vacuum: $E = 0$, so $\Delta - J = 0$. Oscillators at $g_{YM} = 0$: $\Delta - J = 1$. Construct operators out of fields with $\Delta - J = 1$, on top of operator with $\Delta - J = 0$.
- Field with $\Delta = J = 1$: $Z = \Phi^5 + i\Phi^5$: unique! (charged under J). ($\bar{Z}$ has $\Delta = -J = 1$, so $\Delta - J = 2$).
- Fields with $J = 0$ and $\Delta = 1$ (so $\Delta - J = 1$): Φ^m , $m = 1, \dots, 4$ and $D_\mu Z = \partial_\mu Z + [A_\mu, Z]$ (bosonic) and $\chi_{J=+1/2}^a$ (fermionic, 8 comps.; other 8: $\chi_{J=-1/2}^a$).

- Vacuum, with $\mu p^+ = J/R^2$:

$$|0, p^+\rangle = \frac{1}{\sqrt{J} N^{J/2}} \text{Tr} [Z^J].$$

- Oscillators with $n = 0$ (BPS operators, with $\Delta - J$ indep. of g_{YM}), obtained by inserting $a_{0,r}^\dagger = \Phi^r = (D_\mu Z, \Phi^m)$ or $b_{0,b}^\dagger = \chi_{J=-1/2}^a$ in it, e.g.

$$a_{0,r}^\dagger b_{0,b}^\dagger |0, p^+\rangle = \frac{1}{N^{J/2+1} \sqrt{J}} \sum_{l=1}^J \text{Tr} [\Phi^r Z^l \psi_{J=1/2}^b Z^{J-l}].$$

- Excited levels ($n \geq 1$): add momentum wavefunction $e^{\frac{2\pi i n x}{L}}$ around the closed string, so, e.g. $a_{n,4}^\dagger$ insertion is

$$a_{n,4}^\dagger |0, p^+\rangle = \frac{1}{\sqrt{J}} \sum_{l=1}^J \frac{1}{\sqrt{J} N^{J/2+1/2}} \text{Tr} [Z^l \Phi^4 Z^{J-l}] e^{\frac{2\pi i n l}{J}}.$$

- But this vanishes by cyclicity. Nonzero: at least two excitations, so that $P = \sum_n n N_n = 0$, e.g.

$$a_{n,4}^\dagger a_{-n,3}^\dagger |0, p^+\rangle = \frac{1}{\sqrt{J}} \sum_{l=1}^J \frac{1}{N^{J/2+1}} \text{Tr} [\Phi^3 Z^l \Phi^4 Z^{J-l}] e^{\frac{2\pi i n l}{J}}.$$

- These are "BMN operators". Study "dilute gas approx.": few "impurities" among Z 's.

Discretized string action from $\mathcal{N} = 4$ SYM

- Operator-state correspondence in CFT: $z_{a_1 \dots a_m}^{(m)} \sim \partial_{a_1} \dots \partial_{a_m} Z$ on $\mathbb{R}^4 \rightarrow$ KK states on $\mathbb{R}_t \times S^3$, so constant Z : energy 1 = harmonic osc. of $\omega = 1$, with cr.op. $(b^\dagger)^i_j$. Similarly, for Φ , $(a^\dagger)^i_j$, so states

$$|a_l^\dagger\rangle \equiv \text{Tr} \left[(b^\dagger)^l a^\dagger (b^\dagger)^{J-l} \right] |0\rangle.$$

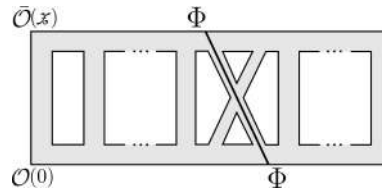
- Interacting Hamiltonian

$$H_{\text{int}} = -g_{YM}^2 \text{Tr} \sum_{I>J} \left\{ [\Phi^I, \Phi^J] [\Phi_I, \Phi_J] \right\},$$

has then term that can act on operators $\mathcal{O}$,

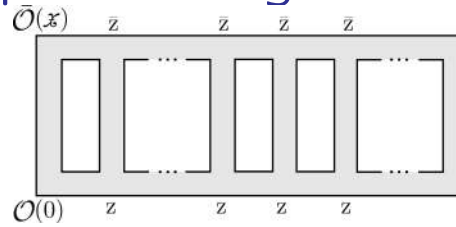
$$H_{\text{int}} = -g_{YM}^2 \text{Tr} \{ [Z, \Phi^m] [\bar{Z}, \Phi^m] \} \rightarrow 2g_{YM}^2 N [b^\dagger, \phi] [b, \phi]; \quad \phi = \frac{a + a^\dagger}{\sqrt{2}},$$

whose action is through Feynman diagrams, as

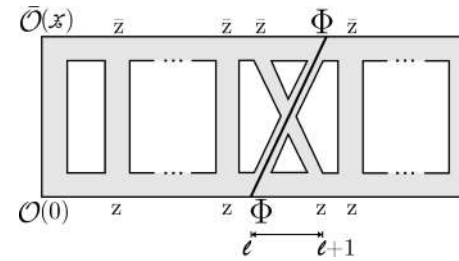
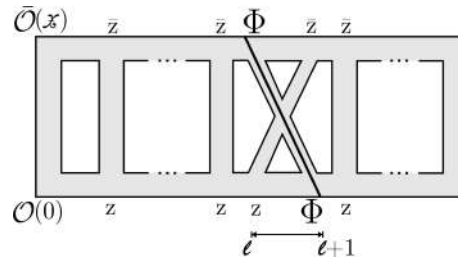


Feynman diagram for the 2-point function of $\mathcal{O}(x)$ at one-loop.

- 't Hooft limit $\Rightarrow$ only planar diagrams.

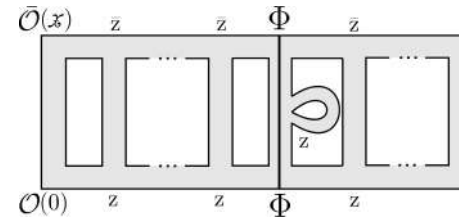
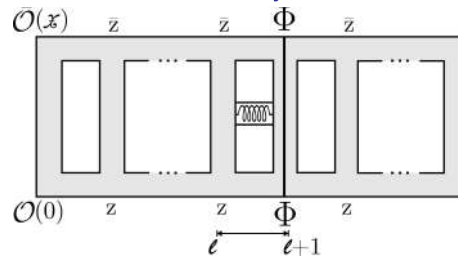


a)



b)

c)



d)

e)

Planar Feynman diagrams for the 2-point function of $\mathcal{O}$. a) The planar tree level diagram. b) Planar one-loop Feynman diagram with hopping from $l + 1$ to l . c) Planar one-loop diagram with hopping from l to $l + 1$. d) One-loop planar diagram with gluon exchange e) One-loop planar diagram with scalar self-energy.

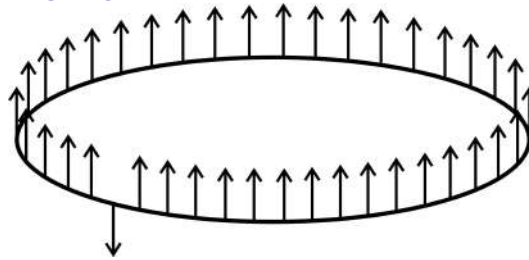
- After a calculation, find the Hamiltonian (when acting on states $\leftrightarrow$ operators)

$$H = \sum_{j=1}^J \frac{a_j^\dagger a_j + a_j a_j^\dagger}{2} + \frac{\lambda}{(2\pi)^2} \sum_{j=1}^J (\phi_j - \phi_{j+1})^2.$$

- Continuum version of Hamiltonian = light-cone string on pp wave,

$$H = \int_0^L d\sigma \frac{1}{2} [\dot{\phi}^2 + \phi'^2 + \phi^2], \quad L = \frac{2\pi J}{\mu\sqrt{\lambda}} = 2\pi\alpha' p^+,$$

so as a discrete "spin chain"



A periodic spin chain of the type that appears in the pp wave string theory. All spins are up, except one excitation has one spin down.

- But: made up of Cuntz oscillators, or rather, indep. Cuntz oscillators at each site:

$$a_i |0\rangle = 0, \quad a_i a_j^\dagger = \delta_{ij}, \quad \sum_{i=1}^n a_i^\dagger a_i = 1 - |0\rangle\langle 0| \rightarrow$$

$$[a_i, a_j] = [a_i^\dagger, a_j] = [a_i^\dagger, a_j^\dagger] = 0, \quad i \neq j$$

$$a_i a_i^\dagger = 1, \quad a_i^\dagger a_i = 1 - (|0\rangle\langle 0|)_i; \quad a_i |0\rangle_i = 0.$$

- Defining Fourier modes for the Cuntz oscillators,

$$a_j = \frac{1}{\sqrt{J}} \sum_{n=1}^J e^{\frac{2\pi i j n}{J}} a_n.$$

and acting on states in the "dilute gas approx.", $|\psi_{\{n_i\}}\rangle = |0\rangle_1 \dots |n_{i_1}\rangle \dots |n_{i_k}\rangle \dots |0\rangle_J$, so that the commutators become almost the usual ones, $[a_n, a_m^\dagger] |\psi_{\{n_i\}}\rangle \simeq (\delta_{nm} + \mathcal{O}(1/J)) |\psi_{\{n_i\}}\rangle$, we can further write superpositions of the left- and right-moving modes, and finally diagonalize by a Bogoliubov transformation,

$$a_n = \frac{c_{n,1} + c_{n,2}}{\sqrt{2}} \quad a_{J-n} = \frac{c_{n,1} - c_{n,2}}{\sqrt{2}}$$

$$\tilde{c}_{n,1} = a_n c_{n,1} + b_n c_{n,1}^\dagger \quad \tilde{c}_{n,2} = a_n c_{n,1} - b_n c_{n,1}^\dagger,$$

to obtain the eigenfrequencies

$$\omega_n = \sqrt{1 + 4|\alpha_n|} = \sqrt{1 + \frac{g_Y^2 M N}{\pi^2} \sin^2 \frac{\pi n}{J}},$$

with the corresponding Fock states

$$c_{n,1/2}^\dagger |0\rangle = \frac{a_n^\dagger \pm a_{J-n}^\dagger}{\sqrt{2}} |0\rangle = \frac{1}{\sqrt{J}} \sum_{j=1}^J \frac{e^{\frac{2\pi i j n}{J}} \pm e^{-\frac{2\pi i j n}{J}}}{\sqrt{2}} a_j^\dagger |0\rangle,$$

- Fock states mapped to the BMN operators

$$\frac{1}{\sqrt{J}} \sum_{l=1}^J \frac{1}{N^{\frac{J}{2}+1}} \text{Tr} [\Phi^1 Z^l \Phi^1 Z^{J-l}] \left[\cos \left(\frac{2\pi i n l}{J} \right) \text{ or } i \sin \left(\frac{2\pi i n l}{J} \right) \right].$$

- Note that for $n \ll J$, both ω_n and the states match the string on pp wave. For $n \sim J$, we also have a match, but not to the pp wave (Penrose limit of $AdS_5 \times S^5$), but a different limit.
- Note that ω_n is valid *to all orders in λ* (even though the Hamiltonian was one-loop, i.e. λ^1), though only for few impurities ($M \ll J$). Why? It seems to resum all interactions.

One loop: spin chain interpretation $\Phi^m \rightarrow a^\dagger$, $Z \rightarrow b^\dagger$: like a spin chain of length J , with spins "up" $|\uparrow\rangle$ for Z and "down" $|\downarrow\rangle$ for Φ . Though until now, only "dilute gas" analysis.

- The interaction Lagrangian (or Hamiltonian, as before)

$$\mathcal{L}_{\text{int}} = 2g_{YM}^2 \text{Tr} [Z, \Phi^m] [\bar{Z}, \Phi^m] = 2g_{YM}^2 (2\text{Tr} [\Phi^m Z \Phi^m \bar{Z}] - \text{Tr} [(Z\bar{Z} + \bar{Z}Z) \Phi^m \Phi^m])$$

leads, through Feynman diagrams for "hopping" acting on operators, to 1-loop 2-point function

$$\langle \mathcal{O}(x) \mathcal{O}^*(0) \rangle = \frac{\mathcal{N}}{|x|^{2J+2}} \left[1 + g_{YM}^2 N I(x) \left(e^{\frac{2\pi i n}{J}} + e^{-\frac{2\pi i n}{J}} \right) \right],$$

where

$$I(x) = \frac{|x|^4}{(4\pi^2)^2} \int d^4y \frac{1}{y^4(x-y)^4} \sim \frac{1}{4\pi^2} \log(|x|\Lambda) + \text{finite.}$$

• Then we can deduce the one-loop ω_n as

$$\begin{aligned} \langle \mathcal{O}(x) \mathcal{O}^*(0) \rangle &= \frac{\mathcal{N}}{|x|^{2J+2}} \left[1 - 2g_{YM}^2 N \left(\cos\left(\frac{2\pi n}{J}\right) - 1 \right) \frac{1}{4\pi^2} \log(|x|\Lambda) \right] \\ &= \frac{\mathcal{N}}{|x|^{2(J+1+(\Delta-J)[g^2 N])}} \\ &\simeq \frac{\mathcal{N}}{|x|^{2(J+1)}} \left[1 + 2(\Delta - J)[g^2 N] \ln(|x|) + \dots \right] \Rightarrow \\ (\Delta - J)_n &= \left[1 + \frac{g_{YM}^2 N}{2\pi^2} \sin^2\left(\frac{\pi n}{J}\right) \right]. \end{aligned}$$

• This matches the first order expansion of the exact Cuntz result.

Spin chains

- Full spin chain: $SO(6)$, for the 6 scalars Φ^I , with operators $\mathcal{O}[\psi] = \psi^{I_1 \dots I_L} \text{Tr} [\Phi_{I_1} \dots \Phi_{I_L}]$, defined through: the identity operator, the trace operator K (from contractions of fields of the same operator) and the permutation operator P (from contractions of fields of different operators, hopping one site),

$$K_{I_l I_{l+1}}^{J_l J_{l+1}} = \delta_{I_l, I_{l+1}} \delta^{J_l, J_{l+1}}, \quad P_{I_l, I_{l+1}}^{J_l, J_{l+1}} = \delta_{I_l}^{J_{l+1}} \delta_{I_{l+1}}^{J_l}.$$

- Renormalization of operators, with anomalous dimension matrix Γ , is

$$\mathcal{O}_{\text{ren}}^A = Z^A_B \mathcal{O}^B, \quad \Gamma = \frac{dZ}{d \ln \Lambda} \cdot Z^{-1},$$

and leads to 2-point functions of eigenvectors of Γ as

$$\langle \mathcal{O}_n^{\text{ren}}(x) \mathcal{O}_n^{\text{ren}}(y) \rangle = \langle Z \cdot \mathcal{O} Z \cdot \mathcal{O} \rangle = \frac{\text{const.}}{|x - y|^{2(L + \gamma_n)}}.$$

- Then for $\mathcal{N} = 4$ SYM at one-loop, one finds the Hamiltonian

$$\begin{aligned} Z_{\dots I_l I_{l+1} \dots}^{J_l J_{l+1} \dots} &= 1 - \frac{g_{YM}^2 N}{16\pi^2} \ln \Lambda \left(\delta_{I_l I_{l+1}} \delta^{J_l J_{l+1}} + 2 \delta_{I_l}^{J_{l+1}} \delta_{I_{l+1}}^{J_l} - 2 \delta_{I_l}^{J_{l+1}} \delta_{I_{l+1}}^{J_l} \right) \Rightarrow \\ \text{" } H \text{"} &= \Gamma = \frac{g_{YM}^2 N}{16\pi^2} \sum_{l=1}^L (K_{l, l+1} + 2 - 2P_{l, l+1}). \end{aligned}$$

$SU(2)$ sector and H_{XXX} from $\mathcal{N} = 4$ SYM

- We can construct a sub-spin chain that is the extension of the dilute gas approx. one, in an $SU(2)$ sector with 2 scalars, corresponding to "spin up" and "spin down",

$$Z = \Phi^1 + i\Phi^2; \quad \text{and} \quad W = \Phi^3 + i\Phi^4,$$

acting on operators (and their generalizations with "magnon" momenta)

$$\mathcal{O}_\alpha^{J_1, J_2} = \text{Tr} [Z^{J_1} W^{J_2}] + \dots (\text{permutations}).$$

- The interaction Hamiltonian in this subsector is

$$H_{\text{int}} = -g_{YM}^2 [Z, W] \text{Tr} [\bar{Z}, \bar{W}],$$

so the renormalization factor and the one-loop Hamiltonian are

$$\begin{aligned} \frac{Z^{J_l J_{l+1}}}{\dots I_l I_{l+1} \dots} &= 1 + \frac{g_{YM}^2 N}{16\pi^2} \ln \Lambda \, 2 \left(\delta_{I_l}^{J_l} \delta_{I_{l+1}}^{J_{l+1}} - \delta_{I_l}^{J_{l+1}} \delta_{I_{l+1}}^{J_l} \right) \Rightarrow \\ H_{\text{planar}}^{(1)} &= \Gamma_{\text{planar}}^{(1)} = \frac{g_{YM}^2 N}{16\pi^2} \sum_{l=1}^L 2 \left(1 - P_{l, l+1} \right). \end{aligned}$$

- A more precise concept of Hamiltonian, extendable to higher loops, is of a *dilatation operator* $\mathcal{D}$, obtained by attaching Feynman diagrams to operators,

$$\mathcal{D} \circ \mathcal{O}_\alpha^{J_1, J_2}(x) = \sum_\beta \mathcal{D}_{\alpha\beta} \mathcal{O}_\beta^{J_1, J_2}(x) ,$$

and can be written in terms of adding and removing fields in the operator, using $\check{Z}_{ij} \equiv \frac{d}{dZ_{ji}}$, so that

$$\mathcal{D}^{(0)} = \text{Tr} (Z\check{Z} + W\check{W}) , \quad \mathcal{D}^{(1)} = -\frac{g_{YM}^2}{8\pi^2} \text{Tr} [Z, W][\check{Z}, \check{W}] .$$

- Then the dilatation operator acts on operators as spin chains as

$$\mathcal{D}_{\text{planar}}^{(1)} = \frac{g_{YM}^2 N}{8\pi^2} \sum_{l+1}^L \left(1_{l, l+1} - P_{l, l+1} \right) ,$$

which is the Heisenberg $XX_{1/2}$ Hamiltonian, with $J = g_{YM}^2 N / (16\pi^2)$. Indeed, that is

$$H = -J \sum_{j=1}^L \vec{\sigma}_j \cdot \vec{\sigma}_{j+1} = -2J \sum_{j=1}^L (P_{j, j+1} - 1) ,$$

where we have used that on the $|\uparrow\rangle, |\downarrow\rangle$ basis on the chain, the permutation operator is $P_{ij} = \frac{1}{2} + \frac{1}{2} \vec{\sigma}_i \cdot \vec{\sigma}_j$.

Coordinate Bethe ansatz

- Denote $|x_1, \dots, x_M\rangle$ state with spins down at positions $x_1, \dots, x_M$ along the chain. Then the "one-magnon" eigenstate of H_{XXX} and its energy are

$$|\psi(p_1)\rangle = \sum_{x=1}^L e^{ip_1 x} |x\rangle, \quad E(p_1) = 8J \sin^2(p_1/2) |\psi(p_1)\rangle.$$

- The 2-magnon state is

$$|\psi(p_1, p_2)\rangle = \sum_{1 \leq x_1 < x_2 \leq L} \psi(x_1, x_2) |x_1, x_2\rangle$$
$$\psi(x_1, x_2) = e^{i(p_1 x_1 + p_2 x_2)} + S(p_2, p_1) e^{i(p_2 x_1 + p_1 x_2)},$$

where $E = E(p_1) + E(p_2)$ and the 2-body S-matrix is

$$S(p_1, p_2) = \frac{\phi(p_1) - \phi(p_2) + i}{\phi(p_1) - \phi(p_2) - i}, \quad \phi(p) = \frac{1}{2} \cot \frac{p}{2} \equiv u.$$

- For M magnons, in terms of $\phi(p) = u =$ rapidities (for true magnon momenta, u called Bethe roots), the energy is

$$E = \sum_{j=1}^M 8J \sin^2 \frac{p_j}{2} = \sum_{j=1}^M 2J \frac{1}{u_j^2 + 1/4}.$$

• For 2 magnons, from periodicity of $\psi(x_1, x_2)$ we have the *Bethe equations*

$$\begin{aligned} e^{ip_1 L} &= S(p_1, p_2) = \frac{\cot p_1/2 - \cot p_2/2 + 2i}{\cot p_1/2 - \cot p_2/2 - 2i} \\ e^{ip_2 L} &= S(p_2, p_1) = \frac{\cot p_2/2 - \cot p_1/2 + 2i}{\cot p_2/2 - \cot p_1/2 - 2i}, \end{aligned}$$

• so $p_1 + p_2 = \frac{2\pi n}{L}$, and for real $p_2 = -p_1$, $p_1 = \frac{2\pi n}{L-1}$. In this case, the 2-magnon wavefunction is

$$|\psi(n)\rangle \equiv |\psi(p_1(n), -p_1(n))\rangle = C_n \sum_{l=1}^L \cos\left(\pi n \frac{2l+1}{L-1}\right) |x_2 + l, x_2\rangle, \quad C_n = 2e^{-\frac{i\pi n}{L-1}},$$

and this corresponds to a $\mathcal{N} = 4$ SYM operator (eigenstate of $\mathcal{D}^{(1)}$) that for $n \ll L, L \rightarrow \infty$ becomes the BMN operator,

$$\begin{aligned} \mathcal{O}_n^{J,2} &= C_n \sum_{l=0}^{L-1} \cos\left[\pi n \frac{2l+1}{L-1}\right] \text{Tr}[W Z^l W Z^{J-l}] \Rightarrow \\ \mathcal{O}_n^{J,2} &\rightarrow C_n \sum_{l=0}^{L-1} \cos \frac{2\pi n l}{L} \text{Tr}[W Z^l W Z^{L-l}]. \end{aligned}$$

• For $M \ll L, n \ll L$, we obtain the spectrum of operators by acting with $a_n^\dagger = \frac{1}{\sqrt{L}} \sum_{l=1}^L e^{\frac{2\pi i n l}{L}} \sigma_l^-$, and, for momenta $p_k \simeq 2\pi n_k/L$, the anomalous dimension in SYM is

$$\gamma = \Delta - L - M = \frac{\lambda}{16\pi^2} \sum_{k=1}^M 8 \sin^2 \frac{p_k}{2} \simeq \frac{\lambda}{8\pi^2} \sum_{k=1}^M p_k^2 = \frac{\lambda}{2L^2} \sum_{k=1}^M n_k^2.$$

- For M magnons, the wavefunction is

$$\psi(x_1, \dots, x_M) = \sum_{P \in \text{Perm}(M)} \exp \left[i \sum_{i=1}^M p_{P(i)} x_i + \frac{i}{2} \sum_{i < j} \delta_{P(i)P(j)} \right], \quad S(p_i, p_j) \equiv e^{i\delta_{ij}},$$

and the Bethe ansatz equations are (again from periodicity)

$$e^{ip_k L} = \prod_{i \neq k, i=1}^M S(p_k, p_i) \Rightarrow \left(\frac{u_k - i/2}{u_k + i/2} \right)^L = \prod_{j \neq k, j=1}^M \left(\frac{u_k - u_j - i}{u_k - u_j + i} \right), \quad k = 1, \dots, M.$$

- Their solutions are called *Bethe roots*, and need not be real, only the energies need be real. Then u_k root implies u_k^* root.
- **Thermodynamic limit** In the limit $L \rightarrow \infty, M \rightarrow \infty$, taking the log of the BEA, and since $p_i \sim 1/L$, $u_i \sim L$, so $xI = u_i/L$ finite, we have

$$\frac{1}{x_i} + 2\pi n_i = \frac{2}{L} \sum_{k \neq i, k=1}^M \frac{1}{x_i - x_k}.$$

Then also $u_k = u_i \pm i$, so $u_k = \text{Re}(u) + ik$ form *Bethe strings*, that curve a bit for $M \rightarrow \infty$ as well. They satisfy (and other equations)

$$2P \int_{\mathcal{C}} dy \frac{\rho(y)}{y - x} = -\frac{1}{x} + 2\pi n_{\mathcal{C}(u)}; \quad x \in \mathcal{C}.$$

Spin chains and Bethe strings from AdS

- Strings moving in $S^3 \subset S^5$ defined by $X^i X^i = 1$, $i = 1, \dots, 4$, with $Z = X^1 + iX^2$, $W = X^3 + iX^4$, and then $SU(2)$ element

$$g = \begin{pmatrix} Z & W \\ -\bar{W} & \bar{Z} \end{pmatrix}, \text{ and matrix currents}$$

$$j_a = g^{-1} \partial_a g \Rightarrow \text{Tr} (j_a)^2 = -2 \sum_{i=1}^4 (\partial_a X^i)(\partial_a X^i),$$

so the string action in conformal gauge, moving in S^3 is

$$S = -\frac{\sqrt{\lambda}}{4\pi} \int_0^{2\pi} d\sigma \int d\tau \left[\frac{\text{Tr} (j_a)^2}{2} + (\partial_a X^0)^2 \right].$$

and has as eq. of m. $\partial_+ j_- + \partial_- j_+ = 0$. We can then check that

$$\begin{aligned} \partial_+ j_- - \partial_- j_+ + [j_+, j_-] &= 0, \quad J_{\pm} = \frac{j_{\pm}}{1 \mp x} \Rightarrow \\ \partial_+ J_- - \partial_- J_+ + [J_+, J_-] &= 0, \quad \forall x. \end{aligned}$$

- That means that J_a is a flat connection, with monodromy

$$\Omega(x) \equiv P \exp \left[- \int_0^{2\pi} d\sigma J_{\sigma} \right] = P \exp \left[\int_0^{2\pi} d\sigma \frac{1}{2} \left(\frac{j_+}{x-1} + \frac{j_-}{x+1} \right) \right].$$

• Define then

$$\text{Tr } \Omega(x) \equiv 2 \cos p(x) = e^{ip(x)} + e^{-ip(x)} \Rightarrow p(x) \simeq -\frac{\pi \kappa}{x \pm 1} + \dots, \quad x \rightarrow \mp 1$$

$$G(x) \equiv p(x) + \frac{\pi \kappa}{x + 1} + \frac{\pi \kappa}{x - 1},$$

$$G(x + i0) - G(x - i0) \equiv 2\pi i \rho(x).$$

• Then $\rho(x)$ satisfies, after rescaling $x \rightarrow 4\pi Lx/\sqrt{\lambda}$,

$$\begin{aligned} 2P \int dy \frac{\rho(y)}{x - y} &= \frac{x}{x^2 - \frac{\lambda}{16\pi^2 L^2}} \frac{\Delta}{L} + 2\pi n_k \\ \int dx \rho(x) &= \frac{M}{L} + \frac{\Delta - L}{2L} \\ \int dx \frac{\rho(x)}{x} &= 2\pi m \\ \frac{\lambda}{8\pi^2 L} \int dx \frac{\rho(x)}{x^2} &= \Delta + L = \frac{\lambda}{8\pi^2} H^{(1\text{-loop})}, \end{aligned}$$

which in the thermodynamic limit $\lambda/L^2 \rightarrow 0$, $\frac{\Delta - L}{L} \rightarrow 0$, $\frac{\Delta}{L} \rightarrow 1$ gives the same equations as for the Bethe strings. Thus each Bethe strings corresponds to an individual macroscopic string in AdS.

Lecture 9

Applications to condensed matter:
AdS/CMT

- AdS/CMT: phenomenological approach. "Top-down": define some known duality, see if physics matches anything. OR: "bottom-up": construct AdS theory that, given holographic map, would imply wanted properties for the field theory, and then calculate other properties.

Gravity dual of Lifshitz points

- CMT: usually nonrelativistic. Construct nonrelativistic gravity dual. E.G.: "Lifshitz scaling":

$$t \rightarrow \lambda^z t, \quad \vec{x} \rightarrow \lambda \vec{x}.$$

z =dynamical critical exponent. Model example:

$$\mathcal{L} = \int d^2x \, dt \left[(\partial_t \phi)^2 - k(\vec{\nabla}^2 \phi)^2 \right].$$

- Then, phenomenological gravity bgr. for Lifshitz scaling,

$$ds_{d+1}^2 = R^2 \left(-\frac{dt^2}{u^{2z}} + \frac{d\vec{x}^2}{u^2} + \frac{du^2}{u^2} \right)$$

(obs.: geodesically incomplete for $z \neq 1$ at $u = \infty$) has scaling invariance

$$t \rightarrow \lambda^z t, \quad \vec{x} \rightarrow \lambda \vec{x}, \quad u \rightarrow \lambda u,$$

with generator (Killing vector)

$$D = -i(zt\partial_t + x^i\partial_i + u\partial_u).$$

- Other generators (Killing vectors)

$$M_{ij} = -i(x^i\partial_j - x^j\partial_i); \quad P_i = -i\partial_i; \quad H = -i\partial_t,$$

forming together the Lifshitz algebra,

$$\begin{aligned} [D, H] &= z\partial_t = izH \\ [D, P_i] &= \partial_i = iP_i; \quad [D, M_{ij}] = 0 \\ [M_{ij}, P_k] &= \delta_k^i\partial_j - \delta_k^j\partial_i = i(\delta_k^i P_j - \delta_k^j P_i) \\ [M_{ij}, M_{kl}] &= i(\delta_{ik}M_{jl} - \delta_{jk}M_{il} - \delta_{il}M_{jk} + \delta_{jl}M_{ik}) \\ [P_i, P_j] &= 0. \end{aligned}$$

- The background is a solution to several *relativistic* actions, e.g.,

$$S = \frac{1}{2\kappa_N^2} \int dt d^D x dr \sqrt{-g} \left[R - 2\Lambda - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{1}{2} m^2 A_\mu A^\mu \right],$$

or in non-relativistic gravity, e.g. Horava gravity.

Gravity dual to Galilean and Schrödinger symmetries

- Larger algebras: -conformal Galilean algebra: M_{ij}, P_i, H, D , but also conserved rest mass, or particle number N and Galilean boosts

$$t \rightarrow t, \quad x_i \rightarrow x_i - v_i t .$$

- For $z = 2$, extra generator C , special conformal generator: *Schrödinger algebra* (symmetry of the Schrödinger equation of a free particle).
- AdS/CFT realization (geometrical): $d + 2$ -dimensional gravity dual (ξ, u extra):

$$ds^2 = R^2 \left(-\frac{dt^2}{u^{2z}} + \frac{d\vec{x}^2}{u^2} + \frac{du^2}{u^2} + \frac{2dt d\xi}{u^2} \right) .$$

- Not time-reversal invariant ($t \leftrightarrow -t$), nonsingular: conformal to pp wave:

$$ds^2 = \frac{R^2}{u^2} \left(-dt^2 u^{2(1-z)} + 2dt d\xi + d\vec{x}^2 + du^2 \right) .$$

- Invariant under scaling

$$t' = \lambda^z t, \quad \vec{x}' = \lambda \vec{x}, \quad u' = \lambda u, \quad \xi' = \lambda^{2-z} \xi ,$$

for generator

$$D = -i(zt\partial_t + x^i\partial_i + u\partial_u + (2-z)\xi\partial_\xi) .$$

- Extra symmetry: Galilean boost K_i ,

$$\vec{x}' = \vec{x} - vt, \quad \xi' = \xi + \frac{1}{2}(2\vec{v} \cdot \vec{x} - v^2 t), \quad K_i = -i(x^i \partial_\xi - t \partial_i).$$

and particle number $N = -i\partial_\xi$, so that the (conformal Galilean) algebra is

$$\begin{aligned} [K_i, P_j] &= \delta_{ij} \partial_\xi = i\delta_{ij} N \\ [D, K_i] &= zt \partial_i - x^i \partial_\xi + (2 - z)x^i \partial_\xi - t \partial_i = (1 - z)iK_i \\ [K_k, M_{ij}] &= t(\delta_{ik} \partial_j - \delta_{jk} \partial_i) + \delta_{jk} x^i \partial_\xi - \delta_{ik} x^j \partial_\xi = i(\delta_{jk} K_i - \delta_{ik} K_j) \\ [K_i, H] &= -\partial_i = -iP_i \\ [D, N] &= (2 - z)\partial_\xi = (2 - z)iN \\ [K_i, N] &= [H, N] = [P_i, N] = [M_{ij}, N] = 0. \end{aligned}$$

- For $z = 2$, extra *special conformal generator* C , for

$$\begin{aligned} u &\rightarrow (1 - at)u, & x^i &\rightarrow (1 - at)x^i \\ t &\rightarrow (1 - at)t, & \xi &\rightarrow \xi - \frac{a}{2}(\vec{x}^2 + u^2). \end{aligned}$$

Then the extra commutation relations for C give the Schrödinger algebra,

$$[D, C] = -2iC, \quad [H, C] = -iD, \quad [M_{ij}, C] = 0 = [K_i, C].$$

- String theory realization of Galilean algebra, by "null Melvin twist" on $AdS_5 \times S^5$ with temperature T ,

$$ds^2 = r^2 \left[-\frac{\beta^2 r^2 f(r)}{k(r)} (dt + dy)^2 - \frac{f(r)}{k(r)} dt^2 + \frac{dy^2}{k(r)} + d\vec{x}^2 \right] + \frac{dr^2}{r^2 f(r)} + \frac{(d\psi + A)^2}{k(r)} + d\Sigma_4^2$$

$$f(r) = 1 - \frac{r_+^4}{r^4}, \quad k(r) = 1 + \frac{\beta^2 r_+^4}{r^2}, \quad T = \frac{r_+}{\pi\beta}.$$

- For $T = 0, k = f = 1$ and KK reducing on ψ and Σ_4 gives $z = 2$ metric (Schrödinger).

Spectral functions

- Retarded Green's functions for observables $\mathcal{O}_A, \mathcal{O}_B$,

$$G_{\mathcal{O}_A \mathcal{O}_B}^R(\omega, k) = -i \int d^{d-1}x \, dt \, e^{i\omega t - i\vec{k} \cdot \vec{x}} \theta(t) \langle [\mathcal{O}_A(t, x), \mathcal{O}_B(0, 0)] \rangle,$$

describe the time evolution of small pert. about equilibrium, in linear response theory,

$$\delta \langle \mathcal{O}_A \rangle(\omega, k) = G_{\mathcal{O}_A \mathcal{O}_B}^R(\omega, k) \delta \phi_{B(0)}(\omega, k).$$

- Retarded ($\exists \theta(t)$) $\Rightarrow$ 2 conditions (close ω contour in complex upper-half plane):

- 1) $G_{\mathcal{O}_A \mathcal{O}_B}^R(\omega, k)$ is analytic in the complex ω plane for $\text{Im}(\omega) > 0$.
- 2) $G_{\mathcal{O}_A \mathcal{O}_B}^R(\omega, k) \rightarrow 0$ for $|\omega| \rightarrow 0$.

- Conditions imply representation as Γ contour integral (real line, closed by semicircle at ∞ in upper-half plane) in ζ or z ,

$$G^R(z) = \oint_{\Gamma} \frac{d\zeta}{2\pi i} \frac{G^R}{\zeta - z},$$

giving the *Kramers-Kronig relations*,

$$\begin{aligned} \text{Re}G^R(\omega) &= P \int_{-\infty}^{+\infty} \frac{d\omega'}{\pi} \frac{\text{Im}G^R(\omega')}{\omega' - \omega} \\ \text{Im}G^R(\omega) &= -P \int_{-\infty}^{+\infty} \frac{d\omega'}{\pi} \frac{\text{Re}G^R(\omega')}{\omega' - \omega}, \end{aligned}$$

and the $\omega \rightarrow 0$ limit gives a thermodynamic "sum rule" (G^R both inside and outside the f),

$$\chi \equiv \lim_{\omega \rightarrow 0 + i0} G_{\mathcal{O}_A \mathcal{O}_B}^R(\omega, x) = \int_{-\infty}^{+\infty} \frac{d\omega'}{\pi} \frac{\text{Im}G_{\mathcal{O}_A \mathcal{O}_B}^R(\omega', x)}{\omega'},$$

while χ is a static thermodynamic susceptibility (like $\chi = \partial D / \partial E$),

$$\chi_{AB} = \frac{\partial \langle \mathcal{O}_A \rangle}{\partial \phi_{B(0)}}.$$

- Spectral function for χ_{AB} is

$$\chi_A = -\text{Im} G_{\mathcal{O}_A \mathcal{O}_B}^R(\omega, \vec{k}) .$$

- Advanced Green's function

$$G_{\mathcal{O}_A \mathcal{O}_B}^A(t, x) = +i\theta(-t)\langle[\mathcal{O}_A(t, x), \mathcal{O}_B(0, 0)]\rangle ,$$

and from it the spectral function $\rho_{\mathcal{O}_A \mathcal{O}_B}(\omega, \vec{k})$,

$$\begin{aligned} \rho_{\mathcal{O}_A \mathcal{O}_B}(t, \vec{x}) &= \langle[\mathcal{O}_A(t, \vec{x}), \mathcal{O}_B(0, 0)]\rangle = i(G_{\mathcal{O}_A \mathcal{O}_B}^R(t, \vec{x}) - G_{\mathcal{O}_A \mathcal{O}_B}^A(t, \vec{x})) \Rightarrow \\ \rho_{\mathcal{O}_A \mathcal{O}_B}(\omega, k) &= \int d^{d-1}x dt e^{i\omega t - i\vec{k}\cdot\vec{x}} \langle[\mathcal{O}_A(t, x), \mathcal{O}_B(0, 0)]\rangle = i(G^R - G^A)(\omega, k) , \end{aligned}$$

since

$$\begin{aligned} G^{R,A}(\omega, \vec{k}) &= \int \frac{d\omega'}{2\pi} \frac{\rho(\omega', \vec{k})}{\omega - \omega' \pm i\epsilon} \Rightarrow \\ \text{Re} G^R(\omega, \vec{k}) &= \text{Re} G^A(\omega, \vec{k}) = P \int \frac{d\omega'}{2\pi} \frac{\rho(\omega', \vec{k})}{\omega - \omega'} \\ \text{Im} G^R(\omega, \vec{k}) &= -\text{Im} G^A(\omega, \vec{k}) = -\frac{1}{2}\rho(\omega, \vec{k}). \end{aligned}$$

Transport

- **Kubo formula for electric conductivity:** In gauge $A_0 = 0$, electric field source $E_j = F_{0j} = -\partial_t \delta A_j$, in momentum space $E_j = -i\omega \delta A_{j(0)}$. Linear response for J_x :

$$\begin{aligned}\langle J_x \rangle &= \sigma E_x = -i\omega \sigma \delta A_{x(0)} \Rightarrow \\ \sigma(\omega, \vec{k}) &= \frac{iG_{J_x J_x}^R(\omega, \vec{k})}{\omega}.\end{aligned}$$

- The (real part of the) DC conductivity is then

$$\sigma(0, \vec{k}) = -\lim_{\omega \rightarrow 0} \frac{\text{Im} G_{J_x J_x}^R(\omega, \vec{k})}{\omega}.$$

- **Kubo formula for shear viscosity:** Shear viscosity, in relativistic theory $\rightarrow$ from expansion of $T_{\mu\nu}$, solving $\nabla_\mu T^{\mu\nu} = 0$ as expansion in derivatives. For dissipative fluid, to first nontrivial order,

$$\begin{aligned}T^{\mu\nu} &= \rho u^\mu u^\nu + P P^{\mu\nu} + \Pi_{(1)}^{\mu\nu} \\ &= \rho u^\mu u^\nu + P(g^{\mu\nu} + u^\mu u^\nu) + 2\eta \sigma^{\mu\nu} - \zeta \theta P^{\mu\nu},\end{aligned}$$

where $P^{\mu\nu} = g^{\mu\nu} + u^\mu u^\nu$ and $\sigma^{\mu\nu}, \theta$ are from decomposition of $\nabla^\nu u^\mu$, in the Landau frame $\pi_{(1)}^{\mu\nu} u_\mu = 0$,

$$\nabla^\nu u^\mu = -a^\mu u^\nu + \sigma^{\mu\nu} + \omega^{\mu\nu} + \frac{1}{d-1} \theta P^{\mu\nu}$$

$$a^\mu = u^\nu \nabla_\nu u^\mu$$

$$\theta = \nabla_\mu u^\mu = P^{\mu\nu} \nabla_\mu u_\nu$$

$$\sigma^{\mu\nu} = \nabla^\mu u^\nu + \nabla^\nu u^\mu - \frac{1}{d-1} \theta P^{\mu\nu}$$

$$\omega^{\mu\nu} = \nabla^{[\mu} u^{\nu]} + \nabla u^{[\mu} a^{\nu]},$$

• For h_{xy} perturbation on fluid at rest,

$$\begin{aligned} T_{xy} &= P h_{xy} + \eta \partial_t h_{xy} + \mathcal{O}(h_{xy}^2) + \mathcal{O}(\partial^2 h_{xy}) \Rightarrow \\ T_{xy}(\omega) &= -\eta i \omega h_{xy} + \mathcal{O}(h_{xy}^2) + \mathcal{O}(\partial^2 h_{xy}), \end{aligned}$$

(ignoring δ fct. coming from const. term), we get the Kubo formula for shear viscosity,

$$\eta(\omega, \vec{k}) = \frac{i G_{T_{xy} T_{xy}}^R(\omega, \vec{k})}{\omega},$$

or, for the (real part of the) static shear viscosity,

$$\eta(0, \vec{0}) = - \lim_{\omega \rightarrow 0} \frac{\text{Im} G_{T_{xy} T_{xy}}^R(\omega, \vec{0})}{\omega}.$$

- **AdS/CFT in Minkowski space at finite temperature** Son, Starinets, 2002. If we can write the on-shell sugra action in asympmt. AdS space as a function of the *boundary value* $\phi_{(0)}$ as the boundary term

$$S_{\text{on-shell}} = \int \frac{d^d k}{(2\pi)^d} \phi_0(-\vec{k}) \mathcal{F}(\vec{k}, z) \phi_0(\vec{k}) \Big|_{z=z_B}^{z=z_H},$$

the the prescription for the retarded Green's function is

$$G^R(\vec{k}) = -2\mathcal{F}(\vec{k}, z)|_{z=z_B}.$$

- **Equivalent formulation:** $S_{\text{ren.}} = S[\partial_\mu \phi] + S_{\text{boundary}}$, so

$$\langle \mathcal{O} \rangle = \lim_{z \rightarrow 0} \left(\frac{R}{z} \right)^\Delta \frac{1}{\sqrt{\gamma}} \left(-\frac{\delta S[\phi_{(0)}]}{\delta \partial_z \phi_{(0)}(z)} - \frac{\delta S_{\text{boundary}}}{\delta \phi_{(0)}(z)} \right),$$

while we saw

$$S_{\text{boundary}} = \frac{\Delta - d}{2R} \int_{z \rightarrow 0} d^d x \sqrt{\gamma} \phi^2$$

$$\phi(z) = \left(\frac{z}{R} \right)^{d-\Delta} \phi_{(0)} + \left(\frac{z}{R} \right)^\Delta \phi_{(2\Delta-d)} + \dots,$$

•Then:

$$\begin{aligned}
 \langle \mathcal{O} \rangle &= - \lim_{z \rightarrow 0} \left(\frac{R}{z} \right)^\Delta \left[\frac{z}{R} \partial_z \phi|_{\phi(0)=0} + \frac{\Delta - d}{2R} 2\phi|_{\phi(0)=0} \right] \\
 &= - \frac{2\Delta - d}{R} \phi(2\Delta - d) \Rightarrow \\
 G_{\mathcal{O}_A \mathcal{O}_B}^R &= \frac{\delta \langle \mathcal{O}_A \rangle}{\delta \phi_{B(0)}} \Big|_{\delta \phi_{(0)}=0} = - \frac{2\Delta_A - d}{R} \frac{\delta \phi_A(2\Delta - d)}{\delta \phi_{B(0)}}.
 \end{aligned}$$

Kubo relations for other transport properties: For $\mu \neq 0$, so charge density $\rho \neq 0$, heat and electric currents mix, so

$$\begin{pmatrix} \langle J_x \rangle \\ \langle Q_x \rangle \end{pmatrix} = \begin{pmatrix} \sigma & \alpha T \\ \alpha T & \bar{\kappa} T \end{pmatrix} \begin{pmatrix} E_x \\ -\frac{\nabla_x T}{T} \end{pmatrix}.$$

•Then similarly, Kubo formulae:

$$\begin{aligned}
 \alpha(\omega) T &= i \frac{G_{Q_x J_x}^R(\omega)}{\omega} \\
 \bar{\kappa}(\omega) T &= i \frac{G_{Q_x Q_x}^R(\omega)}{\omega}.
 \end{aligned}$$

Viscosity over entropy density from dual black holes: Witten metric (AdS-BH in Poincaré c.):

$$ds^2 = \frac{r^2}{R^2} \left(- \left(1 - \frac{r_0^4}{r^4} \right) dt^2 + d\vec{x}_3^2 \right) + \frac{R^2}{r^2} \frac{dr^2}{1 - \frac{r_0^4}{r^4}},$$

and change coordinates $u = r_0^2/r^2$ so that

$$ds^2 = \frac{r_0^2}{R^2} \frac{1}{u} (-f(u) dt^2 + d\vec{x}_3^2) + \frac{R^2}{4} \frac{du^2}{u^2 f(u)}; \quad f(u) = 1 - u^2,$$

giving a perturbation

$$h_{xy}(\vec{x}, u) = \frac{r_0^2}{R^2 u} e^{-i\omega t + i\vec{q} \cdot \vec{x}} \phi_q(u),$$

and vary $\phi_{A(2\Delta-d)}$ in the exact solution w.r.t. it. But: hard. Instead, η at horizon $\simeq$ at boundary, so calculate approx. sol. at horizon,

$$\phi_q^\pm = \phi_0 (1 - u)^{\pm i \frac{\omega}{4\pi T}},$$

and selecting infalling sols. ("−"), we find

$$\begin{aligned} G^R(\omega, \vec{k}) &= -2\mathcal{F}(\omega, \vec{k}, u)_{u=1} \Rightarrow \\ \eta &= -\lim_{\omega \rightarrow 0} \frac{\text{Im} G^R(\omega, \vec{0})}{\omega} = \frac{r_0^3/R^3}{16\pi G_{N,5}} = \frac{\pi}{8} N^2 T^3. \end{aligned}$$

- Since $s = \pi^2/2N^2T^3$, we find (conjectured to be lower bound, but it is not)

$$\frac{\eta}{s} = \frac{1}{4\pi}.$$

The holographic superconductor Gubser, 2008; Hartnoll, Herzog, Horowitz, 2008

- Ingredients: a) AdS_4 background: CFT near transition point. High T_c superconductors (non-Fermi liquids) are 2+1d. b) charge transport: conserved $U(1)$ J_μ , dual to A_μ . c) Temperature, so black hole in AdS_4 . d) symmetry breaking, so $\langle \mathcal{O} \rangle \neq 0$, for a complex field charged under $U(1)$. s wave superconductors $\Rightarrow$ charged scalar ψ .
- Lagrangian for gravity theory ($d = 3$)

$$\mathcal{L} = \frac{1}{2\kappa^2} \left(R + \frac{d(d-1)}{R^2} \right) - \frac{1}{4g^2} F_{\mu\nu}^2 - |(\partial_\mu - iqA_\mu)\psi|^2 - m^2\psi^2 - V(\psi),$$

for $V = 0$, $m^2 R^2 \geq -\frac{d^2}{4R^2} = -9/4$ (BF bound) (scalar stable at ∞).

- We want $\psi \neq 0$ near BH horizon, for $T < T_c$, and $\psi = 0$, $T > T_c$.

- Superconducting ansatz:

$$ds^2 = g_{tt}(r)dt^2 + g_{rr}(r)dr^2 + ds_2^2(r), \quad A_\mu dx^\mu = \Phi(r)dt, \quad \psi = \psi(r).$$

- Then we obtain an *effective mass* in $\mathcal{L}$,

$$-|(\partial_\mu - iqA_\mu)\psi|^2 - m^2|\psi|^2 \rightarrow -g^{rr}|\partial_r\psi|^2 - m_{\text{eff}}^2|\psi|^2, \quad m_{\text{eff}}^2 = m^2 + g^{tt}q^2\Phi^2,$$

but we want $\Phi = 0$ at horizon, yet $m_{\text{eff}}^2 < -9/4$ (BF bound), so unstable at horizon. The scalar operator VEV is

$$\langle \mathcal{O} \rangle = \frac{2\Delta - d}{R} \psi_{(2\Delta-d)},$$

so we need a normalizable mode $\psi_{(2\Delta-d)} \neq 0$ for $T < T_c$.

- Two possible backgrounds: AdS-Reissner-Nordstrom (Gubser),

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2 d\Omega_{2,k}^2, \quad f(r) = k - \frac{2M}{r} + \frac{Q^2}{4r^2} + \frac{r^2}{R^2}$$

$$\Phi(r) = \frac{Q}{r} - \frac{Q}{r_H}, \quad \psi = 0,$$

or neutral AdS-BH (Hartnoll, Herzog, Horowitz), $k = 0$, so $d\Omega_{2,k}^2 = dx^2 + dy^2$, and $Q = 0$, so

$$f(r) = \frac{r^2}{R^2} - \frac{2M}{r}, \quad \Phi = \psi = 0.$$

- The scalar ψ is a probe in background. Boundary conditions

$$\psi = \frac{\psi^{(1)}}{r} + \frac{\psi^{(2)}}{r^2} + \dots, \quad \Phi = \mu - \frac{\rho}{r} + \dots,$$

- but both ψ_1, ψ_2 normalizable. Then, condensates

$$\langle \mathcal{O}_i \rangle = \sqrt{2} \psi^{(i)}, \quad i = 1, 2.$$

and numerically, one finds near $T \simeq T_c$,

$$\begin{aligned} \langle \mathcal{O}_1 \rangle &\simeq 9.3 T_c (1 - T/T_c)^{1/2} \\ \langle \mathcal{O}_2 \rangle &\simeq 144 T_c^2 (1 - T/T_c)^{1/2}, \quad T \simeq 0.118 \sqrt{\rho}. \end{aligned}$$

- **Effective mass at horizon:** needs to be smaller than BF bound *at horizon*, for instability.

$$m_{\text{eff}}^2 = m^2 - \frac{\gamma^2 q^2}{2R^2} < m^2, \quad \gamma^2 = \frac{g^2 2R^2}{\kappa_{N,4}^2}.$$

- Horizon: $AdS_2 \times S^2$, with AdS_2 BF bound $m^2 R_2^2 = m^2 R^2 / 6 \geq -1/4$ (stronger than at infinity, where $m^2 R^2 \geq -9/4$).

- Electric conductivity: Perturbation $\delta A_x = \delta A_x(r)e^{-i\omega t}$, such that at boundary,

$$\delta A_x = \delta A_x^{(0)} + \frac{\langle J_x \rangle}{r} + \dots ,$$

where $\langle J_x \rangle = \delta A_x^{(1)}$ is the normalizable mode. Then

$$\sigma(\omega) = \frac{\langle J_x \rangle}{E_x} = -i \frac{\langle J_x \rangle}{\omega \delta A_x} = -i \frac{\delta A_x^{(1)}}{\omega \delta A_x^{(0)}} ,$$

and numerically, one finds a mass gap: $\sigma = 0$ for $\omega < \omega_g$, and

$$\omega_g \approx (q\langle \mathcal{O} \rangle)^{\frac{1}{\Delta}} , \quad \frac{\omega_g}{T_c} \simeq 8.4 ,$$

approx. matching exp. data. for high T_c supercond.

- BUT: weakly coupled supercond. $\omega_g = 2E_g$ (E_g = energy gap in charged spectrum), while for high T_c (strongly coupled), $E_g \neq \omega_g/2$. Holographic: true, *except* for $\Delta = 1$ or 2 , when $E_g = \omega_g/2$. Puzzle!

Transport properties in strongly coupled systems via AdS/CFT

- Region (“membrane”) near black hole horizon in gravity dual → has fluid properties (gravity/fluid correspondence): “membrane paradigm” .
- Membrane paradigm → calculation: quantities are r -indep. (should be calculated at ∞ , but calculated at horizon). Linear response to perturbations of background $\Rightarrow$ transport.
- Calculate electric and heat conductivities (AdS/CFT: either Kubo formulas, or membrane paradigm: at the horizon)

$$\begin{aligned} j_i &= \sigma_{ij} E^j - \alpha_{ij} (\nabla_j T) \\ Q_i &= T \alpha_{ij} - \bar{\kappa}_{ij} \nabla_j T, \quad \kappa_{ij} = \bar{\kappa}_{ij} - T \alpha_{ik} (\sigma^{-1})_{kl} \alpha_{lj}. \end{aligned}$$

- Also, effect of other parameters on transport (η, ζ for fluid, for instance). But we have
$$\begin{pmatrix} \sigma & \alpha \\ \alpha T & \bar{\kappa} \end{pmatrix} = D \chi_s, ,$$

where D = diffusivity matrix, χ_s = susceptibility matrix, obtained from thermod. pot. Ω (and then we can derive D),

$$\chi_s = \begin{pmatrix} -\frac{1}{V} \frac{\partial^2 \Omega}{\partial \mu^2} \Big|_{B,T} & -\frac{1}{V} \frac{\partial^2 \Omega}{\partial T \partial \mu} \Big|_B \\ -\frac{T}{V} \frac{\partial^2 \Omega}{\partial \mu \partial T} \Big|_B & -\frac{T}{V} \frac{\partial^2 \Omega}{\partial T^2} \Big|_{B,\mu} \end{pmatrix},$$

Transport properties of strongly coupled 2+1d CFTs from properties of 4d AdS-black hole solutions

- Gravity theory in 4d: + scalars (dilaton ϕ , axions χ_1, χ_2) + electromagnetic ($A_\mu \Rightarrow F_{\mu\nu}$) (e.g., L. Alejo, P. Goulart, HN, 2019; D. Melnikov, HN, 2021)

$$S = \int d^4x \sqrt{-g} \left[\frac{1}{16\pi G_N} \left(R - \frac{1}{2} [(\partial\phi)^2 + \Phi(\phi) ((\partial\chi_1)^2 + (\partial\chi_2)^2)] - V(\phi) \right) - \frac{Z(\phi)}{4g_4^2} F_{\mu\nu}^2 - W(\phi) F_{\mu\nu} \tilde{F}^{\mu\nu} \right].$$

- Background solution: black hole $\Rightarrow$ has event horizon at $r = r_H$. $\chi_1 = k_1 x$, $\chi_2 = k_2 y$ breaks translational invariant in 2 spatial directions of field theory.

$$\begin{aligned} ds^2 &= -U(r)(dt + B_1 y dx)^2 + \frac{dr^2}{U(r)} + e^{2V(r)}(dx^2 + dy^2) \\ A_t &= a(r), \quad A_x = -By + (a(r) - \mu)B_1 y \\ \chi_1 &= k_1 x, \quad \chi_2 = k_2 y, \quad \phi = \phi(r) \end{aligned}$$

- Event horizon: $U(r_H) = 0$, $U(r) \simeq (r - r_H)U'(r_H)$, temperature $T = \frac{U'(r_H)}{4\pi}$.
- 4d $A_x = -By$ source gives magnetic field B in 2+1d dimensions. B_1 generates energy magnetization density, $M_E = -\frac{1}{\text{Vol}} \frac{\partial S_E}{\partial B_1} \Big|_{B_1=0}$, μ =chemical potential.
- Then, add perturbations $(-E + \xi a(r))t$ in A_x and $-\xi t U(r)$ in g_{tx} , where $E_i = E\delta_{ix}$ and $\frac{1}{T}\nabla_i T = \xi\delta_{ix}$ is field perturbation $\Rightarrow$ generate response, δ fields, in order to satisfy eqs. of motion.
- Membrane paradigm: r -indep. (electric) currents

$$\begin{aligned}\mathcal{J}^x &= \frac{Z(\phi)}{g_4^2} \sqrt{-g} F^{xr} + 4\sqrt{-g} W(\phi) \tilde{F}^{xr}, \\ \mathcal{J}^y &= \frac{Z(\phi)}{g_4^2} \sqrt{-g} F^{yr} + 4\sqrt{-g} W(\phi) \tilde{F}^{yr} - \xi M(r),\end{aligned}$$

but at ∞ , $M(r) = M$, and we obtain the usual transport currents,

$$\mathcal{J}^i(r = r_H) = \mathcal{J}^i(r \rightarrow \infty) = j^{i(\text{tot})} - \xi M = j^i,$$

while at r_H easier to calculate. Similar for heat currents.

- Then: Find currents as functions of E, ξ : e.g., j_x from $\delta g_{tx}, \delta A_x, \delta h_{ry}$, related from the eqs. of motion to E and $\xi = \frac{|\vec{\nabla} T|}{T}$.

- Then, derive the transport coefficients:

$$\begin{aligned}
\sigma_{xx} &= \frac{e^{2V} k^2 \Phi (2\kappa_4^2 g_4^4 \rho^2 + 2\kappa_4^2 B^2 Z^2 + g_4^2 Z e^{2V} k^2 \Phi)}{4\kappa_4^4 g_4^4 B^2 \rho^2 + (2\kappa_4^2 B^2 Z + g_4^2 e^{2V} k^2 \Phi)^2} \Big|_{r_H}, \\
\sigma_{xy} &= 4\kappa_4^2 B \rho \frac{\kappa_4^2 g_4^4 \rho^2 + \kappa_4^2 B^2 Z^2 + g_4^2 Z e^{2V} k^2 \Phi}{4\kappa_4^4 g_4^4 B^2 \rho^2 + (2\kappa_4^2 B^2 Z + g_4^2 e^{2V} k^2 \Phi)^2} - 4W \Big|_{r_H}, \\
\alpha_{xx} &= \frac{2\kappa_4^2 g_4^4 s \rho e^{2V} k^2 \Phi}{4\kappa_4^4 g_4^4 B^2 \rho^2 + (2\kappa_4^2 B^2 Z + g_4^2 e^{2V} k^2 \Phi)^2} \Big|_{r_H}, \\
\alpha_{xy} &= 2\kappa_4^2 s B \frac{2\kappa_4^2 g_4^4 \rho^2 + 2\kappa_4^2 B^2 Z^2 + g_4^2 Z e^{2V} k^2 \Phi}{4\kappa_4^4 g_4^4 B^2 \rho^2 + (2\kappa_4^2 B^2 Z + g_4^2 e^{2V} k^2 \Phi)^2} \Big|_{r_H}, \\
\frac{\bar{\kappa}_{xx}}{T} &= (2\kappa_4^2) s^2 \frac{g_4^2 [(2\kappa_4^2) B^2 Z + g_4^2 e^{2V} k^2 \Phi]}{(2\kappa_4^2)^2 g_4^4 \rho^2 B^2 + ((2\kappa_4^2) B^2 Z + g_4^2 e^{2V} k^2 \Phi)^2}, \\
\frac{\bar{\kappa}_{xy}}{T} &= (2\kappa_4^2) s^2 \frac{g_4^2 (2\kappa_4^2) g_4^2 \rho B}{(2\kappa_4^2)^2 g_4^4 \rho^2 B^2 + ((2\kappa_4^2) B^2 Z + g_4^2 e^{2V} k^2 \Phi)^2}.
\end{aligned}$$

- Here $\rho = -Ze^{2V} a'$ is field theory charge density ($A_t = a(r) \simeq a'(r_H)(r - r_H) \propto \rho$).

- Find interesting properties of strongly coupled transport, like S-duality, part of $Sl(2, \mathbb{Z})$ group.

- S-duality acting on $\sigma \equiv \sigma_{xy} + i\sigma_{xx}$ as $\sigma \rightarrow \sigma' = -\frac{1}{\sigma}$, or $\sigma'_{xx} = \frac{\sigma_{xx}}{\sigma_{xx}^2 + \sigma_{xy}^2} = \rho_{xx}$, $\sigma'_{xy} = -\frac{\sigma_{xy}}{\sigma_{xx}^2 + \sigma_{xy}^2} = \rho_{xy}$.
- At $\rho = B = 0$, Φ finite, $\sigma_{xx} = Z(r_H)$, $\sigma_{xy} = -4W(r_H) \Rightarrow$ transformation on Z, W that leaves gravitational action invariant:

$$F_{\mu\nu} \rightarrow Z(\phi)\tilde{F}_{\mu\nu} - \bar{W}(\phi)F_{\mu\nu} \equiv Z(\phi)\frac{1}{2}\epsilon_{\mu\nu\rho\sigma}F^{\rho\sigma} - \frac{W(\phi)}{4}$$

$$Z(\phi) \rightarrow -\frac{Z(\phi)}{Z(\phi)^2 + \bar{W}(\phi)^2}, \quad \bar{W}(\phi) \rightarrow \frac{\bar{W}(\phi)}{Z(\phi)^2 + \bar{W}(\phi)^2}.$$

- Transport formulas match 2+1d CMT strongly coupled model for near-transition supercond.-insulator of Hartnoll, Kovtun, Muller, Sachdev (2007), for $\omega \rightarrow \omega + \frac{i}{\tau_{\text{imp}}} \rightarrow \frac{e^2 V k^2}{2\kappa_4^2 s T} \Phi$.
- S-duality extended; also $\rho \rightarrow B$, $B \rightarrow -\rho$.
- Translational invariance breaking $\lambda \propto \Phi$ acts as an RG scale for an RG flow. $\sigma_Q = \frac{Z(r_H)}{g_4^2}$ is the critical point (UV) conductivity.
- A generalized Wiedemann-Franz law $L = \frac{\kappa_{xy}/T}{\sigma_{xy}} \rightarrow \frac{\pi^2}{3} c \frac{g_4^2}{Z} + \mathcal{O}(T)$.
Also $L_{xx} = \frac{\kappa_{xx}/T}{\sigma_{xx}} = L$ if $B = 0$, $\Phi \neq 0$, but very small.

- Toy model in 3d: ABJM model: 3d $\mathcal{N} = 6$ susy CS gauge theory with group $SU(N) \times SU(N)$, gauge fields $A_\mu, \tilde{A}_\mu$, 4 complex bifundamental scalars C^I , 4 fermions ψ^I , $I = 1, 2, 3, 4$, in the global R-symmetry group $SU(4) = SO(6)$ (full R-symm. $SU(4) \times U(1)$), with

$$S = \int d^{2+1}x \left[\frac{k}{4\pi} \epsilon^{\mu\nu\rho\sigma} \text{Tr} \left(A_\mu \partial_\nu A_\rho + \frac{2i}{3} A_\mu A_\nu A_\rho - \tilde{A}_\mu \partial_\nu \tilde{A}_\rho - \frac{2i}{3} \tilde{A}_\mu \tilde{A}_\nu \tilde{A}_\rho \right) \right. \\ \left. - \text{Tr} (D_\mu C_I^\dagger D^\mu C^I) - i \text{Tr} (\psi^{I\dagger} \gamma^\mu D_\mu \psi_I) + V_6(C^I) + \text{Tr} (C C^\dagger \psi \psi^\dagger \text{ term}) \right],$$

and $\mathcal{N} = 6$ enhanced to $\mathcal{N} = 8$ for $k = 1$ or $N = 2$. $\exists$ mass deformation that preserves $\mathcal{N} = 6$.

- Gravity dual of ABJM: string theory in $AdS_4 \times \mathbb{CP}^3$, obtained as $\mathbb{CP}^3 = S^7/\mathbb{Z}_k$ for $k \rightarrow \infty$.
- S^7 defined by constraint $\sum_{i=1}^4 |Z^i|^2 = 1$, obtained as a Hopf fibration with fiber S^1 , over $\mathbb{CP}^3$, S^1 fiber: phase $Z^i \rightarrow e^{i\alpha} Z^i, i = 1, \dots, 4$. Action of $\mathbb{Z}_k$: $Z^i \rightarrow e^{\frac{2\pi i n}{k}} Z^i, n = 0, 1, \dots, k-1$.
- ABJM at finite temperature: AdS_4 -BH, $\times \mathbb{CP}^3$. Hence toy model for 3d CFTs.

Lecture 10

Applications to QCD

- Two large classes of models: "top-down" (dualities derived from decoupled systems of branes) and "bottom-up" (phenomenological models: cook up a gravity dual with desired properties).

Top-down models

- **Finite temperature (Witten) model:** toy model for QCD_3 (pure glue)

$$\begin{aligned}
 ds^2 &= \left(\frac{\rho^2}{R^2} - \frac{R^{n-2}}{\rho^{n-2}} \right) d\tau^2 + \frac{d\rho^2}{\frac{\rho^2}{R^2} - \frac{R^{n-2}}{\rho^{n-2}}} + \rho^2 \sum_{i=1}^{n-1} dx_i^2 \Rightarrow \\
 ds^2 &= \frac{r^2}{R^2} \left[-dt^2 \left(1 - \frac{r_0^n}{r^n} \right) + d\vec{y}_{(n-1)}^2 \right] + R^2 \frac{dr^2}{r^2 \left(1 - \frac{r_0^n}{r^n} \right)},
 \end{aligned}$$

for $n = 3$, and adding S^5 metric. It satisfies the minimal ingredients and general features for QCD-like duals.

- **Cut-off AdS_5 : modified "hard-wall"**. Cut-off at $r_{\min} = R^2 \Lambda_{\text{QCD}}$.
- **Improvement:** cut-off dynamical, as D-brane at $r_{\min}$. Then, modes on D-brane: source pion-like operators. Position: model for the pion, for a precise version of the Froissart bound saturation (with m_π in it).

● **Polchinski-Strassler solution:** gravity dual of $\mathcal{N} = 1^*$ SYM = massive def. of $\mathcal{N} = 4$ SYM. Toy model for $\mathcal{N} = 1$ SYM and QCD. Brane config.: D3-branes "polarizing" ("puffing up", extra space appears) due to nonzero flux. Mass gap appears similarly to finite temp. $AdS_5 \times S^5$.

$$ds_{\text{string}}^2 = Z_x^{-1/2} d\vec{x}_{3+1}^2 + Z_y^{1/2} (dy^2 + y^2 d\Omega_y^2 + dw^2) + Z_\Omega^{1/2} w^2 d\Omega_w^2$$

$$Z_x = Z_y = Z_0 = \frac{R^4}{\rho_+^2 \rho_-^2}; \quad Z_\Omega = Z_0 \left[\frac{\rho_-^2}{\rho_-^2 + \rho_c^2} \right]^2$$

$$\rho_\pm = (y^2 + (w \pm r_0)^2)^{1/2}; \quad R^4 = 4\pi g_s N; \quad \rho_c = \frac{2g_s r_0 \alpha'}{R^2}; \quad r_0 = \pi \alpha' m N$$

$$e^{2\Phi} = g_s^2 \frac{\rho_-^2}{\rho_-^2 + \rho_c^2}.$$

● Metric goes over to $AdS_5 \times S^5$ at large $\rho = \rho_- \simeq \rho_+$. Near-core is $\rho \sim r_0$: typical warp factor $Z^{1/2}$ finite.

● **Klebanov-Strassler solution:** $\mathcal{N} = 1$ susy $SU(N+M) \times SU(N)$, with two chiral bifundamental A_1, A_2 in $(N+M, \bar{N})$ and two B_1, B_2 in $(\bar{(N+M)}, N)$. Brane config.: M "fractional D3-branes" on a conifold point in the near horizon.

• Has "duality cascade": Apply "Seiberg duality (strongly c. $SU(N_c)$ with N_f flavors into weakly c. $SU(N_f - N_c)$ with N_f flavors). Reduce thus gauge group, successively: $SU(N + M) \times SU(N) \rightarrow SU(N) \times SU(N - M) \rightarrow \dots$ minimum groups, at different energy scale $\Rightarrow$ cascade. Metric:

$$ds_{10}^2 = h^{-1/2}(\tau) d\vec{x}^2 + h^{1/2}(\tau) ds_6^2$$

$$ds_6^2 = \frac{1}{2} \epsilon^{4/3} K(\tau) \left[\frac{1}{3K^3(\tau)} (d\tau^2 + (g_5)^2) + \cosh^2\left(\frac{\tau}{2}\right) ((g_3)^2 + (g_4)^2) + \sinh^2\left(\frac{\tau}{2}\right) ((g_1)^2 + (g_2)^2) \right]$$

$$K(\tau) = \frac{(\sinh(2\tau) - 2\tau)^{1/3}}{2^{1/3} \sinh \tau}$$

$$h(\tau) = \alpha \frac{2^{2/3}}{4} \int_{\tau}^{\infty} dx \frac{x \coth x - 1}{\sinh^2 x} (\sinh(2x) - 2x)^{1/3}.$$

• At large τ , log-corrected $AdS_5 \times T^{1,1}$, in terms of $r \sim [\epsilon^2 e^{\tau}]^{1/3}$,

$$ds^2 = h^{-1/2}(r) d\vec{x}^2 + h^{1/2}(r) (dr^2 + r^2 ds_{T^{1,1}}^2)$$

$$h(r) \sim \frac{(g_s M)^2 \ln(r/r_s)}{r^4}$$

$$ds_{T^{1,1}}^2 = \frac{1}{9} \left(d\psi^2 + \sum_{i=1,2} \cos \theta_i d\phi_i \right)^2 + \frac{1}{6} \sum_{i=1,2} (d\theta_i^2 + \sin^2 \theta_i d\phi_i^2)$$

$$= \frac{1}{9} (g_5)^2 + \sum_{i=1}^4 (g_i)^2.$$

- Dilaton approx. const., $\phi = \phi_0$. Again, log-corrected $AdS_5 \times X_5$.
- Log correction related to renormalization of QFT = running coupling const.
- In the QFT IR, at small τ , metric terminates smoothly and warp factor $a_0^{1/2}$ remains finite,

$$ds^2 = a_0^{-1/2} d\vec{x}^2 + a_0^{1/2} \left(\frac{d\tau^2}{2} + d\Omega_3^2 + \frac{\tau^2}{4} ((g_1)^2 + (g_2)^2) \right) .$$

- **Maldacena-Núñez solution:** 4d $\mathcal{N} = 1$ SYM + massive modes. Brane config.: type IIB NS5-branes (S-dual to D5-branes) wrapped on S^2 . String frame metric and dilaton:

$$ds_{10}^2 = ds_{7,string}^2 + \alpha' N \frac{1}{4} (\tilde{w}^a - A^a)^2$$

$$H = N \left[-\frac{1}{4} \frac{1}{6} \epsilon_{abc} (\tilde{w}^a - A^a) \wedge (\tilde{w}^b - A^b) \wedge (\tilde{w}^c - A^c) + \frac{1}{4} F^a \wedge (\tilde{w}^a - A^a) \right]$$

$$ds_{7,string}^2 = d\vec{x}_{3+1}^2 + \alpha' N [d\rho^2 + R^2(\rho) d\Omega_2^2]$$

$$A = \frac{1}{2} [\sigma^1 a(\rho) d\theta + \sigma^2 a(\rho) \sin \theta d\phi + \sigma^3 \cos \theta d\phi] ; \quad a(\rho) = \frac{2\rho}{\sinh 2\rho}$$

$$R^2(\rho) = \rho \coth(2\rho) - \frac{\rho^2}{\sinh^2(2\rho)} - \frac{1}{4}$$

$$e^{2\phi} = e^{2\phi_0} \frac{2R(\rho)}{\sinh(2\rho)} .$$

- 10d sol.: uplift sol. of $\mathcal{N} = 1$ 7d sugra on S^3 transverse to 5-branes. $\tilde{w}^a$: left-inv. forms on S^3 . D5-brane metric is S-dual: $\phi \rightarrow \phi_D = -\phi$, $g_{\mu\nu}^E \rightarrow g_{\mu\nu}^E$, so:

$$ds_{\text{string}}^2 = e^{\phi_D} \left[dx_{(4)}^2 + \alpha' N \left(d\rho^2 + R^2(\rho) d\Omega_2^2 + \frac{1}{4} \sum_a (\tilde{w}^a - A^a)^2 \right) \right]$$

$$e^{2\phi_D} = e^{2\phi_{D,0}} \frac{\sinh(2\rho)}{2R(\rho)}.$$

- QCD string tension $T_s = \frac{e^{\phi_{D,0}}}{2\pi\alpha'}$, and $M_{\text{glueballs}}^2 \sim M_{KK}^2 \sim \frac{1}{R_2^2} \sim \frac{1}{N\alpha'}$, so decoupling of KK states would mean $T_s \ll M_{KK}^2 \rightarrow e^{\phi_{D,0}} N \ll 1$.

- BUT: sugra approx. $\rightarrow$ curvature small in string units $\rightarrow e^{\phi_{D,0}} N \gg 1$: opposite. So can't decouple KK modes.

- UV of QFT: $\rho \rightarrow \infty$,

$$R^2 \simeq \rho; \quad a \simeq 2\rho e^{-2\rho}; \quad \phi \simeq \phi_0 - \rho + \frac{\log \rho}{4} \Rightarrow$$

$$ds^2 = d\vec{x}_{3+1}^2 + \alpha' N \left[d\rho^2 + \rho d\Omega_2^2 + \frac{1}{4} (\tilde{w}^a - A^a)^2 \right]$$

$$= d\vec{x}_{3+1}^2 + \alpha' N \left[\frac{dz^2}{z^2} + (-\log z) d\Omega_2^2 + \frac{1}{4} (\tilde{w}^a - A^a)^2 \right].$$

• But dilaton nontrivial, so is actually equiv. to log-corrected $AdS_5 \times X_5$:

$$\begin{aligned}
 S &= \frac{1}{2\kappa_N^2} \int d^5x \sqrt{-g_5} \left(\int_{X_5} \sqrt{g_{X_5}} \right) g^{\mu\nu} e^{-2\phi} [R_{\mu\nu} - \partial_\mu X \partial_\nu X + \dots] , \\
 ds^2 &= e^{2A(\rho)} d\vec{x}_{3+1}^2 + d\rho^2 + ds_{X_5}^2 = e^{2A(z)} d\vec{x}_{3+1}^2 + \frac{dz^2}{z^2} + ds_{X_5}^2 \Rightarrow \\
 S &= \frac{1}{2\kappa_N^2} \int d^4x d\rho \left(\int_{X_5} \sqrt{g_{X_5}} \right) e^{2(A-\phi)} \delta^{\mu\nu} [R_{\mu\nu} - \partial_\mu X \partial_\nu X + \dots] ,
 \end{aligned}$$

so the condition for log-corr. $AdS_5 \times X_5$ is in fact

$$\phi - \phi_0 - A \xrightarrow{\rho \rightarrow \infty} -\rho (+ \text{log corrections}) = + \log z (+ \text{corrections}) ,$$

and is satisfied ($A = 0$, $\phi = \phi_0 - \rho + \dots = \phi_0 + \log z + \dots$). So UV: OK.

• IR of QFT: at $\rho \rightarrow 0$, the *effective* warp factor $e^{2(A-\phi)}$ is constant,

$$R^2 = \rho^2 + \mathcal{O}(\rho^4); \quad a = 1 + \mathcal{O}(\rho^2); \quad \phi = \phi_0 + \mathcal{O}(\rho^2) .$$

• **Maldacena-Nāstase solution:** analog of Maldacena-Núñez for 3d: 3d $\mathcal{N} = 1$ SYM, with a Chern-Simons coupling, coupled to other massive modes. Brane config.: NS5-branes wrapped on S^3 . CS level k : gravity dual $k = N/2$. Index comp.: $\exists$ unique vacuum, confining. Solution:

$$ds_{10}^2 = ds_{7,string}^2 + \alpha' N \frac{1}{4} (\tilde{w}^a - A^a)^2$$

$$H = N \left[-\frac{1}{4} \frac{1}{6} \epsilon_{abc} (\tilde{w}^a - A^a) \wedge (\tilde{w}^b - A^b) \wedge (\tilde{w}^c - A^c) + \frac{1}{4} F^a \wedge (\tilde{w}^a - A^a) \right] + h$$

$$ds_{7,string}^2 = d\vec{x}_{2+1}^2 + \alpha' N [d\rho^2 + R^2(\rho) d\Omega_3^2]$$

$$A = \frac{w(\rho) + 1}{2} w_L^a$$

$$h = N[w^3(\rho) - 3w(\rho) + 2] \frac{1}{16} \frac{1}{6} \epsilon_{abc} w^a \wedge w^b \wedge w^c ,$$

w^a are left-inv. forms on S^3 for $d\Omega_3^2$, $\tilde{w}^a$ are same for transverse $\tilde{S}^3$ and $w(\rho), R(\rho), \phi(\rho)$ are found numerically.

• QFT UV: large ρ :

$$R^2(\rho) \sim 2\rho; \quad w(\rho) \sim \frac{1}{4\rho}; \quad \phi = -\rho + \frac{3}{8} \log \rho ,$$

so log-corrected $AdS_4 \times X_6$, since

$$\phi - \phi_0 - A \rightarrow -\rho + \log \text{ corrections} .$$

- IR of QFT: $\rho \rightarrow 0$,

$$R^2(\rho) = \rho^2 + \mathcal{O}(\rho^4); \quad w(\rho) = 1 + \mathcal{O}(\rho^2); \quad \phi = \phi_0 + \mathcal{O}(\rho^2),$$

so has finite warp factor, $e^{2(A-\phi)} = e^{-2\phi_0} + \dots$.

- Dynamical susy breaking: Put small nr. n branes ($n \ll N/2$) on noncontractible S^3 , so dual QFT has $k = N/2 + n$: susy unbroken if $n > 0$ (branes), but broken if $n < 0$ (antibranes).

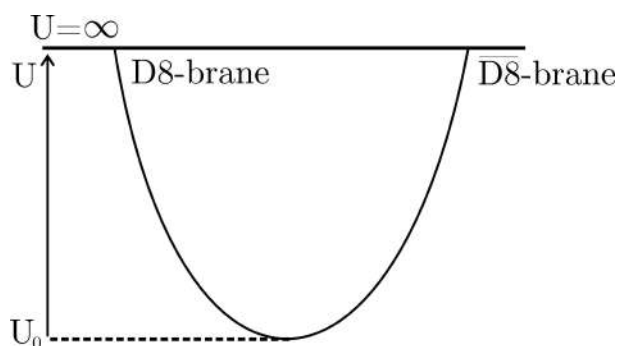
- Sakai-Sugimoto model**: has quarks (fermions in the fundamental) in the probe approx. (so, no back-reaction). Quarks: either fixed D-branes (e.g., branes at orientifold point for $\mathcal{N} = 2$ AdS/CFT), or probe D-branes: here.

- N_c Wick-rotated D4-branes at finite temperature, for gravity dual similar to Witten model,

$$ds^2 = \left(\frac{U}{R}\right)^{3/2} (f(U)d\tau^2 + d\vec{x}_{(4)}^2) + \left(\frac{R}{U}\right)^{3/2} \left(\frac{dU^2}{f(U)} + U^2 d\Omega_4^2\right)$$

$$e^\phi = g_s \left(\frac{U}{R}\right)^{3/4}; \quad F_4 = \frac{2\pi N_c}{V_4} \epsilon_4; \quad f(U) = 1 - \frac{U_{KK}^3}{U^3}.$$

- In this background, N_f D8-brane probes, with transverse coord. U , dep. on worldvol. coord. τ , $U = U(\tau)$. Probe interpreted as $D8 - \bar{D}8 =$ susy breaking, joined in bulk.



The Sakai-Sugimoto model has a probe D8-brane in the gravity dual, starting from infinity and returning to it. At infinity, it looks like a D8-brane/anti-D8-brane pair (parallel branes of opposite orientation).

- Solution for $U(\tau)$ to equations of motion:

$$\tau(U) = U_0^4 f(U_0)^{1/2} \int_{U_0}^U \frac{dU}{\left(\frac{U}{R}\right)^{3/2} f(U) \sqrt{U^8 f(U) - U_0^8 f(U_0)}}.$$

- Modes on D8 couple to mesonic ops. (pion-like), charged under global symm.

- Background related to Witten model for QCD_4 : D4-branes at finite T , (some transf. and) compactify on periodic Euclidean time. Metric at large U (UV of QFT): in terms of $\rho = \sqrt{U}$:

$$ds^2 \sim \rho \left[\rho^2 (f(\rho) d\tau^2 + d\vec{x}^2) + \frac{d\rho^2}{f(\rho)\rho^2} + d\Omega_4^2 \right].$$

- Is conformal factor $\times AdS_6 \times S^4$: OK for compactif. to 4d theory.. Cut-off at finite $U = U_{KK}$ in IR of QFT. Obs.: τ is compact, and also $\tau = \tau(U)$, so D8-brane probes 4d theory.

- **Mass spectra in gravity duals from field eigenmodes.** For a field in AdS space, dual tower of glueball states: tower of discrete modes.

- E.G. Scalar dilaton (massless) Φ dual to $\text{Tr}[F_{\mu\nu}F^{\mu\nu}]$, glueball 0^{++} , and its excited states.

- Field theory mass: from x space or p space 2-point functions of the operator,

$$\begin{aligned} \langle \mathcal{O}(x)\mathcal{O}(y) \rangle &\propto e^{-m_1|x-y|} (+ \# e^{-m_2|x-y|} + \dots). \\ \langle \mathcal{O}(p)\mathcal{O}(-p) \rangle &\sim \sum_j \frac{A_j}{p^2 + m_j^2}. \end{aligned}$$

- In AdS, spectrum of $\vec{k}^2 = -m^2$ for solutions of the free eq. of m. for Φ .
- Infinite discrete spectrum (with no accumulation points) if light takes finite time from boundary to boundary (horizon).
- For an $AdS_{n+1} \times S^m$ space, made non-extremal by a blackening function, we have a finite bd. to bd. time,

$$ds^2 = u^\alpha \left[\left(1 - \frac{u_T^m}{u^m} \right) d\tau^2 - dt^2 + d\vec{x}_{n-2}^2 \right] + \frac{du^2}{u^2 \left(1 - \frac{u_T^m}{u^m} \right)} + d\Omega_m^2.$$

- Defining time of flight variable x by $dx = d\rho \sqrt{g_{\rho\rho}/g_{tt}}$, massless KG eq. $\square\Psi = 0$ becomes 1d QM problem for $E \equiv m^2 = -k^2$ (k is 4d momentum),

$$\left[-\frac{d^2}{dx^2} + (V(x) - E) \right] \tilde{\Psi}(x) = 0.$$

- So $x_{\max}$ finite means $V(x)$ has finite support, so $E_n \equiv m_n^2$ discrete, infinite in nr., with no accumulation points, as for 1d QM box.

- But this is not the only way to obtain mass gap! NS5-branes in flat space have

$$ds_{10,\text{string}}^2 = d\vec{x}_{5+1}^2 + d\rho^2 + d\Omega_3^2, \quad \phi = -\rho.$$

- But $\square$ contains the Einstein metric. So repeating procedure for $g_{\mu\nu}^E$, one finds $V(x) = 1$, with x ranging from 0 to ∞ . So continuous spectrum above a mass gap, though not discrete.
- For Witten's QCD_3 model, one finds x between 0 and $x_{\max} = C$, and $V(0) \rightarrow -\infty$ ($V(x) \simeq -1/(4x^2)$), $V(C) \rightarrow +\infty$ ($V(x) \simeq 15/(4(x-C)^2)$). Then one finds $m^2 R^2 = 6n(n+1)$, so at large n , $E_n = m_n^2 R^2 \propto n^2$, like for particle in a box.
- For Polchinski-Strassler, one finds a finite time of flight at infinity, and near the core, one finds a near-shell approx., with a 5-brane throat, so strictly speaking one cannot regulate the divergence.
- For Klebanov-Strassler, again finite time of flight at infinity, but smooth cut-off at small r (like the plane in spherical coords.), so time of flight is regulated, and spectrum of scalar is discrete.

- For Maldacena-Núñez, $V(\rho) = 4/3 + \dots$ in the IR and $V(\rho) \simeq 1 + 1/(2\rho)$ in the UV, whereas for Maldacena-Nastase, $V(\rho) \simeq 4/(4\rho^2)$ in the IR and $V(\rho) \simeq 1 + 3/(4\rho)$ in the UV. So continuum of states at high energies above a mass gap, though there could be discrete states as well (if there is an energy well at intermediate energies). Modification of the flat space 5-brane spectrum.

- **Mass spectra in gravity dual from mode expansion on probe branes** Sakai-Sugimoto model: Find worldvolume action for $A_M, M = 0, 1, \dots, 4$, for the D8-brane KK reduced on S^4 . Use coordinates (y, z) where the D8-brane is flat, situated at $y = 0$ (extends in z):

$$(y, z) = (\sqrt{U^3 - 1} \cos \tau, \sqrt{U^3 - 1} \sin \tau) .$$

- The DBI+CS action for $A_M = (A_\mu(x^\nu, z), A_z(x^\nu, z))$ is

$$S = \frac{\lambda N_c}{216\pi^3} \int d^4x dz \text{Tr} \left[\frac{1}{2} K^{-1/3} F_{\mu\nu}^2 + K F_{\mu z}^2 \right] + \frac{N_c}{24\pi^2} \int_{M^4 \times \mathbb{R}} \omega_{5,CS}(A), \quad K(z) \equiv 1 + z^2.$$

- Expand in complete and orthonormal sets $\{a_n(z)\}_{n \geq 1}$ and $\{\phi_n(z)\}_{n \geq 0}$ (such that $-K^{1/3} \partial_z (K \partial_z a_n) = \mu_n^2 a_n$),

$$A_\mu(x^\nu, z) = \sum_{n=1}^{\infty} A_\mu^{(n)}(x^\nu) a_n(z)$$

$$A_z(x^\nu, z) = \varphi^{(0)}(x^\nu) \phi_0(z) + \sum_{n=1}^{\infty} \varphi^{(n)}(x^\nu) \phi_n(z),$$

and obtain the kinetic terms

$$S_{D8}^{\text{DBI}} = \int d^4x \text{Tr} \left[(\partial_\mu \varphi^{(0)})^2 + \sum_{n=1}^{\infty} \left(\frac{1}{2} (\partial_\mu A_\nu^{(n)} - \partial_\nu A_\mu^{(n)})^2 + \mu_n^2 (A_\mu^{(n)} - \mu_n^{-1} \partial_\mu \varphi^{(n)})^2 \right) \right] + \text{int},$$

allowing us to identify $\varphi^{(0)}$ with the pion, and $A_\mu^{(n)}$ become massive by eating $\varphi^{(n)}$: vector mesons.

Phenomenological (bottom-up) gauge/gravity duality: AdS/QCD

- **Extended "hard-wall" model.** Extend hard-wall of Polchinski-Strassler. Ehrlich, Katz, Son, Stephanov, 2005. Add gauge fields $A_{L\mu}^a, A_{R\mu}^a$ coupling to currents of $S(N_f)_L \times SU(N_f)_R$ flavor symm., $\bar{q}_L \gamma^\mu T^a q_L$ and $\bar{q}_R \gamma^\mu R^a q_R$, and bifundamental tachyonic scalar $X^{\alpha\beta}$ of $m^2 R^2 = -3 > -4$ (BF bound) coupling to chiral order parameter $\mathcal{O}_X = \bar{q}_R^\alpha q_L^\beta$. Action:

$$S = \int d^5x \sqrt{-g} \text{Tr} \left[-|D_\mu X|^2 + 3|X|^2 - \frac{1}{4g_5^2} (F_{L\mu\nu}^2 + F_{R\mu\nu}^2) \right],$$

where $D_\mu X = \partial_\mu X - iA_{L\mu} X + iX A_{R\mu}$, $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - i[A_\mu, A_\nu]$.

- Boundary conditions: in the **IR**: $(F_L)_{z\mu} = (F_R)_{z\mu} = 0$; in the radial gauge $A_z = 0$ becomes Neumann: $\partial_z A_{L,R\mu} = 0$; also for X (Neumann or Dirichlet). In the **UV** ($z = 0$): $A_{L,R\mu}^a \rightarrow a_{L,R\mu}^a$ (sources for $J_{L,R\mu}^a$), and also ($\mathcal{O}_X$ has $\Delta = 3$ in $d = 4$):

$$X \rightarrow z^{d-\Delta} (X_0 + z^{2\Delta-d} X_{(2\Delta-d)}) = zX_0 + z^3 X_{(2)} = \frac{1}{2} M z + \frac{1}{2} \Sigma z^3,$$

where $X_0 =$ source for $\mathcal{O}_X = M^{\alpha\beta}/2$ (quark mass matrix) and $X_{(2)} = \Sigma^{\alpha\beta}/2$ gives VEV of $\mathcal{O}_X$, $\Sigma^{\alpha\beta} = \langle \bar{q}_R^\alpha q_L^\beta \rangle$.

- If $M = m_q 1l$ and $\Sigma = \sigma 1l$, 4 parameters: m_q, σ, z_m, g_5 and 3 fields: $A_{L\mu}, A_{R\mu}, X$. Can introduce vector $V_\mu = (A_{L\mu} + A_{R\mu})/2$ and axial vector $A_\mu = (A_{L\mu} - A_{R\mu})/2$ coupling to $\bar{q}\gamma_\mu T^a q$ and $\bar{q}\gamma_5\gamma_\mu T^a q$.

- IR+UV conditions imply quantized solutions (discrete spectrum).

- In gauge $V_z(\vec{x}, z) = 0$, in Fourier modes for $\vec{x}$ ($Q^2 = -\vec{q}^2$), we have, from the eqs. of m.,

$$V_\mu(\vec{q}, z) = V(\vec{q}, z)V_{0\mu}(\vec{q}); \quad V(\vec{q}, z = \epsilon) = 1 \Rightarrow V(Q, z) = 1 + \frac{Q^2 z^2}{4} \ln(Q^2 z^2) + \dots$$

- 2-point function for currents:

$$\begin{aligned} \langle J_\mu^a(x) J_\nu^b(0) \rangle &= \frac{\delta^2 S_{\text{sugra}}}{\delta V_{0\mu}^a(x) \delta V_{0\nu}^b(0)} = -\frac{1}{2g_5^2} \frac{\delta^2}{\delta V_{0\mu}^a(x) \delta V_{0\nu}^b(0)} \int_{z=\epsilon} d^4x \left(\frac{1}{z} V_\mu^a \partial_z V^{\mu a} \right) \\ \int d^4x e^{i\vec{q}\cdot\vec{x}} \langle J_\mu^a(x) J_\nu^b(0) \rangle &= \delta^{ab} (q_\mu q_\nu - \vec{q}^2 g_{\mu\nu}) \Pi_V(Q^2) \\ \Pi_V(Q^2) &= -\frac{1}{g_5^2 Q^2} \frac{\partial_z V(\vec{q}, z)}{z} \Big|_{z=\epsilon} = -\frac{1}{2g_5^2} \ln Q^2. \end{aligned}$$

- Compare with perturbative result:

$$\Pi_V(Q^2) = -\frac{N_c}{24\pi^2} \ln Q^2 \Rightarrow g_5^2 = \frac{12\pi^2}{N_c}.$$

- Decay constants into vector meson ρ_n and pol.vector ϵ_n , defined by:

$$\langle 0 | J_\mu^a | \rho_n^b \rangle = F_n \delta^{ab} \epsilon_\mu ,$$

- which implies

$$\begin{aligned} \delta_a^a \epsilon_\mu \epsilon^\mu \sum_n \frac{F_n^2}{m_n^2(\vec{q}^2 - m_n^2 + i\epsilon)} &= \sum_n \langle 0 | J_\mu^a(q) | \rho_n^b \rangle \frac{1}{m_n^2(\vec{q}^2 - m_n^2 + i\epsilon)} \langle \rho_n^b | J_\mu^a(-q) | 0 \rangle \\ &= \frac{1}{\vec{q}^2} \langle 0 | J_\mu^a(q) J_\mu^a(-q) | 0 \rangle = -3\delta_a^a \Pi_v(q^2) , \end{aligned}$$

to be matched against ($\psi_n(z)$ are quantized (discrete) sols. for vector meson states)

$$G(\vec{q}; z, z') = \sum_n \frac{\psi_n(z) \psi_n(z')}{\vec{q}^2 - m_n^2 + i\epsilon} , \Rightarrow \Pi_V(\vec{q}^2) = -\frac{1}{g_5^2} \sum_n \frac{|\psi'(\epsilon)/\epsilon|^2}{(\vec{q}^2 - m_n^2 + i\epsilon) m_n^2} ,$$

leading to

$$F_n^2 = \frac{1}{g_5^2} [\psi'(\epsilon)/\epsilon]^2 \rightarrow \frac{1}{g_5^2} [\psi_n''(0)]^2 .$$

- Other masses, couplings, decay constants can be calculated: fix parameters, then predict others.

- **Soft-wall model for QCD.** Modify background to have good QCD properties for the spectrum. In particular, $m_n^2 \propto n$ (for hard-wall, $m_n^2 \propto n^2$ at large n) and for high spin $S \gg 1$, $m_n^2 \propto S$. More precisely,

$$m_n^2 \sim \sigma n; \quad m_S^2 \sim \sigma S.$$

- Ansatz

$$ds^2 = g_{MN} dx^M dx^N = e^{2A(z)} (\eta_{\mu\nu} dx^\mu dx^\nu + dz^2) = e^{2A(u)} \eta_{\mu\nu} dx^\mu dx^\nu + du^2.$$

- As before, relevant combination is $\Phi(z) - A(z)$, so we want boundary conditions:

$$\text{UV } (z \rightarrow 0) : \Phi(z) - A(z) \sim \log z, \quad \text{IR } (z \rightarrow \infty) : \Phi(z) - A(z) \sim z^2.$$

- Simplest solution:

$$\Phi(z) - A(z) = z^2 + \log z.$$

- Action (has dilaton Φ extra):

$$S = \int d^5x \sqrt{-g} e^{-\Phi(z)} \text{Tr} \left[-|D_\mu X|^2 + 3|X|^2 - \frac{1}{4g_5^2} (F_{L\mu\nu}^2 + F_{R\mu\nu}^2) \right].$$

•Boundary conditions appear because: Schrödinger eq. with potential $V(z)$, for $B(z) = \Phi(z) - A(z)$,

$$-\psi_n'' + V(z)\psi_n = m_n^2\psi_n, \quad V(z) = \frac{1}{4}(B')^2 - \frac{1}{2}B''.$$

•Then $V(z) \propto z^2$ at large z for $E_n = m_n^2 \propto n$, implying $B(z) \propto z^2$; also $B = z^2/z_m^2 + \log z$ gives

$$m_n^2 z_m^2 = E_n = 4(n+1) \Rightarrow V_n(z) = e^{B(z)/2} \psi_n(z) = z^2 \sqrt{\frac{2n!}{(n+1)!}} L_n^1(z^2).$$

•Decay constants become

$$F_n^2 = \frac{1}{g_5^2} [V_n''(0)]^2 = \frac{8(n+1)}{g_5^2}.$$

•To fix Φ and A , not just $\Phi - A$, need higher spin ($S > 2$), $\phi_{M_1 \dots M_S}$, totally symmetric, with gauge invariance $\delta \phi_{M_1 \dots M_S} = D_{(M_1} \xi_{M_2 \dots M_S)}$, and same equations of motion, just with $B = \Phi - (2S - 1)A$. Then again $V(z) \propto z^2$ at large z , but indep. of S , so

$$\Phi \simeq \frac{z^2}{z_m^2}, z \rightarrow \infty, \quad A \propto -\log z, z \rightarrow 0. \quad \text{If } \Phi = \frac{z^2}{z_m^2}, \quad A = -\log z \Rightarrow$$

$$V(z) = \frac{z^2}{z_m^4} + \frac{2(S-1)}{z_m^2} + \frac{S^2 - 1/4}{z^2}, \quad E_n \equiv m_{n,S}^2 z_m^2 = 4(n+S).$$

- **Improved holographic QCD**: engineer a scalar potential in the gravity dual to holographically give wanted running coupling constant.

- BUT: $\lambda(\mu)$ not known, so we need an ansatz: from integrating 2-loop beta function,

$$\mu \frac{d\lambda}{d\mu} = -\frac{d\lambda}{d \log z} = \beta(\lambda) = -b_0 \lambda^2 + b_1 \lambda^3 + b_2 \lambda^4 + \dots \Rightarrow$$

$$\frac{1}{\lambda} \equiv \alpha_s = L - \frac{b_1}{b_0} \log L + \frac{b_1^2}{b_0^2} \frac{\log L}{L} + \mathcal{O}\left(\frac{1}{L^2}\right), \quad L \equiv -b_0 \log(z\Lambda).$$

- Gravity dual (see previous) $A(z) \simeq -\log z$, $z = 1/E$, so $du = e^{A(z)} dz$ gives $u \simeq \log z$, $du = -d \log E$.

- Write potential for $\lambda = N e^\Phi$, and expand

$$V(\lambda) = \sum_{n=0}^{\infty} V_n \lambda^n = V_0 + \frac{V_1}{L} + \frac{V_2}{L^2} + \frac{b_1}{b_0} V_1 \frac{\log L}{L^2} + \mathcal{O}\left(\frac{1}{L^3}\right).$$

- V_i from Einstein equation for action (coming from sugra plus N_f effective $D4 - \bar{D}4$ pairs = $Dp - \bar{D}p$ wrapped on compact space),

$$S = M_{\text{Pl},5}^3 N_c^2 \int d^5 x \sqrt{-g} \left[R - \frac{4}{3} \frac{(\partial_\mu \lambda)^2}{\lambda^2} - V(\lambda) \right].$$

- From the equations of motion written in terms of λ , and then in L , their compatibility requires that

$$V_1 = \frac{8}{9}b_0V_0; \quad V_2 = \frac{23b_0^2 - 36b_1}{3^4}V_0 \Rightarrow V = V_0 \left(1 + \frac{8}{9}b_0\lambda + \frac{23b_0^2 - 36b_1}{3^4}\lambda^2 \right) + \mathcal{O}(\lambda^3),$$

$$ds^2 = \left[1 + \frac{8}{3^2 \log(z\Lambda)} + \frac{4 \left(26 + 9\frac{b_1}{b_0^2} - 18\frac{b_1}{b_0^2} \log(b_0 \log \frac{1}{z\Lambda}) \right)}{3^4 \log^2(z\Lambda)} \right. \\ \left. + \mathcal{O} \left(\frac{\log^2 \log(z\Lambda)}{\log^3(z\Lambda)} \right) \right] \frac{R^2}{z^2} (dz^2 + d\vec{x}^2).$$

- Obs.: we matched the UV asymptotics, but the IR one (most interested in) needs large λ beta function, for which we have an ansatz: extra layer of phenomenology.

Introduction to the AdS/CFT correspondence

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Abstract

The aim of these notes is to give a basic introduction to the AdS/CFT correspondence without assuming a previous exposure to string theory and D-branes. Some minimal knowledge of supersymmetry is required in the second part of the lectures where explicit realizations of the AdS/CFT correspondence are discussed. These notes arise from a Ph. D. course given at EPFL in Lausanne for the Troisième cycle de la physique en Suisse romande.

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1 Introduction

One of the finest achievements of string theory in the last decade is the AdS/CFT correspondence and the use of holography to investigate strongly coupled quantum field theories. One crucial aspect of the correspondence is the possibility of computing quantum effects in a strongly coupled field theory using a classical gravitational theory. This has deep consequences that go far beyond string theory. Originally introduced to study the quantum behaviour of scale invariant theories, the correspondence has been extended to non conformal theories, where it gives an explanation for confinement and chiral symmetry breaking. It has also been used to study non-equilibrium phenomena in strongly coupled plasmas, and, more recently, applied to condensed matter systems. The correspondence also naturally implements the 't Hooft large N expansion, thus providing a verification of many ideas about gauge theories at large N .

These lectures grow from a Ph. D. course given at the EPFL in Lausanne to a

public with minimal previous exposure to string theory and D-brane. The conceptual framework of the correspondence is so simple that it can be discussed without entering deeply in the string realm. We shall introduce the relevant ingredients of string theory and D-branes using an effective theory language. In particular, the first half of these lectures, where the abstract duality between CFTs and gravitational theories in AdS is defined and discussed, uses nothing else than standard quantum field theory and General Relativity. We shall need a little bit of string theory, in particular D-branes, to provide explicit examples of dual pairs.

These lectures provide a very elementary introduction to the correspondence. In particular, we do not cover in details the existing non conformal supersymmetric solutions, the correspondence for conformal theories with less supersymmetry or in dimension different than four, and the inclusion of flavors. The reader may find more details on the basics of the correspondence in the original articles [1–4] and in very good reviews [5–8]. Due to the large numbers of papers in the field, we refer to the list of references in [5] and the other reviews. References to specific papers are given only for specific results and are not exhaustive. A survey of more recent developments can be found in [9–13].

These lectures are organized as follows. In the rest of this long introduction, we introduce the two basic players of the correspondence, the conformal field theories and the gravitational theories with AdS vacuum. We also give a brief review of the large N limit in quantum field theory. General references for this background material are [14, 15]. The reader with some previous knowledge of these subjects is encouraged to start from the main part of these lectures, beginning with section 2, and to return to the introduction for reference.

In section 2, we discuss the conceptual content of the AdS/CFT correspondence without resorting to explicit realizations. We shall explain how to construct consistent correlation functions of a local quantum field theory from the equations of motion of a classical gravitational theory in AdS. We shall also extend the correspondence to the non-conformal case and discuss the holographic description of confinement.

In section 3 and 4 we discuss explicit realizations of the AdS/CFT correspondence, focusing on the best understood example, the duality between $\mathcal{N} = 4$ SYM and $AdS_5 \times S^5$. Section 3 requires a minimal knowledge of supersymmetry, at the level of $\mathcal{N} = 1$ multiplets and superfields. All the necessary ingredients of string theory and D-branes are introduced and discussed at the effective action level. Obviously, an idea of what string theory is will help, but no specific knowledge is required.

1.1 Conformal theories

Theories without scales or dimensionful parameters are classically scale invariant. A simple example is the scalar field with only quartic interaction

$$S = \int dx^4 \left((\partial\phi)^2 + \frac{\lambda}{4!} \phi^4 \right). \quad (1.1)$$

The action is invariant if we simultaneously rescale the space-time coordinates (scale transformation) and the field with a specific weight

$$\phi(x) \rightarrow \lambda^\Delta \phi(\lambda x) \quad (1.2)$$

Δ is called the scaling dimension of the the field and here it coincides with the canonical dimension $\Delta = 1$. The same theory would not be invariant if we add a mass term, as the reader may easily check. Another example of classically scale invariant theory is Yang-Mills coupled to massless fermions and scalars. In all these theories, scale invariance is broken by quantum corrections, but we shall see soon examples of quantum field theories with exact scale invariance.

1.1.1 The conformal group

Invariance under scale transformations typically implies invariance under the bigger group of conformal transformations.

A conformal transformation in a D-dimensional space-time is a change of coordinates that rescales the line element,

$$\begin{aligned} \text{dilatation : } x_\mu &\rightarrow \lambda x_\mu & (dx)^2 &\rightarrow \lambda^2 (dx)^2 \\ \text{conformal transformation : } x_\mu &\rightarrow x'_\mu & (dx)^2 &\rightarrow (dx')^2 = \Omega^2(x) (dx)^2 \end{aligned} \quad (1.3)$$

where $\Omega(x)$ is an arbitrary function of the coordinates. Clearly, scale transformations are a particular case of conformal transformations with constant $\Omega = \lambda$. Conformal transformations rescale lengths but preserve the angles between vectors. At the infinitesimal level $x'_\mu = x_\mu + v_\mu(x)$, $\Omega(x) = 1 + \omega(x)/2$ we easily derive the condition

$$\partial_\mu v_\nu + \partial_\nu v_\mu = \omega(x) \eta_{\mu\nu} \quad (1.4)$$

Taking a trace we obtain $D\omega = 2\partial^\mu v_\mu$ and, substituting this expression in the previous formula, we obtain an equation identifying conformal transformations at the infinitesimal level

$$\partial_\mu v_\nu + \partial_\nu v_\mu - \frac{2}{D} (\partial^\tau v_\tau) \eta_{\mu\nu} = 0. \quad (1.5)$$

It is well known that in two dimensions there are infinite solutions of this equation (given, after Euclidean continuation, by all possible holomorphic functions on a plane) and the conformal group is infinite dimensional. For $D \neq 2$ the number of solutions is smaller and given by at most quadratic functions $v_\mu(x)$. The general solution is indeed

$$\begin{aligned} \delta x_\mu = & \quad a_\mu & P_\mu \\ & \omega_{\mu\nu} x_\nu & J_{\mu\nu} \quad (\omega_{\mu\nu} = -\omega_{\nu\mu}) \\ & \lambda x_\mu & D \\ & (b_\mu x^2 - 2x_\mu(bx)) & K_\mu. \end{aligned} \tag{1.6}$$

We recognize, in the first two lines, translations (whose generator will be denoted by P_μ) and Lorentz transformations (generated by $J_{\mu\nu}$); they are obviously conformal transformations (with $\Omega = 1$) since they leave the line element invariant. The third line corresponds to the dilatation (generated by D). The only novelty is the special conformal transformation (generated by K_μ) given by the fourth line. The corresponding finite transformation is

$$x_\mu \rightarrow \frac{x_\mu + c_\mu x^2}{1 + 2cx + (cx)^2} \tag{1.7}$$

Altogether we have

$$D + \frac{D(D-1)}{2} + 1 + D = \frac{(D+1)(D+2)}{2} \tag{1.8}$$

generators. In fact, one can check that the group is isomorphic to $SO(2, D)$ (for an algebraic proof see below). There is an extra discrete symmetry that acts as a conformal transformation,

$$\begin{aligned} x_\mu &\rightarrow \frac{x_\mu}{x^2} \\ (dx)^2 &\rightarrow x^2(dx)^2. \end{aligned} \tag{1.9}$$

Adding this discrete transformation we obtain the full conformal group $O(2, D)$.

The important point is that, under mild conditions, a scale invariant theory is also conformal invariant. We can easily construct currents associated with the conformal transformations,

$$J_\mu = T_{\mu\nu} \delta x^\nu \tag{1.10}$$

This expression, with some subtleties and redefinitions, can be derived from Noether's theorem [14]. Conservation of the current corresponding to translations requires conservation of the stress energy tensor $\partial^\mu T_{\mu\nu} = 0$ and conservation of the current corresponding to Lorentz transformations is then automatic if $T_{\mu\nu}$ is symmetric. The current for dilation $J_\mu = T_{\mu\nu}x^\nu$ is now conserved if

$$\partial^\mu (T_{\mu\nu}x^\nu) = T_\nu^\nu \equiv 0 \quad (1.11)$$

We see that the condition for scale invariance is the tracelessness of the stress energy tensor. Now, we easily see that in a Poincaré and scale invariant theory (with a symmetric traceless conserved stress energy tensor) the conformal currents are automatically conserved,

$$\partial^\mu (T_{\mu\nu}v^\nu) = \partial^\mu T_{\mu\nu}v^\nu + T_{\mu\nu}\partial^\mu v^\nu = \frac{1}{2}T^{\mu\nu}(\partial_\mu v_\nu + \partial_\nu v_\mu) = \frac{1}{D}\partial^\tau v_\tau T_\mu^\mu \equiv 0 \quad (1.12)$$

The conditions on trace and symmetry properties of the stress energy tensor can be easily realized in most reasonable classical and quantum field theories and, although exotic counterexamples exist, we can safely assume that a scale invariant theory enjoys the full conformal invariance.

In the presence of supersymmetry, the conformal group is enhanced to a supergroup¹ obtained by $O(2, D)$ by adding the supercharges Q^a and the R-symmetry that rotates them. We also need to add the so-called conformal supercharges S^a . These are required to close the superconformal algebra $[K, Q] \sim S$. We shall not use explicitly the algebra of the superconformal group; the reader can find more details in the Appendix.

1.1.2 Conformal quantum field theories

In a quantum theory, conformal invariance is broken by the introduction of a renormalization scale. The Renormalization Group (RG) and the Callan-Symanzik equation can be seen as anomalous Ward identity for dilatations. For example, in a pure Yang-Mills theory, which is classically scale invariant, the gauge coupling runs with the energy scale, a dimensionful parameter is introduced by dimensional transmutation, and the quantum stress energy tensor is not traceless anymore,

$$\begin{aligned} \mu \frac{d}{d\mu} g &= \beta(g) \rightarrow g(\mu), \Lambda_{QCD} \\ T_\mu^\mu &\sim \beta(g) F_{\mu\nu}^2 \end{aligned} \quad (1.13)$$

¹The superconformal group in four dimensions is usually denoted $SU(2, 2|N)$ where N is the number of supersymmetries; $SU(2, 2) \sim SO(4, 2)$.

In a more general theory with gauge fields, fermions and scalars, all dimensionless couplings run with the energy scale. In the following we will denote generically with g the set of couplings of a theory. The classical dimension d of a field will be corrected by the anomalous dimension

$$\Delta = d + \gamma(g), \quad \gamma = \frac{1}{2}\mu \frac{d}{d\mu} \ln Z \quad (1.14)$$

Conformal invariant quantum field theories can be obtained as

- Fixed points of the RG. At points where the beta function vanishes $\beta(g^*) = 0$ the stress energy tensor becomes traceless, the RG equation becomes the Ward identity for dilatations, with a quantum dimension for the fields given by $\Delta = d + \gamma(g^*)$. We can even start with massive theories in the UV and let them flow

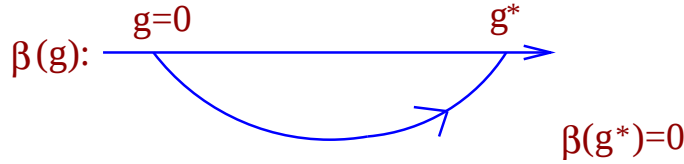


Figure 1: A standard textbook picture for the beta function behaviour near a fixed point.

in the IR. Under certain circumstances, at low energies, we can find IR fixed points.

- Finite theories. Suppose that we have a theory with no divergences at all. In this case $\beta(g) = 0$ for all values of g and there is no RG flow. The theory is conformal also at the quantum level. Since g can have an arbitrary value we have a line (or manifold if there is more than one g) of fixed points. The standard example in this class of theories is $\mathcal{N} = 4$ SYM. As we shall discuss extensively in section 3.1.5, the theory has non-abelian gauge fields, transforming under a group G , coupled to four Weyl fermions and six real scalars, all in the adjoint representation of G . The standard textbook formula for the one loop beta function is

$$\beta(g) = -\frac{g^3}{16\pi^2} \left(\frac{11}{3}c(A) - \frac{2}{3} \sum c(\text{weyl}) - \frac{1}{6} \sum c(\text{scalar}) \right) \quad (1.15)$$

where c denote the second Casimir of the representation of gauge fields, fermions and scalars. Since all the fields transform in the adjoint representation, all the

Casimir are equal and we see that the fermions and scalars balance the negative contribution of the gauge fields

$$-\frac{11}{3} + \frac{2}{3}4 + \frac{1}{6}6 = 0 \quad (1.16)$$

and the one-loop beta function is zero. It can be checked that the theory is finite at three-loops and it is believed to be finite at all orders. It is customary to combine coupling constant and theta angle in a complex parameter

$$\tau = \frac{4\pi i}{g^2} + i\frac{\theta}{2\pi} \quad (1.17)$$

The theory is finite (and therefore conformal) for all value of τ , $\beta(\tau) = \gamma(\tau) = 0$ and we have a complex line of fixed points. The conformal group is enhanced to $SU(2, 2|4)$.

1.1.3 Constraints from conformal invariance

In a conformal invariant theory we have an unitary action of the conformal group on the Hilbert space. The generators P, J, D, K will be represented by hermitian operators. It is a tedious exercise to check that the generators P, J, D, K close the following algebra ($\eta_{\mu\nu} = \text{diag}(-1, 1, \dots, 1)$)

$$\begin{aligned} [J_{\mu\nu}, J_{\rho\sigma}] &= i\eta_{\mu\rho}J_{\nu\sigma} \pm \text{permutation} \\ [J_{\mu\nu}, P_\rho] &= i(\eta_{\mu\rho}P_\nu - \eta_{\nu\rho}P_\mu) \\ [J_{\mu\nu}, K_\rho] &= i(\eta_{\mu\rho}K_\nu - \eta_{\nu\rho}K_\mu) \\ [J_{\mu\nu}, D] &= 0 \\ [D, P_\mu] &= iP_\mu \\ [D, K_\mu] &= -iK_\mu \\ [K_\mu, P_\nu] &= -2iJ_{\mu\nu} - 2i\eta_{\mu\nu}D \end{aligned} \quad (1.18)$$

where, as familiar from quantum field theory courses, the first line is the algebra of the Lorentz group $SO(1, D-1)$, the next three lines state that D is a scalar and P_μ, K_μ are vectors, the next two lines state that P_μ and K_μ are ladder operators for D , increasing and decreasing its eigenvalue, respectively. The last equation states that P and K close on a Lorentz transformation and a dilatation. We can assemble all generators in

$$J_{MN} = \begin{pmatrix} J_{\mu\nu} & \frac{K_\mu - P_\mu}{2} & -\frac{K_\mu + P_\mu}{2} \\ -\frac{K_\mu - P_\mu}{2} & 0 & D \\ \frac{K_\mu + P_\mu}{2} & -D & 0 \end{pmatrix} \quad M, N = 1, \dots, D+2 \quad (1.19)$$

and check that the antisymmetric J_{MN} is a rotation in a $D + 2$ dimensional space with signature $(2, D)$ ($\eta_{MN} = \text{diag}(-1, 1, \dots, 1, -1)$)

$$[J_{MN}, J_{RS}] = i\eta_{MR}J_{NS} \pm \text{permutation} . \quad (1.20)$$

We thus recover algebraically the group $SO(2, D)$.

Particles are usually identified by mass and Lorentz quantum numbers, corresponding to the Casimirs of the Poincaré group. When conformal invariance is present, the mass operator $P_\mu P^\mu$ does not commute anymore with other generators, for example D . Mass and energy can be in fact rescaled by a conformal transformation. If a representation of the conformal group contains a state with given energy, it will contain states with arbitrary energy from zero to infinity obtained by applying dilatations. For this reason the entire formalism of S matrix does not make sense for conformal theories. We need to find different ways of labeling states. In a conformal theory we consider fields with good transformation properties under dilatations. If we set $\lambda = e^\alpha$, $e^{i\alpha D}$ will generate a dilatation. The quantum version of equation (1.2) is $[D, \phi(x)] = i(\Delta + x_\mu \partial^\mu) \phi(x)$ and identifies fields of conformal dimension Δ . We shall be interested in gauge theories and, in this case, the physical objects are gauge invariant operators with given conformal dimension. We can also restrict to fields or operators annihilated (at $x = 0$) by the lowering operator K_μ ; these are called *primary* operators; the others, obtained by applying P_μ and other generators repeatedly, are called descendants. The reader is referred to the appendix for more details. Primary operators are classified according to the dimension Δ and the Lorentz quantum numbers.

Note that there is another possibility of finding good quantum numbers for the conformal group. D and $J_{\mu\nu}$ correspond to the non-compact subgroup $SO(1, 1) \times SO(1, 3)$ of $SO(2, 4)$. Sometimes it is more convenient to use the maximal compact subgroup $SO(2) \times SU(2) \times SU(2) \subset SO(2, 4)$. States are still labeled by three numbers (Δ, j_1, j_2) , now viewed as eigenvalues of the Cartan generators of $SO(2) \times SU(2) \times SU(2)$. The $SO(2)$ generator is $H = (P_0 + K_0)/2$ and it is called the conformal energy. It would seem that its eigenvalues are integer. However, quantum theories strictly realize representations of the covering space of $SO(2, 4)$, obtained by unwinding $SO(2)$ and Δ can assume continuous real values. The physical interpretation of the quantum numbers under the maximal compact subgroup is more evident in the Euclidean version of the theory, since $\mathbb{R}^4$ can be mapped by a conformal transformation to $S^3 \times \mathbb{R}$. In this new description of the theory H becomes an Hamiltonian corresponding to time translation and $SU(2) \times SU(2) = SO(4)$ gives quantum numbers of an expansion

on S^3 . This type of conformal transformation is familiar from two dimensions and corresponds to a radial quantization of the theory.

Conformal invariance gives many constraints on a quantum field theory:

- The Ward identities for the conformal group give constraints on the Green functions. One can always find a basis of primary operators $O_i(x)$, with fixed scale dimension Δ_i . The set of (O_i, Δ_i) gives the spectrum of the CFT. One-, two- and three-point functions are completely fixed by conformal invariance. For example, one-point functions are zero, while two-point functions equal

$$\langle O_i(x) O_j(y) \rangle = \frac{A \delta_{ij}}{|x - y|^{2\Delta_i}} \quad (1.21)$$

The coordinates dependence of 3-point functions is also fixed

$$\langle O_i(x_i) O_j(x_j) O_k(x_k) \rangle = \frac{\lambda_{ijk}}{|x_i - x_j|^{\Delta_i + \Delta_j - \Delta_k} |x_j - x_k|^{\Delta_j + \Delta_k - \Delta_i} |x_k - x_i|^{\Delta_k + \Delta_i - \Delta_j}}. \quad (1.22)$$

- Unitarity of the theory gives bounds restricting the possible dimensions of primary fields. We have inequalities that depend on the Lorentz quantum number $\Delta \geq f(j_1, j_2)$, which are discussed in the Appendix. Three cases will be particularly important for us
 - The dimension of a four-dimensional scalar field must be greater than one, $\Delta \geq 1$, and the saturation of the bound, $\Delta = 1$, implies that the operator obeys free field equations.
 - For a vector field O_μ , $\Delta \geq 3$ and the bound is saturated if and only if the operator is a conserved current $\partial^\mu O_\mu = 0$. Analogously, a spin 2 symmetric operator $O_{\mu\nu}$ has $\Delta \geq 4$, and $\Delta = 4$ corresponds to conservation $\partial^\mu O_{\mu\nu} = 0$. In particular, conserved currents have canonical dimension and are not renormalized.
 - In supersymmetric theories the bounds relate dimension to spin and R symmetry quantum numbers. A typical case in 4d $\mathcal{N} = 1$ supersymmetric theories is the scalar bound $\Delta \geq \frac{3}{2}R$, relating dimension to R-charge, which is saturated by chiral operators. In this case, the saturation of the bound implies that the operator is annihilated by some combinations of the supercharges. This will be further discussed in section 3.3.1

In all cases, the saturation of the bound corresponds to a shortening of the (super) conformal multiplet and some non-renormalization property, which are discussed in the Appendix.

1.2 AdS space

We introduce now the gravitational side of the story.

AdS_5 is the maximally symmetric solution of the Einstein equations in five dimensions with cosmological constant. From

$$S = \frac{1}{16\pi G_5} \int dx^5 \sqrt{|g|} (\mathcal{R} - \Lambda)$$

$$\mathcal{R}_{\mu\nu} - \frac{g_{\mu\nu}}{2} \mathcal{R} = -\frac{\Lambda}{2} g_{\mu\nu} \quad (1.23)$$

we have $\mathcal{R} = \frac{5}{3}\Lambda$ and therefore the Ricci tensor is proportional to the metric

$$\mathcal{R}_{\mu\nu} = \frac{\Lambda}{3} g_{\mu\nu} \quad (1.24)$$

This equation tells us that the solution is an Einstein space. If we further require that

$$\mathcal{R}_{\mu\nu\tau\rho} = \frac{\Lambda}{12} (g_{\mu\tau}g_{\nu\rho} - g_{\mu\rho}g_{\nu\tau}) \quad (1.25)$$

we have a maximally symmetric space. In Euclidean signature, the maximally symmetric solution with positive cosmological constant (and therefore positive curvature) is the sphere S^5 with isometry $SO(6)$ and the one with negative curvature is the hyperboloid H^5 with isometry $SO(1,5)$. In Minkowskian signature, the maximally symmetric solution with $\Lambda > 0$ is called de-Sitter space (dS_5) and the one with $\Lambda < 0$ is called Anti-de-Sitter (AdS_5). All of these spaces can be realized as the set of solutions of a quadratic equation in a six dimensional flat space with suitable signature $\mathbb{R}^{d,6-d}$ ². Let us focus on AdS_5 . We define it as the set of solutions of

$$x_0^2 + x_5^2 - x_1^2 - x_2^2 - x_3^2 - x_4^2 = R^2, \quad \frac{1}{R^2} = -\frac{\Lambda}{12} \quad (1.26)$$

in a flat $\mathbb{R}^{2,4}$ with line element $ds^2 = -dx_0^2 - dx_5^2 + dx_1^2 + dx_2^2 + dx_3^2 + dx_4^2$. It is obvious from this defining equation, that AdS_5 has isometry group $O(2,4)$, identical to the conformal group in four dimensions.

²For example, in Euclidean signature, the sphere S^5 , is defined by a quadratic equation $x_0^2 + x_1^2 + \dots + x_5^2 = R^2$ in $\mathbb{R}^6$ and the hyperboloid H^5 by $x_0^2 - x_1^2 - \dots - x_5^2 = R^2$ in $\mathbb{R}^{1,5}$.

A set of coordinates is given by

$$\begin{aligned} x_0 &= R \cosh \rho \cos \tau \\ x_5 &= R \cosh \rho \sin \tau \\ x_i &= R \sinh \rho \hat{x}_i, \quad \sum_{i=1}^4 \hat{x}_i^2 = 1 \end{aligned} \quad (1.27)$$

and the metric reads

$$ds^2 = R^2 (-\cosh^2 \rho d\tau^2 + d\rho^2 + \sinh^2 \rho d\Omega_3) \quad (1.28)$$

where Ω_3 is the line element of a three-sphere. It is easy to verify that $\rho \in \mathbb{R}^+$ and $\tau \in [0, 2\pi]$ cover the Minkowskian hyperboloid exactly once, and for this reason these coordinates are called *global*. Note that time is periodic and therefore we have close time-like curves. To avoid this we can take the universal cover where $\tau \in \mathbb{R}$: we shall always refer to AdS_5 as this universal cover.

We can find a second set of coordinates given by a four dimensional Lorentz vector x_μ and a fifth coordinate $u > 0$ by a redefinition

$$\begin{aligned} x_0 &= \frac{1}{2u} (1 + u^2(R^2 + \vec{x}^2 - t^2)) \\ x_5 &= R u t \\ x_{1,2,3} &= R u x_{1,2,3} \\ x_4 &= \frac{1}{2u} (1 - u^2(R^2 - \vec{x}^2 + t^2)) \end{aligned} \quad (1.29)$$

which brings the metric to the form

$$ds^2 = R^2 \left(\frac{du^2}{u^2} + u^2(dx_\mu dx^\mu) \right) \quad (1.30)$$

We see that the metric has slices isomorphic to four-dimensional Minkowski space-time, and for this reason these coordinates are called *Poincaré* coordinates. The four dimensional space-time is foliated over u which runs from zero to infinity. The Minkowski metric is multiplied by a warp factor u^2 , whose meaning is that an observer leaving on a Minkowski slice sees all lengths rescaled by a factor of u according to its position in the fifth dimension. The plane $u = \infty$ is referred as the boundary of AdS_5 . Note however that for $u \rightarrow \infty$ the metric ds^2 blows up. Mathematically $u = \infty$ is a conformal boundary (strictly speaking, it is the conformally equivalent metric $d\tilde{s}^2 = ds^2/u^2$ to have a boundary $\mathbb{R}^{1,3}$ at $u = \infty$). The plane $u = 0$ is instead a horizon: the killing vector $\frac{\partial}{\partial t}$ has zero norm at $u = 0$. These coordinates are

convenient since they contain a Minkowski slice, and we shall use them in most of our applications. However, they cover only half of the hyperboloid; $u = 0$ does not correspond to a singularity and the metric can be extended after the horizon (using for example global coordinates).

There are other forms of the metric in Poincaré coordinates that are commonly used. They all differ by a redefinition of the fifth coordinate u . For example with $u = 1/z = e^r$ we have

$$ds^2 = R^2 \left(\frac{dz^2 + dx_\mu dx^\mu}{z^2} \right) = R^2 (dr^2 + e^{2r} dx_\mu dx^\mu) \quad (1.31)$$

The boundary is now at $z = 0$ and $r = \infty$ and the horizon at $z = \infty$ and $r = -\infty$.

As we have already said, the isometry group of AdS_5 is $SO(2, 4)$ which is the same as the conformal group in four dimensions. The AdS/CFT correspondence exploits deeply this fact. It is interesting to compare closely the realization of the two groups. Since the full group is not always manifest in explicit realizations or choices of coordinates, we shall look at particular subgroups,

- The subgroup $SO(2) \times SO(4)$ is manifest when we use global coordinates. In field theory, it is useful for studying quantization on $S^3 \times \mathbb{R}$. We see an explicit copy of S^3 in the metric and a time τ . $SO(2)$ corresponds to the Hamiltonian in field theory and it is time translation in AdS_5 . Notice that in global coordinates the Killing vector $\frac{\partial}{\partial \tau}$ is never vanishing and everywhere defined. Both in field theory and gravity we take time $\tau \in \mathbb{R}$ and we consider the universal cover of $SO(2, 4)$.
- The subgroup $SO(1, 1) \times SO(1, 3)$ is manifest in Poincaré coordinates. The Minkowski slice in the metric with isometry $SO(1, 3)$ can be associated with the four-dimensional space-time where we quantize our field theory. $SO(1, 1)$ is the dilatation in field theory and it is realized as $(u, x_\mu) \rightarrow (\lambda u, x_\mu/\lambda)$.

To conclude this brief excursus on the geometry of AdS_5 let us consider the Euclidean continuation of the metric. This is important because in field theory we shall often perform a Wick rotation to Euclidean signature. We can do this by sending $x_5 \rightarrow -ix_5$, or, in each set of coordinates, $\tau \rightarrow -i\tau$ and $t \rightarrow -it$. The resulting metric is

$$R^2 (\cosh^2 \rho d\tau_E^2 + d\rho^2 + \sinh^2 \rho d\Omega_3) = R^2 \left(\frac{du^2}{u^2} + u^2 (dt_E^2 + d\vec{x}^2) \right) \quad (1.32)$$

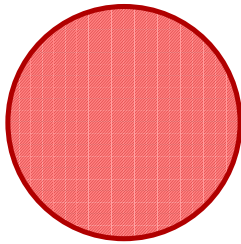


Figure 2: The Euclidean picture of AdS_5 as a five-dimensional ball.

The $u = \infty$ boundary plane $R^{1,3}$ of the Minkowskian version is replaced by $\mathbb{R}^4$. On the other hand, the $u = 0$ plane, which was a plane of null vectors in the Minkowski version, now shrinks to a point.

It is sometime convenient to compactify the boundary of our flat four-dimensional space to S^4 by adding the point $u = 0$ to the boundary $\mathbb{R}^4$. One can show that the space is diffeomorphic to a five-dimensional ball in $\mathbb{R}^5$ with metric

$$\mathbb{R}^5 : \quad y_1^2 + \cdots + y_5^2 \leq R^2, \quad ds^2 = \frac{(dy)^2}{R^2 - |y|^2}$$

Exercise: It is probably instructive for the reader to check these statements in details for AdS_2 , obtained by the previous formulae by neglecting $\vec{x}$. In Poincaré coordinates, by defining $z = t_E + \frac{i}{u}$ we have

$$R^2 \left(\frac{du^2}{u^2} + u^2 dt_E^2 \right) = R^2 \frac{dz d\bar{z}}{(\text{Im} z)^2} \quad (1.33)$$

and we recognize a familiar hyperbolic metric on the upper half-plane. The boundary is the real axis and we can include the point at infinity. With a conformal transformation we can map the half-plane to a disk. The metric will diverge at the circle bounding the disk.

1.3 The large N limit for gauge theories

An $U(N)$ Yang-Mills gauge theory can be simplified in the limit where the number of colors N is large. t'Hooft first proposed to send $N \rightarrow \infty$ and do a systematic expansion in $1/N$. The large N expansion has proved to be useful for various reasons:

- It is a systematic expansion.
- It provides a weakly coupled Lagrangian for mesons and glueballs and it explains $U(1)$ anomalies.

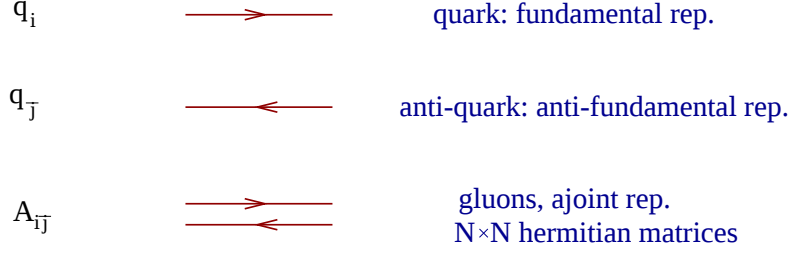


Figure 3: Double-line notation for objects transforming in the fundamental (quarks), anti-fundamentals (anti-quark) and adjoint representation (gluons) of the gauge group.

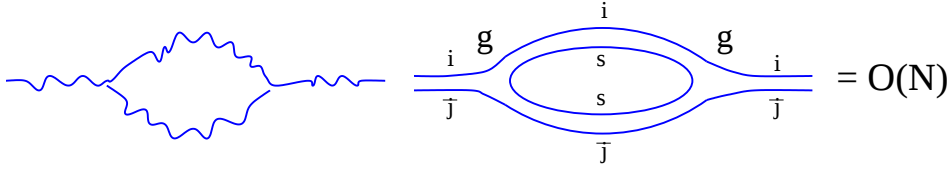


Figure 4: A graph contributing to the gluon self-energy.

- It simplifies the perturbative computation.
- Some QCD models in two dimensions become solvable in the large N limit.

Let us discuss the large N expansion for an $U(N)$ gauge theory

$$L = \text{Tr} (F_{\mu\nu}^2 + L_{\text{matter}}) \quad (1.34)$$

with $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + ig_{YM}[A_\mu, A_\nu]$ and L_{matter} is the matter Lagrangian, which will include fundamental and adjoint fields in our applications. There is a convenient pictorial representation of Feynman graphs in terms of a double line notation, described in Figure 3. Fundamental and anti-fundamental fields can be written as q_i and $q_{\bar{j}}$, respectively, where $i, \bar{i} = 1, \dots, N$ and the bar distinguishes indices transforming in the anti-fundamental representation. Adjoint fields of $U(N)$ can be written as hermitian matrices $A_{i\bar{j}}$ and thought as formal products of a fundamental and anti-fundamental representation. We shall use a Feynman graph notation where oriented lines are associated with indices i and $\bar{j}$ and not with fields. In this way, the propagator for an adjoint field can be then naturally written as a double line.

The Feynman rules follow straightforwardly from the Lagrangian and can be easily understood by looking at explicit examples. Consider the case of the self-energy of a gluon, pictured in Figure 4. The indices at the beginning and end of

a line have been identified, since the kinetic term in the Lagrangian is diagonal on each component of q_i and $A_{i\bar{j}}$. Similarly, indices in the vertices have been contracted according to matrix multiplication $\text{Tr} A^3 = A_{i\bar{j}} A_{j\bar{p}} A_{p\bar{i}}$. We see that the only free index is the internal one s , which may take N different values. The self-energy diverges as $O(N)$. Many other graphs diverge as well. It seems that $N \rightarrow \infty$ is not a sensible limit. However, the self-energy contains powers of the coupling constant and it is of order $O(g_{YM}^2 N)$. If we take the t'Hooft limit

$$\begin{aligned} N &\rightarrow \infty, \\ g_{YM} &\rightarrow 0 \quad \quad x = g_{YM}^2 N \text{ fixed} \end{aligned} \tag{1.35}$$

the self-energy remains finite. The same happens to all other graphs. We shall now see that the t'Hooft limit makes sense for the entire perturbative expansion.

It is better to redefine fields and bring all dependence on g_{YM} in front of the Lagrangian,

$$L = \frac{1}{g_{YM}^2} \text{Tr} (F_{\mu\nu}^2 + \dots) = \frac{N}{x} \text{Tr} (F_{\mu\nu}^2 + \dots) \tag{1.36}$$

This convention will be used in the rest of these notes. The propagators now bring a factor of x/N and all type of vertices a factor of N/x . Let us first restrict to a theory with only adjoints fields. Two simple examples of graphs contributing to the free energy are reported in Figure 5. The first graph is planar, meaning that it can be drawn on a plane. More formally, it can be seen as a triangulation of a sphere. This is explicitly manifest in the double line notation as indicated in the Figure. The second graph instead is not planar; if we insist to draw it on a plane some of its lines will intersect in points which are not vertices of the graph. The best we can do is to consider it as drawn on a torus. Every graph can be drawn without intersecting lines on a Riemann surfaces of Euler characteristic $2 - 2g = F - V + E$ where F is the number of faces of the graph, E is the number of edges and V the number of vertices. g is the genus, or the number of holes, of the Riemann surface. We see from the examples that graphs with different topology have different powers of N . We can derive a general formula, taking into account that we have a factor of N/x for each propagator (E), a factor of x/N for each vertex (V) and a factor of N for each loop (F),

$$x^{V-E} N^{F-V+E} = O(N^{2-2g}) \tag{1.37}$$

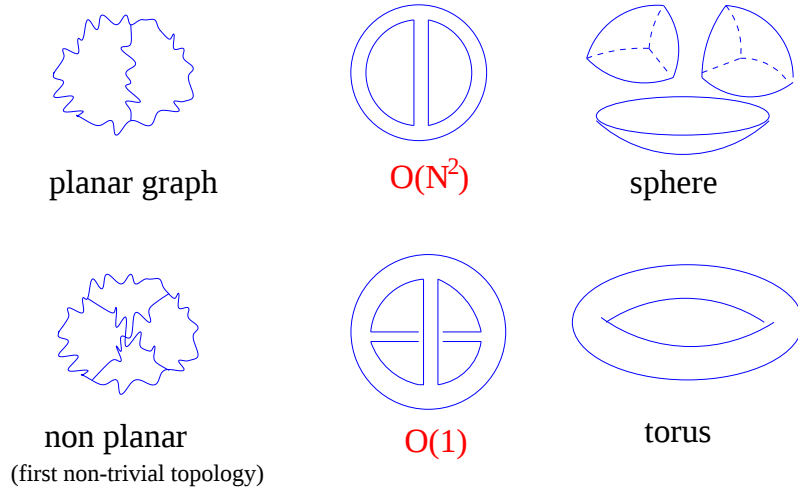


Figure 5: Planar and non-planar graphs and their relation with Riemann surfaces

We see that the t'Hooft expansion organizes graphs according to their topology. The expansion of the free energy in powers of $1/N$ is particularly simple

$$F = \sum_{g=0}^{\infty} N^{2-2g} f_g(x) \quad (1.38)$$

One may worry that the planar graphs give a contribution of order $O(N^2)$ to the free-energy, which seems to diverge. However, this is of the same order of the Lagrangian itself evaluated on a generic configuration

$$\frac{1}{g_{YM}^2} \text{Tr} F^2 = \frac{N}{x} \text{Tr} F^2 \sim O(N^2) \quad (1.39)$$

since the trace of a matrix is of typical order N . So the leading term in the free energy correctly reproduces the behavior of the Lagrangian. The subleading terms are suppressed by powers of $1/N^2$.

We could repeat a similar analysis for Green functions. In these notes we shall be interested in composite operators rather than in elementary fields. These can be written as traces or product of traces of elementary fields. Particularly important for us are the single trace operators, for example $\text{Tr} F_{\mu\nu}^2$. We normalize single traces of products of adjoint fields with a further factor of N

$$O = \frac{1}{N} \text{Tr}(\phi\phi\phi\dots) \quad (1.40)$$

in such a way that they are of order $O(1)$ on a generic field configuration. Connected correlation functions of O have then a $1/N^2$ expansion according to the topology of the graph starting with a leading term of order $O(1)$ given by the planar graphs.

The large N expansion considerably simplifies perturbation theory. For $N \rightarrow \infty$ only the planar graphs (f_0) survive. This has been used to solve some two-dimensional models in the planar limit. However no similar solvable model exists in dimension greater than two. The reader should be alerted that f_0 contains an infinite number of graphs which should be re-summed. The perturbative expansion simplifies in the planar limit but it is not solved in general.

Let us finish this discussion with few observations:

- In most of these notes we are interested in finite theories, where g_{YM} does not run and it is a dimensionless parameter. For theories like QCD, N is the only dimensionless parameter: the coupling constant runs with the scale $g(\mu)$ and it is better traded for the Renormalization Group invariant quantity Λ_{QCD} .
- As we see each graph corresponds to a Riemann surface and it is classified by the genus. The t'Hooft expansion is similar to the world-sheet expansion of string theory. The formula for the free-energy (1.39) is similar to the loop expansion of a string with coupling $g_s = e^\phi = 1/N$.

$$\begin{aligned}
& \textcolor{red}{O(N^2)} \left(\text{planar graphs: } f(x) \right) + \textcolor{red}{O(1)} \left(\text{non-planar graphs} \right) \\
& \textcolor{red}{O(g_s^{-2})} \text{ (circle) } + \textcolor{red}{O(1)} \text{ (torus) } + \dots \textcolor{red}{O(g_s^{-2+2g})} \text{ (genus } g \text{ surfaces)}
\end{aligned}$$

Figure 6: Emergence of a string structure from the Feynman graph expansion in the large N limit.

- We only discussed the case of adjoint fields, which will be the one of most interest for us. Fundamental fields introduce some novelty. It is easy to see that loops of fundamental fields have further powers of $1/N$ with respect to loops of adjoint fields, since a single line instead of a double line is used. In

particular, fundamental fields are suppressed in the planar limit and they enter in the perturbative expansion with odd powers of N . The topology of the graphs is also changed: double lines allow to draw the graphs on closed surfaces, single lines necessarily introduce boundaries. We then obtain an expansion in Riemann surfaces with boundaries. The power of N of a graph is still given by the Euler characteristic which now can be odd. The expansion of a theory with only adjoints is in powers of $1/N^2$, the expansion of a theory with also fundamentals is in powers of $1/N$. Pursuing the analogy with string theory, we see that fundamental fields are associated with open strings. In the context of the *AdS/CFT* correspondence a theory with adjoints is associated with a closed string background while fundamentals requires the introduction of open string (typically in the form of D-branes which introduce the required boundaries).

- Finally a technical remark. In the large N limit $SU(N) \sim U(N)$. However this is only true in the planar limit. The gluon propagator for $SU(N)$ is not diagonal, $\langle A_{i\bar{j}} A_{p\bar{q}} \rangle \sim \delta_{i\bar{q}} \delta_{p\bar{j}} - (1/N) \delta_{i\bar{j}} \delta_{p\bar{q}}$ and the $1/N$ term mixes with the subleading terms of the $1/N$ expansion.

2 The AdS/CFT correspondence

Up to now we saw two apparently different ways of realizing theories with $O(4,2)$ symmetry

- Conformal field theories in four dimensions
- Relativistic gravitational theories in AdS_5

Since the two objects live in space-times with different dimensions it is difficult to imagine a relation between them. However, holography has always been a favorite principle when dealing with gravity. For several reasons, it is thought that in quantum gravity the number of degrees of freedom of a region of space-time grows with the area of the region and not the volume. Here we shall see another form of holography, related to the fact that the boundary of AdS_5 is Minkowski space-time. All dynamics in AdS_5 can be reformulated as a boundary effect and it is captured by a four-dimensional local field theory. We shall set up a correspondence between CFT in four dimensions and gravitational theories in AdS_5 .

The study of this correspondence is the purpose of the rest of these notes. In this Section, we start with the general construction without referring to explicit realizations of the correspondence which will be analyzed in Sections 3 and 4. More precisely, now we shall construct, for every five-dimensional gravitational theory with AdS_5 vacuum, a set of correlation functions for operators which satisfy all constraints required by a four dimensional local field theory with scale invariance. In this Section we work at the level of effective action in 5 dimensions. We deal with a theory that at low energy reduce to General Relativity coupled to gauge and matter fields. We assume the existence of a suitable UV completion of our five dimensional theory although this will not be used. We shall enlight some general aspects of the correspondence which do not refer to explicit realizations. We shall also make clear that AdS_5 is not really necessary. All we need is a space with the topological structure of AdS_5 and a conformal boundary.

2.1 Formulation of the correspondence

To define the correspondence we need a map between the observables in the two theories and a prescription for comparing physical quantities and amplitudes.

We refer to the fields in five dimensions as bulk fields. We may assume that their interaction is described by an effective action

$$S_{AdS_5}(g_{\mu\nu}, A_\mu, \phi, \dots) \quad (2.1)$$

with AdS_5 vacuum. We included the metric, gauge fields and scalars. In most applications this effective action corresponds to some supergravity with also tensor fields and fermions. We assume a potential for the scalar field with a negative value at the minimum thus creating a negative cosmological constant for the AdS_5 vacuum.

We refer to the CFT fields as boundary fields. We call L_{CFT} the four-dimensional Lagrangian. Recall that most often the elementary fields are not observables in the quantum theory, due to gauge invariance. The spectrum is specified by a complete set of primary operators in the CFT.

A basic point of the correspondence is the following statement: a field h in AdS is associated with an operator in the CFT with the same quantum numbers and they know about each other via boundary couplings. More precisely, from the four-dimensional point of view, every operator O can be associated to a source h

$$L_{CFT} + \int d^4x h O \quad (2.2)$$

For the moment $h(x)$ is a four-dimensional background field that is introduced in order to compute correlation functions for the operator O . As usual,

$$e^{W(h)} = \left\langle e^{\int h O} \right\rangle_{QFT} \quad (2.3)$$

define $W(h)$ as the functional generator for connected correlation functions of O

$$\langle O \dots O \rangle_c = \left. \frac{\delta^n W}{\delta h^n} \right|_{h=0} \quad (2.4)$$

We can now think of the source $h(x)$ as the boundary value of a five dimensional field $h(x, x_5)$. For every source configuration $h(x)$ there is an five-dimensional field configuration obtained

$$h(x) \rightarrow \hat{h}(x, x_5) \quad (2.5)$$

by demanding that $h(x, x_5)$ solve the five-dimensional equations of motion derived by

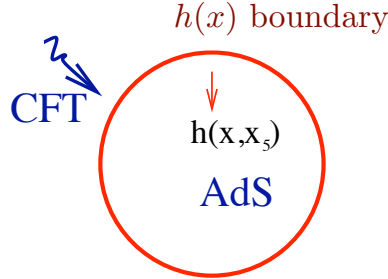


Figure 7: The prescription for extending sources to the bulk. We use the Euclidean version for clarity.

S_{AdS} . The extension from boundary to bulk is unique if we impose suitable boundary conditions at the horizon.

The fundamental statement of the AdS/CFT correspondence is now:

$$e^{W(h)} = \left\langle e^{\int h O} \right\rangle_{QFT} = e^{S_{AdS_5}(\hat{h})} \quad (2.6)$$

where on the left hand side we have a functional depending on an arbitrary four dimensional (off-shell) configuration $h(x)$ and on the right hand side we have the (on-shell) value of the five dimensional Lagrangian, evaluated on the solution of the equations of motion that reduce to $h(x)$ at the boundary. Since the knowledge of $W(h)$ for all possible sources of composite operators determines completely the CFT,

the previous formula states the required equivalence between the CFT and the five-dimensional theory. The precise meaning of this formula will be explored in the rest of these notes. However, a few immediate comments are in order.

- The equation of motions in AdS are second order and we need to specify two boundary conditions in order to find a unique solution. Both boundary conditions require some care. First of all, we cannot simply set $\hat{h}(x, \text{boundary}) = h(x)$ since solutions of the equations of motion diverge or vanish at the boundary. The typical example is the metric; as we saw, the metric blows up at the boundary. The right condition at the AdS boundary is of the form $\hat{h}(x, x_5) \sim f(x_5)h(x)$ and it will be extensively discussed in the next Section. The second boundary condition is to be imposed in the interior of AdS. We shall mainly work with the Euclidean version of the theory where things simplify. In the Euclidean, AdS_5 is a ball and the center of the space is just a point. We shall require regularity of the solution at the center of the ball. Working instead in Minkowski, and using Poincaré coordinates, we have to impose a suitable condition at the horizon in order to have an unique extension. This typically requires keeping only incoming waves from the horizon.
- The prescription, as stated, uses the AdS effective action. When we have a consistent UV completion of the five dimensional theory, the right hand side can be interpreted also at the quantum level. In examples where AdS_5 is embedded in a full string theory background, the right-hand side of the last equation is replaced by some string theory S-matrix element for the state h .
- We should again stress that we used the AdS equations of motion. An off-shell theory in four dimensions corresponds to an on-shell theory in five dimensions. This is a general feature of all the AdS/CFT inspired correspondences.
- We haven't said how to map CFT operators to fields in the bulk. This will be specified by the details of the two theories and will be available when we have a constructive way of determining dual pairs. This is provided by string theory and will be discussed in Sections 3 and 4, where we shall also see how to map observables in specific examples. For the moment, let us notice that the field that couples to an operator can be often found using symmetries. h and O have the same $O(2, 4)$ quantum numbers. In particular, there are obvious couplings in the case of conserved currents: we introduce a background gauge field by

covariantizing the boundary action. The natural linearized couplings

$$L_{CFT} + \int d^4x \sqrt{g} (g_{\mu\nu} T_{\mu\nu} + A_\mu J_\mu + \phi F_{\mu\nu}^2 + \dots) \quad (2.7)$$

suggest that the operator associated with the graviton is the stress-energy tensor and the operator associated with a gauge fields in AdS is a current. Note that conservation of stress-energy tensor or currents are associated with the gauge invariance at the level of the sources (meaning that the field theory functional $W(A_\mu, g_{\mu\nu})$ is a gauge invariant functional of A_μ and $g_{\mu\nu}$). This obviously extends by consistency to the bulk theory. We see from this a general fact: global symmetries in the CFT correspond to gauge symmetries in AdS. We couldn't refrain from adding a particular scalar operator which is present in all gauge theories: $O = \text{Tr} F_{\mu\nu}^2$. We coupled it to a source ϕ ; the corresponding bulk field, in many explicit examples, is the string theory dilaton.

2.2 Physics in the bulk

One surprising thing about the correspondence is the existence of a fifth radial coordinate in the gravitational picture which is needed for the holographic interpretation. Let us see its role in more details.

A crucial ingredient in all the models obtained by the *AdS/CFT* correspondence is the identification of the radial coordinate in the supergravity solution with an energy scale in the dual field theory. The identification between radius and energy follows from the form of the *AdS* metric in Poincaré coordinates (we put $R = 1$ when no confusion is possible)

$$ds^2 = \frac{dx_\mu^2 + dz^2}{z^2}. \quad (2.8)$$

A dilatation $x_\mu \rightarrow \lambda x_\mu$ in the boundary *CFT* corresponds in *AdS* to the $SO(4, 2)$ isometry

$$x_\mu \rightarrow \lambda x_\mu, \quad z \rightarrow \lambda z. \quad (2.9)$$

We see that we can roughly identify $u = 1/z$ with an energy scale μ . The boundary region of *AdS* ($z \ll 1$) is associated with the UV regime in the *CFT*, while the horizon region ($z \gg 1$) is associated with the IR. This is more than a formal identification: as we shall see, holographic calculations of Green functions or Wilson loops associated with a specific reference scale μ are dominated by bulk contributions from the region $u = \mu$.

Let us consider now the dynamics of bulk fields in AdS_5 . For the purposes of the correspondence, the boundary values of fields are arbitrary functions of four-spacetime coordinates x_μ , while the profile of fields in the fifth directions is set on-shell by the equations of motion. For simplicity, let us consider the case of a massive field ϕ in AdS_5 dual to some operator O in the CFT. We consider the Euclidean continuation of the metric (2.8). Given a scalar field $\phi(z, x_\mu)$ with action

$$S \sim \int dx^5 \sqrt{g} (g^{mn} \partial_m \phi \partial_n \phi + m^2 \phi^2) = \int dz dx \frac{1}{z^5} (z^2 (\partial_z \phi)^2 + z^2 (\partial_\mu \phi)^2 + m^2 \phi^2)$$

the equation of motion reads

$$\partial_z \left(\frac{1}{z^3} \partial_z \phi \right) + \partial_\mu \left(\frac{1}{z^3} \partial^\mu \phi \right) = \frac{1}{z^5} m^2 \phi \quad (2.10)$$

Look first at the z behaviour. Consider first a mode independent of x_μ . The equation reduces to

$$z^5 \partial_z (z^{-3} \partial_z \phi) = m^2 \phi \quad (2.11)$$

which has two independent power-like solutions $\phi \sim z^\Delta$ with

$$m^2 = \Delta(\Delta - 4) \quad (2.12)$$

Since under a dilation $z \rightarrow \lambda z, x_\mu \rightarrow \lambda x_\mu$, Δ corresponds to a scaling dimension for the field and, as we shall see, will be identified with the conformal dimension of the dual operator O .

Denoting simply by Δ the largest solution of the quadratic equation (2.12), we find the near boundary behaviour of an on-shell field

$$\phi \sim \phi_0 z^{4-\Delta} + \phi_1 z^\Delta \quad (2.13)$$

The coefficient ϕ_0 and ϕ_1 correspond to the two linearly independent solutions of the second order equation of motion. They can be distinguished by the fact that the solution corresponding to ϕ_0 is not normalizable at the boundary,

$$\int dx^5 \sqrt{g} |\phi|^2 = \infty \quad (2.14)$$

while the one corresponding to ϕ_1 is. If we now include the x_μ dependence the previous behaviour is modified to

$$\phi(z, x_\mu) \sim (\phi_0(x) z^{4-\Delta} + O(z)) + (\phi_1(x) z^\Delta + O(z)) \quad (2.15)$$

where we can still identify the coefficients $\phi_{0,1}(x)$ of the two linearly independent solutions, which will still grow as $z^{4-\Delta}$ and z^Δ with corrections depending on both z and x_μ .

Now we are ready to discuss the boundary conditions to be imposed on a field near the boundary. We see that the leading term of a solution of the equation of motion can be singular if $\Delta > 4$ or vanishes if $\Delta < 4$. It approaches a constant only in the case $\Delta = 4$. In order to have a consistent prescription we need to impose at the boundary $z = 0$

$$\phi(z, x_\mu) \rightarrow z^{4-\Delta} \phi_0(x_\mu). \quad (2.16)$$

$\phi_0(x)$ is the boundary value of our field to be identified with the source of the dual operator O . Once the value of $\phi_0(x)$ is specified, we have a unique regular solution that extends to all of AdS_5 . In particular $\phi_1(x)$ will be determined as a functional of $\phi_0(x)$ by imposing the equations of motion and regularity at the center.

Let us see explicitly how it works in the simple case of a massless scalar field $m^2 = 0$. In this case $\Delta = 4$. It is convenient to perform a Fourier decomposition of modes on $\mathbb{R}^4$. The Fourier mode $\phi_p(z)e^{ipx}$ satisfies

$$z^5 \partial_z (z^{-3} \partial_z \phi_p(z)) - p^2 z^2 \phi_p(z) = 0 \quad (2.17)$$

which, with $\phi_p = (pz)^2 y(pz)$, reduces to a Bessel equation

$$(pz)^2 \frac{d^2 y}{d(pz)^2} + (pz) \frac{dy}{d(pz)} - (4 + (pz)^2) y = 0 \quad (2.18)$$

whose general solution is $A_p I_2(pz) + B_p K_2(pz)$ ³. Correspondingly $\phi_p \sim B_p(1 + \dots) + A_p(z^4 + \dots)$ as expected from (2.15) for a field with $\Delta = 4$. K_2 is the non-normalizable solution and I_2 the normalizable one. Since $I_2(pz)$ is exponentially growing for large z , regularity of the solution at the center requires $A_p = 0$ and we are left with $K_2(pz)$, which is exponentially vanishing for large z .

We can now go further and actually compute the effective action $W(\phi_0)$ depending on the boundary conditions ϕ_0 , identified with the field theory source. For $\Delta = 4$, (2.16) set the asymptotic value of the solution equal to ϕ_0 . In general computation in AdS_5 , various quantities in the game diverge for $z \rightarrow 0$ and it is convenient to introduce a cut off and impose boundary conditions at $z = \epsilon$. At the end of the computation one sends ϵ to zero. This allows to keep track of local divergent pieces of

³ $x^2 y'' + xy' - (x^2 + a^2) = 0$ with $a \in \mathbb{N}$ has solutions $I_a(x) \sim J_a(ix)$ and $K_a(x)$, with asymptotic behaviour, $I_a(x) \sim x^a + \dots$ and $K_a(x) \sim \frac{1}{x^a} (1 + \dots + c_a x^{2a} \log x)$ for $x \rightarrow 0$ and $I_a(x) \sim \frac{e^x}{\sqrt{x}}$ and $K_a(x) \sim \frac{e^{-x}}{\sqrt{x}}$ for $x \rightarrow \infty$.

the effective action and it is a general prescription for computing correlation functions in AdS_5 . So we impose

$$\phi_p(z = \epsilon) \equiv \phi_p^0 \quad (2.19)$$

so that the solution is

$$\phi_p(z) = \frac{(pz)^2 K_2(pz)}{(p\epsilon)^2 K_2(p\epsilon)} \phi_p^0 \quad (2.20)$$

$W(\phi_0)$ is now obtain by evaluating the five-dimensional Lagrangian on the solution of the equations of motion. The computation can be simplified using a standard trick: by integrating by parts

$$S_{AdS} \sim \int_{\text{boundary}} \sqrt{g} \phi \partial^n \phi + \int \sqrt{g} \phi (-\square + m^2) \phi \quad (2.21)$$

the second term is zero on the equations of motion and the action reduces to a boundary contribution. In our case,

$$S_{AdS} \sim \left. \frac{\phi \partial_z \phi}{z^3} \right|_{z=\epsilon} \quad (2.22)$$

and inserting the solution of the equations of motion

$$\phi(z, x) = \int dp^4 e^{ipx} \phi_p(z) \quad (2.23)$$

we obtain

$$W(\phi_0) = S_{AdS} \sim \int dp dp' \delta(p + p') \phi_p^0 \phi_{p'}^0 \frac{1}{z^3} \partial_z \log \phi_p(z) \Big|_{z=\epsilon} \quad (2.24)$$

from which we see that

$$\begin{aligned} \langle O(p) O(p') \rangle &= \frac{\delta W}{\delta \phi_p^0 \delta \phi_{p'}^0} \sim \delta(p + p') p^4 \frac{1}{(pz)^3} \frac{d}{d(pz)} \log \phi_p(z) \Big|_{z=\epsilon} \\ &\sim p^4 \log(p\epsilon) + \sum_k \frac{1}{\epsilon^k} (\text{polynomial in } p) + O(\epsilon) \end{aligned}$$

since $\phi_p(z) = a_0 + a_2(pz)^2 + a_4(pz)^4 + c_4(pz)^4 \log pz + \dots$. We see that there are divergent terms in ϵ but they are local polynomials in p . These analytic terms are irrelevant in a quantum field theory computation since they can be reabsorbed by local counterterms. In the $\epsilon \rightarrow 0$ limit the relevant contribution is

$$\langle O(p) O(p') \rangle \sim p^4 \log(p\epsilon) \quad (2.25)$$

which after Fourier transform to coordinate space becomes

$$\langle O(x) O(x') \rangle = \frac{1}{(x - x')^8} \quad (2.26)$$

in agreement with CFT expectations for an operator of dimension four.

An analogous computation can be performed for $m^2 \neq 0$ and $\Delta = 4$. In this case the boundary condition (2.16) requires

$$\phi_p(z = \epsilon) \equiv \phi_p^0 \epsilon^{4-\Delta} \quad (2.27)$$

so that $\phi(z) \sim z^{4-\Delta}$ for $z \rightarrow 0$ and the solution is

$$\phi_p(z) = \frac{(pz)^2 K_{\Delta-2}(pz)}{(p\epsilon)^2 K_{\Delta-2}(p\epsilon)} \phi_p^0 \epsilon^{4-\Delta} \quad (2.28)$$

As before

$$\langle O(p)O(p') \rangle \sim p^{2\Delta-4} + \text{analytic terms} \quad (2.29)$$

which after Fourier transform to coordinate space becomes

$$\langle O(x)O(x') \rangle = \frac{1}{(x-x')^{2\Delta}} \quad (2.30)$$

Let us note that the introduction of a cut-off ϵ is not only a matter of convenience, since the $\epsilon \rightarrow 0$ limit does not commute with other expansions performed to obtain the result. In particular, the cut-off prescription is the right one for obtaining the right normalization of the two point functions consistent with Ward identities when vector fields are included [16].

This computation confirms the interpretation of Δ as the conformal dimension. There are various observations to be made about this identification and its relation with the mass in AdS_5

$$R^2 m^2 = \Delta(\Delta - 4) \quad (2.31)$$

where we have restored the AdS radius,

- $m^2 \geq 0$ only for $\Delta \geq 4$. There are certainly theories where $\Delta < 4$. In fact the unitary bound is $\Delta \geq 1$. Operators with $\Delta < 4$ correspond to fields with negative mass in AdS_5 . However they are not tachions. Energy is positive as long as the Breitenlohner-Freedman bound $m^2 R^2 \geq -4$ is satisfied [17]; the curvature gives indeed a positive contribution to the energy of a scalar field propagating in AdS. The minimal value $m^2 R^2 = -4$ corresponds to $\Delta = 2$.
- Still a puzzle. Unitary bound requires $\Delta \geq 1$. Using masses greater than $-4/R^2$ we can obtain all operators with $\Delta \geq 2$. What happens to the operators with $1 \leq \Delta < 2$? Recall that we choosed Δ as the largest solution of equation

(2.31). This because typically only the largest solution is greater than the unitary bound. However, precisely for $-4 \leq m^2 R^2 \leq -3$, equation (2.31) has two different solutions satisfying the unitary bound, one with $1 \leq \Delta \leq 2$ and one with $2 \leq \Delta \leq 3$. One then has two different choices for imposing boundary conditions: they amount to choice ϕ_0 or ϕ_1 as boundary value of the bulk field. The two different choices lead to correlation functions for two different operators, one with $1 \leq \Delta \leq 2$ and one with $2 \leq \Delta \leq 3$ [18].

- The relation between mass and conformal dimension for fields of arbitrary spin is

scalar ϕ	$(j_1, j_2) = (0, 0)$	$m^2 = R^2 \Delta(\Delta - 4)$
vector A_μ	$(j_1, j_2) = (\frac{1}{2}, \frac{1}{2})$	$m^2 = R^2(\Delta - 1)(\Delta - 3)$
symm $g_{\mu\nu}$	$(j_1, j_2) = (1, 1)$	$m^2 = R^2 \Delta(\Delta - 4)$
antisymm $B_{\mu\nu}$	$(j_1, j_2) = (1, 0) + (0, 1)$	$m^2 = R^2(\Delta - 2)^2$
spin $\frac{1}{2} \psi$	$(j_1, j_2) = (\frac{1}{2}, 0) + (0, \frac{1}{2})$	$m = R(\Delta - 2)$
spin $\frac{3}{2} \psi_\mu$	$(j_1, j_2) = (\frac{1}{2}, 1) + (1, \frac{1}{2})$	$m = R(\Delta - 2)$

The mass is a function of the three quantum numbers (Δ, j_1, j_2) , or more geometrically, of the Casimirs of the conformal group $O(2, 4)$. For a scalar field, the mass just corresponds to the quadratic Casimir. In general, mass is an ambiguous concept in AdS because of the coupling to the curvature (for a scalar ϕR^2). We choosed a definition that is consistent with the dual interpretation in terms of conformal fields. For example a massless graviton and a massless gauge field correspond to operators with dimension four and three, respectively. This is consistent with the fact that conformal invariance requires conserved currents to have canonical dimension.

2.3 Construction of correlation functions

We now sketch the general construction of n -point correlation functions for a CFT with a gravitational dual given by an effective action in AdS_5 . For all reasonable S_{AdS} , we can inductively construct a set of Green functions that satisfy all requirements of a local quantum field theory. We shall perform the computation in a theory with AdS_5 vacuum, but it will be clear from the construction how to extend the prescription to other spaces that are have the same topological structure of AdS_5 . Our aim is to

provide a general picture and not a working technology. For this the reader is referred to the very good existing reviews [7, 8].

We shall focus for simplicity on a set of scalar fields ϕ_i with mass m_i interacting through a local Lagrangian $L_{AdS}(\phi)$. We denote with ϕ_i^0 the boundary value of the fields obtained by imposing (2.16) where Δ_i satisfies (2.12); ϕ_i^0 is identified on the quantum field theory side with the source for the dual operators O_i of dimension Δ_i . The CFT generating function is given by evaluating the bulk action on the solution of the equations of motion with the prescribed boundary conditions. A n-point function is obtained by differentiating the on-shell bulk action with respect to the sources ϕ_i^0 and setting $\phi_i^0 = 0$ afterwards

$$\langle O_1 \cdots O_n \rangle = \frac{\delta^n S}{\delta \phi_1^0 \cdots \delta \phi_n^0} \Big|_{\phi_i^0=0}. \quad (2.32)$$

Since on-shell fields vanish when the sources ϕ_i^0 are turned off, it is obvious that an interaction in the bulk action with more than n fields will not contribute to this derivative. Therefore, in order to compute n-point functions we can keep in the action only the terms with at most n-fields.

We shall now consider in turn 1-, 2- and 3-point functions. Up to order $n = 2$ we just need to keep the quadratic terms in the Lagrangian. The equations of motion are then a set of Klein-Gordon equations in the bulk, assuming that the kinetic terms are canonically normalized. To extend a field ϕ with mass m and conformal dimension Δ from the boundary to the bulk we need a Green function or boundary-to-bulk propagator as shown in Figure 8:

$$\phi(z, x_\mu) = \int dx'_\mu K(z, x'_\mu - x_\mu) \phi^0(x'_\mu) \quad (2.33)$$

which satisfies

$$\begin{aligned} (-\square + m^2) K &= 0 \\ K &\rightarrow z^{4-\Delta} \delta(x_\mu - x'_\mu) \quad z \rightarrow 0 \end{aligned} \quad (2.34)$$

It is easy to find a solution working in Euclidean space and treating AdS_5 as a ball in $\mathbb{R}^5$. More precisely, we compactify the boundary to S^4 by adding the point $z = \infty$. We saw in the previous section that the Klein-Gordon equation has a particular x_μ -independent solution z^Δ (see (2.10) and discussion thereafter). This solution is zero on most of the boundary ($z = 0$) and infinity at a particular point ($z = \infty$); it looks like a delta function corresponding to the insertion of a source at $z = \infty$. Since the

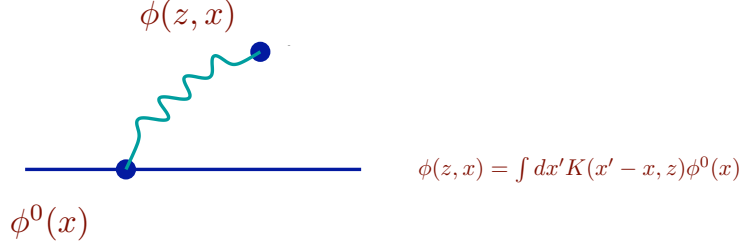


Figure 8: The boundary-to-bulk propagator.

equations of motion are covariant under the Euclidean conformal group $O(1, 5)$ we can find the generic Green function (2.34) by mapping $z = \infty$ to a generic point on the boundary by a conformal transformation. In particular we can send it to $(z = 0, x_\mu = 0)$ by

$$\begin{aligned} z &\rightarrow \frac{z}{z^2 + x_\mu^2} \\ x_\mu &\rightarrow \frac{x_\mu}{z^2 + x_\mu^2} \end{aligned}$$

obtaining the required Green function

$$K(z, x_\mu) = c \frac{z^\Delta}{(z^2 + x_\mu^2)^\Delta} \quad (2.35)$$

where c is some normalization constant. The value of c is not particularly important for us but it can be found by computing

$$\begin{aligned} \int dx K(z, x) \phi^0(x) &= c z^{4-\Delta} \int dx \frac{z^{2\Delta-4}}{(z^2 + x^2)^\Delta} \phi^0(x) \\ &= c z^{4-\Delta} \int dy \frac{\phi^0(zy)}{(1+y^2)^\Delta} \rightarrow c \int \frac{dy}{(1+y^2)^\Delta} z^{4-\Delta} \phi^0(0), \quad z \rightarrow 0 \end{aligned}$$

so that $c^{-1} = \int \frac{dy}{(1+y^2)^\Delta}$.

The solution of the equations of motion is then given by equation (2.33). We can expand the solution in powers of z ,

$$\phi(z, x_\mu) \sim \phi_0(x)(z^{4-\Delta} + O(z)) + \phi_1(x)(z^\Delta + O(z)) \quad (2.36)$$

where

$$\phi_1(x) = c \int dx' \frac{\phi_0(x')}{|x - x'|^{2\Delta}}. \quad (2.37)$$

As in the previous section, the value of the on-shell action reduces to a boundary term

$$S_{AdS} \sim \int_{\text{boundary}} \sqrt{g} \phi \partial^n \phi + \int \sqrt{g} \phi (-\square + m^2) \phi \quad (2.38)$$

since the second term is zero on the equations of motion and we obtain

$$S_{AdS} \sim \frac{1}{z^3} \phi \partial_z \phi \Big|_{\text{boundary}} \quad (2.39)$$

By inserting (2.36) we see that we have some divergent terms proportional to $\phi^0(x)^2$ or other local functions of ϕ_0 ⁴. These are contact terms which can be reabsorbed by local counterterms and we disregard them. The finite contribution is non local and given by

$$S_{AdS} \sim \int dx \phi^0(x) \phi^1(x) \sim \int dx dx' \frac{\phi^0(x) \phi^0(x')}{|x - x'|^{2\Delta}} \quad (2.40)$$

The first thing we note is that in general the 1-point function is given by

$$\langle O(x) \rangle = \frac{\delta S_{AdS}}{\delta \phi^0(x)} \Big|_{\phi^0=0} = \phi^1(x) \Big|_{\phi^0=0} \quad (2.41)$$

the normalizable term in the solution of the equations of motion. Since ϕ^1 is proportional to ϕ^0 , the 1-point function vanishes identically. This is in agreement with conformal invariance since an operator with non zero dimension cannot have non zero vacuum expectation value without breaking the symmetry under dilations. However the identification of the normalizable term in the solution with a VEV [18, 19] is important in application of the AdS/CFT correspondence to non conformal theories.

The 2-point function is instead given by

$$\langle O(x) O(x') \rangle = \frac{\delta^2 S_{AdS}}{\delta \phi^0(x) \delta \phi^0(x')} \Big|_{\phi^0=0} = \frac{1}{(x - x')^{2\Delta}} \quad (2.42)$$

as prescribed by conformal invariance.

For computing n-point function we need to keep up to the n-adic terms in the action,

$$S_{AdS} = \int dx^5 \left(\frac{1}{2} \sum_i (\partial \phi_i)^2 + \frac{m_i^2}{2} \phi_i^2 + \sum_{k=3}^n \lambda_{i_1 \dots i_k} \phi_{i_1} \dots \phi_{i_k} \right) \quad (2.43)$$

We are considering for simplicity fields with canonic kinetic terms, no higher derivatives interactions and we are neglecting couplings to other gauge and gravity fields.

⁴All the subleading terms in 2.36 with powers of z greater than $z^{4-\Delta}$ contribute divergent terms. However one can check that all these terms are local functions of ϕ^0 . ϕ^1 and its subleading terms are instead non local functions of the source $\phi_0(x)$.

All these other ingredients can be incorporated in the AdS/CFT correspondence without any conceptual effort (but with some technical effort). The equations of motion now have higher order terms and cannot be solved exactly. We can however set a perturbation expansion. The typical equation $(-\square + m^2)\phi = \lambda\phi^n$ can be solved in power series of λ if we know the Green functions for the Klein-Gordon equation in the bulk. At order λ^0 we know that the solution is

$$\phi(z, x)^{\text{zero}} = \int K(z, x - x')\phi^0(x')$$

where K is the boundary-to-bulk propagator (2.34). At $O(\lambda)$ we have

$$\phi^{\text{one}}(z, x) = \lambda \int G(z - z', x - x')(\phi^{\text{zero}}(z', x'))^n$$

where now G is a bulk-to-bulk Green function, that is the solution of

$$(-\square + m^2)G(z - z', x - x') = \frac{1}{\sqrt{g}}\delta(z - z')\delta(x - x') \quad (2.44)$$

with (z, x) and (z', x') arbitrary points in the bulk. Explicit expressions for G can be found in [7]. We can then reinsert ϕ^{one} in the right-hand side of the equation of motion and determine ϕ^{two} . This sets a perturbative expansion for the solution. Since all ϕ are functions of the source ϕ^0 we can stop the expansion after a finite number of steps: all contributions containing more than n powers of ϕ^0 do not give contribution to equation (2.32). One can even set a graphical Feynman description of this perturbative series: the n points on the boundary are connected by boundary-to-bulk propagators to points in the bulk where we insert vertices of the Lagrangian; vertices in the bulk are connected to each other by bulk-to-bulk propagators and we integrate on their position as in Feynman rules.

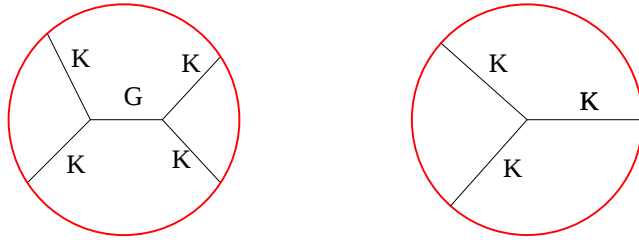


Figure 9: Contributions to 4-point and 3-point functions in a theory with cubic interaction. K are boundary-to-bulk and G bulk-to-bulk propagators. 3-point functions are exhausted by the contribution shown.

For the simple case of a 3-point function, the three external points can only connect a cubic vertex in the bulk. All other graphs will give higher powers of ϕ^0 .

There is no need for bulk-to-bulk propagators. If the cubic interaction is $\lambda_{ijk}\phi_i\phi_j\phi_k$ we obtain

$$\langle O_i(x_i)O_j(x_j)O_k(x_k) \rangle \sim \frac{\lambda_{ijk}}{|x_i - x_j|^{\Delta_i + \Delta_j - \Delta_k} |x_j - x_k|^{\Delta_j + \Delta_k - \Delta_i} |x_k - x_i|^{\Delta_k + \Delta_i - \Delta_j}}$$

in agreement with the requirements of conformal invariance that fix the coordinate dependence of the 3-point function.

Starting with 4-point functions, we would need bulk-to-bulk propagators. The functional dependence a 4-point function is not fixed by conformal invariance but must be consistent with channel decomposition and the OPE expansion in the CFT. Actual computations indeed confirm this: the correlation functions obtained by the AdS/CFT prescription satisfy all requirements of a consistent local quantum field theory.

Let us finish this section with some comments,

- As we saw, divergent contributions appear in the correlation function computation. Local terms in $W(\phi^0)$ can be always eliminated by local counterterms. However, this requires at least regularizing the effective action with a cut-off as we did in section 2.2. This can be done in general and in accord with all Ward identities in the presence of gauge fields. And this should be done in a real calculation. In fact, the two point functions computed as we did, without introducing a cut-off, lead to a wrong normalization. For the purposes of these notes, this is an irrelevant detail. As already said, the reader interested in actual calculation technology should refer to the existing literature.
- The same computation can be done for vector, spinor and tensor fields. Each requires its own propagators. We again refer to the literature for explicit expressions. For example, the two-point function for a massless vector field would give

$$\langle J_\mu(x)J_\nu(x') \rangle = \frac{\delta_{\mu\nu}}{|x - x'|^6} - 2 \frac{(x - x')_\mu(x - x')_\nu}{|x - x'|^8} \quad (2.45)$$

consistent with a field of dimension 3, the canonical dimension of a current (recall $m^2 R^2 = (\Delta - 1)(\Delta - 3)$ for vector fields). The tensor structure shows that the current is conserved. We should stress again a basic fact about vector fields: gauge invariance in the bulk corresponds to a global symmetry in the boundary with conserved currents. Gauge invariance requires zero mass in the bulk and therefore $\Delta = 3$; conformal invariance now implies that a vector operator J_μ with $\Delta = 3$, which saturates the unitary bound, is conserved $\partial^\mu J_\mu = 0$.

2.4 Wilson loops

In gauge theories, another natural observable is the Wilson loop, defined, for every closed contour C and representation R of the gauge group, by the path ordered integral of the holonomy of the gauge field along the path

$$W_R(C) = \text{Tr}_R P e^{i \int_C A_\mu^a T^a dx^\mu} \quad (2.46)$$

where T^a are the generators in the representation R . It has the following intuitive interpretation. Given a pure gauge theory, we introduce external massive sources (quarks) transforming in a representation R of the gauge group. The loop C corresponds to the propagation of a quark-antiquark pair along C , from its creation to its disappearing and measures the free energy of this configuration. For a rectangular Wilson loop in Euclidean space with length L in space and height T in time,

$$W_R(C) = e^{-TE_I(L)} \quad (2.47)$$

where $E_I(L)$ is the energy of a pair of quarks at distance L .

The Wilson loop is a signal for confinement if it grows as (the exponential of) the area of the loop C . In a confining theory indeed external quarks have an energy which grows linearly with distance $E = m_q + m_{\bar{q}} + E_I$ with $E_I = \tau L$ since they are connected by a color flux tube, or string, with tension τ . It follows that, for a rectangular loop $W = e^{-TE_I} = e^{-\tau TL}$, and more generally

$$W(C) \sim e^{-\tau A(C)} \quad (2.48)$$

where $A(C)$ is the area of the loop, or equivalently the area of the world-sheet for the propagation of the string. In this picture, the quarks are considered as external non dynamical sources (for example quarks with a very large mass) and $W(C)$ just captures the dynamics of the gauge fields in the theory. Confinement in QCD is indeed a property of the glue vacuum.

We can define an analogous quantity in AdS. An external source is inserted at the boundary and we may attach to it a string. This is very natural in the explicit realizations of the AdS/CFT correspondence where the gravitational background is embedded in a string vacuum. We are lead to consider a string whose endpoint lies on a contour C on the boundary. The natural action for the string is the Nambu-Goto action $\int dx^2 \sqrt{g}$, which is proportional to its area. We can now define a very natural observable in AdS

$$W(C) = (\text{minimal area surface with boundary } C) \quad (2.49)$$

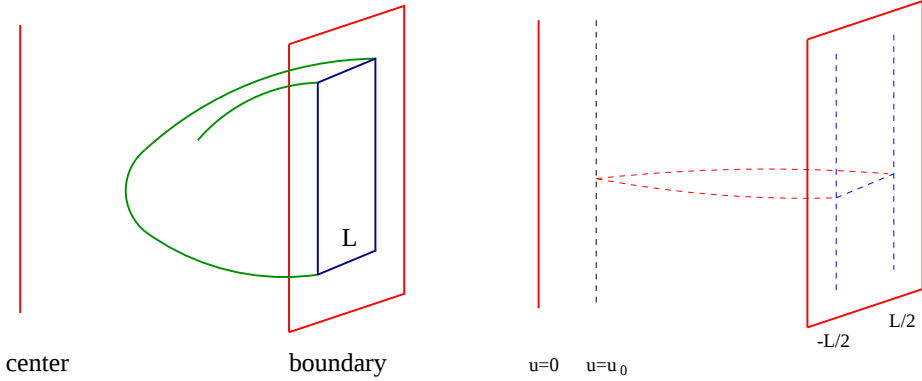


Figure 10: The string world-volume that minimizes the area in AdS. It enters deeply into the space.

This is identified with the value of some Wilson loop in the dual CFT. We shall see in section 3 and 4 that this identification is very natural in explicit realizations of AdS/CFT correspondence.

In a flat space-time, the surface of minimal area with rectangular C would lie entirely on the boundary, giving an obvious confining behaviour $S \sim LT$, good for a confining theory, not certainly for a CFT. However, things are different in AdS. The point is that the AdS metric diverges on the boundary $u = \infty$:

$$ds^2 = u^2(dx_\mu dx_\mu) + \frac{(du)^2}{u^2} \quad (2.50)$$

(as usual we put $R = 1$ when possible) and it is energetically favorable for the string to enter inside AdS. As in figure 10, the string will penetrate deeply in the interior of the space where the gravitational interaction is weaker.

Choose a parameterization of the world-sheet by coordinates σ and τ . The string world-sheet in AdS will be given by an embedding $X^M(\sigma, \tau)$ and the action is

$$S = \int_C d\sigma d\tau \sqrt{\det(g_{MN} \partial_a X^M \partial_b X^N)}. \quad (2.51)$$

We can perform a simple calculation for a time invariant configuration of two external sources separated by a distance L . In this case, as in Figure 10, we can choose $t = \tau, x = \sigma, U = U(\sigma) \equiv U(x)$ and we obtain the action

$$S \sim \int_{-L/2}^{L/2} \int_0^T dt \sqrt{(\partial_x u)^2 + u^4} \sim T \int_{-L/2}^{L/2} \sqrt{(\partial_x u)^2 + u^4} \quad (2.52)$$

We are taking T very large in order to have a very large strip and not to bore about the bottom and top of the rectangle. To find the minimal area is just a classical

exercise with the Euler-Lagrange equations. In particular, since the problem is time invariant, the energy

$$H = \frac{\delta L}{\delta(\partial_x u)} \partial_x u - L = \frac{u^4}{\sqrt{(\partial_x u)^2 + u^4}} \quad (2.53)$$

is conserved. Its value can be computed at the turning point of the string, which by symmetry is at $x = 0$. Since at the turning point $u'(0) = 0$ we have $H = u(0)^2$. We obtain a differential equation for u

$$u' = u^2 \sqrt{\frac{u^4}{u(0)^4} - 1} \quad (2.54)$$

which we can solve by

$$x = \int_0^x dx = \frac{1}{u(0)} \int_1^{u/u(0)} \frac{dy}{y^2 \sqrt{y^4 - 1}} \quad (2.55)$$

First of all, note that, at the boundary $u = \infty$, $x = L/2$ and we obtain a relation between L and the turning point

$$L/2 = \frac{1}{u(0)} \int_1^\infty \frac{dy}{y^2 \sqrt{y^4 - 1}} \sim \frac{1}{u(0)} \quad (2.56)$$

The action evaluated on the solution reads

$$S = T \int_{-L/2}^{L/2} \sqrt{(\partial_x u)^2 + u^4} = 2Tu(0) \int_1^\infty \frac{y^2 dy}{\sqrt{y^4 - 1}} \quad (2.57)$$

This integral is linearly divergent. The interpretation of this divergence is that we are really computing the energy of a pair of quarks including their large renormalized self-energy $m_q + m_{\bar{q}} + E_I$. The energy of a single quark can be estimated by a long linear string from $u = \infty$ to $u = 0$. We are only interested in the potential energy of the sources, therefore we subtract two linearly divergent contributions and we obtain a finite result

$$S = 2Tu(0) \int_1^\infty \left(\frac{y^2 dy}{\sqrt{y^4 - 1}} - 1 \right) \sim Tu(0) \sim T \frac{1}{L} \quad (2.58)$$

With more effort, Wilson loops can be computed for more general contours. Let us make some observations.

- We see that the result for a Wilson loop is consistent with conformal invariance: by dimensional reasons, in absence of dimensionful parameters, the potential energy should go like $1/L$. We shall see later what happens in backgrounds which holographically realize confinement. If we restore factors of R and the tension τ of the string, the final result is $E_I \sim (\tau R^2)/L$. We shall discuss again this behaviour in section 3.3.4.
- We see from equation (2.56) that for large separation between the sources the turning point goes to the center of AdS. As we already anticipated, this has a very natural holographic interpretation: probing large distances in quantum field theory means probing the horizon. More generally, from $L \sim 1/u(0)$, we see that field theory UV computations ($L \ll 1$) take contributions from region with large u , while IR computations ($L \gg 1$) from region with small u , according to the interpretation of u as an energy scale.
- The regularization of the action is similar in spirit to the elimination of local terms in the holographic computation of correlation functions. We see that, even if the theory in AdS is classical, we need to consistently implement a regularization and a sort of renormalization of physical quantities.

2.5 Weyl anomaly

A very interesting object to compute with the AdS/CFT correspondence is the Weyl anomaly. Conformal invariance breaks when the CFT is coupled to an external metric, or equivalently when the theory is defined on a curved background. In fact $\langle T^\mu_\mu \rangle \neq 0$ when $g_{\mu\nu} \neq 0$. By general covariance, one can prove that

$$T^\mu_\mu = -a E_4 - c I_4 \quad (2.59)$$

where E_4, I_4 are the two invariants we can make with the Riemann tensor

$$\begin{aligned} E_4 &= \frac{1}{16\pi^2} (R_{\mu\nu\tau\rho}^2 - 4R_{\mu\nu}^2 + R^2) \\ I_4 &= -\frac{1}{16\pi^2} \left(R_{\mu\nu\tau\rho}^2 - 2R_{\mu\nu}^2 + \frac{1}{3}R^2 \right) \end{aligned} \quad (2.60)$$

(and cannot be reabsorbed with local counterterms). c and a are called central charges. They generalize the familiar central charge c of two dimensional conformal field theories. It is known that c cannot be zero in a unitary two-dimensional

CFT, and the same is true of a and c in unitary four dimensional CFTs. For example, in free theories, we have a formula in terms of the number N_i of fields of spin i :

$$c = \frac{12N_1 + 2N_{1/2} + N_0}{120} \quad a = \frac{124N_1 + 11N_{1/2} + 1N_0}{720} \quad (2.61)$$

The non-vanishing of the trace of the stress-energy tensor is equivalent to the fact that the functional

$$e^{W(g)} = \left\langle e^{\int dx \sqrt{g} g_{\mu\nu} T^{\mu\nu}} \right\rangle \quad (2.62)$$

is not invariant under Weyl rescaling $\delta_\lambda g_{\mu\nu} = \lambda g_{\mu\nu}$. In fact $\langle T^\mu_\mu \rangle = \delta_\lambda W$. This looks like something that is amenable to an holographic computation: the AdS/CFT correspondence in fact just computes the functional $W(g)$ for external sources. In fact, starting with the Einstein action with cosmological constant

$$S = \int dx^5 \sqrt{g} (\mathcal{R} - \Lambda) \quad (2.63)$$

we can compute $W(g)$ using holography. After regularizing and removing divergences, we can compute $\langle T^\mu_\mu \rangle$ and we reproduce the functional form (2.59) predicted by conformal invariance [20]. This is technical calculation that it is too long to report here but it is very instructive and it is strongly suggested as a complementary reading.

As a surprising result of this computation, we have the prediction that for all CFT described by AdS [20]:

$$c = a \quad (2.64)$$

and the common value is determined in terms of the cosmological constant

$$a = c \sim R^3 \sim (\Lambda)^{-3/2} \quad (2.65)$$

Equation (2.64) is the first result that restrict the class of theories that have a weakly coupled holographic description. Only theories with $c = a$ can have a dual description based on an effective Lagrangian for Einstein gravity coupled to other fields.

2.6 Which quantum field theory?

Given an effective Lagrangian for gravity coupled to other fields and an AdS_5 vacuum we have constructed a set of correlation functions satisfying the axioms of a local conformal field theory. What kind of theory is it? Moreover, is it true that all four

dimensional CFTs have a holographic dual? We already saw that there are restrictions. If we assume an effective weakly coupled Lagrangian that reduces to Einstein gravity the central charges of the CFT are equal: $c = a$. This is not a general features of CFTs, as the free field theory case clearly shows. Moreover, we considered only theories with maximal spin equal to two. Therefore all CFT operators should have maximal spin two. Again, it is very easy to find theories where conformal operators have arbitrarily high dimensions, as the free field theory case shows. We see that, in this way, we can describe completely only theories with very few operators with low spin. We shall see in the next section that in the specific realizations of the AdS/CFT correspondence these restrictions are naturally implemented and without any contradiction. But this restricts the class of CFTs with a weakly coupled gravitational description.

We could include higher derivative interactions and higher spin fields in our effective Lagrangian and hope to obtain a complete description of any CFT. However, we typically face problems with ghosts and consistency of higher spin theories. For example, the free field theory case, where we have an infinite number of conserved currents of arbitrary spin, shows clearly that we should deal with a theory with infinite massless higher spin fields in the bulk. All these theory typically make sense only if embedded in a consistent string background. We can say that every CFT has a holographic dual, but this is a string theory dual with all complications about string theory. In particular there could be no regime where the gravitational dual is weakly coupled; in this case, we will not be able to compute CFT correlators using a classical theory.

2.7 The non-conformal realm

Holography, that works so well for conformal gauge theories, certainly has to play a role in the description of non conformal, realistic theories. Pure glue theories (with no supersymmetry or $\mathcal{N} = 1$) confine, have a mass gap and a discrete spectrum of massive glueballs. We will now describe how these features can be realized through a gravitational dual.

In the description of non conformal theories we have to give up the AdS form of

the metric for some more general metric with four-dimensional Poincaré invariance ⁵,

$$ds^2 = e^{2A(z)} (dz^2 + dx_\mu dx^\mu) . \quad (2.66)$$

We shall study only the case where the metric (2.66) is asymptotic to AdS_5 for small z . We also assume that it is everywhere regular. Since the space has a boundary isomorphic to $\mathbb{R}^{1,3}$, we can apply most of the technology of the AdS/CFT correspondence.

With respect with the conformal case, we also give up the association of bulk fluctuations with conformally invariant operators. Fluctuations of the background are now associated with bound states of the dual gauge theory. In QCD-like theories these would be glueballs and mesons. Baryons, as usual in large N Lagrangian, will appear as solitonic objects, typically wrapped branes. Other non-perturbative objects that characterize the dynamics of strongly coupled gauge theories, like monopoles, flux tubes or domain walls, typically appear as bound states made with extended objects, strings or branes.

2.7.1 Confining theories

The computation of correlation functions and Wilson loop proceeds exactly as in the conformal case. The results however are different.

- **Confinement:** In general, a criterion for confinement is the following: the warp factor $e^{2A(z)}$ multiplying the four-dimensional part of the metric must be bounded above zero. We can see this using a Wilson loop. As in the conformal case, we introduce heavy external sources at the boundary of AdS and study their energy by analyzing a string connecting them. Since the metric is still blowing up at the boundary and decreasing in the interior, the string will find energetically favorable to reach the IR part of the background. In particular, the string will minimize its energy by reaching the point z_0 where the warp factor has a minimum. For large L , the minimal energy configuration consists of three straight segments: two long strings at fixed x_μ connecting the boundary to the point z_0 , and a string at fixed z_0 stretching for a distance L along the four-dimensional spacetime directions as in Figure 11. The infinite energy of

⁵In explicit realizations, one typically considers higher dimensional metrics, but, for simplicity, in this section we restrict to the five-dimensional case; all the results can be extended to the higher dimensional case.

the long string from $z = 0$ to $z = z_0$ is interpreted as the bare mass of the external source. All the relevant contribution to the potential energy between two external sources is then due to a string localized at $z = z_0$ and stretched in the x direction. The total energy

$$E(L) = m_q + m_{\bar{q}} + e^{2A(z_0)}\tau L \quad (2.67)$$

then gives the linear increasing potential characteristic of confinement. We denoted with τ the tension of the string; in any explicit realization this will be fixed by the string scale M_s . The theory has stable finite tension strings, which

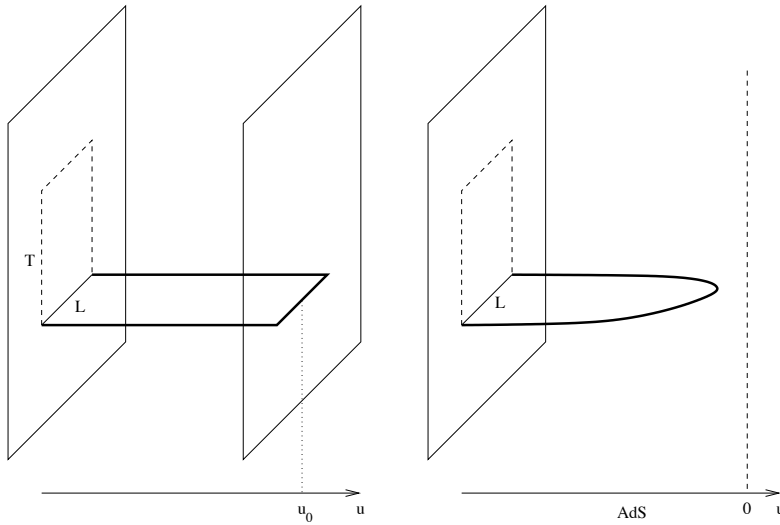


Figure 11: The quark-antiquark potential in the confining and in the conformal case.

we identify with the color flux tube of the dual gauge theory. They will live in the region of the solution where the warp factor has its minimum value e^{2A_0} and their tension will be given by $e^{2A_0}\tau$. This situation should be contrasted with the *AdS* case, where the metric vanishes in the IR at $z = \infty$: the vanishing of the metric is responsible for the $1/L$ behaviour of the Wilson loop.

- **The Glueball Spectrum:** In a theory with mass gap and discrete spectrum, we expect poles in the two point functions corresponding to the physical states (mesons, glueballs,...)

$$\langle \phi(k)\phi(0) \rangle = \sum_i \frac{A_i}{k^2 + M_i^2} \quad (2.68)$$

where M_i^2 are the masses of the glueballs. We use for convenience the Euclidean version of the theory. The poles appear for unphysical values of $k^2 = -M_i^2$.

The correlation functions can be computed as in the conformal case by evaluating the action on the equation of motion for a field ϕ dual to the operator O . In the case of a minimally coupled scalar field, the equation of motion is

$$\partial_z (e^{3A(z)/2} \partial_z \phi) + \partial_\mu (e^{3A(z)/2} \partial^\mu \phi) = e^{5A(z)/2} m^2 \phi \quad (2.69)$$

At small z the theory is asymptotically AdS and the solution still behaves as $\phi_k = z^{4-\Delta}(A(k) + O(z)) + z^\Delta(B(k) + O(z))$, where $R^2 m^2 = \Delta(\Delta - 4)$. The contribution proportional to A is non-normalizable while the one proportional to B is normalizable. Regularity of the solution for large z determines $B(k)$ and the subleading terms as functions of $A(k)$ exactly as in the conformal case. By normalizing the field

$$\hat{\phi}_k(z) = \phi_k^0 \left[z^{4-\Delta}(1 + O(z)) + \frac{B(k)}{A(k)} z^\Delta(1 + O(z)) \right] \quad (2.70)$$

with the value of the boundary source, we see that 2-point Green functions have poles if and only if $A(k) = 0$; or in other words if there exist normalizable regular solutions of the equations of motion. In the physical region $k^2 \geq 0$ there are no such solutions⁶. However nothing prevents the existence of solutions in the unphysical region $k^2 < 0$. The corresponding values of k^2 determines the masses of bound states through $M^2 = -k^2$.

We have thus reduced the problem of finding the spectrum to that of finding the regular normalizable solutions of the equations of motion. We can further reduce this problem to a Schroedinger-like one. It is easy to see that the equation of motion of a massive scalar field with mode expansion

$$\phi(x_\mu, z) = \phi(z) e^{ikx}, \quad k^2 = -M^2 \quad (2.71)$$

can be written as

$$-\psi'' + \left(\frac{9}{4}(A')^2 + \frac{3}{2}A'' + m^2 e^{2A} \right) \psi \equiv E\psi, \quad E = -k^2 \quad (2.72)$$

where $\phi = e^{-3A/2} \psi$. We see that the spectrum is given by the positive eigenvalues $E = M^2 = -k^2$ of a Schoedinger equation. Boundary conditions for this

⁶In fact the action reduces to a boundary term as in (2.40) that vanishes for $A(k) = 0$; on the other hand the Euclidean action is definite positive for $k^2 \geq 0$ and can vanish only for a solution which is identically zero. Recall that, in the case $-4 \leq R^2 m^2 \leq 0$, the action is still positive definite due to the effect of curvature.

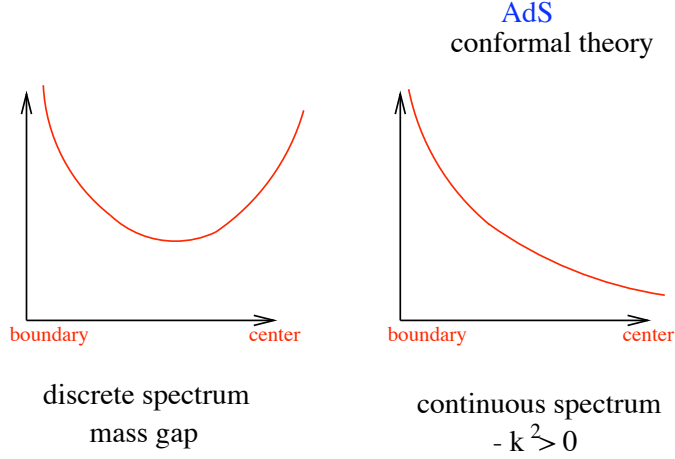


Figure 12: The potential in the confining and conformal case.

equation should be worked out from the original problem but are straightforward. Normalizability of ϕ

$$\int \sqrt{g} |\phi|^2 \sim \int \frac{|\psi|^2}{z^2} dz \quad (2.73)$$

requires vanishing of ψ at the boundary $z = 0$. Conditions at $z = \infty$ are imposed by the regularity of the solution.

In the case of AdS_5 , the potential in the Schoedinger equation is $\sim 1/z^2$, and we have a continuum spectrum starting from zero (see Figure 12), appropriate for a confining theory. On the other hand, the typical potential for a confining theory coincides with that of AdS_5 only for small z , and it goes to a constant value or blows up for large z ; it has therefore a discrete spectrum M_i^2 bounded from below which gives the glueball masses. This happens in particular in the case where $A(z)$ is finite and bounded above zero.

We shall discuss explicit examples of regular backgrounds based on string theory in section 4.1.

Exercise: A very simple example to which apply the previous discussion is given by a slice of AdS_5 . A bounded warp factor is obtained by introducing a boundary brane at a finite value z_0 and imposing Dirichlet or neumann boundary conditions. The reader is invited to work out such a model and to compute the spectrum.

2.7.2 The RG flow

In general, the background (2.66) should solve the equations of motion of a local five-dimensional Lagrangian, which will be used for all computations based on holography. In this section we shall discuss general properties of Poincaré-four invariant solutions of an arbitrary five-dimensional theory. Simple exercises in classical General Relativity lead to interesting conclusions.

Start with a local five dimensional gravitational theory

$$\mathcal{L} = \sqrt{-g} \left[-\frac{\mathcal{R}}{4} - \frac{1}{2} G_{ab} \partial \phi_a \partial \phi_b + V(\phi) \right] \quad (2.74)$$

where, for simplicity, we have only considered scalar fields. In order to simplify the equations of motion it is useful to write the metric in the coordinates

$$ds^2 = dy^2 + e^{2Y(y)} dx_\mu dx^\mu. \quad (2.75)$$

AdS_5 is recovered for $Y(y) \sim y/R$. An easy computation, that the reader is strongly advised to perform, shows that all the Einstein and scalar equations of motion following from eq. (2.74) can be deduced from the effective Lagrangian

$$\mathcal{L} = e^{4Y} \left[3 \left(\frac{dY}{dy} \right)^2 - \frac{1}{2} G_{ab} \frac{d\varphi^i}{dy} \frac{d\varphi^j}{dy} - V(\varphi) \right], \quad (2.76)$$

supported by the zero energy constraint

$$3(Y')^2 - \frac{1}{2} G_{ab} (\varphi^i)' (\varphi^j)' + V(\varphi) = 0. \quad (2.77)$$

In particular, there are only two independent equations that comes from Einstein and scalar equations of motion

$$\begin{aligned} (G_{ij} \varphi_j')' + 4G_{ij} Y' \varphi_j' &= \frac{\partial V}{\partial \varphi_i}, \\ 6(Y')^2 &= G_{ij} \varphi_i' \varphi_j' - 2V. \end{aligned} \quad (2.78)$$

An obvious solution of these equations with a boundary is obtained at the critical points of the potential $\frac{\partial V}{\partial \varphi_i} = 0$; if the critical value of V is negative, we obtain an AdS_5 solution of the equations of motion with constant scalar fields. In fact $Y(y) = y/R$ where $1/R^2 = -V_{\text{crit}}/3$. The value of V at the critical point determines the cosmological constant and the radius of AdS . By the general discussion given in this section, we expect that the gravitational theory in this vacuum describes a dual CFT.

Other solutions with a boundary can be obtained as follows. Start with a five-dimensional theory with a critical point of the potential which, without loss of generality, we take at $\varphi_i = 0$. This AdS vacuum corresponds to some dual CFT. We can expand the action to quadratic order around the AdS vacuum and read the masses m_i^2 of the scalar fields; call Δ_i the dimension of the dual operator, obtained using $R^2 m^2 = \Delta(\Delta - 4)$. We now look for more general solutions with asymptotics: $Y(y) \rightarrow y/R$ and $\varphi_i(y) \rightarrow 0$ for $y \rightarrow \infty$. Since asymptotically the background is AdS_5 , the scalar fields behave as

$$\varphi_i(y) = A_i e^{-(4-\Delta_i)y} + B_i e^{-\Delta_i y}, \quad (2.79)$$

for large y . According to the general rules of the AdS/CFT correspondence, A_i is associated with a source for the dual operator O_i , while as seen in equation (2.41), B_i can be associated with the 1-point function for O_i . We are then naturally led to the following interpretation of the new solutions. We associate solutions behaving as $e^{-(4-\Delta)y}$ with deformations of the original CFT with the operator O_i

$$L_{CFT} \rightarrow L_{CFT} + \int dx^4 A_i O_i \quad (2.80)$$

On the other hand, solutions asymptotic to $e^{-\Delta y}$ (the subset with $A_i = 0$) are associated with a different vacuum of the theory, where the operator O_i has a non-zero VEV proportional to B_i [18, 19].

Solutions of this type can be interpreted as the gravitational dual of a Renormalization Group flow (RG) in quantum field theory. The deformation of the original CFT breaks conformal invariance and induces a running of deformation parameters and coupling constants with the scale. In the gravitational description this RG flow is described by the non-zero profile of the scalar fields. The energy scale can be roughly identified with the fifth dimensional coordinate, with the region of large y corresponding to the UV region. The functions $\varphi_i(y)$ then encode the running with the scale of the deformation parameters.

Interesting is the case where a CFT perturbed by a relevant operator O_i flows in the IR to another fixed point. The supergravity description corresponds to the case where the potential V has another critical point for non-zero value of the scalar field φ_i . The $5d$ description of the RG flow between the two CFTs is a kink solution, which interpolates between the two critical points [21]. Explicitly this is given by a solution with asymptotics: $Y(y) \rightarrow y/R_{UV,IR}$ for $y \rightarrow \pm\infty$; $\varphi_i(y) \rightarrow 0$ for $y \rightarrow \infty$, while $\varphi_i(y) \rightarrow \varphi_{i,IR}$ for $y \rightarrow -\infty$. We associate larger energies with increasing y . Most of the solutions however leaves a critical point and goes to infinity in the space

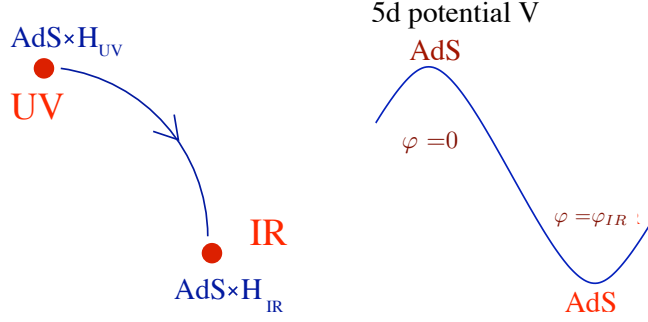


Figure 13: Schematic picture of the RG flow.

of φ_i . They correspond to theories which are non-conformal in the IR. Unfortunately most of these solutions have naked singularities.

Now some technical observations.

- Some of the squared masses around a critical point can be negative. As we have already said, due to the non-zero curvature, these modes are not *tachionic*: recall that a mode is stable iff $m^2 R^2 \geq -4$ (see section 2.2). We have
 - Negative mass modes $\rightarrow$ Relevant operators ($\Delta < 4$)
 - Massless modes $\rightarrow$ Marginal operators ($\Delta = 4$)
 - Positive mass modes $\rightarrow$ Irrelevant operators ($\Delta > 4$)

In particular, the critical point of V does not need to be a minimum. In the interesting cases of CFT with relevant operators, it is a maximum, with some unstable directions associated with fields with $0 \geq R^2 m^2 \geq -4$. In particular the flow between two CFTs induced by a relevant operator interpolates between a maximum and a minimum as in Figure 13; the deformation must be relevant in the UV to leave the critical point and irrelevant in the IR to reach a new one.

- The problem of finding solutions simplify for potential of simple form, occurring typically if some supersymmetry is present [22]. If the potential V can be written in terms of a superpotential W as

$$V = \frac{1}{8} G^{ij} \frac{\partial W}{\partial \varphi^i} \frac{\partial W}{\partial \varphi^j} - \frac{1}{3} |W|^2. \quad (2.81)$$

we can find particular solutions by reducing the second order equations to first order ones. It is easy to check that a solution of

$$\begin{aligned}\frac{d\varphi^i}{dy} &= \frac{1}{2}G^{ij}\frac{\partial W}{\partial\varphi^j}, \\ \frac{dY}{dy} &= -\frac{1}{3}W.\end{aligned}\tag{2.82}$$

also solve the second order equations of motions. In supersymmetric theories the reduction from second to first order is obtained by looking at the vanishing of the supersymmetry variations instead than the equations of motion. It is possible to show that for various gauged supergravity supersymmetric solutions follow from a superpotential, even if this is not a general statement.

The existence of a holographic RG flow has striking consequences for conformal field theories at strong coupling. The most remarkable one is the possibility of proving a c -theorem for all theories with an AdS dual. As discussed in section 2.5, there are two central charges a, c in the superconformal algebra in four dimensions. It has been conjectured in [23] that, for general four dimensional conformal field theories, a is decreasing along the RG flow and thus it is the candidate central charge for a c -theorem in four dimensions. There are examples that, on the contrary, show that c , which enters in the OPE of two stress energy tensors, is not in general decreasing. For theories with an AdS dual $a = c$ and we can recover the value from the cosmological constant $a = c \sim R^3 \sim (\Lambda)^{-3/2}$. It is possible to extrapolate the value of the central charge all along an RG flow. We can define a c -function that is monotonically decreasing [21, 22] and reduces to the previous result at the fixed points. The obvious choice

$$c(y) \sim (Y')^{-3}\tag{2.83}$$

makes the job. The monotonicity of c can be easily checked from the equations of motion (2.78) and the boundary conditions of the flow [21]. The equations of motion indeed give

$$Y'' = -\frac{4}{3}G_{ab}\varphi'_a\varphi'_b\tag{2.84}$$

In a consistent theory, the kinetic terms for scalar fields need to be positive definite to avoid ghosts. The previous equation then shows the decreasing of the c -function for all sensible theories. More generally, without resorting on a specific Lagrangian, we can reduce the c -theorem to a positivity condition for energy [22]. The equations of motion for five dimensional gravity coupled to matter with a stress energy tensor $T_{\alpha\beta}$ give

$$-2Y'' \sim (T_0^0 - T_r^r)\tag{2.85}$$

The c theorem is then equivalent to the weak positive energy condition,

$$\xi^\alpha \xi^\beta T_{\alpha\beta} \geq 0 \quad \xi \text{ null vector} \quad (2.86)$$

that is expected to hold in all physically relevant supergravity solutions. Let us stress that the value of c is well defined only at a fixed point, where it represents a central charge. In QFT, the value of c along the flow is scheme dependent. Similarly, in supergravity there are several possible definitions of monotonic functions interpolating between the central charges at the fixed points.

2.7.3 Some remarks

Let us finish the discussion of non-conformal models with some general comments.

- It is very easy to find solutions with Poincaré invariance and a boundary, as we discussed above. However, it is very difficult to find regular solutions without naked singularities. Regularity is a basic requirement in the AdS/CFT correspondence. We can have a five dimensional effective theory with singularities but these should be resolved when the model is embedded in a consistent string theory. Singular solutions are nevertheless often used, in particular in phenomenological applications like AdS/QCD, since in most cases correlation functions computed through holography give finite results despite the singularity.
- The most successful regular solutions describing confining theories in string theory arise directly from ten dimensional constructions. Most of what we say, from the criteria for confinement to the picture of an RG flow, straightforwardly extend to higher dimensional solutions.
- We only consider asymptotically AdS solutions. In this framework we can only describe field theories that become conformal in the UV. This condition can be relaxed in specific constructions. In particular, the best understood example of a dual for a $\mathcal{N} = 1$ gauge theory, the Klebanov-Strassler solution [24], is based on a background which is asymptotic to the AdS_5 metric only up to logarithmic corrections depending on the radial coordinate. Other regular solutions, for example the Maldacena-Nunez one [25], are based on even more exotic form of holography for NS branes.

3 Explicit examples: the conformal case

In this section we analyze in details the founding example of the AdS/CFT correspondence, the duality between $\mathcal{N} = 4$ SYM and the type IIB string background $AdS_5 \times S^5$. We shall be able to give a precise map between observables in the two theories and apply and extend the general results of section 2.

The basic tool for providing dual pairs is the ability of realizing gauge theories on the world-volume of D-branes in string theory. D-branes are extended objects with non zero tension and they modify the surrounding space-time. The gravity dual is obtained by taking a near-brane (or near-horizon) limit of the D-brane geometry.

We shall work at the level of effective actions without requiring any specific knowledge of string theory. All the necessary ingredients will be reviewed in the following.

3.1 String theory, supergravity and D-branes

In this section, we give a quick presentation of the string theory ingredients of the game. The following is not intended as a tutorial in string theory, but just a basic description of the context and players, and an attempt to describe them in terms of effective field theories. We refer to [26–29] for more details on string theory, supergravity, p- and D-branes.

A consistent quantization of gravity is obtained by replacing elementary particles with extended objects. The fundamental ingredient of the theory is a string with tension $M_s^2 = \frac{1}{2\pi\alpha'}$. The vibrations of this extended object give rise, upon quantization, to some massless modes and a tower of particles of mass $m^2 \sim \frac{1}{\alpha'}$. The connection with gravity comes from the fact that the spectrum of a closed string contains a massless tensor of rank two, which can be decomposed in a symmetric tensor, an antisymmetric tensor and a scalar field,

$$g_{\mu\nu}, \quad B_{\mu\nu}, \quad \phi \tag{3.1}$$

In particular, the massless spin 2 can be identified with the graviton. The scalar ϕ has also an important role and it is called the dilaton.

It will be important in the following to consider also open strings. The quantization of an open string contains at the massless level a tensor of rank one, that is a vector field A_μ . This will be extremely important for the constructions involving D-branes.

There is a variety of consistent string and superstring theories. At low energy, the effective action of the massless modes of a closed string reduces to General Relativity coupled to matter fields or to supergravity when supersymmetry is present. We shall often ignore the fermionic part of the effective action, thus avoiding all technical details about supergravity theories.

One peculiar point of supergravities in higher dimensions is the presence of antisymmetric tensor fields with a gauge invariance, generalizations of the electromagnetic field. These are given by multi-indices potentials $C_{\mu_1 \dots \mu_k}$ invariant under

$$C_{\mu_1 \dots \mu_k} \rightarrow C_{\mu_1 \dots \mu_k} + \partial_{\{\mu_1} \Lambda_{\mu_2 \dots \mu_k\}} \quad (3.2)$$

which generalizes gauge transformations. The gauge invariant object is the curvature

$$F_{\mu_1 \dots \mu_{k+1}} = \partial_{\{\mu_1} C_{\mu_2 \dots \mu_{k+1}\}} \quad (3.3)$$

and the corresponding kinetic term is

$$\int dx^{10} F_{\mu_1 \dots \mu_{k+1}} F^{\mu_1 \dots \mu_{k+1}} \quad (3.4)$$

We shall see that these antisymmetric fields play an important role in the correspondence.

3.1.1 Type IIB supergravity

There are two consistent maximally supersymmetric string theories and both live in 10 dimensions. They are called type II string theories. Maximal supersymmetry means $\mathcal{N} = 2$ in ten dimensions, or better the existence of 32 real supercharges⁷. At low energy, we can consider a supersymmetric effective action for the massless fields of string theory. Supersymmetry restricts both the field content and the effective action up to two derivatives. In fact, there are only two supermultiplets with 32 supercharges and the corresponding supergravity is uniquely fixed by supersymmetry. They are obviously called type II supergravities, the non-chiral A and the chiral B⁸. We are mostly interested in type IIB supergravity.

⁷Counting in terms of real supercharges allows to compare theories in different dimensions and avoid complications related to the properties of spinors that wildly depend on the dimension of space-time. For example a $\mathcal{N} = 1$ theory in four dimensions, where the supersymmetric parameter is a complex Weyl fermion with four real components, has 4 supercharges. The maximally supersymmetric $\mathcal{N} = 4$ gauge theory in four dimensions has 16 supercharges.

⁸Focusing on bosons, type II supergravity has a common set of NS-NS fields $(g_{\mu\nu}, \phi, B_{\mu\nu})$, and a set of R-R fields, consisting of antisymmetric tensors C_k , for all odd k in type IIA and for all

The supersymmetry type IIB multiplet contains the following bosonic fields: a metric $g_{\mu\nu}$, a scalar ϕ , called the dilaton, a 2-indices antisymmetric tensor $B_{\mu\nu}$, and three 0-,2-,4-, potentials (C_0, C_2, C_4) . Supersymmetry requires that C_4 is self-dual ($F_5 = *F_5$). The fermionic content is given by a spin 3/2 and a chiral spin 1/2 fermion. The action is completely fixed by supersymmetry and reads ⁹

$$\begin{aligned}
S_{IIB} = & \frac{1}{(2\pi)^7(\alpha')^4} \int dx^{10} \sqrt{g} e^{-2\phi} \left(R + 4(\partial\phi)^2 - \frac{1}{12} H_{\mu\nu\tau}^2 \right) \\
& - \frac{\sqrt{g}}{2} \left((F_\mu)^2 + \frac{\tilde{F}_{\mu\nu\tau}^2}{3!} + \frac{\tilde{F}_{\mu\nu\tau\rho\sigma}^2}{5!} \right) \\
& - \frac{1}{2} \epsilon_{\mu_1 \dots \mu_{10}} C_{\mu_1 \dots \mu_4} H_{\mu_5 \dots \mu_7} F_{\mu_8 \dots \mu_{10}} + \text{fermions}
\end{aligned} \tag{3.5}$$

where we denoted with $H_{\mu\nu\tau}$ the curvature for $B_{\mu\nu}$ and by $\tilde{F}_{\mu_1 \dots \mu_{p+1}}$ the modified curvature $F_{\mu_1 \dots \mu_{p+1}} - B_{\{\mu_1 \mu_2} F_{\mu_3 \dots \mu_{p+1}\}}$ ¹⁰. We haven't written the fermionic part of the Lagrangian since we shall not need it and it is completely fixed by supersymmetry. We shall not use much the bosonic Lagrangian itself. The full action is invariant under 32 supersymmetries given by two Majorana-Weyl parameters in 10 dimensions, $\epsilon = \epsilon_1 + i\epsilon_2$.

Equation (3.5) is the effective action of type IIB string theory at low energy, restricted to massless modes and two-derivatives. We can identify two parameters. One is $2\pi\alpha' = 1/M_s^2$ which determines the string mass M_s and control the masses of the string modes, $m^2 \sim 1/\alpha'$. A second one is implicitly given by the vacuum expectation value of the dilaton $g_s = \langle e^\phi \rangle$ which can be arbitrary since there is no potential for ϕ . g_s determines the string coupling constant and controls string interactions and quantum corrections. Both parameters can correct the effective

even k in type IIB. The names NS and R refers to Neveu-Schwartz and Ramond, respectively, and reflect the explicit world-sheet construction. In generic dimensions the electric-magnetic duality $F_{\mu\nu} \rightarrow \epsilon_{\mu\nu\tau\rho} F^{\tau\rho}$ is generalized by $F_{\mu_1 \dots \mu_{k+1}} = \epsilon_{\mu_1 \dots \mu_{10}} F^{\mu_{k+2} \dots \mu_{10}}$ ($F_{k+1} = *F_{9-k}$) so if we have a potential C_k with k-indices we also have its magnetic dual C_{9-k} . Obviously, they are mutually non local and they cannot appear simultaneously in a local Lagrangian. When two potentials C_k are electric-magnetically dual only the one with the lower number of indices is included in the type II Lagrangian.

⁹Actually, type IIB supergravity has equations of motion and no Lagrangian, due to the presence of a self-dual potential. (3.5) is nevertheless a good approximation to a Lagrangian: its equation of motion (with unconstrained C_4) are the equations of motion of type IIB supergravity to which we need to add the self-duality constraint $F_5 = *F_5$.

¹⁰These couplings are required by supersymmetry and implies that the gauge invariance $\delta B_{\mu\nu} = \partial_{[\mu} \Lambda_{\nu]}$ should be combined with the transformation $\delta A_{\mu_1 \dots \mu_p} = \Lambda_{\{\mu_1} F_{\mu_2 \dots \mu_p\}}$. None of these details will be used in the following.

action. If we integrate out the massive fields we get higher derivative corrections weighted by powers of α' . If we add the quantum corrections we get corrections (with arbitrary derivatives) weighted by powers of g_s . Schematically ¹¹

$$\begin{aligned} \int dx^{10} \sqrt{g} e^{-2\phi} \mathcal{R} + \dots &+ \sqrt{g} e^{-2\phi} (\alpha' \mathcal{R}^2 + \dots) \\ &+ \sum_{g=1}^{\infty} e^{(2g-2)\phi} \sqrt{g} (\dots) \end{aligned}$$

Any physical quantity has an expansion of form

$$\sum_{g=0}^{\infty} g_s^{2g-2} f_g\left(\frac{\alpha'}{R^2}\right) \quad (3.6)$$

Note that, in order to have a well defined dimensionless expansion parameter, we weighted α' with the typical radius R of the space-time in the vacuum where the computation is performed.

String theory gives an explicit way of computing all these corrections. The loop expansion for a string is given by fattened Feynman graphs, represented by Riemann surfaces where the genus is the number of loops (see figure 6). At each order in perturbation theory we have precisely one graph, which can be computed using the world-sheet formulation of string theory and the technology of two-dimensional conformal field theories. The result is a non-trivial function of α' encoding the effect of all massless and massive string modes.

3.1.2 D-branes

Type II supergravities (and string theories) have solitonic extended charged objects, called p-branes, which generalize strings and membranes. They are massive objects that extend in p-spacelike directions and interact with the gravitational and gauge fields through the coupling

$$S_{\text{p-brane}} = \tau \int dx^{p+1} \sqrt{g} + q \int dx^{p+1} A_{p+1} \quad (3.7)$$

where the integral is taken on the world-volume of the brane. The first term is the generalization of the Nambu-Goto action for strings and it defines the tension τ of the brane. A massive object in General Relativity moves by minimizing the volume

¹¹The RR fields kinetic term enters the action without a factor of $e^{-2\phi}$, in order to simplify gauge couplings. It should be consider at all effect part of the tree level action.

swept. The second term is the coupling to the A_{p+1} antisymmetric tensor field; it generalizes the coupling of a charged particle to a vector, which is given by the integral of the one-form potential on the world-line of the particle,

$$q \int dt \vec{A} \vec{v} = q \int dx^\mu A^\mu \equiv q \int A_1. \quad (3.8)$$

The parameter q is the p-brane charge. As in electro-magnetism, it is useful to measure the charge by the flux on a sphere surrounding the source. For an extended object with $p+1$ space-time directions, the transverse space is $\mathbb{R}^{9-p}$ and the correct definition is ¹²

$$\int_{S^{8-p}} *F_{p+2} = N \quad (3.9)$$

The flux N is proportional to the charge q by α' and numerical factors. N is more useful than q because N is an integer by the Dirac quantization condition, similarly to what happens in electro-magnetism.

We can see p-branes as solitons. In fact, there exist classical solutions of type II supergravity with sources corresponding to massive charged p-branes, or, in other words, solutions of the coupled action

$$S_{II}(g, B, \phi, F) + S_{\text{p-brane}} \quad (3.10)$$

These solutions are called black p-brane and generalize rotating black holes. They have $SO(1, p) \times SO(9-p)$ isometry (corresponding to an object extending in the first $p+1$ coordinates of space-time), N units of flux for the A_{p+1} potential as in equation (3.9), and a finite energy per unit volume in the p space-like directions, which we can identify with the tension of the brane

$$\mathcal{E} = \frac{E}{\text{Vol}_d} = \tau. \quad (3.11)$$

The black p-brane solutions are very similar to Kerr black holes, with a singularity surrounded by an inner and outer horizon. As in the case of rotating black holes, we avoid naked singularities only if the energy density is greater than the charge,

$$\mathcal{E} = \tau \geq \frac{N}{(2\pi)^2 g_s (\alpha')^{(p+1)/2}}. \quad (3.12)$$

When the bound is saturated, we speak of extremal p-branes. They are particularly important because the saturation of the bound is equivalent to the preservation of part of the supersymmetries of the theory. An extremal p-brane preserves exactly

¹²As we see from integrating the equation of motion $d * F_{p+2} \sim q \delta$ on the trasverse space.

half of the 32 supersymmetries of type II supergravity. For this reason it is called a BPS object ¹³. The extremal p-brane lying in the first $p + 1$ directions (including time) corresponds to the solution

$$\begin{aligned} ds^2 &= H^{-1/2}(r)dx_\mu^2 + H^{1/2}(r)dy^2 \\ A_{0\dots p} &= H(r) \\ e^\phi &= g_s H(r)^{(3-p)/4} \\ H(r) &= 1 + \frac{cg_s N(\alpha')^{(7-p)/2}}{r^{7-p}} \end{aligned} \tag{3.13}$$

with c a numerical factor. y are directions transverse to the brane and $r^2 = \sum_i y_i^2$ is the radial distance. Note that, perhaps not unexpectedly, $H(y_i)$ is the solution of the Laplace equation in the transverse space $\mathbb{R}^{9-p}$ with a delta function source at the origin. The solution has an $SO(1, p) \times SO(9 - p)$ isometry group and preserves 16 supersymmetries. One can check all these statements by explicit computations (which we shall not report here).

The horizon of an extremal p-brane collapses on the singularity. Something very special happens for 3-branes: it is good exercise in General Relativity to check that the metric for an extremal 3-brane is actually completely regular. Note also that the dilaton becomes constant.

There is a generalization of the extremal solution where H is replaced by a more general harmonic function

$$H(y_i) = 1 + cg_s(\alpha')^{(7-p)/2} \sum_{a=1}^N \frac{1}{|y - y_a|^{7-p}} \tag{3.14}$$

This solution has still charge N , an energy density that saturates the unitary bound (3.12) and it preserves 16 supercharges. While (3.13) corresponds to a brane of charge N sitting at the point $\vec{y} = 0$, (3.14) corresponds to N branes of unit charge sitting at the arbitrary points $\vec{y}_a$. This is called a multi-center solution. The extremality of the multi-center solution has an important consequence: p-branes do not exert force on each other. They can be separated and moved around in space-time without any cost in energy. Typically, if we put two massive objects with the same charge at a certain distance, they will attract or repulse by a combination of gravitational and gauge interaction. However, for extremal p-branes the energy density of a configuration is

¹³In the language of supersymmetry the inequality follows from the supersymmetry algebra in presence of central charges (given by the charge of the p-brane) and its saturation is the BPS condition.

proportional to its charge. Since charge is additive, the energy density of a system of identical extremal branes is equal to the sum of the energy density of the single branes: the potential energy vanishes. In fact, precisely by equality of tension and charge, the gravitational attraction is compensated by the gauge repulsion. This phenomenon is typical of BPS objects and also happens in the physics of four-dimensional monopoles in supersymmetric gauge theories.

In string theory extremal p-branes have an explicit characterization as planes where open strings can end. These objects are called D-branes and we shall mostly use this terminology in the following.

3.1.3 Collective coordinates

Solitonic objects usually carry collective coordinates. The same happens for p-branes: there are fields leaving on their world-volume.

Some of these collective coordinates are related to fluctuations of our extended object in the transverse directions. We can easily see where they come from if we look closely to the Nambu-Goto action. The position of the brane in space-time is specified by a set of functions $X^M(x^a)$ where x^a are coordinates on the world-volume while X^M are the coordinates of the ten-dimensional space-time. The functions $X^M(x^a)$ describe the embedding and the shape of the D-brane in space-time. The metric on the brane is given by

$$g_{MN}dX^M dX^N = g_{MN}\partial_a X^M \partial_b X^N dx^a dx^b \quad (3.15)$$

and the Nambu-Goto action reads

$$\int dx^{p+1} \sqrt{g} = \int dx^{p+1} \sqrt{\det_{ab} (\partial_a X^M \partial_b X^N g_{MN})} \quad (3.16)$$

For a p-brane embedded in the first $p+1$ coordinates and sitting at a point $(\phi^{p+2} \dots \phi^{10})$ we can set $X^a = x^a$, $a = 0, \dots, p+1$ and $X^i = \phi^i$, $i = p+2, \dots, 10$. To study fluctuations, we allow the positions ϕ^i to vary slowly as functions of the x^a . Computing the induced metric and expanding it in powers of the fluctuations we see that

$$\int \sqrt{\det_{ab} (g_{ab} + g_{ij} \partial_a \phi^i \partial_b \phi^j)} = \int \sqrt{g} + \frac{1}{2} \sqrt{g} g_{ij} \partial_a \phi^i \partial^a \phi^j + \dots \quad (3.17)$$

At the order of two derivatives we obtain the standard kinetic term for $9-p$ scalar fields leaving on the world-volume of the brane. Obviously, this result was expected: it is the only term with two derivatives that is covariant under space-time symmetries. The scalar fields parametrize the position of the brane in space-time.

In addition to scalar fields parametrizing the $9-p$ transverse positions, there are other fields on the brane. The nature of these modes and their effective Lagrangian are difficult to find for generic p-branes. However in the case of extremal p-branes supersymmetry comes to a rescue. Since an extremal brane preserves half of the supersymmetry of the background, the world-volume theory should have 16 supercharges. Fortunately, there is exactly one supermultiplet with 16 supercharges in any dimensions and this is the vector supermultiplet. It is not a coincidence that a vector multiplet contains exactly $9-p$ scalar fields. This uniquely fixes the world-volume fields leaving on an extremal p-brane. The Lagrangian is also fixed by supersymmetry at the level of two derivatives. It can be obtained by the dimensional reduction of ten dimensional YM theory, as we shall see explicitly.

The previous argument is confirmed by an explicit calculation in string theory, where the world-volume modes can be studied and incorporated by allowing open strings to end on a D-brane. By quantizing open strings ending on extended planes we indeed find massless excitations corresponding to a vector multiplet with 16 supercharges. We also find a tower of massive string modes with squared masses of order $1/\alpha'$. The effective action for world-volume fields and the interaction with the background can be determined by the open plus closed string perturbative expansion. In particular one finds a nice generalization of (3.7) (for a brane of unit charge),

$$\frac{1}{(\alpha')^{(p+1)/2}} \int dx^{p+1} \left(e^{-\phi} \sqrt{\det(g + (2\pi\alpha'F + B_2))} + e^{2\pi\alpha'F+B_2} \wedge \sum_k C_k \Big|_{p+1\text{-form}} \right) \quad (3.18)$$

The first term is called the Dirac-Born-Infeld action and generalizes the Nambu-Goto action by including gauge fields and their coupling to the bulk field $B_{\mu\nu}$; the second one is called the Wess-Zumino term and generalizes the coupling to A_{p+1} to the other RR forms¹⁴. The dilaton factor comes from the tension of a D-brane (see equation (3.12)). By expanding up to two-derivatives in the gauge fields the previous expression, similarly to what we did for the scalar fields, we would get the effective action for gauge fields. Doing it explicitly for the case of a D3-brane in type IIB we obtain

$$\int dx^4 e^{-\phi} F_{\mu\nu}^2 + C_0 \epsilon^{\mu\nu\tau\rho} F_{\mu\nu} F_{\rho\sigma} \quad (3.19)$$

which is indeed the action for a gauge vector. The background value of the dilaton fixes the coupling constant while the background value of the scalar C_0 fixes the theta angle. Note that the expansion in derivatives is equivalent to an expansion in α' .

¹⁴In the Wess-Zumino term only the term in the expansion of $e^{F+B} \wedge \sum C$ with $p+1$ indices, which can be integrated on the world-volume, should be kept.

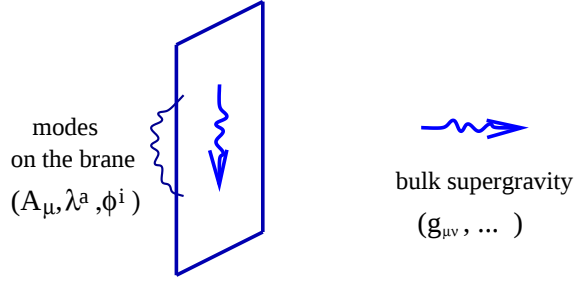


Figure 14: The coupled brane-bulk system.

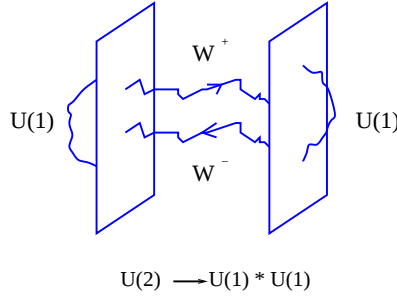


Figure 15: Non-abelian gauge symmetries are naturally realized on the world-volume of coinciding D-branes. When D-branes coincide, open strings connecting different branes become massless giving an enhanced non-abelian symmetry.

3.1.4 Non-abelian gauge fields from D-branes

It will be of utmost importance for us that a set of N Dp-brane carries on its world-volume a YM theory with gauge group $U(N)$. Most of the recent interest in D-branes comes from this fact.

The solution (3.13) with charge N can be interpreted as the superposition of N elementary Dp-brane with unit charge. It is in fact the limit of the more general solution (3.14) when all the D-brane positions coincide. We already know that D-branes carry vector multiplets. In string theory these multiplets come from the quantization of open strings ending on the D-branes as in Figure 15. Open strings ending on the same D-brane give rise to a massless vector multiplet. On the other hand, we also have open strings connecting different D-brane are these give rise to massive vector multiplets, since the string, which has a tension $1/\alpha'$, has a non zero length. The mass of these modes is then proportional to the distance between branes: $m \sim |\vec{\phi}_1 - \vec{\phi}_2|/\alpha'$. When D-branes coincide, however, the masses of these vector multiplets vanish. By counting the number of open strings that become massless, we obtain N^2 vector fields,

consistent with an enhanced $U(N)$ symmetry (see Figure 15 for the $N = 2$ case).

We shall see now that this spectrum of states is well reproduced by a world-volume analysis. The bosonic part of the maximally supersymmetric YM theory in $p + 1$ dimensions is obtained by dimensional reduction from super YM in ten. The bosonic part of the 10-dimensional theory is remarkably simple

$$- \int dx^{10} \text{Tr} F_{\mu\nu}^2, \quad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + i[A_\mu, A_\nu] \quad (3.20)$$

The dimensional reduction is obtained by considering configurations that depend only on the first $p + 1$ coordinates. We also split the ten dimensional vector in a $p + 1$ dimensional vector A_μ , $\mu = 0, \dots, p + 1$ and $9 - p$ scalars $\phi^i = A_i$, $i = p + 2, \dots, 10$. Since all derivatives in the $9 - p$ trasverse directions vanish, we obtain a very simple action for the scalar fields ϕ^i

$$\sum_{i=p+2}^{10} \text{Tr} (\partial_\mu \phi^i + i[A_\mu, \phi^i])^2 + \sum_{i,j=p+2}^{10} \text{Tr} [\phi^i, \phi^j]^2 \quad (3.21)$$

The scalar potential is a sum of squares, as required by supersymmetry, and its vacua are given by the configurations

$$[\phi^i, \phi^j] = 0 \quad (3.22)$$

In a $U(N)$ theory the ϕ^i are hermitian matrices. Since they commute, they can be simultaneously diagonalized by a gauge transformation. In the vacuum corresponding to diagonal matrices ¹⁵ $\phi^i = \text{diag}\{\phi_1^i, \dots, \phi_N^i\}$ the gauge group $U(N)$ is broken to $U(1)^N$. In fact, thinking of A_μ^{pq} as an N by N hermitian matrix, it is easy to see that the diagonal components remain massless (giving the $U(1)^N$) while the off-diagonal ones aquire a mass

$$|A_\mu^{pq}|^2 \sum_{i=p+2}^{10} |\phi_p^i - \phi_q^i|^2 \quad (3.23)$$

The diagonal VEV for the ϕ^i can be interpreted as the position of the N D-branes in space-time. There is a one-to-one correspondence between the set of vacua of the world-volume theory and the minimal energy configuration of a set of D-branes in space-time. Moreover, the mass of the gauge fields in any given vacuum nicely fits with the space-time description: the diagonal gauge fields correspond to open strings connecting the same D-brane and the off-diagonal ones to open strings connecting different D-branes with a mass proportional to the distance. Finally, we see what

¹⁵Due to a residual discrete gauge symmetry corresponding to the Weyl group of $U(N)$, a permutation of the eigenvalues give a gauge equivalent configuration.

happens when all the D-branes coincide. In field theory, we have an inverse Higgs mechanism. Some $W^\pm$ bosons become massless enhancing the symmetry to $U(N)$. In the space-time picture the open string connecting different branes become massless.

We see that the set of vacua of a $U(N)$ world-volume field theory matches the space-time expectation for a set of N BPS objects.

3.1.5 A closer look to D3-branes

We shall be interested in particular in D3-branes. As already mentioned the supergravity solution is regular and the dilaton is constant. On the field theory side, the world-volume theory on N D3-branes is $\mathcal{N} = 4$ SYM with gauge group $U(N)$ (16 real supercharges). The theory contains a gauge field, six scalar fields ϕ^i (parametrizing the six transverse directions) and four Weyl fermions ψ_a . A D3-brane has a $SO(1,3) \times SO(6)$ global symmetry. The first factor is the Lorentz group while the second is the $SU(4)_R \sim SO(6)$ R-symmetry of the theory, which rotates the four supercharges Q_α^a . Scalar and fermions transform in the $\underline{6}$ and $\underline{4}$ representation of the R-symmetry group, respectively. Although we shall not use it much, we report the Lagrangian for completeness

$$S = \frac{1}{g_{YM}^2} \int dx^4 \text{Tr} \left(-\frac{F_{\mu\nu}^2}{2} - i\bar{\psi}^a \not{D} \psi_a - (D_\mu \phi^i)^2 + C_i^{ab} \psi_a [\phi^i, \psi_b] + [\phi^i, \phi^j]^2 \right)$$

with C_i^{ab} Clebsh-Gordan coefficients for the decomposition $\underline{4} \times \underline{4} \rightarrow \underline{6}$. The supersymmetry transformations are generated by Q_α^a acting on fields as $\delta^a \chi = [\epsilon^\alpha Q_\alpha^a + \bar{\epsilon}^{\dot{\alpha}} \bar{Q}_\alpha^a, \chi]$ and sending

$$\begin{aligned} \phi^i &\rightarrow C^{iab} \psi_{\alpha b} \\ \psi_{\beta b} &\rightarrow F_{\mu\nu}^+ \sigma_{\alpha\beta}^{\mu\nu} \delta_b^a + [\phi^i, \phi^j] \epsilon_{\alpha\beta} C_{ijb}^a \\ \bar{\psi}_\beta^b &\rightarrow C_i^{ab} \bar{\sigma}_{\alpha\dot{\beta}}^\mu D_\mu \phi^i \\ A_\mu &\rightarrow \sigma_{\mu\alpha}^{\dot{\beta}} \bar{\psi}_\beta^a \end{aligned} \tag{3.24}$$

We see that the scalar potential agrees with the general discussion in the previous section. The form of the Lagrangian makes manifest the $SU(4)_R$ symmetry. There is a simpler form of this Lagrangian using the language of $\mathcal{N} = 1$ supersymmetry. The $\mathcal{N} = 4$ multiplet can be decomposed in a $\mathcal{N} = 1$ gauge multiplet W_α , containing A_μ and ψ^4 , and three complex chiral fields Φ_i , containing $\phi^i + i\phi^{i+3}$ and

ψ^i for $i = 1, 2, 3$. The Lagrangian is then

$$\frac{1}{g_{YM}^2} \int d^4x \left(\int d\theta^4 (W_\alpha^2 + \sum_{i=1}^3 \bar{\Phi}_1 \Phi_i) + \int d\theta^2 \epsilon_{ijk} \Phi_i \Phi_j \Phi_k \right) \quad (3.25)$$

A disadvantage of this formulation is that only a $U(1)_R \times SU(3)$ subgroup of the R-symmetry $SU(4)_R$ is manifest.

From the explicit expression (3.19), we know that the kinetic term of a D3-brane should be multiplied by $1/g_s$. We are thus led to the identification of the Yang-Mills coupling on a D3-brane with the constant value of the dilaton $g_{YM}^2 \sim g_s$. More generally, from the same equation, we identify

$$\tau = \frac{\theta}{2\pi} + \frac{4\pi i}{g_{YM}^2} = \frac{C_0}{2\pi} + \frac{i}{g_s} \quad (3.26)$$

where we for reference we reintroduced the correct numerical factors. As discussed above there is a moduli space of vacua where the scalar fields ϕ^i are diagonal with arbitrary entries. These are flat directions of the gauge theory.

More importantly, the theory is conformal. As already discussed in section 1.1.2, the beta function vanishes up to three-loops and it is believed to vanish at all orders. All correlation functions of elementary fields are finite at all order, when a suitable regularization is used. With supersymmetry the conformal group is enhanced to $SU(2, 2|4)$ obtained by $O(2, 4)$ by adding the global $SU(4)_R$ generators, the four Weyl supercharges Q_α^a , $a = 1, \dots, 4$ and the four so-called conformal supercharges S_α^a . These are required to close the superconformal algebra $[K, Q] \sim S$. In particular, if we include the S generators, we collect 32 real supersymmetry generators, the 16 supercharges and the 16 conformal supersymmetries. We shall not use explicitly the superconformal group; the reader can find more details about it in the Appendix.

3.2 The near-horizon geometry

The original Maldacena conjecture states that $\mathcal{N} = 4$ SYM is dual to the type IIB string background $AdS_5 \times S^5$. The conjecture arised from the observation that the two theories can be obtained by the same decoupling limit $\alpha' \rightarrow 0$, performed on the world-volume theory and on the back-reacted metric in space-time.

$\mathcal{N} = 4$ SYM theory can be realized on the world-volume of N parallel D3-branes in Type IIB, as depicted in Figure 15 and 16. The D3 branes interact with the bulk fields which lives in 10 dimensions. If we expand in powers of α' the leading terms in

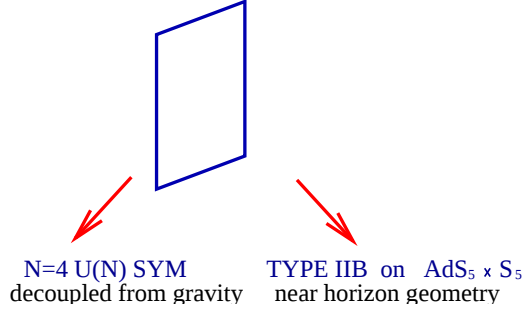


Figure 16: The two limits.

the action for the coupled brane/bulk system are

$$\frac{1}{g_s} \int d^4x F_{\mu\nu}^2 + \frac{1}{\alpha'^4} \int d^{10}x \sqrt{g} R e^{-2\phi} + \dots \quad (3.27)$$

where $g_{YM}^2 \sim g_s$. We see that, in the limit $\alpha' \rightarrow 0$, we can turn off the interaction with the bulk: the brane theory decouples from the bulk and reduces to $\mathcal{N} = 4$ YM. The limit should be performed in a careful way in order to keep field theory quantities finite. The mass of a gauge boson in the Higgs phase of $\mathcal{N} = 4$ YM is given, in the D3 brane description, by the mass of a stretched open string $m = \Delta r / \alpha'$, where Δr is the distance between branes. If we want to keep these masses finite we take the limit

$$\begin{aligned} \alpha' &\rightarrow 0 \\ g_s &\quad \text{fixed} \\ N &\quad \text{fixed} \\ \phi^i &= \frac{r^i}{\alpha'} \quad \text{fixed} \end{aligned} \quad (3.28)$$

In other words, we zoom on the region containing the branes.

On the other hand, a D3-brane is a solution of the equation of motion of type IIB supergravity and deforms the background as in equation (3.13),

$$\begin{aligned} ds^2 &= H^{-1/2} dx_\mu dx^\mu + H^{1/2} (dr^2 + r^2 \Omega_5^2) \\ F^{(5)} &= \text{flux of a charged object} \\ e^\phi &= g_s \equiv \frac{g_{YM}^2}{4\pi} \\ H &= 1 + \frac{g_{YM}^2 N \alpha'^2}{r^4} = 1 + \frac{g_{YM}^2 N}{\alpha'^2 \phi^4} \end{aligned}$$

This is the background induced by a tensionful 3-brane charged under the field C_4 ; far from the brane the deformation vanishes and the metric becomes flat. If we take the limit (3.28) on the solution we obtain

$$ds^2 \sim \alpha' \left\{ R^2 \frac{(d\phi)^2}{\phi^2} + \frac{\phi^2}{R^2} (dx)^2 + R^2 \Omega_5 \right\}, \quad R^2 = \alpha' \sqrt{g_{YM}^2 N}$$

which we recognize as the product of two Einstein spaces of constant curvature: $AdS_5 \times S^5$. Starting from an asymptotically flat space-time and performing a limit we have obtained a curved space. This is possible because the limit amounts to discard the constant term in $H(r)$ thus decoupling the asymptotically flat part of the metric. Equivalently, we are taking $r \rightarrow 0$ and zooming on the region where the branes sit. For this reason this procedure is called a near-horizon limit.

This observation led Maldacena to formulate his conjecture and propose the equivalence between $\mathcal{N} = 4$ SYM and the type IIB string background $AdS_5 \times S^5$.

A first obvious check concern symmetries on the two sides. These are reported in the following self-explaining table:

$\mathcal{N} = 4$ SYM	Type IIB on $AdS_5 \times S^5$
Conformal group $O(4, 2)$	Isometry group $O(4, 2)$ of AdS_5
Supersymmetries = 16 Q + 16 S	32 supersymmetries
R-symmetry $SU(4)_R$	Isometry of S^5 : $SO(6) = SU(4)$

In a word, the symmetry on both sides is the superconformal group $SU(2, 2|4)$. The only subtle point in these identifications regards supersymmetry. The D3 brane solution has 16 supercharges. However its near horizon region has an enhanced supersymmetry: one can check that $AdS_5 \times S^5$ is a maximally supersymmetric background of type IIB. On the field theory side, the 16 supercharges of the YM theory are enhanced to 32 by adding the conformal supersymmetries.

Since S^5 is compact, we can perform a dimensional reduction and think about the gravity theory as an effective five-dimensional theory with an infinite numbers of fields and an AdS_5 vacuum.

It is interesting to compare parameters in the two theories. The radius of both AdS_5 and S^5 is given by $R^2 = \alpha' \sqrt{x}$, where $x = g_{YM}^2 N$ is the t'Hooft coupling. This suggests some relation to the large N limit. In fact, the CFT has two dimensionless parameters, x and N and these can be matched with the two string parameters, g_s and α' as follows

$$\begin{aligned} 4\pi g_s &= \frac{x}{N} \\ \frac{R^2}{\alpha'} &= \sqrt{x} \end{aligned} \tag{3.29}$$

The dual string theory is useful when it is weakly coupled, reducing to an effective supergravity. This happens if $g_s \rightarrow 0$ (string loops suppressed) and $\alpha'/R^2 \rightarrow 0$ (higher derivatives terms and massive string modes suppressed). This regime corresponds to the large N limit and the strong coupling ($x \rightarrow \infty$) of the CFT.

- We see that the t'Hooft limit $N \rightarrow \infty, x = g_{YM}^2 N$ fixed, is naturally implemented in the correspondence.
- We also see that we have a *duality*: the weak coupling regime of the gravity theory corresponds to the strong coupling regime of the CFT. This is what makes the correspondence useful. We can learn everything about the weak coupling of $\mathcal{N} = 4$ SYM using Feynman graphs and perturbative expansion.

Let us finish this section with some comments.

- We can apply the methods of section 2 and compute Green functions of the CFT using a classical gravitational theory. This method will work in the large N limit and strong coupling. The correspondence is valid for all N and x , but for computations at finite N and x we need the full string theory. We can organize a t'Hooft expansion in $1/N$ and $1/x$, as in Figure 17. The CFT expansion in $1/N$ is the loop expansion of string theory; both are organized as a Riemann surface expansion. The expansion in $1/x$ is the expansion in higher derivatives. In field theory, at each order in $1/N$, we need to re-sum the infinite graphs of given topology obtaining an highly non trivial function $f_g(x)$ of the t'Hooft coupling. In the limit $x \rightarrow \infty$, $f_0(x)$ is computable in supergravity, the corrections in $1/x$ corresponding to the higher derivatives corrections of string theory. The correspondence is useful at the moment only at large N and strong coupling. To go beyond the planar limit is probably hopeless, since it requires computing string loops. On the other hand, to go beyond the strong

$$\begin{array}{c}
\text{CFT} \\
\langle O O \rangle = \left(\text{planar graphs: } f(x) \right) + \frac{1}{N^2} \left(\text{non-planar graphs} \right) \\
\\
\text{AdS} \\
\langle h h \rangle = \text{world-sheet corrections} + \text{higher derivatives terms}
\end{array}$$

Figure 17: The string expansion versus the the large N expansion

coupling involves computing world-sheet corrections, which are, in principle, more tractable than loop corrections. In flat space, for example, all the α' corrections are computable. In the *AdS* case, the analogous computation is made difficult, but perhaps not impossible, by the presence of RR-fields.

- Taking a near horizon limit is a general method for obtaining the gravity dual of the gauge theory living on a set of D-branes. This works every time the limit is able to decouple consistently brane and bulk physics. For example, we can find many other examples with less supersymmetry by placing D-branes in curved geometries or considering complicated set of intersecting branes. Some examples will be briefly discussed in the following.

3.3 Matching the spectrum

We shall now give a map between observables in the dual theories. In both case we shall classify them in terms of the quantum numbers of the superconformal group: the dimension Δ , the spin (j_1, j_2) and the R-symmetry representation.

3.3.1 The field theory side

The theory is finite and (using a suitable regularization) elementary fields are not renormalized. However, gauge invariance requires observables to be composite operators. As well-known, composite operators need additional renormalization: Green functions have short-distance singularities when two or more elementary fields coin-

cide. The conformal dimension of a composite operator can be renormalized

$$\Delta_O = \text{canonical dim} + \gamma_O(g_{YM}) \quad (3.30)$$

where the anomalous dimension $\gamma_O = \Lambda \frac{\partial}{\partial \Lambda} \log Z_O$ is obtained from the wave-function renormalization of the operator: $O_{\text{ren}} = Z_O^{-1} O_{\text{bare}}$.

Some particular operators are protected by the superconformal algebra and their dimension is not renormalized; this certainly happens to the conserved currents (and their partners under supersymmetry) that have canonical dimension. The multiplets of conserved currents are special cases of the so-called short multiplets, which contain less states than a generic representation, as discussed in the Appendix. They saturate an unitary bound and, as a consequence, the dimension of the operators in such multiplets are uniquely determined by the Lorentz and R-symmetry quantum numbers.

- In $\mathcal{N} = 4$ SYM we have various conserved currents, the stress-energy tensor $T_{\mu\nu}$, the supercharges Q_α^a and the $SU(4)_R$ R-currents $(R_b^a)_\mu$. They all belong to the same supermultiplet

$$(Tr\phi_i\phi_j - \text{trace}, \dots, Q_\alpha^a, \dots, (R_b^a)_\mu, \dots, T_{\mu\nu}) \quad (3.31)$$

whose lowest component is a scalar field. Since the currents are conserved and have canonical dimensions, the same is true for their supersymmetric partners.

- The relevant short multiplets of $\mathcal{N} = 4$ SYM are the generalization of the chiral multiplets of $\mathcal{N} = 1$ supersymmetry. Recall that in $\mathcal{N} = 1$ a multiplet is chiral when is annihilated by half of the supersymmetries (let's say the $\bar{Q}$); the corresponding superfields depends only on the θ coordinates (and not $\bar{\theta}$) and it is shorter than a generic multiplet. A chiral multiplet satisfies the unitary bound

$$\Delta = \frac{3}{2}R \quad (3.32)$$

where R is the R-charge. This bound implies a non-renormalization theorem: Δ and R can be re-normalized but their ratio is not. The analogous case in $\mathcal{N} = 4$ is a multiplet annihilated by half of the supercharges. It has maximum spin two with a scalar lowest state transforming in the symmetric traceless representation of rank k of $SO(6) \sim SU(4)_R$. The dimension of the lowest state is fixed by the shortening condition to be $\Delta = k$ ¹⁶. Since k is an integer, it follows that

¹⁶A more general short multiplet with lowest scalar state transforming in the $[p, k, p]$ representation of $SU(4)$ with dimension $\Delta = k + 2p$ plays a role in analyzing multi-trace operators.

the dimension of the lowest state (and of all its partners under supersymmetry) cannot be renormalized. In the following, the words *short or chiral* in $\mathcal{N} = 4$ will refer to this particular multiplet.

An example of chiral multiplet A_k of $\mathcal{N} = 4$ SYM is obtained by applying supersymmetries to the operator

$$Tr \phi_{\{i_1} \cdots \phi_{i_k\}} - \text{traces} \quad (3.33)$$

of canonical dimension k . A_2 corresponds to the supercurrent multiplet. Quite remarkably, one can prove that the A_k are short multiplets with protected dimensions and that they are the only single-trace short multiplets of $\mathcal{N} = 4$ SYM.

All single-trace operators not lying in one of the A_k are not protected by the superconformal symmetry and are in general renormalized, unless some miracle or some dynamical reason intervenes.

- **Example:** Consider the quadratic operators made with the six scalar fields of $\mathcal{N} = 4$ SYM: $Tr \phi_i \phi_j$. These operators are symmetric in i and j due to the cyclic property of the trace. We have 21 independent operators. In terms of $SU(4)_R$ representations, the trace $Tr \sum_i \phi_i \phi_i$ is a singlet and the symmetric traceless part spans a 20. An explicit computation shows that the singlet is renormalized while the 20 is not,

$$\begin{aligned} \underline{20} : \quad & Tr \phi_i \phi_j - \frac{\delta_{ij}}{6} \sum_i \phi_i \phi_i \quad \Delta = 2. \\ \underline{1} : \quad & \sum_i \phi_i \phi_i \quad \Delta = 2 + O(g_{YM}) \end{aligned} \quad (3.34)$$

The scalar operator in the 20 is not renormalized since it belongs to the short multiplet A_2 ; it is the lowest component of the supermultiplets of currents. On the other hand, the trace part does not belong to any multiplet of currents and there is no reason why it should not receive corrections. $Tr \phi_i \phi_i$ is the lowest component of a long unprotected multiplet, called the Konishi multiplet, and its dimension is renormalized.

- One can check the non-renormalization of A_k just by using the $N = 1$ subalgebra and its bounds. The $\mathcal{N} = 4$ Lagrangian is written in $\mathcal{N} = 1$ notations in (3.25). It has a $U(1)_R$ symmetry which give charge 1 to θ and charge $2/3$ to Φ_i . As we said, in $\mathcal{N} = 1$ supersymmetry, the chiral multiplets are short and their lowest component saturates the unitary bound $\Delta = 3R/2$. The 20 multiplet,

when decomposed in $\mathcal{N} = 1$ supersymmetry, contains among other things the chiral multiplet $\text{Tr}\Phi_i\Phi_j$ which is protected and has dimension $\Delta = 2$ since its R-charge is $R = 4/3$. By supersymmetry the entire $\mathcal{N} = 4$ multiplet has protected dimension. On the other hand, the singlet $\text{Tr}\phi_i\phi_i$ corresponds in $\mathcal{N} = 1$ notations to $\bar{\Phi}_i\Phi_i$ which is not chiral and not protected. Analogously the multiplet A_k contains, after decomposition, the chiral multiplet $\Phi_{\{i_1}\cdots\Phi_{i_k\}}$ with protected dimension k . Note however that the products $\phi_{i_1}\cdots\phi_{i_k}$, where the indices are not symmetrized, are not protected. In fact, although $\Phi_{i_1}\cdots\Phi_{i_k}$ looks as chiral multiplets, they are not. In fact, by the equations of motion of $\mathcal{N} = 4$ SYM we see that

$$[\Phi_i, \Phi_j] \sim \epsilon_{ijk} D^2 \bar{\Phi}_k \quad (3.35)$$

so that the anticommutator of two scalars is not the lowest component of a chiral field, but rather a descendant under supersymmetry of something else. The non-renormalization condition only applies to the lowest component of a chiral field.

3.3.2 The gravity side

Since S^5 is compact we can perform a KK reduction of the ten-dimensional fields to five-dimensions. This is similar to the KK reduction of fields on a circle given by expansion in Fourier modes. If we have a space-time $\mathbb{R}^{1,3} \times S^1$, with a circle of radius R , the equations of motion of a five-dimensional massless field can be diagonalized by considering Fourier modes $\phi(x, y) = \sum_k \phi_k(x) e^{iky/R}$ with $k \in \mathbb{Z}$,

$$-\square_5 \phi(x_\mu, y) = -\square_4 \phi(x_\mu, y) - \partial_y^2 \phi(x_\mu, y) = \sum_k e^{iky/R} \left(-\square \phi_k + \frac{k^2}{R^2} \phi_k \right) \quad (3.36)$$

We thus obtain an infinite tower of KK modes with mass k^2/R^2 .

The situation is similar on S^5 . We need to diagonalize the Laplacian on S^5 and this is done using spherical harmonics. The bosonic massless modes in ten-dimensions are

$$(g_{\mu\nu}, B_{\mu\nu}, C_{\mu\nu}, \phi, C_0, A_{\mu\nu\rho\sigma}^+) \quad (3.37)$$

They give rise to five-dimensional modes by expansion on spherical harmonics on the sphere: for a scalar for example

$$\phi(x, y) = \sum_I \phi^I(x) Y_I(y) \quad (3.38)$$

where Y_I are the eigenfunctions of the scalar Laplacian on S^5 .

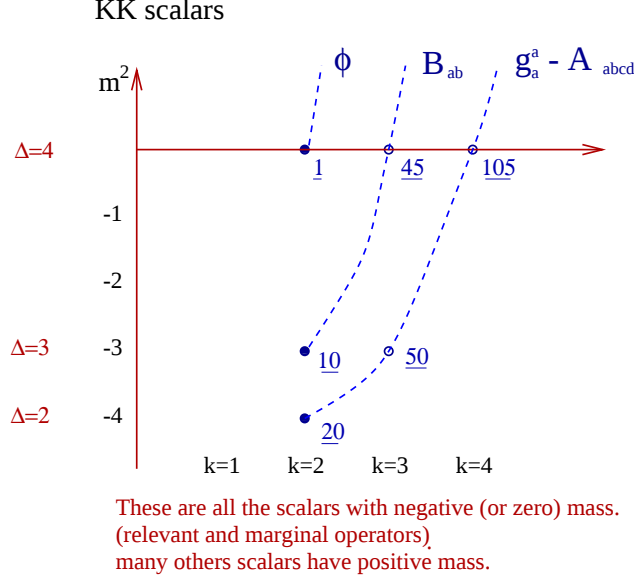


Figure 18: The lines connect states belonging to the same KK tower. Multiplets of supersymmetry are arranged in vertical lines. There is one multiplet for all integers $k \geq 2$.

The KK spectrum was computed and organized in supersymmetry multiplets in the eighties [30, 31]. In figure 18 the reader can find the lowest scalar excitations. For each state we indicated the $SO(6)$ representation. The mass can be read on the vertical axis. Modes on the same vertical line belong to the same supermultiplet. One can show that the entire spectrum consists of a series of short multiplets A'_k of the superconformal algebra, labelled by an integer $k \geq 2$. The lowest state in A'_k is a scalar in the k -fold symmetric representation of $SO(6)$ with mass $m^2 = k(k-4)/R^2$ and the maximum spin in the multiplet is two. For example, $k=2$ is the graviton multiplet,

$$(g_{\mu\nu}, \psi_\mu^i : \text{in the } \underline{4}, A_\mu : \underline{15}, B_{\mu\nu} : \underline{6} + \bar{\underline{6}}, \lambda : \underline{4} + \underline{10}_c, \text{scalars} : \underline{1}_c + \underline{10}_c + \underline{20}) \quad (3.39)$$

Note that even if this is called the massless multiplet, only the gauge fields are strictly massless: the mass is not a Casimir of the superconformal group and varies inside a multiplet.

Let us make some observations.

- In the classical KK expansion on a circle, the zero-modes are massless and separated from massive modes by a quantity of order $1/R^2$. We can decouple massive modes by taking large R . Due to the non-zero curvature, in AdS there

is no separation: all the KK modes have a mass of the same order of the zero-modes. In fact strictly massless modes are those with gauge invariance (graviton, etc...). But as we already noticed even some of their susy partners are massive.

- Even if we cannot decouple the massive multiplets A'_k with $k \geq 3$ by taking the internal manifold large, it make sense to write an effective action for the massless multiplet A'_2 . This is the $\mathcal{N} = 8$ gauge supergravity in five-dimensions. It is believed to be a consistent truncation of the theory in the sense that every solution of its equations of motion can be uplifted to a solution of the ten-dimensional equations of motion.

3.3.3 The comparison

There is a complete correspondence between the KK spectrum and the short multiplets of $\mathcal{N} = 4$ SYM: In each case we have precisely one short multiplet of the superconformal algebra for every $k \geq 2$. The massless multiplet A'_2 on the gravity side, which contains the gravitons, the gravitino and the $SO(6)$ gauge fields, corresponds to the supercurrent multiplet on the field theory side. This is the usual manifestation of the fact that gauge symmetries in the bulk correspond to global symmetries in the boundary. The multiplets A'_k corresponds to the field theory multiplet A_k with lowest component

$$Tr \phi_{\{i_1} \cdots \phi_{i_k\}} - \text{traces} . \quad (3.40)$$

It is an useful exercise to check explicitly that the two multiplets contains fields with the same quantum numbers and compare their masses and dimensions using the standard AdS rules discussed in section 2.2.

Exercise: identify all scalars in Figure 18, by using susy, mass/dimension relations and $SU(4)$ quantum numbers. The result is (schematically)

SU(4) rep.	operator	multiplet/dim.
<u>20</u>	$Tr \phi_{\{i} \phi_{j\}} - \text{traces}$	$A_2 \quad \Delta = 2$
<u>50</u>	$Tr \phi_{\{i} \phi_j \phi_{k\}} - \text{traces}$	$A_3 \quad \Delta = 3$
<u>10_c</u>	$Tr \lambda_a \lambda_b + \phi^3$	$A_2 \quad \Delta = 3$
<u>105</u>	$Tr \phi_{\{i} \phi_j \phi_k \phi_{p\}} - \text{traces}$	$A_4 \quad \Delta = 4$
<u>45_c</u>	$Tr \lambda_a \lambda_b \phi_i + \phi^4$	$A_4 \quad \Delta = 4$
<u>1_c</u>	on – shell Lagrangian	$A_2 \quad \Delta = 4$

Up to now, we have discussed the KK modes. What happens to the string modes? In the supergravity limit, all stringy states are very massive and decouple. In the CFT these states correspond to operators with anomalous dimension $x^{1/4}$

$$m^2 = \frac{\Delta(\Delta - 4)}{R^2} \sim \frac{1}{\alpha'} = \frac{\sqrt{x}}{R^2} \quad \rightarrow \quad \Delta \sim x^{1/4}. \quad (3.41)$$

Since they have infinite dimension, they should decouple from all the OPE and Green functions. We then have a very strong prediction: in $\mathcal{N} = 4$ SYM at strong coupling only the set of protected operators A_k have finite dimensions; all the other non protected operators have an anomalous dimension that diverges at strong coupling as $x^{1/4}$. In particular all multiplets containing spin greater than two and many other including the Konishi multiplet should have infinite dimension. We thus have a very large separation between a set of operators with maximum spin two and all the other operators in the theory. This separation, as discussed in section 2.6, is necessary for every CFT with a weakly coupled dual. All quantum field theory indications are consistent with the fact that this separation actually occurs for $\mathcal{N} = 4$ SYM at strong coupling.

Let us finish with some important comments.

- Some of the KK modes have negative mass. The corresponding mode is stable if $m^2 R^2 \geq -4$ (Breitenloner-Freedman bound). In $\mathcal{N}=4$, all operators have $\Delta \geq 2$ and therefore $m^2 \geq -4$. Stability is a consequence of supersymmetry. Non-supersymmetric solutions might be unstable and the corresponding CFT non-unitary.
- The separation of scales is a property of strong coupling. In $\mathcal{N} = 4$ SYM at weak coupling all operators, including the non protected ones, have small dimensions, computable in power series of $x = g_{YM}^2 N$. The dual description of this weakly coupled regime requires a stringy limit where the radius of AdS is small; in this situation many stringy states have low masses and correctly reproduce the CFT spectrum at weak coupling, including operators with arbitrary spin.
- The exact correspondence between KK modes and protected operators is a peculiarity of $\mathcal{N} = 4$ SYM and it does not extend to theories with less supersymmetry. The reason why the KK spectrum of $\mathcal{N} = 4$ SYM contains only chiral multiplets of the form discussed is that all other short, semishort and long multiplets of the $\mathcal{N} = 4$ superconformal algebra contain fields with spin greater than two which cannot appear in supergravity. The $\mathcal{N} = 1$ superalgebra

has even long multiplets with maximum spin two. And in fact, in examples of $\mathcal{N} = 1$ dual pairs there are KK modes corresponding to non-protected operators with finite dimension.

- We discussed only single trace operators. Multi-trace operators correspond in the AdS/CFT correspondence to multi-particles states in the bulk.

There is a final important remark. It is believed that the correspondence applies to an $\mathcal{N} = 4$ SYM theory with $SU(N)$ gauge group. The extra $U(1)$ factor leaving on the D3 branes is IR free and its dynamics is not captured by the duality. A first indication that the gauge group is $SU(N)$ comes from the KK spectrum. In a $U(N)$ theory $\text{Tr}\phi_i$ is a gauge invariant operator with protected dimension. However, the corresponding KK mode can be gauged away and it is not a physical state [30]. The full A'_1 multiplet is absent from the KK spectrum. Further strong indications that the correspondence describes $SU(N)$ gauge theories come from dual pairs with $\mathcal{N} = 1$ supersymmetry where one can identify states dual to baryonic operators, which would be not gauged invariant in a $U(N)$ theory.

3.3.4 Correlation functions, Wilson loops and all that.

The methods developed in section 2 can be applied to $\mathcal{N} = 4$ SYM.

- Green function for operators in A_k can be computed at strong coupling using supergravity as explained in section 2. As a curious result, 3-point functions for the lowest component of A_k , which in principle depends multiplicatively by an arbitrary function of the t'Hooft coupling x , agree with the free-field theory result [32]. This is a hint for some non-renormalization theorem.
- In $\mathcal{N} = 4$ SYM the computation of Wilson loops can be seen in a better light: external sources can be naturally obtained by separating one brane from the other and breaking the group $U(N) \rightarrow U(N-1) \times U(1)$ as in Figure 19. At large N , $U(N-1) \sim U(N)$. Since the near-horizon focuses on the stack of branes, moving away one of them corresponds to pull it to the boundary. The $W^\pm$ bosons then play the role of external sources: they are described by long and very massive strings connecting the boundary with the bulk and they transform in the fundamental representation of the unbroken $U(N-1) \sim U(N)$ group. This interpretation makes clear why, in the prescription described in section 2.4, we renormalize the Wilson loop by subtracting the mass of two very long

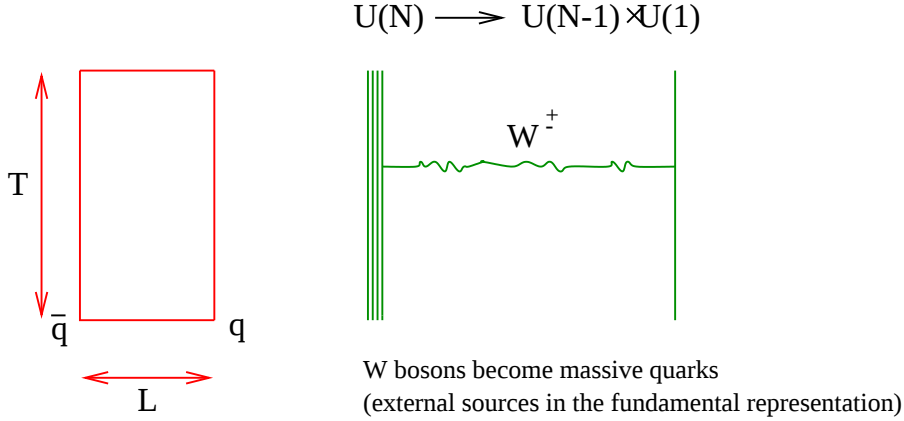


Figure 19: In a systems of branes, external sources in the fundamental representation of the gauge group can be obtained by moving one brane far away from the others

strings: this is the mass of the external quarks. In section 2.4 we obtained the result $E \sim \frac{R^2}{L} \sim \frac{\sqrt{g_{YM}^2 N}}{L}$. The coefficient in this formula is a genuine strong coupling result, as the presence of a square root indicates. At weak coupling we would find $E = \frac{g_{YM}^2 N}{L}$. There is no non-renormalization theorem for the basic Wilson loop.

- $SU(4)$ anomaly: in $\mathcal{N} = 4$ SYM the global $SU(4)_R$ symmetry is anomalous, since there four Weyl fermions ψ_a transforming in the $\underline{4}$ of $SU(4)$, a chiral representation. Interestingly we can see the anomaly in the dual supergravity. It is enough to consider the five-dimensional effective action for the massless multiplet on AdS_5 which is $\mathcal{N} = 8$ gauged supergravity. It contains a Chern-Simon coupling for the $SU(4)$ gauge fields $L_{CS} = \int A \wedge dA \wedge dA$. Under a gauge transformation $A \rightarrow A + d\Lambda$, this gives a boundary term:

$$L_{CS} \rightarrow \int_{AdS} d(\Lambda F \wedge \tilde{F}) = \int_{\partial AdS} \Lambda F \wedge \tilde{F} \quad (3.42)$$

The field theory functional $W(A)$, obtained by evaluating the on-shell five-dimensional action, is not invariant under gauge transformations, but exhibits the standard non-abelian anomaly.

3.4 Other $\mathcal{N} = 1$ CFTs from D-branes

In this section we present a brief detour on the other $\mathcal{N} = 1$ CFTs that admit an AdS dual. The content requires some more advanced knowledge of $\mathcal{N} = 1$ supersymmetry. The results of this section will not be used in the following.

3.4.1 Deformation of $\mathcal{N} = 4$

We can obtain other CFT by deforming $\mathcal{N} = 4$ SYM. In field theory, it is difficult to say when a deformed Lagrangian is conformal or, more generally, flow to a conformal theory in the IR. However we have some tools for dealing with supersymmetric theories.

- The Leigh and Strassler argument (LS) [33]: the understanding of this argument requires some advance knowledge of supersymmetry but the result is very easy to manage. Consider a $\mathcal{N} = 1$ gauge theory with chiral matter fields transforming in some representations R_a and a superpotential W . Anomalies and supersymmetry relate the $\mathcal{N}=1$ gauge beta function to the dimension of matter fields via,

$$\beta(g) \sim 3 \left[T(\text{adj}) - \sum T(R_a)(3 - 2\Delta_a) \right] \quad (3.43)$$

where a runs over chiral matter fields and T denotes the second Casimir. The requirements for conformal invariance are exhausted by the vanishing of the previous expression combined with the fact that the sum of dimensions of the fields appearing in each term of the superpotential is three.

We can also obtain information from the dual supergravity side, when the operator used for deforming the Lagrangian is dual to a KK mode. In this case we can use the results of section 2.7.2. In particular we can study the deformations of $\mathcal{N} = 4$ SYM by all the modes contained in Table 2 using supergravity.

Let us discuss first the marginal case. We see from Table 2 that we have marginal modes transforming in the 1, 45 and 105. When added to the original Lagrangian as deformation, these operators preserve conformal invariance at first order in h_i . We are interested in exactly marginal operators. that preserve conformal invariance for all values of the parameters. They provide lines of fixed points continuously connected to the original CFT. It is impossible to study this problem in full generality, however:

- The scalar 1_c is an example of exactly marginal deformation: it preserves $\mathcal{N}=4$ supersymmetry and the corresponding perturbation of the $\mathcal{N}=4$ theory is simply a change in the complexified coupling constant. This corresponds to an exactly marginal deformation, because the $\mathcal{N}=4$ Yang-Mills theory is conformal for each value of the coupling. Its supergravity description is extremely simple and it corresponds to vary the constant value of the dilaton and axion C_0 .

- There is an exactly marginal deformation of $\mathcal{N} = 4$ SYM preserving $\mathcal{N} = 1$ supersymmetry given by the superpotential

$$\mathcal{W} = h\epsilon_{ijk}\text{Tr}\Phi_i\Phi_j\Phi_k + Y_{ijk}\text{Tr}\Phi_i\Phi_j\Phi_k, \quad (3.44)$$

In this equation, Φ_i are the three adjoint chiral fields of $\mathcal{N} = 4$ SYM in $\mathcal{N} = 1$ notations and Y_{ijk} is a generic symmetric tensor transforming in the $\underline{10}$ of $SU(3)$. By expanding in components we see that the deformations in the Lagrangian are cubic and quartic terms contained in the $\underline{45}$ and $\underline{105}$. The $\mathcal{N}=4$ theory is recovered for $Y_{ijk} = 0$ and $h = g_{YM}$. The 11 complex parameters in the superpotential can be reduced to 4 independent ones by using the (complexified) $SU(3)$ global symmetry. A convenient parametrization of the superpotential is

$$h\text{Tr}(e^{i\pi\beta}\Phi_1\Phi_2\Phi_3 - e^{-i\pi\beta}\Phi_1\Phi_3\Phi_2) + h'\text{Tr}(\Phi_1^3 + \Phi_2^3 + \Phi_3^3) \quad (3.45)$$

There is a particular relation between the four complex parameters g_{YM}, h, h', β for which the theory is superconformal. In the specific case, the gauge beta function vanishes if

$$\Delta_{\Phi_1} + \Delta_{\Phi_2} + \Delta_{\Phi_3} = 3 \quad (3.46)$$

On the other hand, the requirement that the superpotential terms have formal dimension three, gives the condition $\Delta_{\Phi_i} = 1$ for all $i = 1, 2, 3$. Compressively, the equations for the vanishing of all beta functions are satisfied if

$$\Delta_{\Phi_i}(g_{YM}, h, h', \beta) = 1 \quad (3.47)$$

Using the obvious permutation symmetry among the Φ_i we conclude that we have a single equation for four unknown. This yields a three-dimensional complex manifold of fixed points. Although we will be never able to solve equation (3.47) but in perturbation theory, the LS argument is strong enough to conclude the existence of the manifold of fixed points. The supergravity dual of this manifold is obtained by deforming $AdS_5 \times S^5$ with the corresponding KK modes. From Table 2 and figure 18 we see that we need a deformation of the metric but also the inclusion of a non-zero background for the antisymmetric complex two-form of type IIB. A perturbative analysis of the supergravity equations of motion, performed up to second order, reveal that it is possible to find an AdS_5 solution with these new modes turned on, in agreement with the fact that the corresponding deformations are exactly marginal. The solution with arbitrary parameters is only known at few orders in perturbation theory [21, 34]. For the case $h' = 0$, called the β deformation of $\mathcal{N} = 4$ SYM however a full supergravity description exists [35].

Now we turn to the relevant deformations of $\mathcal{N} = 4$ SYM. We see from Table 2 that we have relevant modes in the 20, 50 and 20_c. In particular, we can describe almost all massive deformations of $\mathcal{N} = 4$. The only exception corresponds to the mass term $\sum \phi_i \phi_i$ which is dual to a string mode. We can use supergravity to study the existence of possible IR fixed points. They should correspond, as discussed in section 2.7.2, to kink solutions of type IIB supergravity that connect two different AdS_5 vacua. The existence of these kink solutions is better investigated in the context of a five-dimensional reduction of the theory. We do not have a full consistent 5d Lagrangian containing all relevant states in Table 2. However we do have a consistent Lagrangian for the massless multiplet A'_2 , $\mathcal{N} = 8$ gauged supergravity. The scalar sector of the gauged supergravity contains the complexified dilaton 1_c, the 20 and 10_c, for a total of 42 scalars. In particular it describes all KK modes dual to mass terms. As discussed in section 2.7.2, IR fixed points correspond to the critical points of the gauged supergravity potential. The full potential is too hard to study, however:

- All critical points with at least $SU(2)$ symmetry have been classified [36]. A central critical point, with $SO(6)$ symmetry and with all the scalars λ_a vanishing, corresponds to the unperturbed $\mathcal{N} = 4$ YM theory. There are three $N=0$ theories with residual symmetry $SU(3) \times U(1)$, $SO(5)$ and $SU(2) \times U(1)^2$. They correspond to non-zero VEV for some of the scalars in the 10, 20, and 10 + 20, respectively. Then there is an $\mathcal{N} = 2$ point (this corresponds to a $\mathcal{N} = 1$ field theory) with symmetry $SU(2) \times U(1)$, obtained giving VEV to scalars in the 10 + 20. The three $N=0$ theories are unstable and correspond to non-unitary CFTs. The $\mathcal{N} = 2$ theory is stable by supersymmetry. The central charges of these theories can be computed from the value of the cosmological constant at the critical points.
- The supersymmetric IR point can be explicitly identified in field theory [22, 37]. It is obtained by adding a mass term for one of the chiral fields of $\mathcal{N} = 4$ when written in $\mathcal{N} = 1$ notations,

$$h \text{Tr} \Phi_3 [\Phi_1, \Phi_2] + m^2 \Phi_3^2 \quad (3.48)$$

Integrating out the massive field Φ_3 , we obtain an $SU(N)$ theory with two adjoint fields $\Phi_i, i = 1, 2$ and a superpotential

$$\lambda \text{Tr} [\Phi_1, \Phi_2]^2, \quad \lambda \sim \frac{h^2}{m^2} \quad (3.49)$$

It is easy to show that the theory flows to an IR fixed points. The vanishing of the gauge beta function and the requirement of scaling invariance of the

superpotential give the same constraint

$$2(\Delta_{\Phi_1} + \Delta_{\Phi_2}) = 3 \quad (3.50)$$

Since the theory has an $SU(2)$ symmetry rotating Φ_1 and Φ_2 , the fields Φ_i have the same dimensions and the previous equation reduces to a single constraint

$$\Delta_{\Phi}(g_{YM}, h) = 3/4 \quad (3.51)$$

With two couplings, g_{YM} and λ , we have a line of fixed points. Using some more sophisticated methods, one can even compute the central charge of the IR fixed point: $\frac{a_{IR}}{a_{N=4}} = \frac{27}{32}$ [22]. It is probably superfluous to say that it corresponds exactly with the supergravity prediction based on gauged supergravity: $\left(\frac{\Lambda_{UV}}{\Lambda_{IR}}\right)^{3/2} = \frac{27}{32}$. The supergravity dual of the Renormalization Group flow from $\mathcal{N} = 4$ to the IR fixed point was found in [22] using five dimensional gauged supergravity and succesively uplifted to ten dimensions [38]; as expected from Table 2 and Figure 1, the solution is a warped AdS_5 compactification on a squashed sphere with non zero B field.

3.4.2 Other $\mathcal{N} = 1$ theories.

We can obtain an infinite class of $\mathcal{N} = 1$ theories by taking a near horizon limit of systems of D3-branes in curved geometries. Since the AdS/CFT correspondence focuses on the near brane region and every smooth manifold is locally flat, we will find new models only when the branes are placed at a singular point of the transverse space [39–41]. An interesting class of theories makes use of conifold singularities. We place branes at the singularity of Ricci-flat manifolds C_6 whose metric has the conical form

$$ds_{C_6}^2 = dr^2 + r^2 ds_{M_5}^2. \quad (3.52)$$

One can prove that C_6 is Ricci-Flat iff M_5 is a five-dimensional Einstein manifold [40, 42]. The AdS/CFT correspondence is then formulated with the background $AdS_5 \times M_5$, which is the near horizon geometry of the previous metric. The gauge theory is maximally supersymmetric $\mathcal{N} = 4$ SYM with gauge group $U(N)$ in the case $M_5 = S^5$ and a less supersymmetric gauge theory depending on M_5 otherwise.

- Famous examples of this construction are orbifolds on $\mathcal{N} = 4$ SYM or the Klebanov-Witten theory, pictured in Figure 20. They give examples of the so-called quiver gauge theories, with product of $U(N)$ gauge groups (where N is the number of branes) and bi-fundamental or adjoint matter fields.

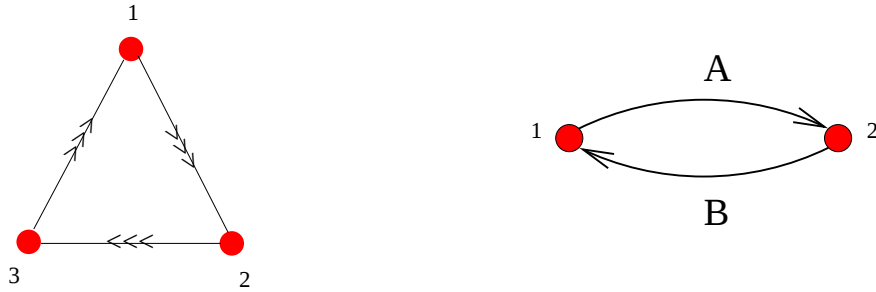


Figure 20: The quiver corresponding to $M_5 = S^5/\mathbb{Z}^3$ (the orbifold $C_6 = \mathbb{C}^3/\mathbb{Z}_3$) and $M_5 = T^{1,1}$ (the conifold $C_6 : xy = zw$ in $\mathbb{C}^4$).

- The correspondence between M_5 and CFTs can be worked out explicitly when the cone $C_6 = C(M_5)$ is a toric manifold using Tiling techniques [43, 44]¹⁷.
- The five-manifold M_5 has typically non trivial topology, with two and three cycles. Three cycles lead to the existence of baryons in these theories [48] and indicate that the gauge group is $SU(N)$. Their existence is also the base for the most succesful attempts to describe confining theories using the AdS/CFT correspondence [24].

We shall not discuss further this general class of theories since they have a very rich structure that will take us too far.

Other examples of $\mathcal{N} = 1$ CFTs can be obtained by considering flux compactifications of type IIB. Much less is known about this class of backgrounds, in principle very large; the typical example in this class is the warped compactification corresponding to the IR point of the $\mathcal{N} = 1$ adjoint deformation of $\mathcal{N} = 4$ SYM [38]. Other classes of (quite exotic) $\mathcal{N} = 2$ CFTs are obtained from M theory compactifications on Riemann surfaces [49].

4 Explicit examples: the non-conformal case

One aim of the correspondence is the description of non-conformal theories that reduce at low energy to pure YM theories. We will now describe as a simple toy model for

¹⁷Toric roughly means that there are three $U(1)$ isometries. The corresponding quivers can be drawn as a tiling of a two-dimensional torus. One can show that these quivers are conformal (using a LS argument), that the field theory mesonic moduli space is $(\text{Sym}C_6)^N$ [45] and that the spectrum computed via a-maximization [46] coincide with the prediction of supergravity [47].

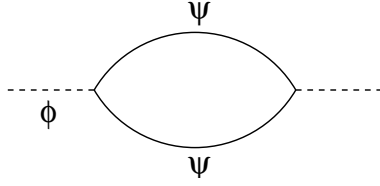


Figure 21: Scalars get mass through Yukawa couplings.

confining theories based on a regular background obtained using D-branes at finite temperature [2]. We shall explicitly confirm the picture of confinement discussed in section 2.7.1.

4.1 The back three-brane

We shall study euclidean $\mathcal{N} = 4$ SYM with gauge group $SU(N)$ at finite temperature T . This can be obtained by compactifying the euclidean time on a circle of radius $R_0 \sim 1/T$. In theories at finite temperature, the fermions have antiperiodic boundary conditions along S^1

$$\begin{aligned} \psi(y) &= -\psi(y + 2\pi R_0) \rightarrow \psi = \sum_{\mathbb{Z}+1/2} \psi_k e^{iky/R} \\ m_{\psi_k}^2 &= \frac{k^2}{R_0^2} > 0 \quad (|k| > 0). \end{aligned} \quad (4.1)$$

Conformal invariance is broken by the compactification and supersymmetry by the boundary conditions. The fermions get masses through these boundary conditions and the scalars get masses through loops of fermions. For $R_0 \rightarrow 0$ all fermions and scalars decouple and we are left with pure YM in three-dimensions. From

$$\int d^4x \frac{1}{g_4^2} F_{\mu\nu}^2 = \int d^3x \frac{2\pi R_0}{g_4^2} F_{\mu\nu}^2 \quad (4.2)$$

we see that the three-dimensional coupling is given by $\frac{1}{g_3^2} = \frac{2\pi R_0}{g_4^2}$. In order to keep the three-dimensional coupling finite in the decoupling limit, we need to send $R_0 \rightarrow 0$, and $g_4 \rightarrow 0$ with g_3 fixed. In this limit, one obtains a non-supersymmetric and non-conformal YM theory in three-dimensions. Such a theory confines, has a mass gap and a discrete spectrum of massive glueballs.

This model can be studied using a weakly coupled supergravity dual. We consider a set of N D3-branes defined on a space-time with topology $\mathbb{R}^3 \times S^1$. The

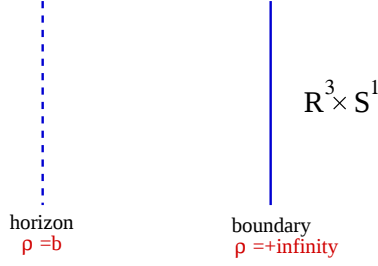


Figure 22: The black brane describing $\mathcal{N} = 4$ SYM at finite temperature.

solution for an Euclidean black three-brane with this boundary topology is known and its near horizon geometry is given by:

$$ds^2 = R^2 \left[\left(u^2 \sum_{i=1}^3 dx_i^2 + u^2 \left(1 - \frac{u_0^4}{u^4} \right) d\tau^2 \right) + \frac{du^2}{u^2 \left(1 - \frac{u_0^4}{u^4} \right)} + d\Omega_5 \right] \quad (4.3)$$

where, as usual, $R = \sqrt{4\pi g_s N \alpha'}$. The geometry has an horizon at $u = u_0$. If we expand the metric around $u \sim u_0$, we obtain

$$ds^2 \sim R^2 \left(4u_0(u - u_0)d\tau^2 + \frac{d(u - u_0)^2}{4u_0(u - u_0)} + \dots \right) = R^2(4u_0^2\rho^2 d\tau^2 + d\rho^2 + \dots) \quad (4.4)$$

with $\rho = \sqrt{(u - u_0)/u_0}$. This becomes the two-dimensional flat metric in polar coordinates only if τ is an angular variable with radius $R_0 = \frac{1}{2u_0}$. For all other choices of periodicity, the metric would have a conical singularity at $u = u_0$. Moreover the metric admits only one spin structure where fermions change sign along S^1 . One way of see it is to consider that τ is not a good coordinate everywhere; near u_0 it corresponds to the polar angle in the plane $(u - u_0, \tau)$ whose value is not defined at the origin. In local cartesian coordinates, the wave functions for a fermion is single valued; however, it is easy to see that if we coordinate transform it to polar coordinates the same wave-function will change sign in a loop around the origin. For all these reasons, the black three-brane solution is the natural candidate for the gravitational dual of $\mathcal{N} = 4$ SYM with gauge group $SU(N)$ on $\mathbb{R}^3 \times S^1$.

There is a refined and more complicated version of this construction that gives pure YM in four-dimensions [2].

- **Confinement:** As discussed in section 2.7.1 , the criterion for confinement is the following: the warp factor e^{2A} multiplying the space-time part of the metric must be bounded above zero. In the black brane example, the warp factor reaches its minimum $e^{2A_0} = \sqrt{4\pi g_s N} u_0^2$ at the horizon u_0 . The theory has then

stable strings with finite tension; they live in the region of the solution where the warp factor has its minimum value e^{2A_0} and their tension will be given by $\frac{e^{2A_0}}{2\pi\alpha'}$.

- **Glueball Spectrum:** The masses of bound states can be extracted from correlation functions of gauge invariant operators evaluated at large distance,

$$\langle O(x)O(y) \rangle \sim \sum a_i e^{-M_i|x-y|}, \quad |x-y| \gg 1 \quad (4.5)$$

More efficiently, as discussed in section 2.7.1, we can extract M_i from the normalizable solutions of the equations of motion. We expand a massive field in Fourier modes both on S^1 and on the three-dimensional space-time

$$\phi(x, \tau, u) = \phi(u) e^{in\tau} e^{ikx} \quad (4.6)$$

For simplicity, we consider the lowest states and we keep only the zero-mode on S^1 ($n=0$). The action in the black three-brane background is

$$S = \int_{u_0}^{\infty} du u^3 \left\{ u^2 \left(1 - \frac{u_0^4}{u^4} \right) \left(\frac{\partial \phi}{\partial u} \right)^2 + m^2 \phi^2 + \frac{k^2}{u^2} \phi^2 \right\}$$

The glueball spectrum is obtained by finding the normalizable solutions of this equation which are regular in the IR. The corresponding values $M^2 = -k^2$ give the masses of the bound states. As discussed in section 2.7.1, with a redefinition of fields and coordinates ($u \rightarrow z, \phi \rightarrow \psi$), we can always reduce the problem to a Schroedinger equation

$$-\psi'' + V(z)\psi = -k^2\psi, \quad E = -k^2 = M^2 \quad (4.7)$$

The potential $V(z)$ behaves as $1/z^2$ near the boundary and goes at a finite value at $z_0 = z(u_0)$. Regularity and normalizability of the solution requires vanishing conditions at the two boundaries. We then face a standard Schoedinger problem for a well with impenetrable bareers. We find a discrete spectrum $E_i = -K_i^2 = M_i^2$ of glueball bounded from below.

Exercise: Show that the right change of coordinates is $z = \int_u^{\infty} \frac{du}{\sqrt{u^4 - u_0^4}}$ and study the resulting potential $V(z)$. Examine closely the boundary conditions. Recall that u is a polar coordinate: a zero mode, independent of the angle τ , is a smooth function in the origin only if expands in even powers of u . Check that all normalizable solutions corresponds to negative k^2 and thus positive masses.

- **Decoupling YM_3 :** The theory on the black three-brane is $\mathcal{N} = 4$ SYM in the UV and flows to pure YM in three dimensions in the IR. We would like to take a limit where the low energy YM theory decouples from the *CFT*. Since the YM coupling constant

$$\frac{1}{g_3^2 N} = \frac{2\pi R_0}{g_4^2 N} . \quad (4.8)$$

should remain finite in the decoupling limit, we need to take $R_0 \rightarrow 0$ with $x = Ng_4^2 \rightarrow 0$. Unfortunately, this is a limit where we can not trust supergravity. In fact, we have a weakly coupled gravitational background only when $x = Ng_4^2 \gg 1$ (recall that the coupling constant on a D3 brane is related to the string coupling by $g_4^2 = 4\pi g_s$). In order to describe the pure YM theory in three-dimensions we need to use a strongly coupled string background. The supergravity solution describes a theory that reduces in the IR to pure three dimensional YM theory but has an UV completion which is a strongly coupled four-dimensional conformal gauge theory.

We might consider the supergravity solution as a description of pure YM with a finite cut-off $\Lambda \sim M$. The situation is similar, in spirit, to a lattice computation at strong coupling. To remove the cut-off, it would be sufficient to re-sum all world-sheet α' corrections in the string background. World-sheet corrections are, in principle, more tractable than loop corrections. In flat space, for example, all the α' corrections are computable. In the *AdS* case, the analogous computation is made difficult by the presence of RR-fields and at the moment it is outside our technical abilities.

Quite surprisingly, although the theory is not really pure YM, a numerical evaluation of the ratio of glueball masses for the black D3 brane give results in good quantitative agreement with the lattice results for the three dimensional YM theory [50].

4.2 Examples of non conformal theories with gravitational dual

Various methods have been used in order to construct string duals of non-conformal gauge theories. Here we list the main ones and we refer to the reviews [11, 13] and references therein for a detailed discussion. There are very few solutions dual to non conformal theories which are completely regular.

- Finite temperature: Historically this was the first example of non-conformal gauge/gravity duals. We already briefly discussed the case of $\mathcal{N} = 4$ SYM at finite temperature. This method necessarily implies a dimensional reduction from an higher dimensional theory and the resulting theory is typically not supersymmetric.
- Deformations: we can break conformal invariance by deforming the AdS background. The gauge theory becomes conformal in the UV and the gravity background will asymptote to $AdS_5 \times H$. Most of the massive deformations of $\mathcal{N} = 4$ SYM can be studied in this way. One notable example is the $\mathcal{N} = 1$ massive deformation of $\mathcal{N} = 4$ SYM, which is called the $\mathcal{N} = 1^*$ theory. We still do not have a complete supergravity description of this theory. A solution can be obtained as a RG flow in five-dimensional gauged supergravity [51] but it is singular even when lifted to 10 dimensions [52]. A 10 dimensional approach has been pursuit in [53]; only the UV and IR behaviours of the solution are known.
- Fractional and Wrapped Branes: this method applies to branes wrapping vanishing cycles in singular internal geometries (fractional branes) or non trivial cycles in regular ones (wrapped branes). It has been the most successful way of obtaining regular solutions dual to $\mathcal{N} = 1$ confining YM theories. The Klebanov-Strassler solution [24] uses fractional branes and it is based on a $SU(N + M) \times SU(N)$ gauge theory with bi-fundamental fields in four dimensions. It is asymptotic in the UV to a logarithmic deformations of AdS_5 . The Maldacena-Nunez solution [25] uses wrapped branes and decompactifies to higher dimensions in the UV. Both theories are believed to reduce to an $SU(M)$ SYM in the IR. The gravity description is based in both cases on a background with bounded warp factor which is geometrically $\mathbb{R}^7 \times S^3$ in the IR. The spontaneous breaking of chiral symmetry, the existence of domain walls and confining strings can be explicitly verified in these backgrounds and are related to the behaviours of various types of D-branes.

4.3 The decoupling problem

All the weakly coupled supergravity solutions considered so far describe YM theories with many non decoupled massive modes.

- For example, the YM_3 of the black-brane example becomes $\mathcal{N} = 4$ SYM in the UV. As already discussed, the supergravity limit is valid only when the $\mathcal{N} = 4$

SYM is strongly coupled; in the same regime we cannot decouple the UV and IR regimes.

- Analogously pure $\mathcal{N} = 1$ SYM can be embedded in a massive deformation of $\mathcal{N} = 4$ (the $\mathcal{N} = 1^*$ theory) and have a supergravity description. Even in this case we cannot decouple the massive modes without leaving the supergravity regime. A mass deformation m indeed induces a dimensional scale $\Lambda \sim m e^{-1/Ng_{YM}^2}$. Once again the decoupling limit is $m \rightarrow \infty$, $x = Ng_{YM}^2 \rightarrow 0$, with Λ fixed, and this is the opposite of the supergravity limit.
- A similar argument, although more complicated, applies to the Klebanov-Strassler solution, where one can decouple the pure YM theory in the IR from its complicated UV completion only at the price of leaving the supergravity regime.

The general lesson is that supergravity describes gauge theories which are strongly coupled at all scales. We learned indeed that the curvature of the background is inversely proportional to the strength of the coupling constant. When the curvature is small the coupling constant should be large. Within this general picture, QCD at large N , which is asymptotically free and becomes weakly coupled in the UV, cannot have a supergravity description; the QCD dual is a string theory on a strongly curved background. This expectation is also strengthened by simple considerations about the spectrum of bound states. In QCD we expect Regge trajectories of bound states with masses linearly increasing with the spin. On the other hand, in any effective supergravity limit of string theory we only have states with maximum spin two; all the other stringy states are separated by a very large energy gap. This clearly does not fit QCD expectations. The path to the real large N QCD involves quantizing tree level string theory on a strongly coupled background. As we already mention, we are not yet able to solve string theory on curved backgrounds, but this is a technical problem that we might hope to overcome in the future.

However, by engineering gauge theories on D-branes, we can construct many stringy-inspired modifications of pure gauge theories and we have a large moduli space of QCD-like theories. Some of these theories correspond to gauge theories with weakly coupled dual backgrounds. These theories have field content different from QCD or modified couplings and Lagrangian. In particular, they substantially differ from QCD in the UV where they are strongly coupled. Typically new massive fields are added in such a way that the UV completion has a weakly coupled holographic dual. The theory may become conformal or decompactify in the UV. The IR behavior instead can be very similar to that of QCD. We can use these exactly solvable theories as

extremely efficient toy models for studying the strong dynamics of non abelian gauge theories, from confinement to chiral symmetry breaking.

Acknowledgments

I would like to thank the audience of these lectures in EPFL and in particular R. Rattazzi and M. Redi for many interesting comments, interruptions and attempts to cure my natural sloppiness. This work is supported in part by INFN and MIUR under contract 2007-5ATT78-002.

A Appendix: The superconformal group

In this Appendix we include few information about representations of the (super) conformal group in four dimensions. For more information the reader is referred to [54–56] and references therein.

The CFT fields and operators are usually classified by quantum numbers under the non-compact sub-group $SO(1, 2) \times S(1, 3) \subset SO(2, 4) \sim SU(2, 2)$. As one can see from the algebra, P_μ and K_μ act as raising and lowering operators for D . The lowest state of a representation will be annihilated by K . Every irreducible (infinite dimensional) representation is then specified by an irreducible representation of the Lorentz group with definite conformal dimension and annihilated by K_μ . In terms of the operator in $x = 0$, stabilized by the subalgebra $K_\mu, D, M_{\mu\nu}$, we define a primary conformal field, in the (j_L, j_R) representation of the Lorentz group, by

$$\begin{aligned} [D, O_{(j_L, j_R)}(0)] &= i\Delta O_{(j_L, j_R)}(0) \\ [K_\mu, O_{(j_L, j_R)}(0)] &= 0. \end{aligned} \tag{A.1}$$

The descendents $\partial \dots \partial O_{(j_L, j_R)}(0)$ reconstruct the operator by Taylor expansion and fill an infinite dimensional representation specified by three numbers (Δ, j_L, j_R) , corresponding to the conformal dimension and the Lorentz quantum numbers of the primary operator.

Since the operators $O_{(j_L, j_R)}(0)$ lead to non normalizable states when applied to the vacuum ¹⁸, it is useful to consider the maximal compact subgroup $SO(2) \times$

¹⁸This is necessary, since otherwise these states would realize a finite dimensional unitary representation of the non compact group corresponding to the stability algebra at $x = 0$ which is forbidden

$SO(4) \subset SO(2, 4)$. Using this compact subgroup, we can study and classify unitary representations of the conformal group using states with finite norm. The choice of this subgroup is also natural in AdS_5 . In this picture, we classify states though the eigenvalues of $H = (P_0 + K_0)/2$ and $SO(4) = SU(2) \times SU(2)$, identified with (Δ, j_L, j_R) respectively. This choice of quantum numbers is also useful when, upon Euclidean continuation, we radially quantize the theory on $S^3 \times \mathbb{R}$.

The superconformal group $SU(2, 2|N)$ corresponding to a theory with $\mathcal{N}$ supersymmetries is obtained by adding $\mathcal{N}$ supercharges Q_α^a , $\mathcal{N}$ superconformal charges S_a^α and the generators of a $U(\mathcal{N})$ global symmetry R_b^a . Some of the relevant commutation relations are

$$\begin{aligned}
[D, Q_\alpha^a] &= \frac{i}{2} Q_\alpha^a \\
[D, S_a^\alpha] &= -\frac{i}{2} S_a^\alpha \\
[K_\mu, Q_\alpha^a] &= -(\sigma_\mu)_{\alpha\dot{\alpha}} \bar{S}_a^{\dot{\alpha}} \\
[P_\mu, S_a^\alpha] &= (\bar{\sigma}_\mu)^{\dot{\alpha}\alpha} \bar{Q}_{\dot{\alpha}a} \\
[Q_\alpha^a, \bar{Q}_{\dot{\alpha}b}] &= 2\delta_b^a (\sigma^\mu)_{\alpha\dot{\alpha}} P_\mu \\
[\bar{S}^{\dot{\alpha}a}, S_b^\alpha] &= 2\delta_b^a (\bar{\sigma}^\mu)^{\dot{\alpha}\alpha} K_\mu \\
\{Q_\alpha^a, S_b^\beta\} &= -i\delta_b^a (\sigma^\mu \bar{\sigma}^\nu)_{\alpha}^{\beta} J_{\mu\nu} - 2i\delta_b^a \delta_\alpha^\beta D - 4\delta_\alpha^\beta R_b^a
\end{aligned} \tag{A.2}$$

where $(Q_\alpha^a)^\dagger = \bar{Q}_{a\dot{\alpha}}$, $(S_a^\alpha)^\dagger = \bar{S}^{a\dot{\alpha}}$. The first two lines specify the dimension of the charges; notice that Q is a raising operator while S is a lowering operators for D . The next two lines clarify why we need to introduce S to close the algebra. Next we have the standard commutation relation defining supersymmetry and after that its conformal partner. There are other equations that specify the R-symmetry quantum numbers of the various quantities,

$$[R_b^a, Q_\alpha^c] = \delta_b^c Q_\alpha^a - \frac{1}{4} \delta_b^a Q_\alpha^c \quad [R_b^a, S_c^\alpha] = -\delta_c^a S_b^\alpha + \frac{1}{4} \delta_b^a S_c^\alpha \tag{A.3}$$

The R_b^a are generators of $U(\mathcal{N})$ and close the corresponding algebra. For $\mathcal{N} = 4$, the trace R_a^a decouples from the algebra and become an outer automorphism. Correspondingly the R-symmetry is $SU(4)$.

Superconformal representation have a lowest state which is annihilated by both the lowering operators K and S ; it is identified by the quantum numbers Δ, j_L, j_R by group theory; this is also the reason why the necessary i in the D commutator in A.1 does not contradict the hermicity of D .

under the conformal group and $R, a_1, \dots, a_{N-1}$ under the R-symmetry $U(1) \times SU(N)$. The infinite dimensional representation is obtained by acting on the lowest state with an infinite numbers of P and Q s. Since the Q are fermionic charges, only a finite numbers of applications of Q s give a non-vanishing result. We can keep track of the action of the supercharges by introducing superfields $\Phi_{(j_L, j_R, R, a)}(x, \theta_\alpha^a, \bar{\theta}_\alpha^a)$ where the θ^a are superspace coordinates in the $\mathcal{N}$ of $SU(\mathcal{N})$; the expansion in θ has generically $2^{4N}r$ components, where r is the dimension of the Lorentz and R-symmetry representation, with a generic spin range of $\Delta j_L = \Delta j_R = N/2$.

Unitarity imposes bounds on the representations. These are obtained by imposing that all the states in a representation have positive norm. The saturation of the bounds corresponds to representations with null states which can be removed leaving a shorter representation. The vanishing of a particular state implies some differential constraints on the primary field (it is annihilated by some polynomial in P and Q) which indeed make the representation shorter.

For the conformal group non trivial operators must satisfy,

$$\begin{aligned} \Delta &\geq 1 + j_R & j_L &= 0, & \text{or } (j_R \rightarrow j_L) \\ \Delta &\geq 2 + j_L + j_R & (j_L j_R &\neq 0) \end{aligned} \quad (\text{A.4})$$

The two unitarity thresholds are satisfied by free massless fields and conserved tensors field, respectively. The saturation of the bounds corresponds in fact to

$$\begin{aligned} \partial^2 \Phi_{(0, j_R)} &= 0 \\ \partial^{\alpha_1 \dot{\alpha}_1} O_{\alpha_1 \dots \alpha_{2j_L}, \dot{\alpha}_1 \dots \dot{\alpha}_{2j_R}} &= 0 \end{aligned} \quad (\text{A.5})$$

The first bound for $j_R = 0$ say that the dimension of a scalar primary operator is greater than one and it can be one if and only if the field is free. The second bound for $j_L = j_R = 1/2$ says that a spin one operator J_μ has dimension greater than 3 and equal to 3 if and only if it is a conserved current, $\partial^\mu J_\mu = 0$. Analogously, a spin two operator $T_{\mu\nu}$ ($j_L = j_R = 1$) has dimension greater than 4 and equal to 4 if and only if it is conserved.

The superconformal representations are richer but similar in spirit. Shortening now may correspond to annihilation of the primary operator by some supersymmetry charge or some combinations of charges. An example that can be familiar to the reader is that of a chiral superfields in $\mathcal{N} = 1$ supersymmetry $\bar{D}_\alpha \Phi = 0$. Chiral superfields are annihilated by half of the supersymmetries and the corresponding multiplet are short, depending only by θ and not by $\bar{\theta}$ (modulo subtleties). This

shortening corresponds to a saturation bound fixing the dimension of the operator in terms of its R-charge

$$\Delta = \frac{3}{2}R \quad (\text{A.6})$$

where, as customary, we normalized the R charge of Q to one. While the generic scalar superfield has a spin range of 1, a chiral superfields has spin range $1/2$.

Just for curiosity, we can report the full unitarity bounds for non-trivial $SU(2, 2|1)$ representations,

$$\begin{aligned} a) \quad & \Delta \geq 2 + 2j_R + \frac{3}{2}R \geq 2 + 2j_L - \frac{3}{2}R \quad (\text{or } j_L \rightarrow j_R, R \rightarrow -R) \\ b) \quad & \Delta = \frac{3}{2}R \geq 2 + 2j_L - \frac{3}{2}R, \quad j_R = 0 \quad (\text{or } j_L = 0, j_R = j, R \rightarrow -R). \end{aligned}$$

where R is the $U(1)$ R-charge. We will not discuss in detail all the varieties of shortening. The chiral superfield is contained in case b) as well as some generalizations. Case a) contains conserved supercurrents

$$D^{\alpha_1} L_{\alpha_1 \dots \alpha_{2j_L} \dot{\alpha}_1 \dots \dot{\alpha}_{2j_R}} = \bar{D}^{\alpha_1} L_{\alpha_1 \dots \alpha_{2j_L} \dot{\alpha}_1 \dots \dot{\alpha}_{2j_R}} = 0 \quad (\text{A.7})$$

with $\Delta = 2 + j_L + j_R$ and various other shortenings (for example semiconserved superfields ($\bar{D}^{\alpha_1} L_{\alpha_1 \dots \alpha_{2j_L} \dot{\alpha}_1 \dots \dot{\alpha}_{2j_R}} = 0$)). All these short as well as some long representation appear in the KK spectrum in generic $\mathcal{N} = 1$ AdS_5 background.

The $\mathcal{N} = 4$ superalgebra $SU(2, 2|4)$ has analogously a variety of representations with different unitary bounds. The lowest state is characterized by $(\Delta, j_L, j_R, p, k, q)$, where p, k, q are the Dynkin labels of an $SU(4)$ representation. They correspond to $SU(4)$ Young tableaux with rows of length $p, p+k, p+k+q$ starting from the bottom. With this notation, the fundamental $\underline{4}$ is $[1, 0, 0]$, the anti-fundamental $\bar{\underline{4}}$ is $[0, 0, 1]$ and the $\underline{6}$ (antisymmetric of $SU(4)_R$ and vectorial for $SO(6)$) is $[0, 1, 0]$. A generic long representation has now spin range 4. Short multiplets of $\mathcal{N} = 4$ have

$$\Delta = 2p + k, \quad p = q, \quad j_L = j_R = 0 \quad (\text{A.8})$$

Consider first the case $p = 0$. These are the so-called chiral primary operators and generalize the chiral operators of the $\mathcal{N} = 1$ case. They are $\frac{1}{2}$ BPS representations, with maximum spin two and range of dimensions $\Delta_{\max} - \Delta_{\min} = 4$. The case $k = 1$ has a further shortening and has maximum spin 1; it is called a singleton representation. It can be described by a superfield $W_{[AB]}(x, \theta, \bar{\theta})$, $A = 1, \dots, 4$ which satisfies

$$\begin{aligned} W_{[AB]} &= \frac{1}{2} \epsilon_{ABCD} \bar{W}_{[CD]} \\ \mathcal{D}_{\alpha A} W_{[BC]} &= \mathcal{D}_{\alpha [A} W_{BC]} \end{aligned} \quad (\text{A.9})$$

In a suitable harmonic superspace it becomes a twisted chiral superfield. W contains, as first component, a set of six scalars $\phi_{[AB]}$ in the 6 of $SU(4)$, which will be denoted also $\phi_l, l = 1, \dots, 6$. The superfield itself will be also denoted W_l . The full multiplet contains six scalars, four Weyl fermions and a gauge vector; this is the field content of the elementary fields in $\mathcal{N} = 4$ SYM. All other representations with $k \geq 2$ have maximum spin 2 and can be constructed by tensor products of singletons as $W^k = W_{\{l_1 \dots l_k\}} - \text{traces}$. One can show that these multiplets are short. Their dimension is then $\Delta = k$. In the main text, we called them A_k . The special case $k = 2$ has a further shortening with respect to $k > 2$ since it contains conserved currents. The series of multiplets A_k with $k \geq 2$ is in one-to-one correspondence with the KK spectrum on S^5 . $k = 1$ does not appear indicating that the theory is $SU(N)$ instead of $U(N)$. We could have predicted from the very beginning that all the KK modes correspond to short multiplets because their maximal spin range is 2. Representations with $p \neq 0$ are $\frac{1}{4}$ BPS with $\Delta_{\max} - \Delta_{\min} = 6$ and appear when multi-trace operators in CFT (or multi-particle states in the bulk) are considered.

There are also more complicated semishort multiplets of the $\mathcal{N} = 4$ superconformal algebra, which play a role in the operator product expansion of chiral primary operators.

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A hint of AdS/CFT

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ABSTRACT: The aim of these lecture notes is to give a hint of the theoretical concepts underlying the AdS/CFT correspondence to the 2-3 year bachelor students. The knowledge of classical electrodynamics, theoretical mechanics with action principle and the classical gravity is assumed.

We introduce classical Abelian and non-Abelian gauge field theories, basic features of the string theory, including the spectra of open and closed strings. Then we turn to D-branes and construct the holographic duality between gravity on $\text{AdS}_5 \times \text{S}_5$ and large N SYM theory. We discuss the possible generalizations of the duality to less symmetric cases, including finite temperature and chemical potential and introduce the applications to Condensed Matter theory.

These lectures are a part of the LION Summer School: Modern Physics at all Scales, held in Leiden University 21-31 July 2019.

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1 Introduction

AdS/CFT duality (gauge/string duality, holography) is lying on a forefront of the modern theoretical physics. This deepest concept relies on an unexpected point of view which was provided by string theory and it allows to relate gauge field theories and the theory of gravity in nontrivial way. This duality provides unique tools to address the dynamics of the quantum gauge theories in the regime of strong coupling, where the other approaches fail. On the other hand, it can unveil the properties of the quantum gravity, in case when the dual field theory is accessible to study. Among its applications are the treatment of the physics of hadrons and that of the strongly correlated quantum systems, like high temperature superconductors. Nowadays gauge/string duality is treated as a well developed set of rules which allows multitude of applications in diverse subjects of physics. However, knowledge of the origin and the underlying principles of the duality is crucial in order to use this powerful tool properly. The aim of these notes is to give a hint of these principles and provide a starting point for the education of this spectacular concept in the modern theoretical physics.

2 Gauge fields

2.1 Classical electrodynamics as Abelian gauge field theory

Let's start from the concept of gauge field theory. The familiar theory of electrodynamics is described by the field strength tensor [1]

$$F_{i0} = -F_{0i} = E_i \quad (2.1)$$

$$F_{ij} = \epsilon_{ijk} H_k. \quad (2.2)$$

Its equations of motion – the Maxwell equations, follow from the action principle

$$S = -\frac{1}{4} \int d^4x F_{\mu\nu} F^{\mu\nu}, \quad (2.3)$$

where the dynamical variables are the components of the vector potential

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu. \quad (2.4)$$

The crucial property of the action of electrodynamics is its *gauge symmetry*: the action is symmetric with respect to the *local* (coordinate dependent) transformation of the vector potential of the form

$$A_\mu(x) \rightarrow A_\mu(x) + \partial_\mu \lambda(x). \quad (2.5)$$

The gauge invariance plays a fundamental role in the physics of electromagnetic interaction, since it is tightly related to the conservation of charge and restricts significantly the possible form of the equations of motion. Indeed, how to we couple the shift-symmetric field A_μ to matter, keeping the gauge symmetry intact? For this we need to construct a *charged* matter.

For a simple example consider the complex scalar field Φ with the action [2]

$$S \equiv \int d^4x \mathcal{L} \quad (2.6)$$

$$\mathcal{L} = -\partial^\mu \Phi^\dagger \partial_\mu \Phi - m^2 \Phi \Phi^\dagger + O[(\Phi \Phi^\dagger)^2] \quad (2.7)$$

This action is invariant under global (coordinate independent) phase rotations of the complex scalar

$$\Phi \rightarrow e^{I\alpha} \Phi \quad \delta\Phi = i\delta\alpha \Phi \quad (2.8)$$

$$\Phi^\dagger \rightarrow e^{-I\alpha} \Phi^\dagger \quad \delta\Phi^\dagger = -i\delta\alpha \Phi^\dagger \quad (2.9)$$

This is a $U(1)$ -transformation group: the transformation which keeps an absolute value of a *single* complex number intact. The action (2.6) can only contain the terms which are invariant under this $U(1)$ symmetry. Therefore **the symmetry restricts the possible form of the action.**

What is a physical consequence of the global $U(1)$ symmetry in the action? The equations of motion coming from variation of the action (2.6) are

$$\frac{\delta S}{\delta \Phi^\dagger} = \frac{\partial \mathcal{L}}{\partial \Phi^\dagger} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial \partial_\mu \Phi^\dagger} \right) = 0 \quad (2.10)$$

On the other hand the variation of the Lagrangian reads [2](Ch.22)

$$\delta\mathcal{L} = \frac{\partial\mathcal{L}}{\partial\Phi^\dagger}\delta\Phi^\dagger + \frac{\partial\mathcal{L}}{\partial\partial_\mu\Phi^\dagger}\delta\partial_\mu\Phi^\dagger + c.c. \quad (2.11)$$

Using the equations of motion we rewrite this as

$$\delta\mathcal{L} = \partial_\mu \left(\frac{\delta\mathcal{L}}{\partial(\partial_\mu\Phi^\dagger)}\delta\Phi^\dagger \right) + \frac{\delta S}{\delta\Phi^\dagger}\delta\Phi^\dagger + c.c. \quad (2.12)$$

The term in parenthesis is called *the Noether current*. Its fundamental feature is that since on the solutions to the equations of motion the second term vanishes, in case when the Lagrangian has a *global* symmetry $\delta\mathcal{L}/\delta\alpha = 0$, Noether current is conserved:

$$\partial_\mu j^\mu = 0 \quad (2.13)$$

$$j^\mu = \frac{\delta\mathcal{L}}{\partial(\partial_\mu\Phi^\dagger)} \frac{\delta\Phi^\dagger}{\delta\alpha} + \frac{\delta\mathcal{L}}{\partial(\partial_\mu\Phi)} \frac{\delta\Phi}{\delta\alpha} = i \left(\partial^\mu\Phi\Phi^\dagger - \partial^\mu\Phi^\dagger\Phi \right) \quad (2.14)$$

Rewriting this in terms of space and time components we arrive at the continuity equation for the electromagnetic current, where j^0 is a charge density.

$$\frac{\partial}{\partial t}j^0(x) + \nabla \cdot \mathbf{j}(x) = 0 \quad (2.15)$$

We see that **the global U(1) symmetry describes the matter with electric charge and ensures its conservation.**

Once we have a charged matter, we can couple it to the electromagnetic field. We write

$$S = \int d^4x \left[-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + gA_\mu j^\mu - \partial^\mu\Phi^\dagger\partial_\mu\Phi - m^2\Phi\Phi^\dagger \right], \quad (2.16)$$

with g being the coupling constant. Now when we apply a gauge transformation (2.5), the Lagrangian changes by

$$\delta\mathcal{L}_\lambda = gj^\mu\partial_\mu\lambda, \quad (2.17)$$

however, if we simultaneously perform a *coordinate dependent* $U(1)$ -transformation (2.8) with parameter $\alpha(x)$, then the kinetic term of the scalar field will result in the same extra term

$$\delta\mathcal{L}_\alpha = - \left(\frac{\delta\mathcal{L}}{\partial(\partial_\mu\Phi^\dagger)} \frac{\delta\Phi^\dagger}{\delta\alpha} + \frac{\delta\mathcal{L}}{\partial(\partial_\mu\Phi)} \frac{\delta\Phi}{\delta\alpha} \right) \partial_\mu\alpha(x) \equiv -j^\mu\partial_\mu\alpha(x). \quad (2.18)$$

The action of the electrodynamics coupled to a charged matter remains gauge invariant if the transformations of the gauge field are linked to the simultaneous transformations of the matter field

$$A_\mu \rightarrow A_\mu + g\partial_\mu\lambda(x), \quad \Phi \rightarrow e^{i\lambda(x)}\Phi \quad (2.19)$$

In this way the electrodynamics can be built as **U(1) gauge field theory**.

This whole construction gets even more transparent when one introduces the gauge covariant derivative:

$$D_\mu\Phi = \partial_\mu\Phi + igA_\mu\Phi, \quad (2.20)$$

which conveniently encapsulates the current and kinetic terms of the action

$$D_\mu\Phi D^\mu\Phi^\dagger = gA_\mu j^\mu - \partial^\mu\Phi^\dagger\partial_\mu\Phi \quad (2.21)$$

2.2 Non-Abelian gauge field theories

In the previous section we considered a $U(1)$ gauge field theory. As mentioned before, $U(1)$ is a group of transformation leaving the absolute value of a single complex number intact. The question arises: can we generalize the notion of the gauge field theory to a more complicated groups[1]?

Suppose now the scalar field is not a single complex number, but rather a set of 2 complex numbers with independent phases.

$$\Phi = \begin{pmatrix} \Phi_1 \\ \Phi_2 \end{pmatrix} \quad (2.22)$$

Now the global $U(1)$ transformation (2.8) is substituted by $SU(2)$ – the transformation which keeps the norm of the 2-component vector invariant.

$$\Phi \rightarrow e^{\alpha^1 \sigma_1 + \alpha^2 \sigma_2 + \alpha^3 \sigma_3} \cdot \Phi \quad (2.23)$$

$$\delta \Phi = (\delta \alpha^1 \sigma_1 + \delta \alpha^2 \sigma_2 + \delta \alpha^3 \sigma_3) \cdot \Phi. \quad (2.24)$$

Here σ_i are the Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \quad (2.25)$$

Which form a basis in the $su(2)$ Lie group and $e^{\alpha^i \sigma_i}$ is a generic element of the $SU(2)$ group: a group of 2×2 hermitian matrices with unit determinant.

One says that the 2-component complex vector Φ_i is in a *fundamental* representation of the group $SU(2)$, while the transformation matrix $e^{\alpha^i \sigma_i}$, which has 3 real parameters, is in the *adjoint* representation of the group. In general the $SU(N)$ group has N -dimensional complex valued fundamental representations and $N^2 - 1$ -dimensional real valued adjoint representations. The adjoint representation can also be realized by the hermitian $N \times N$ matrices with unit determinant.

By a direct analogy with the previous considerations (2.19) we deduce the form of the gauge field in this case. Now the gauge field A must acquire values in the adjoint representation of the gauge group, i.e. now the gauge field is a $SU(2)$ matrix, parametrized by the 3 components.

$$(A_\mu)_{\alpha\beta} = A_\mu^i (\sigma_i)_{\alpha\beta}, \quad i = 1 \dots 3, \quad \alpha, \beta = 1 \dots 2. \quad (2.26)$$

When writing a Lagrangian for the gauge field theory with larger gauge groups one has to be careful since the matrix-valued fields don't commute anymore. These theories are called non-Abelian, in contrast to the Abelian $U(1)$ case, which commutes. Nonetheless the Lagrangian turns out to be quite similar to the one we saw previously (2.16).

$$S = \int d^4x \left[-\frac{1}{4} \text{Tr}[F_{\mu\nu} F^{\mu\nu}] - D^\mu \Phi D_\mu \Phi^\dagger - m^2 \Phi \Phi^\dagger \right] \quad (2.27)$$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + g_{YM} [A_\mu, A_\nu] \quad (2.28)$$

$$D_\mu \Phi = \partial_\mu \Phi + g_{YM} A_\mu \Phi \quad (2.29)$$

This action is invariant with respect to non-Abelian gauge transformations (*check*)

$$\Phi \rightarrow \omega \Phi, \quad \omega \equiv e^{\alpha^i \sigma_i} \quad (2.30)$$

$$A_\mu \rightarrow \omega A_\mu \omega^{-1} + \omega \partial_\mu \omega^{-1} \quad (2.31)$$

This field theory with non-Abelian group is often called Yang-Mills gauge field theory. The major difference with the Abelian one is seen in the expression for the gauge field strength, which now contains a nonlinear term. This renders the equations of motion of this theory nonlinear even in case without matter and when the coupling constant is large, it turns into major obstacle, since the perturbation theory is not applicable and the exact solution to the nonlinear equations is absent.

The Yang-Mills $SU(2)$ theory has originally been introduced in order to describe protons and neutrons in the nucleus. Even though that particular approach didn't work, now the Yang-Mills theory is underlying the modern theory of the strong interactions: the Quantum Chromodynamics [3], where the gauge group is $SU(3)$, and 8 different types of gluons couple together 3 colors of quarks. The QCD is known to be particularly hard to solve in the low energy limit, where the coupling constant is large, the quarks are confined and the perturbative treatment fails.

Quite remarkably, it is in the applications to QCD, where the first versions of the String theory have been introduced back in 1960s.

3 Strings

3.1 String theory: open and closed strings

Now we turn to the String theory, which is a conceptual basis of the AdS/CFT correspondence. In a nut shell, the string theory as a theory of 1-dimensional objects propagating in time. While the usual particle is represented by a world line in the space-time, the string spans the two-dimensional surface, extended in both spatial and time directions – the world sheet. There are two types of strings: the open ones with free ends and the closed ones with two ends joined together. The quantum strings have a particular finite tension $\mathcal{T}$, which sets the mass scale of the theory. Due to this tension the strings tend to shrink to zero size (apart from the remaining quantum fluctuations) and the lowest energy states of them behave as point like particles. The consistent theory of super strings¹ can be formulated in 9+1 dimensional target spacetime

Similarly to the usual mechanical string, the quantum strings support the vibrational modes. This is a new crucial feature as compared to the ordinary particle, which doesn't provide enough degrees of freedom for those vibrations. Remarkably the **lowest energy excitations of the closed strings appear as the gravitons** from the point of view of the target space, where the string propagates. On the other hand, the **lowest energy excitation of the open string is a gauge vector particle** [4], see Table 1.

¹This involves a further generalization of the symmetry of the theory: the supersymmetry, which we will not explain here.

	Closed strings	Open strings
Lowest excitation	graviton	gauge vector boson
Effective theory	Gravity	Gauge field theory

Table 1: Low energy excitations in the string theory.

An important aspect is related to the open string. Unlike a closed one, **the open string has to end somewhere**. The end points of the open strings can be attached to the additional extended multidimensional objects of the string theory: *the D-branes*. One can think of a D-brane as a multidimensional analogue of the plane in 3 dimensions. A plane has 2 internal directions and one direction in the space is perpendicular to it. Similarly, a D3-brane is an object with $3 + 1$ dimensional world volume and 6 directions transverse to it. If one end of an open string is attached to the D3 brane, it can propagate along the $3 + 1$ internal directions, but it can not leave the brane and move along the 6 transverse directions. **An open string with both ends attached to the brane has a lowest energy excitation which looks like a gauge vector particle living in the world volume of this brane.**

3.2 Two points of view on a stack of D3-branes

It gets even more interesting when we consider a stack of coincident N D3 branes [5, 6]. In order to keep track on which end of the string is attached to which particular brane, one has to introduce an additional index for each end – the Chan-Paton factor. The open string on a stack of N branes carries two such indices, ranging from 1 to N . These indices are inherited by the gauge vector particle as well and we recognize the familiar structure of the non-Abelian $SU(N)$ gauge potential (c.f. (2.26)), which carries two indices in the fundamental representation, which are equivalent to one adjoint. Hence we arrive to a crucial finding: **the open strings attached to a stack of N D3-branes realize the $SU(N)$ Yang-Mills gauge field theory as their low energy effective theory.**

However, if N is large enough, one can look at the stack of D3-branes from a completely different point of view. The D3-branes, much like the strings themselves, have a finite tension. Therefore they carry the energy density and if one puts many of them in the same place, they will affect the curvature of the outer $9+1$ dimensional space, in complete analogy to the usual black hole, sourced by a point-like mass sitting in its center. The only difference is the dimensionality of the outer space and the extra dimensions of the branes themselves. The resulting metric of the curved 10-dimensional space, affected by the stack of D_p branes is [6, 7].

$$ds^2 = H_p^{-1/2} dx \cdot dx + H_p^{1/2} dy \cdot dy, \quad H_p(r) = 1 + \left(\frac{r_p}{r}\right)^{7-p}, \quad (3.1)$$

where in our case we will focus of $p = 3$, $dx \cdot dx$ is the $(3+1)$ -dimensional metric along the world volume of the branes and $dy \cdot dy = dr^2 + r^2 d\Omega_5^2$ is the metric of the 6 perpendicular directions (r being the radial distance to the stack in this transverse space).

Let us focus on the near horizon asymptotic form of the D3-brane metric (3.1). We set $r_3 = R$ and consider $r \rightarrow 0$. In this case the metric reads

$$ds^2 \sim (r/R)^2 dx \cdot dx + (R/r)^2 dr^2 + R^2 d\Omega_5^2. \quad (3.2)$$

Changing the variables $z = R^2/r$ it acquires the form

$$ds^2 \sim R^2 \frac{dx \cdot dx + dz^2}{z^2} + R^2 d\Omega_5^2. \quad (3.3)$$

The second part of the metric is 5-dimensional sphere S_5 in transverse directions to the stack of the branes, while the first part is a metric of the 5-dimensional anti-de-Sitter space, the multidimensional analogue of a hyperbolic plane. The $z = 0$ is the asymptotic boundary of this hyperbolic space, which is isomorphic to 3+1 dimensional Minkowski space where the gauge theory is defined. It is usually stated that the gauge theory lives “on the boundary”, while gravity lives in the “bulk” of the AdS.

Now we recall that the gravitational background is the coherent collection of the gravitons – the low energy excitations of the *closed* strings. At this point we arrived to the second part of the correspondence: **the stack of the D3 branes can be seen from the point of view of the closed strings as the source producing the gravitational $\text{AdS}_5 \times S_5$ background.**

4 AdS/CFT

We just showed that the same object: the stack of N D3-branes can be described either from the point of view of the closed strings as the $SU(N)$ Yang-Mills theory, or, from the point of view of the closed strings, as the theory of gravity in the $\text{AdS}_5 \times S_5$ background. The AdS/CFT correspondence states that these theories are equivalent, since they describe the same object in string theory. Let us take a look on how this correspondence work.

4.1 Symmetries

The most powerful matching principle behind the correspondence is again the principle of the symmetries. The global symmetries in the gauge field theory correspond to the isometries of the theory of gravity [6].

So far we skipped completely the subject of a supersymmetry, which is beyond the scope of the current overview, however for completeness we have to mention that all the theories participating in the correspondence are actually extended by the supersymmetry. In case of the Yang Mills theory of the branes, it has 4 supercharges. These 4 supercharges form $SU(4)$ global symmetry group, which is isometric to $SO(6)$ – the group of rotations in 6-dimensional space. The rotations of a unit vector in 6D is in turn equivalent to translations of its endpoint along the 5-dimensional sphere. Therefore we arrive to the conclusion that **the supersymmetry of the gauge theory is reflected in the S_5 part of the geometry on the gravity side of the correspondence.**

Another feature of the gauge theory on the boundary is its scale invariance. The gauge symmetry and the supersymmetry turn out to be so restrictive, that it is impossible to

introduce any dimensionful parameter in the theory, without breaking the symmetries. The scale invariance together with the Lorentz invariance on the brane form conformal symmetry group (hence the name: conformal field theory). The scale invariance is also reflected in the geometry of the AdS space. Consider the metric

$$ds^2 = \frac{1}{z^2} (dx^\mu dx_\mu + dz^2). \quad (4.1)$$

The scale transformation in the boundary consists of rescaling all the coordinates

$$x_\mu \rightarrow \lambda x_\mu \quad (4.2)$$

However, this can be *undone* in the bulk, by rescaling the holographic coordinate $z \rightarrow \lambda z$, due to the denominator in the metric. After this rescaling the metric returns to its original shape, therefore **the scale (conformal) invariance is the isometry of the AdS space.**

4.2 Coupling constant: gauge/gravity duality

The crucial and the most useful feature of the correspondence is that it allows to treat the strongly coupled gauge theory by means of the weakly curved gravity and vice versa the weakly coupled field theory allows one to get insight over the quantum gravity [6, 7]. The coupling constants in the Yang-Mills and string theory are proportional:

$$g_{YM}^2 = 4\pi g_s, \quad (4.3)$$

however the effective coupling constant, which enters the leading diagrams of the Yang-Mills is the so called 't Hooft coupling

$$\lambda = g_{YM} N^2. \quad (4.4)$$

When N is large, the 't Hooft constant can be large even if g_{YM} is small, giving the weakly coupled string theory on the dual side.

The curvature radius of both sphere and AdS in (3.3) is also related to N , roughly because the curvature is supported by the number of D3 branes:

$$R = \lambda^{1/4} l_s, \quad (4.5)$$

where l_s is the quantum string length scale. **The gravity in the bulk is weakly curved as long as $R \gg l_s$, meaning $\lambda \gg 1$. On the other hand the quantum corrections to the gravity treatment are suppressed as long as $g_s \sim \lambda^2/N^4 \ll 1$, meaning $N \gg 1$.**

4.3 Breaking symmetries: applications

Having this powerful tool at hand one would like to apply the gauge/gravity duality to many unsolved problems in physics, which were previously inaccessible due to the strong coupling constant in the corresponding gauge field theory. One example is QCD, which was mentioned already, the other one in the strongly correlated quantum systems in Condensed

matter [8]. In both case before applying the duality one has to get rid of the excessive amount of symmetry, which is present in the stringy construction, but will be broken in the applications. For instance QCD is not supersymmetric and Condensed Matter systems break Lorentz invariance by finite temperature, chemical potential and the crystal lattice. Construction of such type of models is extremely hard from the string theory “top-down” point of view. However keeping in mind the general principles, one can try to construct the dual backgrounds “bottom-up” with desired features and produce the duality inspired phenomenological models in this way. Let’s discuss briefly how these holographic models can be constructed.

In the previous section we identified the S_5 part of the gravity background as corresponding to the supersymmetry. Therefore in order to break it in the holographic construction it is enough to choose a point on this sphere and completely suppress the dynamics along these directions. Therefore for the non-supersymmetric boundary theories in 3+1 dimensional Minkowsky space-time one is left with 4+1 dimensional bulk AdS spacetime with only one extra dimension (4.1). It gets particularly clear now why these models are called holographic.

The other generalization is the introduction of finite temperature in the boundary. From the geometrical point of view the finite temperature can be seen as the compactification of the Euclidean time direction. Since the bulk and boundary theories share the same time coordinate, the gravitational dual to the thermal field theory would have a compactified time at asymptotic infinity as well. Those familiar with the Hawking radiation would immediately recognize the gravitational solution which has a compact Euclidean time direction at infinity: this is a black hole. In complete analogy **the gravitational dual to the thermal field theory is a black hole in AdS space-time**

$$ds^2 = \frac{1}{z^2} \left(-f(z)dt^2 + \frac{dz^2}{f(z)} + dx_i^2 \right), \quad f(z) = 1 - (z/z_0)^4 \quad (4.6)$$

Here the temperature is related to the horizon radius as

$$\frac{1}{T} = \pi z_0. \quad (4.7)$$

One can show furthermore[8] that the finite chemical potential on the boundary would correspond to the charge of the dual black hole, turning it into AdS–Reissner-Nordstrom solution. In case one wishes to introduce the effects of the crystal lattice, one can consider the spatially modulated chemical potential, which produces the spatially modulated black holes as the solutions to the corresponding Einstein equations. In this way **the AdS/CFT duality relates the physics of the strongly correlated quantum materials to the physics of black holes in the curved auxiliary spacetime.**

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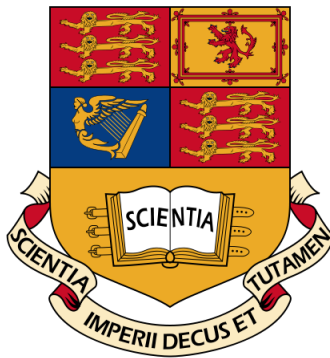
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The AdS-CFT Correspondence: *A Review*

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September 2014, Imperial College London

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Submitted in partial fulfilment of the requirements for the degree of Master of Science of Imperial
College London

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To Vlad

Abstract

This paper reviews the AdS/CFT Correspondence, introduced by Juan Maldacena in 1997, relating Type IIB string theory on $AdS_5 \times S^5$ to $\mathcal{N} = 4$ Super Yang-Mills theory in four dimensions. We provide a comprehensive study of the fundamental aspects of the conjecture, including: supersymmetry, conformal symmetry, geometry of AdS space, and string theory. By looking at a configuration of parallel D3-branes we are able to realise the correspondence and relate the dynamics of the bulk string theory to two- and three-point correlation functions within Super Yang-Mills theory.

1 Introduction

When string theory was introduced in the 1960s as a theory to describe the strong force, it was plagued by serious pitfalls, and had to be abandoned in favour of the formidable QCD. However, one of the problems, a massless spin-two particle, revealed itself to be the graviton and allowed the theory to rise from the ashes as a consistent description of quantum gravity. It was not until thirty years later that its intentions have come full-circle - string theory and the strong force appear to be dual. In 1997, this conjecture was brought to the world's attention by Maldacena[16]. Focussing on extended objects within string theory, known as D-branes, he was able to use Type IIB string theory propagating on an $AdS_5 \times S^5$ background to describe Super Yang-Mills, a conformal theory living in four-dimensional flat space. This was dubbed the *AdS/CFT Correspondence*. The correspondence focuses on the principle of holography, where the information contained within a $(d + 1)$ -dimensional gravity theory can be described completely by a d -dimensional field theory living at the boundary. Through this, we can relate the observables of both theories - in particular, the fields propagating in the bulk of string theory and the correlation functions of Super Yang-Mills. The correspondence remains a conjecture as this gauge-gravity duality is an example of a strong-weak duality, where the valid regime for the gauge theory is not reliable to describe the string theory, and vice versa, rendering it very difficult to prove. Despite this, the duality allows us to perform strong coupling calculations in Super Yang-Mills that would be very difficult otherwise[1]. Although this isn't a realistic theory, the fundamental ideas are important in the understanding of strongly-coupled quantum field theories.

The correspondence draws upon many areas of physics, and this paper aims to explore each of these components in order to provide the reader with a firm background understanding and alluring relations. We begin by looking at supersymmetry, important as both Super-Yang Mills and string theory are supersymmetric. By studying the structure of the algebra and representations, we are able to find the content of the $\mathcal{N} = 4$ Super Yang-Mills theory, which is invariant under the conformal and superconformal algebras seen in Section 3. Following this, we look to the stringy side, studying the geometrical properties of AdS , before moving onto the spectrum of string theory in Section 5. Here we introduce the D-branes by looking at them as endpoints of open strings, and geometrically engineer the configuration of branes to give rise to an $SU(N)$ gauge theory on their worldvolume. The second viewpoint is introduced in Section 6, looking at the branes as solutions of ten-dimensional supergravity, whose presence deforms the geometry nearby to look like $AdS \times S^5$. Finally, everything is brought together in the form of the correspondence in section 7. By looking at the limits of validity and dynamics of string theory, we are able to derive the correlation functions of Super Yang-Mills from the fields propagating in the bulk.

2 Supersymmetry and $\mathcal{N} = 4$ Super Yang-Mills

Physicists strive for more by looking for less, forcing separate parts to fit inside one whole. The Poincaré Group, our beloved spacetime symmetry group, was thought to be exempt from closing into an algebra alongside the internal symmetries by the Coleman-Mandula theorem, where they declared in their 1967 paper that there was no way to combine spacetime and internal symmetries in anything but a trivial way[7]. However, not content with this, Haag, Lopuszanski and Sohnius found

that we could include spacetime symmetries if we allowed both commutators and anticommutators in the algebra[11]. Thus, we could make a fermionic extension to the Poincaré group known as the Super-Poincaré group, whose algebra we discuss below.

Why is supersymmetry important? It can be used as a potential answer to some of the major obstacles seen today in theoretical research:

- it can provide an answer to the hierarchy problem, where fundamental masses do not equal experimental masses
- it can be used to extend the standard model, with aim to combine $U(1) \times SU(2) \times SU(3)$ into a larger symmetry group, such as $SU(5)$
- extensions of the standard model include SUSY, such as dark matter and string theory.

One of the consequences of supersymmetry (SUSY) is an equivalence of mass between bosons and fermions. Of course, this isn't seen in nature, which means it is a symmetry that must be broken at suitably low energies. Attention was turned to finding SUSY at the LHC, and the failure to do so yet has led to many people abandoning the theory. However, for the course of this paper we hold strong our faith. Below we discuss the SUSY algebra and its representations, leading us to $\mathcal{N} = 4$ Super Yang-Mills theory.

2.1 Supersymmetry Algebra

The SUSY algebra is a fermionic extension of the Poincaré, introducing fermionic generators Q_α and $\bar{Q}_{\dot{\alpha}}$, transforming as spinors in the $(\frac{1}{2}, 0)$ and $(0, \frac{1}{2})$ representations respectively. These generators act on bosons and fermions, flipping their nature, and are key within SUSY as they allow for both to be placed on a par. The symmetry group enjoys what is known as a graded Lie algebra, due to its inclusion of both bosons and fermions. We can define the algebra vector space V spanning $V = V_B \oplus V_F$, with V_B the bosonic vector space and V_F the fermionic vector space. The composition law is given as:

$$[, \} = V \times V \rightarrow V$$

$$\begin{aligned} [V_B, V_B] &= [V_B, V_B] \in V_B \\ [V_B, V_F] &= [V_B, V_F] \in V_F \\ [V_F, V_F] &= \{V_F, V_F\} \in V_B \end{aligned}$$

This forms the super-Poincaré algebra, and it is generated by

- Lorentz Transformations $M_{\mu\nu}$
- Translations P_μ
- Supercharges $Q_\alpha, \bar{Q}_{\dot{\alpha}}$

The next step is to look at the underlying structure of the SUSY algebra. This is important as we need to see what the Casimirs of the group are, and thus how we may label and build the irreducible representations (irreps). The simplest case is $\mathcal{N} = 1$ SUSY, where we have only one copy of each supercharge. We also focus on the four-dimensional representations, as the end goal is to be able to represent four-dimensional Super Yang-Mills theory.

$$[\mathbf{Q}_\alpha, \mathbf{M}_{\mu\nu}]$$

The supercharges are spinors, so transform as a spinor under the Lorentz group. Infinitesimally the transformation reads

$$Q_\alpha \rightarrow Q'_\alpha = \exp(-\frac{1}{2}\omega^{\mu\nu}\sigma_{\mu\nu})^\beta_\alpha Q_\beta \simeq (\mathbb{I} - \frac{i}{2}\omega^{\mu\nu}\sigma_{\mu\nu})^\beta_\alpha Q_\beta.$$

They are also operators, with transformation

$$Q'_\alpha = U^\dagger Q_\alpha U \simeq (1 + \frac{i}{2}\omega^{\mu\nu}M_{\mu\nu})Q_\alpha(1 - \frac{i}{2}\omega^{\mu\nu}M_{\mu\nu}).$$

Comparing the RHS from both equations, we find

$$-\frac{i}{2}\omega^{\mu\nu}(\sigma_{\mu\nu})^\beta_\alpha Q_\beta = -\frac{i}{2}\omega^{\mu\nu}(Q_\alpha M_{\mu\nu} - M_{\mu\nu}Q_\alpha),$$

giving the commutator between the Lorentz generator and supercharges as $[Q_\alpha, M_{\mu\nu}] = (\sigma_{\mu\nu})^\beta_\alpha Q_\beta$. Comparing to the structure of the Lie algebra above, it's as expected: the commutation relation between a boson and fermion gives a fermion.

$$[\mathbf{Q}_\alpha, \mathbf{P}_\mu]$$

From the structure of the Lie algebra, the result of this commutator must be a fermionic generator. We can also look at the indices and deduce that there must be one free spinor and one Lorentz index. The only object that we have at our disposal is the Pauli matrix $\sigma^\mu_{\alpha\dot{\alpha}}$ and we make the ansatz

$$[Q_\alpha, P_\mu] = c(\sigma_\mu)_{\alpha\dot{\alpha}}\bar{Q}^{\dot{\alpha}},$$

where the Pauli matrix changes the handedness of the spinor. The constant c is still undetermined - just as a regular algebra respects the Jacobi identity, our super-algebra respects the super-Jacobi identity[4], whose form follows from the composition law above. For the case of P^μ and Q_α :

$$[P_\mu, [Q_\alpha, P_\nu]] + [P_\nu, [P_\mu, Q_\alpha]] + [Q_\alpha, [P_\nu, P_\mu]] = 0.$$

The third term vanishes as translations commute, leaving

$$\begin{aligned} [P_\mu, c(\sigma_\nu)_{\alpha\dot{\alpha}}\bar{Q}^{\dot{\alpha}}] - [P_\nu, c(\sigma_\mu)_{\alpha\dot{\alpha}}\bar{Q}^{\dot{\alpha}}] &= cc * [(\sigma_\nu)_{\alpha\dot{\alpha}}(\sigma_\mu)^{\dot{\alpha}\beta}Q_\beta - (\sigma_\mu)_{\alpha\dot{\alpha}}(\sigma_\nu)^{\dot{\alpha}\beta}Q_\beta], \\ &= cc * (\sigma_\nu\sigma_\mu - \sigma_\mu\sigma_\nu)^\beta_\alpha Q_\beta, \\ &= 0. \end{aligned}$$

The term within the brackets satisfies the Dirac algebra and is non-zero, implying that the constant $c = 0$, and $[Q_\alpha, P_\mu] = 0$.

$$\{\mathbf{Q}_\alpha, \mathbf{Q}_\beta\}$$

Using a similar method as above, we postulate that we need two free spinor indices on the RHS. Because of the Coleman-Mandula theorem, the only generator that can be placed on the RHS is $M_{\mu\nu}$ [4], so that $\{Q_\alpha, Q_\beta\} = k\sigma^{\mu\nu}_{\alpha\beta}M_{\mu\nu}$, where $\sigma^{\mu\nu} = \frac{i}{4}(\sigma^\mu\bar{\sigma}^\nu - \sigma^\nu\bar{\sigma}^\mu)$. As translations and Lorentz transformations do not commute, by the super-Jacobi identity this anticommutator must vanish.

$$\{Q_\alpha, \bar{Q}_{\dot{\alpha}}\}$$

Recalling that the supercharges $Q_\alpha, \bar{Q}_{\dot{\alpha}}$ transform in the left and right hand representations respectively, the product of the commutator should be in the $(\frac{1}{2}, \frac{1}{2})$ representation, i.e. a Lorentz vector. Within our algebra, the object we have is P_μ , and again we can look at the required indices to find

$$\{Q_\alpha, \bar{Q}_{\dot{\alpha}}\} = 2\sigma_{\alpha\dot{\alpha}}^\mu P_\mu, \quad (1)$$

with 2 being a canonical choice[19]. This is a very interesting outcome, two SUSY transformations results in a translation in spacetime. SUSY is infact a spacetime symmetry; it knows all about spacetime!

2.2 Consequences of the Supersymmetry algebra

The structure of the super-Poincaré algebra encodes three main important features of supersymmetry. These are[30]:

- Positivity of Energy
- Equal Number of Bosons and Fermions
- Equal Masses for Bosons and Fermions

2.2.1 Positivity of Energy

Beginning with the identity (1)

$$\bar{\sigma}^{\nu\alpha\dot{\beta}}\{Q_\alpha, \bar{Q}_{\dot{\beta}}\} = 2\bar{\sigma}^{\nu\alpha\dot{\beta}}\sigma_{\alpha\dot{\beta}}^\mu P_\mu,$$

we then use the Dirac algebra to turn this into a metric

$$\bar{\sigma}^{\nu\alpha\dot{\beta}}\sigma_{\alpha\dot{\beta}}^\mu = Tr(\bar{\sigma}^\nu \sigma^\mu) = \frac{1}{2}Tr(\bar{\sigma}^\nu \sigma^\mu - \bar{\sigma}^\mu \sigma^\nu) = \frac{1}{2}Tr(\eta^{\nu\mu}\mathbb{I}) = 2\eta^{\nu\mu}.$$

Inserting this into the above equation

$$\bar{\sigma}^{\nu\alpha\dot{\beta}}\{Q_\alpha, \bar{Q}_{\dot{\beta}}\} = 4\eta^{\nu\mu}P_\mu = P^\nu. \quad (2)$$

We define energy to be $P^0 = E$, and can assess the expectation value of an arbitrary state $|\psi\rangle$ to be in this energy state E . We also note that $\bar{Q}_{\dot{\alpha}} = (Q_\alpha)^\dagger$, i.e. the Hermitian conjugate.

$$\langle\psi|E|\psi\rangle = \frac{1}{4}\sum_{\alpha}(\langle\psi|Q_\alpha Q_\alpha^\dagger|\psi\rangle + \langle\psi|Q_\alpha^\dagger Q_\alpha|\psi\rangle) \quad (3)$$

$$= \frac{1}{4}\sum_{\alpha}(|Q_\alpha^\dagger|\psi\rangle|^2 + |Q_\alpha|\psi\rangle|^2) \geq 0, \quad (4)$$

by the positivity of the Hilbert Space. The energy of any non-vacuum state is positive-definite, and the vanishing of the vacuum state is a necessary condition for the existence of a unique vacuum.

2.2.2 Equal Number of Bosons and Fermions

Defining a fermionic number operator $\nu_F = (-1)^{N_F}$, where

$$\begin{aligned} N_F |B\rangle &= (\text{even number}) |B\rangle \\ N_F |F\rangle &= (\text{odd number}) |F\rangle, \end{aligned}$$

gives

$$\nu_F = \begin{cases} +1 & \text{bosonic state} \\ -1 & \text{fermionic state} \end{cases}$$

This operator commutes with the supercharges. Using the SUSY identity and taking the trace

$$\begin{aligned} \text{Tr}(\nu_F \{Q_\alpha, \bar{Q}_{\dot{\alpha}}\}) &= \text{Tr}(\nu_F Q_\alpha \bar{Q}_{\dot{\alpha}} + \nu_F \bar{Q}_{\dot{\alpha}} Q_\alpha) \\ &= \text{Tr}(-Q_\alpha \nu_F \bar{Q}_{\dot{\alpha}} + Q_\alpha \nu_F \bar{Q}_{\dot{\alpha}}) \\ &= 0. \end{aligned}$$

where we've made use of the cycle property of the trace. Looking at the RHS of $\nu_F \{Q_\alpha, \bar{Q}_{\dot{\alpha}}\} = 2\nu_F \sigma_{\alpha\dot{\alpha}}^\mu P_\mu$, this implies that $\text{Tr} \nu_F = 0$. The trace of the fermionic operator can be expressed as

$$\begin{aligned} \text{Tr} \nu_F &= \sum_{\text{bosons}} \langle B | \nu_F | B \rangle + \sum_{\text{fermions}} \langle F | \nu_F | F \rangle \\ &= \sum_{\text{bosons}} \langle B | B \rangle - \sum_{\text{fermions}} \langle F | F \rangle. \\ &= 0 \end{aligned}$$

Thus, for a given representation, known as a multiplet, the number of bosons and fermions will be the same; we must have the same degrees of freedom for either within the same multiplet.

$$\sum_{\text{bosons}} \langle B | B \rangle = \sum_{\text{fermions}} \langle F | F \rangle. \quad (5)$$

2.2.3 Equal Masses

The mass operator $P_\mu P^\mu = P^2 = -\mathcal{M}^2$ is a Casimir of the super-Poincaré group, and the mass of particles in any given representation of SUSY is the same[30]. As a Casimir, P^2 commutes with all other generators of the algebra. We consider the supercharge Q_α , which transforms bosons into fermions and vice versa, i.e. for bosonic state $|B\rangle$ and fermionic state $|F\rangle$

$$Q_\alpha |B\rangle = |F\rangle.$$

We have $[P^2, Q_\alpha] = 0$, giving

$$\begin{aligned} [P^2, Q_\alpha] |B\rangle &= 0 \\ P^2 Q_\alpha |B\rangle - Q_\alpha P^2 |B\rangle &= 0 \\ P^2 |F\rangle - m_F^2 |B\rangle &= 0 \\ m_F^2 |F\rangle - m_B^2 |B\rangle &= 0 \end{aligned}$$

SUSY thus says that the mass of the fermions must be equal to the mass of the bosons in a given representation

$$m_F^2 = m_B^2. \quad (6)$$

Of course, this symmetry isn't realised in nature, so must be spontaneously broken at low enough energies. One may also ask about the spin of the particles, do we have the same spin in a given representation? The Poincaré group has $W^2 = W^\mu W_\mu$ as a Casimir, where W_μ is the Pauli-Lubanski vector that describes spin, and thus by Wigner's classification it is possible to label a representation by its spin. However, it is not a Casimir of the super-Poincaré group[5], and thus different particles within the same representation will have different spins.

2.3 Representations of Supersymmetry

Using the analysis and consequences from the previous section, the representations of SUSY may be built. In the Poincaré group, we classify particles by their properties. The super-Poincaré group is a larger group, but still contains the Poincaré group as a subgroup. Using Wigner's classification, we can define a superparticle, which will be a collection of particles, known as a *multiplet*[5]. Initially, the simplest case of $\mathcal{N} = 1$ is focused upon, before making the natural progression to $\mathcal{N} \geq 1$, known as extended SUSY.

2.3.1 $\mathcal{N} = 1$ Supersymmetry

Wigner's Classification may be used to find the unitary irreducible representations (irreps) of SUSY, adopting the usual method of moving into the rest frame by setting $P^2 = 0$. For the purposes of this paper, we focus on the massless representations, with rest frame momentum $P^\mu = (E, 0, 0, E)$. Once again, using equation (1) gives

$$\{Q_\alpha, \bar{Q}_{\dot{\alpha}}\} = 2E \left[\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \right]_{\alpha\dot{\alpha}} = 4E \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}_{\alpha\dot{\alpha}}.$$

From the positivity of energy, we see that half of the operators are trivially realised. The remaining half can be used to suggestively define raising and lowering operators

$$a = \frac{1}{\sqrt{4E}} Q_\alpha \quad a^\dagger = \frac{1}{\sqrt{4E}} \bar{Q}_{\dot{\alpha}},$$

analogous to those found in quantum theory, where $\{a, a^\dagger\} = \mathbb{I}$. Different particles within the same representation will have different spin (or helicity for the massless case), and we can use these eigenvalues as a basis to create the multiplet of particles. The spin generators are defined by $J_i = \epsilon_{ijk} M_{jk}$. Choosing arbitrary $J_3 = M_{12}$ and using the SUSY algebra

$$\begin{aligned} [M_{12}, Q_2] &= i(\sigma_{12})_2^2 Q_2 \\ &= \frac{1}{2}(\sigma_3)_2^2 Q_2 \\ &= -\frac{1}{2} Q_2, \end{aligned}$$

indicates that the supercharge lowers the helicity by half. Here we used $\sigma_{12} = \frac{1}{4}(\sigma_1 \bar{\sigma}_2 - \sigma_2 \bar{\sigma}_1) = -\frac{i}{2}\sigma_3$. Similarly, $\bar{Q}_{\dot{\alpha}}$ raises the helicity by half; these are the building blocks of the representations. We define a vacuum state $|\lambda_0\rangle$, known as the Clifford vacuum, such that it is annihilated by the supercharge Q_α

$$Q_\alpha |\lambda_0\rangle = 0. \quad (7)$$

The eigenvalue of this state, λ_0 , is the state of minimal helicity, upon which we build our representations. For $\mathcal{N} = 1$ SUSY we only have one copy of the supercharges, and thus may only have two states in the representation

$$\begin{aligned} & |\lambda_0\rangle \\ \bar{Q}_{\dot{\alpha}} |\lambda_0\rangle & \equiv |\lambda_0 + \frac{1}{2}\rangle, \end{aligned}$$

differing by $\frac{1}{2}$ in helicity. Dependent on the choice of minimal helicity, we can have various multiplets, including:

- $\lambda_0 = 0$: within this multiplet lies a scalar and fermion of helicity $+\frac{1}{2}$, and is known as the *chiral multiplet*.
- $\lambda_0 = \frac{1}{2}$: within this multiplet lies a fermion and vector field, and is known as the *vector multiplet*. Particles within the same representation must transform similarly, and as this multiplet contains the vector field, it transforms in the adjoint representation.
- $\lambda_0 = \frac{3}{2}$: within this multiplet we have a gravitino and graviton, and is known as the *gravity multiplet*.

We're restricted to a maximum helicity of 2, as there is no known consistent local interacting theory that contains spins greater than this. Finally, we must consider the CPT invariance of the multiplet. Under this symmetry, the helicity changes sign, and therefore it would be broken without a symmetric distribution of helicity around 0[5]. We know that this symmetry isn't broken in nature, and must include the CPT conjugate multiplet to give

$$\begin{aligned} & \left(0, \frac{1}{2}\right) \oplus \left(-\frac{1}{2}, 0\right) \\ & \left(\frac{1}{2}, 1\right) \oplus \left(-1, \frac{1}{2}\right) \\ & \left(\frac{3}{2}, 2\right) \oplus \left(-2, -\frac{3}{2}\right) \end{aligned} \tag{8}$$

2.3.2 Extended Supersymmetry

We now turn our attentions to extended $\mathcal{N}$ SUSY, where we have more than one copy of a supercharge Q_{α}^I , $I = 1, \dots, \mathcal{N}$. Using the previous method, we can define the Clifford vacuum $|\lambda_0\rangle$, and build representations using the annihilation and creation operators

$$\begin{aligned} a^I &= \frac{1}{\sqrt{4E}} Q_{\alpha}^I \\ a^{I\dagger} &= \frac{1}{\sqrt{4E}} \bar{Q}_{\dot{\alpha}}^I. \end{aligned}$$

Because there are $\mathcal{N}$ copies, we can build upon the Clifford vacuum in various ways

$$\begin{aligned}
a^{1\dagger} |\lambda_0\rangle &= |\lambda_0 + \frac{1}{2}\rangle^1 \\
a^{2\dagger} a^{1\dagger} |\lambda_0\rangle &= |\lambda_0 + 1\rangle^{12} \\
&\dots \\
a^{N\dagger} \dots a^{1\dagger} |\lambda_0\rangle &= |\lambda_0 + \frac{\mathcal{N}}{2}\rangle^{1\dots\mathcal{N}}.
\end{aligned}$$

The operators are antisymmetric in these indices, and for a given helicity $\lambda_0 + \frac{k}{2}$ there are $\binom{\mathcal{N}}{k}$ options[4]. The total number of states is therefore given by the binomial formula

$$(x+y)^\mathcal{N} = \sum_{k=0}^{\mathcal{N}} \binom{\mathcal{N}}{k} x^{\mathcal{N}-k} y^k,$$

applying this to our case

$$2^\mathcal{N} = (1+1)^\mathcal{N} = \sum_{k=0}^{\mathcal{N}} \binom{\mathcal{N}}{k} 1^{\mathcal{N}-k} 1^k = \sum_{k=0}^{\mathcal{N}} \binom{\mathcal{N}}{k}, \quad (9)$$

gives a total of $2^\mathcal{N}$ states in a representation, and by the consequence of SUSY, $2^{\mathcal{N}-1}$ are bosons and the other $2^{\mathcal{N}-1}$ are fermions. What multiplets do we have for $\mathcal{N} > 1$ SUSY?

2.3.3 $\mathcal{N} = 2$ Representations

We have four possible representations:

- $\lambda_0 = -\frac{1}{2} \rightarrow (-\frac{1}{2}, 0, 0, \frac{1}{2}) \oplus (-\frac{1}{2}, 0, 0, \frac{1}{2})$: the *hypermultiplet*, containing two Weyl fermions and two complex scalars
- $\lambda_0 = 0 \rightarrow (0, \frac{1}{2}, \frac{1}{2}, 1) \oplus (-1, -\frac{1}{2}, -\frac{1}{2}, 0)$: the *gauge* or *vector multiplet*, containing one vector, two Weyl fermions, and a complex scalar
- $\lambda_0 = \frac{1}{2} \rightarrow (\frac{1}{2}, 1, 1, \frac{3}{2}) \oplus (-\frac{3}{2}, -1, -1, -\frac{1}{2})$: the *gravitino multiplet*, containing one gravitino, two vectors, and a Weyl fermion.
- $\lambda_0 = 1 \rightarrow (1, \frac{3}{2}, \frac{3}{2}, 2) \oplus (-2, -\frac{3}{2}, -\frac{3}{2}, -1)$: the *graviton multiplet*, containing one graviton, two gravitini and a vector.

2.3.4 $\mathcal{N} = 4$ Representations

Here we have only one possible representation, the $\mathcal{N} = 4$ *gauge multiplet*, which contains the following

λ	-1	$-\frac{1}{2}$	0	$\frac{1}{2}$	1
#	1	4	6	4	1

(10)

This multiplet is known as the $\mathcal{N} = 4$ Super Yang-Mills multiplet[8].

2.4 $\mathcal{N} = 4$ Super Yang-Mills Theory

$\mathcal{N} = 4$ Super Yang-Mills Theory is a non-Abelian theory - as all fields are within the same multiplet, they cannot escape the gauge field, and must all transform within the adjoint representation, with gauge group $SU(N)$.

$$\mathcal{L} = \text{tr} \left\{ \frac{1}{2g^2} F_{\mu\nu} F^{\mu\nu} + \frac{\theta_I}{8\pi^2} F_{\mu\nu} \tilde{F}^{\mu\nu} - \sum_a i \bar{\lambda}^a \bar{\sigma}^i D_\mu \lambda_a - \sum_i D_\mu X^i D^\mu X^i + \sum_{a,b,i} g C_i^{ab} \lambda_a [X^i, \lambda_b] + \sum_{abi} g \bar{C}_{iab} \bar{\lambda}^a [X^i, \bar{\lambda}_b] + \sum_{abi} g \bar{C}_{iab} \bar{\lambda}^a [X^i, \bar{\lambda}^b] + \frac{g^2}{2} \sum_{ij} [X^i, X^j]^2 \right\}, \quad (11)$$

where D_μ is just the usual gauge covariant derivative, of the form $D_\mu = \partial_\mu + i[A_\mu, \cdot]$. Here, $\tilde{F}_{\mu\nu}$ is the hodge dual of the field strength $\tilde{F}_{\mu\nu} = \epsilon_{\mu\nu\rho\sigma} F^{\rho\sigma}$, and appears in the Lagrangian as an instanton term, controlled by the instanton angle θ_I [3], which under $\theta_I \rightarrow \theta_I + 2\pi$ remains invariant. The number of degrees of freedom in the table denotes the transformation properties of the fields under the $SU(4)_R$ symmetry: the vector A_μ transforms in the singlet representation $\underline{1}$, the fermions λ_α in the (anti)fundamental representations $\underline{4}, \bar{\underline{4}}$, and the scalars X_i in the adjoint representation $\underline{6}$. The algebra of $SU(4)$ is given by the Dynkin Diagram A_3



Figure 1: Dynkin diagram A_3 representing the Lie algebra of $SU(4)$

whose representations can be classified in terms of Dynkin labels $[n_1, n_2, n_3]$. The dimensions of the representations are given by Weyl's dimension formula:

$$\dim[n_1, n_2, n_3] = \frac{1}{12} (n_1 + 1)(n_2 + 1)(n_3 + 1)(n_1 + n_2 + 2)(n_2 + n_3 + 2)(n_1 + n_2 + n_3 + 3) \quad (12)$$

- The scalars are represented by $[010]$
- The fermions are represented by $[001]$

The transformations under SUSY are given as

$$\begin{aligned} \delta X^i &= [Q_\alpha^a, X^i] = C^{iab} \lambda_{\alpha b} \\ \delta \lambda_b &= \{Q_\alpha^a, \lambda_{\beta b}\} = F_{\mu\nu}^+ (\sigma^{\mu\nu})^\alpha{}_\beta \delta_b^a + [X^i, X^j] \epsilon_{\alpha\beta} (C_{ij})^a{}_b \\ \delta \bar{\lambda}_{\dot{\beta}}^b &= \left\{ Q_\alpha^a, \bar{\lambda}_{\dot{\beta}}^b \right\} = C_i^{ab} \bar{\sigma}_{\alpha\dot{\beta}}^\mu D_\mu X^i \\ \delta A_\mu &= [Q_\alpha^a, A_\mu] = (\sigma_\mu)_\alpha{}^{\dot{\beta}} \bar{\lambda}_{\dot{\beta}}^a. \end{aligned}$$

where the constants $(C_{ij})^a{}_b$ are related to the Clifford Dirac matrices of $SO(6)_R$ [8]. It's clear to see that, due to the action of the supercharge, a boson will always transform to a fermion, and vice versa. Because the action must be dimensionless, we can deduce the dimensions of the fields and the couplings (in terms of mass dimension).

$$S = \int d^4x \mathcal{L}$$

$$\left[\int d^4x \right] = -4 \implies [\mathcal{L}] = 4.$$

From this we see deduce that $[X] = 1$ and the coupling constant g is dimensionless. Thus there is no running of the coupling, and our Super Yang-Mills theory is scale invariant, which is quite unique amongst field theories. The relationship between scale and conformal invariance is a difficult one; whilst there is no proof that they're equivalent, it is believed that if a theory demonstrates scale invariance, it will also be invariant under the conformal group, becoming a conformal field theory[21]. This is the case for $\mathcal{N} = 4$ Super Yang-Mills theory.

3 Conformal and Superconformal Symmetry Group

3.1 The Conformal Group

The conformal group in Euclidean space is one whose transformations preserve the angles of the space. We can extend this to Minkowski spacetime, where the conformal group preserves the form of the metric up to a scale factor, thus preserving the causal structure of the spacetime. We define the conformal transformation map from a spacetime (M, g) to $(M, \tilde{g})$, where M is the manifold and g is the metric, such that the metric transforms as $g_{\mu\nu} \rightarrow \tilde{g}_{\mu\nu} = \Lambda^2(x)g_{\mu\nu}$, where $\Lambda^2(x)$ is a smooth function that is positive everywhere.

The conformal group is a bosonic extension of the Poincaré group, and is generated by[1]:

- Lorentz transformations $M_{\mu\nu}$
- Translations P_μ
- Scaling D
- Special Conformal transformations K_μ

As we did for the super-Poincaré group, we can study the structure of the algebra by looking at the infinitesimal transformations, beginning with scalings and translations. We consider the product of two transformations - first translations and then scaling, and vice versa. The transformations are:

1. $x_\mu \rightarrow x'_\mu = (x_\mu + \epsilon_\mu); x'_\mu \rightarrow x''_\mu = (1 + \lambda)(x_\mu + \epsilon_\mu) = x_\mu + \lambda x_\mu + \epsilon_\mu + \lambda \epsilon_\mu$
2. $x_\mu \rightarrow x'_\mu = x_\mu + \lambda x_\mu; x'_\mu \rightarrow x''_\mu = x_\mu + \lambda x_\mu + \epsilon_\mu,$

which can be infinitesimally expressed as

1. $U(\lambda T) = \mathbb{I} + i\lambda D - i\eta^{\nu\mu}(\epsilon_\nu + \lambda \epsilon_\nu)P_\mu$
2. $U(T\lambda) = \mathbb{I} + i\lambda D - i\eta^{\nu\mu}\epsilon_\nu P_\mu.$

Each transformation has an associated unitary operator; scalings given by $U(\lambda) = e^{i\lambda D}$ and translations by the familiar $U(T) = e^{-i\epsilon^\mu P_\mu}$. Treating these infinitesimally also, we find

$$U(\lambda)U(T) - U(T)U(\lambda) = \eta^{\nu\mu}\lambda\epsilon_\nu[D, P_\mu]$$

$$U(\lambda T) - U(T\lambda) = -i\eta^{\nu\mu}\lambda\epsilon_\nu P_\mu.$$

Equating the RHS of each equation gives

$$[D, P_\mu] = -iP_\mu, \quad (13)$$

and with similar analysis we also find

$$[D, K_\mu] = iK_\mu. \quad (14)$$

The interesting representations to focus on are those build around the eigenfunctions of the scaling operator D , with eigenvalue $-i\Delta$, where Δ is the scaling dimension[1]. In this representatuion, P_μ raises the state and K_μ lowers it. Wtih this structure, we define the *primary operator* to be the state which is annihilated by K_μ , and upon which all other states in the representation may be built upon. The full structure of the algebra is given by

$$\begin{aligned} [M_{\mu\nu}, P_\mu] &= -i(\eta_{\mu\nu}P_\nu - \eta_{\nu\rho}P_\mu) & [M_{\mu\nu}, M_{\rho\sigma}] &= -i(\eta_{\mu\rho}M_{\nu\sigma} - \eta_{\mu\sigma}M_{\nu\rho} - \eta_{\nu\rho}M_{\mu\sigma} + \eta_{\nu\sigma}M_{\mu\rho}) \\ [M_{\mu\nu}, K_\rho] &= -i(\eta_{\mu\rho}K_\nu - \eta_{\nu\rho}K_\mu) & [P_\mu, K_\mu] &= 2iM_{\mu\nu} - 2i\eta_{\mu\nu}D. \end{aligned}$$

The algebra may be packaged in a suggestive way by defining new generators[1]

$$J_{\mu\nu} = M_{\mu\nu}, \quad J_{\mu d} = \frac{1}{2}(K_\mu - P_\mu), \quad J_{\mu(d+1)} = \frac{1}{2}(K_\mu + P_\mu), \quad J_{(d+1)d} = D,$$

and defining the matrix

$$J_{ab} = \begin{pmatrix} M_{\mu\nu} & \frac{1}{2}(K_\mu - P_\mu) & \frac{1}{2}(K_\mu + P_\mu) \\ -\frac{1}{2}(K_\mu - P_\mu) & 0 & D \\ -\frac{1}{2}(K_\mu + P_\mu) & D & 0 \end{pmatrix} \quad (15)$$

The matrix may look familiar - it is a $(d+2) \times (d+2)$ antisymmetric matrix with the Lorentz subgroup. We see that the conformal group is isomorphic to $SO(2, d)$; or in four dimensions $SO(2, 4)$. This is a key point, and should be noted - we will see that this is also the isometry group of AdS_5 . It teases us with a possible relation between a $(d+1)$ -dimensional gravity theory in AdS space and a d -dimensional conformal field theory. The full beauty will be revealed in the following sections.

3.2 The Superconformal Group

We have seen both a fermionic and bosonic extension of the Poincaré group, the former in the guise of the super-Poincaré group and the latter in the Conformal group. The natural question from here is to ask whether there is a way to merge the extensions into one encompassing algebra (as I mentioned, physiscists always like lesser parts!). This is possible, and is called the Superconformal Group. The algebra inherits the generators from both the SUSY algebra and Conformal group, and we find two additional generators that the algebra closes on: another fermionic generator S_α , and its hermitian conjugate, and R -symmetry, which is no longer an automorphism of the group[8, 1]. As this is incorporated into the algebra, we must also absorb the symmetry group, $SU(4) \sim SO(6)$, consequently giving the superconformal group as $SO(4, 2) \times SU(4) \equiv SU(2, 2 | 4)$, with $SO(4, 2) \sim SU(2, 2)$. The structure of the algebra can be given schematically as[1]

$$\begin{aligned} [K, \bar{Q}] &\sim S & [K, Q] &\sim \bar{S} & [D, Q] &\sim -\frac{i}{2}Q \\ [D, S] &\sim \frac{i}{2}S & [P, S] &\sim Q & \{S, S\} &\sim K & \{Q, S\} &\simeq M + D + R \end{aligned}$$

Once again, the interesting representations are those based around the eigenfunctions of the scaling generator D . Analogous to the conformal group, the state that is annihilated by S_α is defined to be the

superprimary operator, which is the state with lowest scaling dimension. We build the remaining states of the representation by repeated action with the supercharges Q_α , which always raises the scaling dimension, and so it is not possible to write the superprimary operator in terms of a commutator with Q_α . So what options do we have for a superprimary operator in the Super Yang-Mills theory? Looking back at the SUSY transformation of the fields, we see that we cannot use terms including the derivatives of X^i , fermions, or commutators of X^i , as all these can be expressed in terms of a commutator of Q_α . What we are left with is the symmetric part of the X^i , and consider the simplest form, known as a single trace operator

$$O^{i_1 \dots i_n} = sTr(X^{i_1} \dots X^{i_n}), \quad (16)$$

where sTr is the symmetrised trace, and the index $i_j, j = 1 \dots N$ runs over the $SO(6)_R$ fundamental representation that the scalars transform in.

We may also have some superprimary operators that are annihilated by some of the supercharges, and these are known as *chiral superprimary operators*, and they form the basis of a chiral, or 'BPS' multiplet. With fewer supercharges, we have fewer building blocks for our representation, and therefore the multiplets become shorter. These multiplets are known as *chiral* multiplets. To determine whether we have a chiral multiplet we look at the superprimary operator and its transformation under the R -symmetry group. The scalars are in the $\underline{6}$ of $SU(4)_R$, characterised by the representation $[010]$. We can consider the symmetrised product of two scalars $Sym^2[010] = [020] + [000]$ or, in terms of dimensions, $\frac{(6 \cdot 7)}{2} = \underline{21} \rightarrow \underline{20} + \underline{1}$. The $\underline{1}$ rep corresponds to $sTr(X^i X^i)$, and to check for null vectors, we look at the commutation relations between this and the supercharges, as for chiral multiplets, we know the superprimary operator is annihilated by some of the supercharges. Schematically, we find $[Q_\alpha, Tr(X^i X^i)] \sim Tr(\lambda_\alpha X^j)$, $\underline{4} \times \underline{1} \rightarrow \underline{4}$, so the representation sizes tie out on either side. However, if we look at the $\underline{20}$ rep, which corresponds to the symmetric traceless product $sTr(X^{\{i} X^{j\}})$, we find $[Q_\alpha, sTr(X^{\{i} X^{j\}})] \sim Tr(\lambda_\alpha X^j)$, where the LHS should give $\underline{4} \times \underline{20} \rightarrow \underline{60} + \underline{20}$, but we only see the $\underline{20}$ on the RHS[1]. Therefore we see that we have null vectors, corresponding to a short multiplet. It turns out that the symmetric traceless product of scalars will always be the superprimary of a chiral multiplet.

3.3 Properties of Correlation Functions

The aim of the correspondence is to relate the observables of both the gravity and conformal field theories, for which the latter includes correlation functions. In section 7, the gravity theory dynamics of the bulk field in AdS_5 can be shown to determine the two- and three-point correlation functions in the Super Yang-Mills theory, which is invariant under the conformal group. As this group has a high degree of symmetry, including

- Translational invariance
- Scaling
- Special Conformal,

it imposes restrictions on the forms of the correlations functions. Infact, for two- and three-point functions, the symmetry completely determines the structure. It is important to ensure that the correspondence produces the results that obey the conformal group.

3.3.1 Two-Point Functions

The two-point functions may be represented by $\langle \mathcal{O}(x_1) \mathcal{O}(x_2) \rangle = g(x_1, x_2)$. The forms imposed by the symmetries are:

- Translational invariance: as this is an invariant under $f(x) = x + a$, g must be a function of the difference between two vectors

$$g(x_1, x_2) = g(x_1 - x_2).$$

- Scaling: consider two operators of conformal dimension Δ_1, Δ_2 . The two-point function must be invariant under rescalings $f(x) = \lambda x$:

$$\langle \lambda^{\Delta_1} \mathcal{O}(\lambda x_1) \lambda^{\Delta_2} \mathcal{O}(\lambda x_2) \rangle = \lambda^{\Delta_1 + \Delta_2} g(\lambda(x_1 - x_2)) = g(x_1 - x_2).$$

due to the previous restriction by translational invariance[13]. This forces us to choose the form

$$g(x_1 - x_2) = \frac{d_{12}}{(x_1 - x_2)^{\Delta_1 + \Delta_2}},$$

where d_{12} is known as the structure constant.

- Special Conformal Transformations: these are a combination of translations and inversions. We have already restricted the form of the two-point function with translational invariance, so just need to consider the restrictions imposed by inversions $f(x) = -\frac{1}{x}$.

$$\langle \frac{1}{x_1^{2\Delta_1}} \frac{1}{x_2^{2\Delta_2}} \mathcal{O}(-\frac{1}{x_1}) \mathcal{O}(-\frac{1}{x_2}) \rangle = \frac{1}{x_1^{2\Delta_1} x_2^{2\Delta_2}} \frac{d_{12}}{\left(-\frac{1}{x_1} + \frac{1}{x_2}\right)^{\Delta_1 + \Delta_2}} = \frac{d_{12}}{(x_1 - x_2)^{\Delta_1 + \Delta_2}}.$$

To obey the restrictions of scaling, the conformal weights of the two operators must be the same, $\Delta_1 = \Delta_2 \equiv \Delta$, giving

$$\langle \mathcal{O}(x_1) \mathcal{O}(x_2) \rangle = \frac{d_{12}}{(x_1 - x_2)^{2\Delta}}. \quad (17)$$

This is the expected form of the two-point function for an operator of conformal weight Δ .

3.3.2 Three-Point Functions

The three point functions may be represented by $\langle \mathcal{O}(x_1) \mathcal{O}(x_2) \mathcal{O}(x_3) \rangle = g(x_1, x_2, x_3)$. The forms imposed by the symmetries are

- Translational Invariance:

$$g(x_1, x_2, x_3) = g((x_1 - x_2)(x_2 - x_3)(x_1 - x_3)).$$

- Scaling invariance:

$$\langle \lambda^{\Delta_1} \mathcal{O}(\lambda x_1) \lambda^{\Delta_2} \mathcal{O}(\lambda x_2) \lambda^{\Delta_3} \mathcal{O}(\lambda x_3) \rangle = \lambda^{\Delta_1 + \Delta_2 + \Delta_3} g(\lambda(x_1 - x_2)(x_2 - x_3)(x_1 - x_3)) = g((x_1 - x_2)(x_2 - x_3)(x_1 - x_3)),$$

which tells us that the three-point function must be of the form

$$\frac{C_{123}}{(x_1 - x_2)^a (x_2 - x_3)^b (x_1 - x_3)^c},$$

where C_{123} is the structure constant, and $a + b + c = \Delta_1 + \Delta_2 + \Delta_3$ to cancel out the λ in the numerator.

- Finally the special conformal transformations impose our choices of a, b and c and we find

$$\begin{aligned}
a &= \Delta_1 + \Delta_2 - \Delta_3 \\
b &= \Delta_2 + \Delta_3 - \Delta_1 \\
c &= \Delta_1 + \Delta_3 - \Delta_2,
\end{aligned}$$

which gives the conformal structure of the three-point function as

$$\langle \mathcal{O}(x_1) \mathcal{O}(x_2) \mathcal{O}(x_3) \rangle = \frac{C_{123}}{|\vec{x}_1 - \vec{x}_2|^{\Delta_1 + \Delta_2 - \Delta_3} |\vec{x}_1 - \vec{x}_3|^{\Delta_1 + \Delta_3 - \Delta_2} |\vec{x}_2 - \vec{x}_3|^{\Delta_2 + \Delta_3 - \Delta_1}}. \quad (18)$$

We will see in section 7 that the forms of the correlation functions from the gravity theory exactly match what we would expect for correlation functions these conformal dimensions.

3.4 't Hooft Limit and Large N

The origins of string theory lie far away from where they are now. The theory was originally formulated to describe the strong force, but the presence of a massless spin-two particle and the demanding requirement that we live in ten dimensions meant it was abandoned in favour of QCD, which has held up very well ever since. In his 1974 paper, 't Hooft showed insight into a link between strings and QCD that brought the theory back to its roots. It is an asymptotically free theory, so whilst it is easy to describe the high-energy behaviour of the strong interactions with a gauge theory, this becomes more difficult at lower energies due to the increasing strength of the coupling constant, meaning that perturbative expansions can't be used. However, 't Hooft looked to expanding the $SU(3)$ colour group to arbitrary N colours with $SU(N)$ gauge group, looking at the topological properties of the Feynman diagrams. By defining a fixed coupling constant $\lambda = g^2 N$, the 't Hooft coupling, whilst this limit is taken, we can then define a perturbative expansion in $1/N$ [27].

We may think of some generic non-Abelian theory for some field Φ_i^a , where a is in the adjoint representation of $SU(N)$, and i denotes some particular field (i.e. a scalar, quark, gauge). We assume the Lagrangian includes three-point and four-point interactions, and by rescaling the fields $\tilde{\Phi} = g\Phi$ can write a schematic Lagrangian as

$$\mathcal{L} \sim \frac{1}{g^2} \left(Tr(\tilde{\Phi}_i \tilde{\Phi}_i) + Tr(\tilde{\Phi}_i \tilde{\Phi}_j \tilde{\Phi}_k) + Tr(\tilde{\Phi}_i \tilde{\Phi}_j \tilde{\Phi}_k \tilde{\Phi}_l) \right) \quad (19)$$

To acquire the Feynman diagrams for this theory, double-line notation is used. The fields in the adjoint representation are considered as the product between the fundamental and anti-fundamental representations, and thus we label the field as Φ_j^i , with one index in each representation, and draw this on the diagram as two double lines in opposing directions. Considering the Feynman diagrams as a topological simplicial triangulation, we have the contributions[18]:

- propagators contribute $\frac{\lambda}{N}$, which we denote E , the edges of double lines
- vertices contribute $\frac{N}{\lambda}$, denoted V
- loops contribute N , denoted F , as the loops form faces

The contribution to each diagram is given as $N^{F+V-E} \lambda^{E-V}$. However, in terms of topological expansions, the quantity $F + V - E$ is equal to Euler's characteristic, χ which for closed oriented surfaces can be written as $\chi = 2 - 2g$, g being the genus of the surface[20].

Doing a perturbative expansion over the diagrams gives

$$\sum_{g=0}^{\infty} N^{2-2g} f_g(\lambda), \quad (20)$$

where $f_g(\lambda)$ is some polynomial. As we take the large N limit, $N \rightarrow \infty$, the graphs with the lowest genus dominate, indicating that the theories for large N should be simpler than, say, $SU(3)$. This

is the same perturbative expansion as in closed oriented string theory, if we make the identification $g_s = \lambda/N$, where g_s is the string coupling. We see a mapping between the gauge theory and string theory, adding another tantalising element to Maldacena's conjecture.

4 Anti de-Sitter space

Anti de-Sitter space, known as AdS , is a Lorentzian analogue of Euclidean hyperbolic space, holding constant negative curvature. The correspondence uses the idea of the holographic principle, which states that some $(d + 1)$ -dimensional theory can be encoded entirely at the d -dimensional boundary. To make the connection between the gravity and conformal field theories, we must understand the way AdS behaves both in the bulk and at the boundary, where the conformal field theory lives. This chapter looks at the conformally compactified structure of both flat space and AdS , in order to make the necessary geometric relation between the two. Within this chapter we follow the method of [9]

4.1 Conformal Compactification

The method of conformal compactification is useful for the investigation of the causal structure of spacetime. In this, infinite distances are brought to a finite coordinate distance, allowing us to probe the structure at the boundaries of the spacetime. Integral to this method is the idea of a conformal transformation, which takes a spacetime equipped with a metric g to another, equipped with a metric $\tilde{g}$, related to the original metric by

$$\tilde{g}_{\mu\nu} = \Omega^2(x)g_{\mu\nu}, \quad (21)$$

where $\Omega(x)$ is known as the conformal factor, and is taken as $\Omega^2(x) > 0$ everywhere within the original coordinate series. In order to bring the diverging distances in the metric to within a finite distance, the factor must simultaneously decrease, eventually reaching zero on the conformal boundary only defined for $\tilde{g}_{\mu\nu}$. Conformal transformations preserve the causal structure of spacetime, such that vectors which were defined to be timelike/null/spacelike with the original metric, continue to be so with the conformally transformed metric. As mentioned in the previous section, one of the points at the heart of the correspondence is the equivalence between the isometry group of AdS_{d+1} and the conformal symmetry of flat Minkowski space $\mathbb{R}^{1,d-1}$. To see this, we begin by looking at the conformal structure of Minkowski space.

4.1.1 Conformal Structure of 2D Minkowski Space, $\mathbb{R}^{1,1}$.

The metric for 2D Minkowski space is

$$ds^2 = -dt^2 + dx^2, \quad (22)$$

with the coordinates defined over the range $-\infty < t, x < \infty$. With these coordinates ranges, there is no way to represent the spacetime on a finite piece of paper, the infinite points are not defined within the spacetime - to do this we must compactify. The first step is to transform into Eddington-Finkelstein coordinates, based on null geodesics (lightcone coordinates)

$$\begin{aligned} t + x &\equiv v && \text{ingoing EF coordinate} \\ t - x &\equiv u && \text{outgoing EF coordinate.} \end{aligned}$$

With this coordinate transformation, the metric reads

$$ds^2 = -dudv.$$

We can then transform into trigonometric coordinates $u = \tan \tilde{u}$ and $v = \tan \tilde{v}$, such that

$$-\infty < u, v < \infty \quad \leftrightarrow \quad -\frac{\pi}{2} < \tilde{u}, \tilde{v} < \frac{\pi}{2}$$

$$\frac{du}{d\tilde{u}} = \frac{d}{d\tilde{u}}(\tan\tilde{u}) = \frac{d}{d\tilde{u}}\left(\frac{\sin\tilde{u}}{\cos\tilde{u}}\right) = \frac{\cos^2\tilde{u} + \sin^2\tilde{u}}{\cos^2\tilde{u}} = \frac{1}{\cos^2\tilde{u}},$$

giving

$$ds^2 = -\frac{1}{\cos^2\tilde{u}\cos^2\tilde{v}}d\tilde{u}d\tilde{v}.$$

Making the conformal transformation $\tilde{g}_{\mu\nu} = (\cos\tilde{u}\cos\tilde{v})^2 g_{\mu\nu}$, we define a new metric $\tilde{ds}^2 = -d\tilde{u}d\tilde{v}$. The range has been brought to within a finite coordinate distance with the use of trigonometric coordinates. The conformally transformed metric is regular at $\pm\frac{\pi}{2}$, and can be added as points to create a new spacetime, denoted $\tilde{M}$. The original Minkowski spacetime is a subset of this spacetime, which we define as the conformal compactification of M .

4.1.2 Conformal Structure of Higher-Dimensional Minkowski Space, $\mathbb{R}^{1,d-1}$.

The previous example can be extended to higher dimensional Minkowski space, with metric

$$ds^2 = -dt^2 + dr^2 + r^2 d\Omega_{d-2}^2.$$

$d\Omega_{d-2}^2$ is the spatial metric defined on the $d-2$ sphere. We follow the same logic as before, but with an additional constraint - we have introduced a radial coordinate, r , which is defined only for $r \geq 0$. Consequently, we find that $v \geq u$, and thus $\tilde{v} \geq \tilde{u}$. The conformally transformed metric now reads

$$\tilde{ds}^2 = -4d\tilde{u}d\tilde{v} + \sin^2(\tilde{v} - \tilde{u})d\Omega_{d-2}^2.$$

However, this is not the most useful form, and to make the conformal structure upon compactification manifest, we make the transformation $\tau = \tilde{v} + \tilde{u}$ and $\chi = \tilde{v} - \tilde{u}$.

$$\begin{aligned} d\tau &= d\tilde{v} + d\tilde{u} \\ d\chi &= d\tilde{v} - d\tilde{u} \\ -d\tau^2 &= (d\tilde{v} + d\tilde{u})^2 \\ d\chi^2 &= (d\tilde{v} - d\tilde{u})^2 \\ -d\tau^2 + d\chi^2 &= -4d\tilde{u}d\tilde{v} \\ ds^2 &= -d\tau^2 + d\chi^2 + \sin^2\chi d\Omega_{d-3}^2. \end{aligned}$$

This gives the topology of $\mathbb{R} \times S^{d-2}$, which is the same geometry as Einstein's Static Universe[29]. This will be important when we look at the conformal compactification of AdS_{d+1} in the next subsection.

4.1.3 Conformal Structure of Anti-de Sitter Space

AdS_d can be represented by the hyperboloid embedded in $d+1$ dimensional space[22]

$$\begin{aligned} X_0^2 + X_d^2 - \sum_{i=1}^{d-1} X_i^2 &= R^2 \\ ds^2 &= -dX_0^2 - dX_d^2 + \sum_{i=1}^{d-1} dX_i^2, \end{aligned}$$

where we have defined some radius of curvature, R . Orthogonal transformations preserve the magnitude of a vector, and the embedding of AdS_d gives the same equation as the invariant of a rotation in

$d + 1$ dimensional space, making the $SO(2, d - 1)$ symmetry manifest. The coordinates that satisfy the above are

$$\begin{aligned} X_0 &= R \cosh \rho \cos \tau \\ X_d &= R \cosh \rho \sin \tau \\ X_i &= R \sinh \rho \Omega_i, \end{aligned}$$

where we constrain Ω_i by imposing $\sum_i \Omega_i^2 = 1$. With this choice of coordinates, we find that the differentials are

$$\begin{aligned} dX_0(\rho, \tau) &= R \sinh \rho \cos \tau d\rho - R \cosh \rho \sin \tau d\tau \\ dX_d(\rho, \tau) &= R \sinh \rho \sin \tau d\rho + R \cosh \rho \cos \tau d\tau \\ dX_i(\rho, \Omega_i) &= R \cosh \rho \Omega_i d\rho + R \sinh \rho d\Omega_i. \end{aligned}$$

So, our metric becomes (by noting $\Omega_i d\Omega_i = 0$)

$$ds^2 = R^2 (-\cosh^2 \rho d\tau^2 + d\rho^2 + \sinh^2 \rho d\Omega_{d-2}^2). \quad (23)$$

These coordinates are called global coordinates, and cover the whole space, with $\rho \geq 0$ and $0 \leq \tau < 2\pi$. This is quite a suggestive coordinate system, and by making the transformation $\tan^2 \theta = \sinh^2 \rho$ we see

$$\begin{aligned} \cosh^2 \rho - \sinh^2 \rho &= 1 \\ \frac{\sin^2 \theta + \cos^2 \theta}{\cos^2 \theta} &= \tan^2 \theta + 1 \\ \tan^2 \theta + 1 &= \sinh^2 \rho + 1 \\ \frac{1}{\cos^2 \theta} &= \cosh^2 \rho, \end{aligned}$$

which gives the metric

$$ds^2 = \frac{R^2}{\cos^2 \theta} (-d\tau^2 + d\theta^2 + \sin^2 \theta d\Omega_{d-2}^2).$$

The prefactor can be removed by a conformal transformation, and the conformally transformed metric is

$$\tilde{ds}^2 = -d\tau^2 + d\theta^2 + \sin^2 \theta d\Omega_{d-2}^2. \quad (24)$$

The original global coordinate ρ was defined for $\rho \geq 0$, and therefore $0 \leq \theta < \frac{\pi}{2}$. However, as previously seen in the Minkowski case, the metric is now regular for $\theta = \frac{\pi}{2}$ and the space can be conformally compactified to include this point - we can now see that AdS_d is conformal to one-half of the Einstein Static Universe. Just like flat space, if we can conformally compactify a spacetime to have the same conformal structure as AdS then we call it Asymptotically AdS , or $AAdS$. If we take the conformally compactified AdS_d to the boundary at $\theta = \frac{\pi}{2}$ we see that we have the topology $\mathbb{R} \times S^{d-2}$, which is the same as the conformally compactified Minkowski Space on $\mathbb{R}^{1, d-2}$. This is one of the key points of the AdS/CFT correspondence - when we take AdS_5 to the boundary, it looks like Minkowski in $(3 + 1)$ -dimensions, where the conformal field theory, $\mathcal{N} = 4$ Super Yang-Mills, lives.

4.1.4 Poincaré Coordinates

Another useful paramaterisation of the embedding of AdS_d is given by Poincaré coordinates, defined as the set $(\vec{x}, t, u)$ and $u > 0$. The coordinates cover half of the space, and are given by

$$\begin{aligned} X_0 &= \frac{1}{2u}(1 + u^2(R^2 + \vec{x}^2 - t^2)) \\ X_i &= Rux_i \\ X_d &= Rut \\ X_{d-1} &= \frac{1}{2u}(1 - u^2(R^2 - \vec{x}^2 + t^2)), \end{aligned}$$

with metric

$$ds^2 = R^2 \left(\frac{du^2}{u^2} + u^2(-dt^2 + d\vec{x}^2) \right).$$

This can be brought into a slightly nicer form by equating $u = \frac{1}{z}$,

$$ds^2 = \frac{R^2}{z^2} (dz^2 + (-dt^2 + d\vec{x}^2)). \quad (25)$$

The conformal boundary is now at $z = 0$. This form of the metric is of particular use when studying the behaviour of the fields in the 'bulk' of AdS and their asymptotic behaviour. The boundary of the space lives at $u \rightarrow \infty$, corresponding to $z \rightarrow 0$. Obviously this creates a divergent metric - the volume of the space is infinite, and thus suffers from IR divergences. As we will see, the AdS/CFT correspondence is a duality, and thus 'long' in one corresponds to 'short' in another. Through this we are able to express this IR divergence in terms of the UV entropy divergences that Super Yang-Mills suffers at arbitrarily small distances. Not only that, but renormalising one theory results in the renormalisation of the other.[26] This is the idea of *holographic* renormalisation.

5 String Theory and D-branes

In the formulation of the AdS/CFT correspondence we consider Type IIB String Theory. The theory is one of closed oriented superstrings that contain the theory of gravity in the bulk of spacetime, and open strings ending on dynamical hypersurfaces called D-branes. D-branes are a crucial element, as their presence within the theory can distort the spacetime to look like *AdS* nearby. They also contain gauge theories on their worldvolume, providing the link between both sides of the correspondence. To study the massive degrees of freedom within this theory, we use Wigner's classification. As we'll be dealing with the low-energy dynamics of the theory, we consider here just the massless representations, those of the Type IIB supergravity theory, characterised by the little group in 10 dimensions, $SO(8)$.

5.1 Spectroscopy - Closed Strings

We begin by looking at the closed strings of Type IIB theory. Although the spectrum cannot be obtained directly from dimensional reduction, it is still instructive to do so; finding the Type IIA spectrum, and using tensor products to deduce the spectrum of Type IIB.

5.1.1 Supergravity in 11 Dimensions

In 11 dimensions the supergravity multiplet consists of

- a graviton, $g_{\mu\nu}$: rank 2 symmetric traceless tensor with $\frac{10 \cdot 9}{2} - 1 = 44$ degrees of freedom
- a gravitino, ψ_μ^α : Majorana spin- $\frac{3}{2}$ vector-spinor with $8 \cdot 2^4 = 128$ degrees of freedom
- third-rank antisymmetric tensor (three-form) $C_{\nu\mu\rho}$ with $\frac{9 \cdot 8 \cdot 7}{3 \cdot 2} = 84$ degrees of freedom

The graviton and three-form make up the required 128 bosonic degrees of freedom, and the gravitino the 128 fermionic degrees of freedom, satisfying the requirement from supersymmetry that, within a given multiplet, there should be an equal number of bosonic and fermionic degrees of freedom. To classify these irreps, we look to the little group $SO(9)$ represented by the Dynkin diagram B_4 .

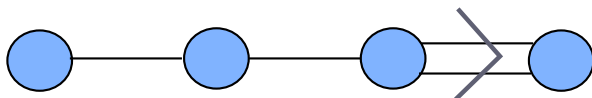


Figure 2: Dynkin diagram B_4 , representing the Lie algebra of $SO(9)$

This diagram is classified by four labels $[n_1, n_2, n_3, n_4]$. In the highest weight representation, which is chosen because the weights are non-degenerate and thus uniquely define the irrep, there are four basic representations of $SO(9)$, given by

1. $[1000]_9$ vector representation, of dimension 9.
2. $[0100]_9$ second-rank antisymmetric tensor representation, of dimension $\binom{9}{2} = 36$
3. $[0010]_9$ third-rank antisymmetric tensor representation, of dimension $\binom{9}{3} = 84$
4. $[0001]_9$ spinor representation, of dimension $2^4 = 16$

Using the basic representations, and Weyl's dimensional formula, we can represent the graviton by $[2000]_9$, the gravitino by $[1001]_9$ (with one vector and one spinor label), and the three-form by $[0010]_9$.

5.1.2 Dimensional Reduction to Type IIA SUGRA

Now the 11 dimensional spectrum has been classified, we can use branching rules for the irreps to dimensionally reduce to Type IIA SUGRA in 10 dimensions. To classify the degrees of freedom in 10 dimensions, we look at the algebra for $SO(8)$, represented by the Dynkin diagram D_4

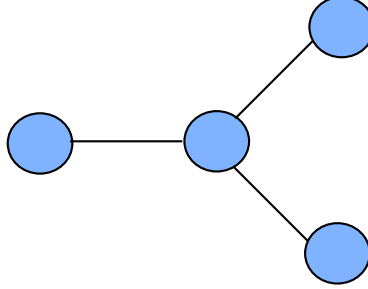


Figure 2: Dynkin diagram D_4 , representing the Lie algebra of $SO(8)$

Here, the basic representations are given by

1. $[1000]_8$ vector representation: dimension = 8
2. $[0100]_8$ adjoint representation: dimension $\binom{8}{2} = 28$
3. $[0010]_8$ spinor s representation: dimension $2^3 = 8$
4. $[0001]_8$ spinor s' representation: dimension $2^3 = 8$

The s and s' spinor representations are of different chiralities, and have the same dimension as the vector representation. This is unique to D_4 , the most symmetrical Dynkin diagram, and is known as triality. These three vector spaces can be permuted by an S_3 permutation arbitrarily[10], which will be useful for the classification of irreps. With knowledge of the dimensionalities of the $SO(8)$ basic representations, we can now look at how $SO(9)$ embeds itself into $SO(8)$. For the basic representations, the branching rules give

- $[1000]_9 = [1000]_8 + [0000]_8 \rightarrow \underline{\mathbf{9}} = \underline{\mathbf{8}} + \underline{\mathbf{1}}$
- $[0100]_9 = [0100]_8 + [1000]_8 \rightarrow \underline{\mathbf{36}} = \underline{\mathbf{28}} + \underline{\mathbf{8}}$
- $[0010]_9 = [0010]_8 + [0100]_8 \rightarrow \underline{\mathbf{84}} = \underline{\mathbf{56}} + \underline{\mathbf{28}}$
- $[0001]_9 = [0010]_8 + [0001]_8 \rightarrow \underline{\mathbf{16}} = \underline{\mathbf{8}} + \underline{\mathbf{8'}}$

To find the closed string spectrum of Type IIA, we dimensionally reduce the 11D SUGRA spectrum - the three-form reduction is already given above by $[0010]_9$ branching rules. For the graviton $[2000]_9$ and gravitino $[1001]_9$

- $[2000]_9 = [2000]_8 + [1000]_8 + [0000]_8 \rightarrow \underline{\mathbf{44}} = \underline{\mathbf{35}} + \underline{\mathbf{8}} + \underline{\mathbf{1}}$
- $[1001]_9 = [1001]_8 + [1010]_8 + [0001]_8 + [0010]_8 \rightarrow \underline{\mathbf{128}} = \underline{\mathbf{56}} + \underline{\mathbf{56'}} + \underline{\mathbf{8}} + \underline{\mathbf{8'}}$

where we've distinguished between the two chiralities. Using the above, the spectrum of type IIA supergravity is

$g_{\mu\nu}$	$B_{\mu\nu}$	ϕ	$A_{1\mu}$	$A_{3\mu\nu\rho}$	$\psi_{\pm\alpha}^\mu$	$\lambda_{\pm\alpha}$
<u>44</u>	<u>28</u>	<u>1</u>	<u>8</u>	<u>56</u>	<u>56, 56'</u>	<u>8, 8'</u>

This theory contains fermions of both chiralities, making Type IIA a non-chiral theory. To determine the Type IIB spectrum, we begin by looking at tensor products.

5.1.3 Tensor Products and Type IIB Spectrum

A complete decription of tensor products is given by Slansky. Useful for our purposes is that for two highest weights a_i and b_i , the tensor product leads with the weight $a_i + b_i$ [25]. Sometimes the rules may not illuminate the entire tensor product, but we can often use informed guessing to complete this, such as using the knowledge of the degrees of freedom of the basic and composite representations. We are interested in $SO(8)$, and first consider $[1000]_8 \otimes [1000]_8$. Using the aforementioned rule, the leading term gives the graviton

$$[1000]_8 \otimes [1000]_8 = [2000]_8 + \dots$$

Comparing the degrees of freedom of both sides give $\underline{8} \times \underline{8} = \underline{35}$, indicating that some are missing, so the tensor product must give further irreps. The product of two vectors gives a matrix, which can be broken down into a second-rank antisymmetric tensor, symmetric-traceless, and the trace. Therefore we postulate that this tensor product gives

$$[1000]_8 \otimes [1000]_8 = [2000]_8 + [0100]_8 + [0000]_8$$

and the degrees of freedom now read $\underline{8}_v \times \underline{8}_v = \underline{35} + \underline{28} + \underline{1}$, agreeing. Next we consider the tensor product of a vector and a spinor $[1000]_8 \otimes [0001]_8$. The leading term must be a vector-spinor object, and the only one of this kind at our disposal is the gravitino $[1001]_8$ - comparing the degrees of freedom for the leading term gives $\underline{8}_v \times \underline{8}_s = \underline{56}$, and thus we are missing an $\underline{8}$. However, for $SO(8)$ three basic representations have this dimension; which representation do we choose? Using triality, we find that the result is a spinor of the opposite chirality, $\underline{8}_{s'}$.

$$[1000]_8 \otimes [0001]_8 = [1001]_8 + [0010]_8 \quad \underline{8}_v \times \underline{8}_s = \underline{56} + \underline{8}_{s'}$$

Finally, we consider the tensor product of two spinors of opposite chirality $[0010]_8 \otimes [0001]_8$. The leading term is $[0011]_8$, but looking at the degrees of freedom we find that we are missing an $\underline{8}$ - again using triality we can determine that this is the $\underline{8}_v$ of the vector representation:

$$[0010]_8 \otimes [0001]_8 = [0011]_8 + [1000]_8 \quad \underline{8}_s \times \underline{8}_{s'} = \underline{56} + \underline{8}_v$$

By looking at these tensor products we are able to find all of the simple and composite irreps of Type IIA SUGRA, and we may express them in a factorised form as

$$([1000]_8 + [0010]_8) ([1000]_8 + [0001]_8)$$

Spinors of both chiralities are used for this non-chiral theory - Type IIB is chiral, so it is natural to consider the chiral tensor product factorisation $([1000]_8 + [0001]_8)^2$, which gives us the Type IIB spectrum

- $[2000]_8$: graviton
- $[0100]_8$: NSNS two-form
- $[0000]_8$: dilaton
- $2[1001]_8$: gravitini
- $2[0010]_8$: spinors
- $[0000]_8$: RR zero-form
- $[0100]_8$: RR two-form
- $[0002]_8$: self-dual four-form

5.1.4 Brane Spectroscopy

By considering an extension of electrostatics and charge conservation, we can see how branes enter into the theory. For the AdS/CFT Correspondence, we will be particularly interested in Dp -branes, which carry charges under the Ramond-Ramond forms[24]. Initially, one can consider electromagnetism in $(3 + 1)$ -dimensions. We have a gauge field, which is a one-form $A \equiv A_\mu dx^\mu$ with a two-form field strength $F \equiv F_{\mu\nu} dx^\mu \wedge dx^\nu$. Looking at the differential form of Maxwell's equation for an electric and magnetic source, we have

$$\begin{aligned} d * F &= \delta^{(3)} Q_e \\ dF &= \delta^{(3)} Q_m, \end{aligned}$$

representing an electric and magnetic point charge respectively, localised in three spatial dimensions. The Hodge star $*$ takes an r -form to a $(D - r)$ -form, and the exterior derivative takes an r -form to an $(r + 1)$ -form. For the electric source equation example above, $*F$ is a two form, and taking the exterior derivative gives a three-form $\delta^{(3)}$ on the RHS.

Type IIB theory has an NS-NS two-form, as well as an RR zero-form, two-form and self-dual four-form. Extending the $(3 + 1)$ Maxwell case, where $D = 10$, we consider the NS-NS two-form $B_{\mu\nu}$, with corresponding field strength $H^{(3)} = dB \rightarrow d * H^{(3)} = \delta^{(8)} Q_e$, representing an electric charge localised in 8-dimensions and spanning one dimension. This is the fundamental string, denoted F_1 . Similarly $dH^{(3)} = \delta^{(4)} Q_m$, which is a magnetic charge localised in 4 spatial dimensions and spanning five dimensions, which we call the NS5-brane. We can take the RR forms also :

$$\begin{aligned} C^{(0)} \begin{cases} d * F^{(1)} = \delta^{(10)} Q_e & \text{D(-1)-brane, instanton} \\ dF^{(1)} = \delta^{(2)} Q_m & \text{D7-brane} \end{cases} \\ C^{(2)} \begin{cases} d * F^{(3)} = \delta^{(8)} Q_e & \text{D1-brane} \\ dF^{(3)} = \delta^{(4)} Q_m & \text{D5-brane} \end{cases} \\ C^{(4)} \begin{cases} d * F^{(5)} = \delta^{(6)} Q_e & \text{D3-brane} \\ dF^{(5)} = \delta^{(6)} Q_m & \text{D3-brane.} \end{cases} \end{aligned}$$

The self-duality of the four-form results in the D3-brane carrying both electric and magnetic charge; it is a dyonic object. This is the brane spectrum of Type IIB theory.

5.2 Open String Spectrum and D-branes

We have obtained the massless spectrum of the closed strings, which give rise to Type IIB supergravity. However, the correspondence involves a gauge theory living on a D-brane, and we do not see gauge fields in the supergravity theory - where do these additional degrees of freedom come from? These can be found upon the quantisation of the open strings propagating in the theory[28]. To be a theory relating to nature, the open superstrings should contain both fermions $\psi, \bar{\psi}$ and bosons X , and the action describing the dynamics (given in the superconformal gauge) is written as

$$S \sim \int d\sigma d\tau \left(\partial_a X^\mu \partial_a X^\mu - i \bar{\psi}^\mu \rho^a \partial_a \psi^\mu \right),$$

whose equations of motion yield the familiar Klein-Gordon and Dirac equations in two dimensions. As the open strings have endpoints, they are subject to boundary conditions, both bosonic and fermionic for the superstring theory. Varying the action with respect to X^μ yields the boundary condition

$$\delta S \sim \int dt [\delta X^\mu \partial_a X_\mu]_0^\pi,$$

which we require to vanish. Therefore, we can impose two boundary conditions

$$\begin{aligned}\partial_a X^\mu(0, t) &= 0 \quad \text{Neumann} \\ \delta X^\mu(0, t) &= 0 \quad \text{Dirichlet}.\end{aligned}$$

(26)

The Neumann boundary condition restricts the end points of the string to only move freely in a plane. The latter, the Dirichlet condition, is more interesting. It implies that the end points of the strings are fixed at some particular position in time. It is this hypersurface that we call the D-brane. We may also see the need for this object within the theory by looking at Gauss' Law for the open string propagating in ten-dimensional space

$$Q = \int_{S^7} \vec{E} \cdot d\vec{a}.$$

Taking the integral over S^7 gives us the total charge contained. We can first consider the string ending at some point in spacetime. If we imagine moving the sphere along the string until falling off the endpoint, we see that the charge is not conserved. Therefore, an open string simply cannot end, it must end on an object, namely the D-brane. In addition to this, the point charge acts as a source for a $U(1)$ gauge theory on the worldvolume of the brane. There are also fermionic boundary conditions known as Neveu-Schwarz and Ramond boundary conditions, which are given in terms of the chirality of the fermions

$$\begin{aligned}\psi_-^\mu(0, t) &= \xi_1 \psi_+^\mu(0, t) \\ \psi_-^\mu(\pi, t) &= \xi_2 \psi_+^\mu(\pi, t),\end{aligned}$$

(27)

where $\xi_1, \xi_2 = \pm 1$. The Ramond boundary condition is periodic, and the Neveu-Schwarz is anti-periodic[17]. Upon quantisation, these boundary conditions give rise to half-integer (Neveu-Schwarz) and integer (Ramond) mode expansions, which lead to spacetime bosons and fermions respectively. Both of these sectors taken separately would of course not constitute a natural theory, and certain other issues arise, such as the existence of a tachyon. However, we can use the a process derived by Gliozzi-Scherk-Olive, known as the GSO projection, to combine fermions and bosons and eradicate the issues[6], leaving us with the massless string spectrum of

- a gauge field A_μ with 8 degrees of freedom, given by the $\mathbf{8}_v$ representation of $SO(8)$
- a Weyl spinor, λ_α , which may either be in the $\mathbf{8}_s$ or $\mathbf{8}_{s'}$ representations of $SO(8)$

Thus, the string carries 8 bosonic and 8 fermionic degrees of freedom, and gives rise to fields that are seen in gauge theories. In fact, by imposing Dirichlet conditions on the open string in $9 - p$ directions, we can describe the dynamics of the D-brane from the excitations of the open string. To do this, we consider Lorentz invariance in 10 dimensions, given by the group $SO(9, 1)$. The presence of a D-brane breaks this invariance to

$$SO(9, 1) \supset SO(p, 1) \times SO(9 - p)_R,$$

where $SO(p, 1)$ is Lorentz on the Dp -brane and $SO(9 - p)_R$ is the rotational symmetry in the $(9 - p)$ -dimensions transverse to the brane. By Goldstone's Theorem, for every broken generator there should

be a corresponding massless mode, and so we should expect $(9 - p)$ scalars. The massless degrees of freedom within the vector multiplet can be classified using the little group, which for 10 dimensional massless theory is $SO(8)$, and decomposes in the presence of a D p -brane to

$$SO(8) \supset SO(p - 1) \times SO(9 - p)_R.$$

We can begin by considering the vector multiplet in $(9 + 1)$ -dimensions, where no decomposition is required. The bosonic degrees of freedom are given by the gauge field, with $[1000]_8$, and the 8 fermionic degrees of freedom are given by the gaugino, $[0001]_8$. The D9-brane fills all of spacetime, and thus $SO(9, 1)$ is preserved. As there are no broken generators, there are no additional massless scalars. For the AdS/CFT correspondence, we are interested in a gauge theory in $(3 + 1)$ -dimensions, and so it is natural to consider $p = 3$, a D3-brane. The little group is broken to

$$SO(8) \supset SO(2) \times SO(6),$$

where $SO(2)$ is the little group on the worldvolume of the brane, and $SO(6)_R \sim SU(4)_R$ is the R-symmetry group, which crucially is the same R-symmetry group as $\mathcal{N} = 4$ SYM in $d = 4$. We can use the branching rules from dimensional reduction to find how the $(9 + 1)$ -dimensional vector multiplet embeds itself into the separate products, maintaining the 8 bosonic and fermionic degrees of freedom. Doing this, we find

$$\begin{aligned} [1000]_8 &= [100]_6 q^0 + [000]_6 (q^2 + q^{-2}) \\ [0001]_8 &= [010]_6 q^1 + [001]_6 q^{-1}. \end{aligned}$$

(28)

Using $SO(2) \sim U(1)$, this representation has been labelled by the Abelian charge. By looking at the $SO(6)$ product with the q^n , the way that the $U(1)$ fields transform under the R-symmetry is manifest. Thus, on the world volume of the D3-brane, there is:

- A $U(1)$ gauge field, transforming as a singlet under $SO(6)$
- 6 scalars, transforming in the $\underline{6}$ of $SO(6)$
- 4 complex Weyl fermions (gauginos), in the $\underline{4}$ and $\bar{\underline{4}}$ of $SO(6)_R$

This is the massless $U(1)$ abelian gauge theory on the D3-brane. We are looking to realise higher gauge groups, in particular the $SU(N)$ gauge group of $\mathcal{N} = 4$ Super Yang-Mills. Consider N parallel branes, positioned arbitrarily at $\vec{x}_i$, $i = 1, \dots, N$. With this configuration we can engineer different gauge theories. In the simplest case, we may have strings that begin and end on the same brane, each contributing a $U(1)$ gauge theory, with resultant $U(1)^N$ gauge group. However, we can also consider strings that stretch between two distinct branes. These are oriented, and one string has $U(1) \times U(1)$ charge $(1, -1)$, whilst the other has charge $(-1, 1)$. The mass of the states is $m_n \sim \frac{n}{L} + TL$, where the tension $T = l_s^{-2}$, with l_s the string length[12]. We see that the zero mode mass goes as $m_0 \sim TL$, so as the strings are brought to be coincidental, $L \rightarrow 0$, these modes become massless. For N coincidental branes, there are N^2 massless multiplets, giving the enhanced symmetry of $U(N)$. As we're interested in the dynamics of the gauge theory, we can ignore the $U(1)$ symmetry that corresponds to the position of the branes in spacetime, and thus this configuration gives rise to a $U(N)/U(1) \sim SU(N)$ gauge theory.

Considering the bosonic part of the theory, our massless excitations are the gauge field A_μ and scalars ϕ^I (in the $\underline{6}$ of $SO(6)$), both transforming in the adjoint representation. The effective action for an observer on the brane is given by

$$S \sim \int_{WV} d^4\xi Tr \left(\frac{1}{4} F_{ab} F^{ab} + \frac{1}{2} D_a \phi^I D_b \phi^I - \frac{1}{4} \sum_{I \neq J} Tr[\phi^I, \phi^J]^2 \right), \quad (29)$$

where the first two terms are the kinetic terms of the gauge and scalar fields respectively, and the third term is the scalar potential. The integral is taken over the worldvolume of the brane, parameterised by coordinates ξ . We can extend this to include the fermionic sector, and thus find that the low-energy dynamics of N parallel D3-branes is given exactly by the $\mathcal{N} = 4$ Super Yang-Mills theory in $d = 4$ with $SU(N)$ gauge group.

6 Black Holes and p-branes

In the previous section we looked at D-branes as objects within string theory that carry the Ramond-Ramond charges, harbouring a gauge theory on their worldvolume. Below we are led to the second viewpoint, where D-branes are seen as solutions to the supergravity equations, deforming the geometry around them into AdS space. Both analyses from the crux of Maldacena's conjecture. It is natural to begin looking at the simplest example of a brane - the black hole - before extending to p -spatial dimensions, known as a p -brane. We consider the Schwarzschild black hole, which is non-rotating and uncharged.

6.1 Schwarzschild Metric

The Schwarzschild metric is a solution to Einstein's field equations in a vacuum, $R_{\mu\nu} - \frac{1}{2} \mathcal{R} g_{\mu\nu} = 0$. It describes the behaviour of a gravitational field outside some spherically symmetric mass. The metric is given by[9]

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2 d\Omega_2^2, \quad (30)$$

where r is a radial coordinate measured from the origin at $r = 0$, and $d\Omega_2^2 = d\theta^2 + \sin\theta d\phi^2$ is the spatial metric for a two-sphere. Birkhoff's theorem states that the Schwarzschild solution is the unique spherically symmetric solution to the Einstein equations. We can note that the metric has no explicit dependence on t and admits a killing vector $\xi = \partial_t$. Consequently, this means that any spherically symmetric solution must be static, or independent of time. One interesting consequence of Birkhoff's theorem is that the solution does not emit gravitational radiation[14].

The metric is given by

$$g_{\mu\nu} = \begin{pmatrix} -(1 - \frac{2M}{r}) & 0 & 0 & 0 \\ 0 & (1 - \frac{2M}{r})^{-1} & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2 \sin^2\theta \end{pmatrix}$$

It looks as though something funny may happen at $r = 2M$, indeed we find that at this radius the metric is not invertible and blows up - this is called the *event horizon* of the black hole. In order to try and understand the physicality of this, we can consider a particle on the surface of a ball of pressureless dust which is collapsing from rest at a radius R_{max} . The collapse is radially inwards ($d\theta = d\phi = 0$) and is spherically symmetric, so its exterior can be described by the Schwarzschild metric (the interior of the ball may have more complicated dynamics, but we do not consider them here). How long does it take for the particle to fall to a radius of $r = 2M$? The total time is expressed as

$$T_{2M} = \int_0^{t_{2M}} dt.$$

To evaluate this, we consider the action of a massive particle

$$S = \int d\tau \mathcal{L} = \int d\tau \left(- \left(1 - \frac{2M}{r} \right) \left(\frac{dt}{d\tau} \right)^2 + \left(1 - \frac{2M}{r} \right)^{-1} \left(\frac{dr}{d\tau} \right)^2 - m^2 \right),$$

where τ is the proper time. Using the Euler-Lagrange equations

$$\frac{d}{d\tau} \left(\frac{\partial \mathcal{L}}{\partial (\frac{dt}{d\tau})} \right) - \frac{\partial \mathcal{L}}{\partial t} = 0,$$

we find

$$\frac{d}{d\tau} \left(\left(1 - \frac{2M}{r} \right) \left(\frac{dt}{d\tau} \right) \right) = 0.$$

The result defines a constant of motion along the worldline of the particle

$$\epsilon = \left(1 - \frac{2M}{r} \right) \frac{dt}{d\tau}.$$

The initial condition for the particle was to fall from rest, $\frac{dR_{max}}{d\tau} = 0$, and for a massive particle we have $ds^2 = -d\tau^2$

$$\begin{aligned} \rightarrow \quad & \left(\frac{dt}{d\tau} \right)^2 \left(1 - \frac{2M}{R_{max}} \right) = 1 \\ & 1 - \epsilon^2 = \frac{2M}{R_{max}}. \end{aligned}$$

Using these and solving the equations of motion for R , we find that the total time taken for a particle to fall to a radius of $2M$ is

$$T = \int_0^{t_{2M}} dt = -\epsilon \int_{R_{max}}^{2M} \frac{dR}{\left(1 - \frac{2M}{R} \right)} \left(\frac{2M}{R} - \frac{2M}{R_{max}} \right)^{-1/2}.$$

So as $r \rightarrow 2M$, the total time diverges and it seems as though the particle takes an infinite amount of time to reach the event horizon! Of course, we know this is not the case - so what is happening? It turns out that this is due to the choice of coordinates, t is a bad coordinate at $r = 2M$. Out towards $r \rightarrow \infty$, the metric is asymptotically flat, and t acts as though it measures what we call time. However, as we approach the horizon, this is no longer the case - coordinates have no physical meaning, they are just a parameterisation of spacetime. We can define a new coordinate r_* via the relation

$$dr_*^2 = \left(1 - \frac{2M}{r} \right)^{-2} dr^2.$$

And go into Eddington-Finkelstein coordinates once more

$$\begin{aligned} v &= t + r_* \\ u &= t - r_*. \end{aligned}$$

Choosing the ingoing coordinate v , the Schwarzschild metric in terms of (v, r, θ, ϕ) becomes

$$ds^2 = - \left(1 - \frac{2M}{r} \right) dv^2 + 2dvdr + r^2 d\Omega_2^2. \quad (31)$$

If we do the same analysis as before, taking the radius to $r = 2M$, the (v, r) part of the metric reads

$$g_{\mu\nu} |_{r=2M} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

which is invertible. Therefore the metric is perfectly regular at $r = 2M$; nothing special happens. This is an example of a coordinate singularity, where the singularity of the metric can be remedied by moving into another coordinate system. This is in contrast to a curvature singularity, which will exist in all coordinate bases, and has physical meaning within the spacetime. In the Schwarzschild example, a curvature singularity exists at $r = 0$; it is here that coordinate-invariant measures such as the Kretschmann scalar (the square of the Riemann tensor) diverge[2].

6.2 Reissner-Nordström Metric

In this example, we consider gravity coupled to an electric field, whose dynamics can be described by the Einstein-Maxwell action

$$S = \frac{1}{16\pi G} \int \sqrt{-g}(R - F_{\mu\nu}F^{\mu\nu})d^4x, \quad (32)$$

where R is the Ricci scalar and $F_{\mu\nu}$ is the electromagnetic field strength. The equations of motion here also admit a spherically symmetric solution, describing a non-rotating black hole parameterised by a source charge Q and mass M . This solution is known as the Reissner-Nordström metric

$$ds^2 = -(1 - \frac{2M}{r} + \frac{Q^2}{r^2})dt^2 + (1 - \frac{2M}{r} + \frac{Q^2}{r^2})^{-1}dr^2 + r^2d\Omega_2^2.$$

In reality, these black holes are unlikely to exist - the universe is approximately neutral, so if a black hole were to become charged, it would quickly neutralise by interacting with the matter surrounding it. However, as we will see, it is instructive to study this metric in order to aid our understanding of p -branes in string theory. The coefficient

$$(1 - \frac{2M}{r} + \frac{Q^2}{r^2})$$

is a polynomial in r and may be factorised in the form $\Delta(r) = (r - r_-)(r - r_+)$. Solving the quadratic equation $r^2 - 2Mr + Q = 0$ gives us two roots

$$r = r_{\pm} = M \pm \sqrt{M^2 - Q^2}. \quad (33)$$

As there are two parameters, Q and M , we have three situations:

1. $M < Q$: this is the superextremal case; the roots become imaginary and there are no real solutions. Thus, there is no event horizon surrounding the black hole and we have a naked singularity, observable to the outside universe, at $r = 0$. However, it is strongly believed that naked singularities do not exist in nature, proposed in the conjecture known as Cosmic Censorship by Penrose[23]. Indeed, if they were to exist, then they would be apparent for us to observe, which has not been the case.
2. $M > Q$: this is the subextremal case. We have two real roots and two separate event horizons; r_+ is the outer horizon, r_- is the inner horizon. Although not physical, the subextremal case has a few interesting properties - the inner horizon where an observer falling over may see the infinite history of the universe, and internal infinities.
3. $M = Q$: this is the extremal case, and the most interesting for the purposes of this thesis. Here, both horizons converge into a single horizon at $r = M$.

6.3 Extremal Reissner-Nordström Metric

This metric has a coordinate singularity at $r = M$. We can make a change of coordinates, defining $r = M + \bar{r}$, such that

$$\left(1 - \frac{M}{r}\right) = \left(1 - \frac{M}{M + \bar{r}}\right) = \left(\frac{M + \bar{r}}{\bar{r}}\right)^{-1} = \left(1 + \frac{M}{\bar{r}}\right)^{-1},$$

with the metric taking on the form

$$ds^2 = -\left(1 + \frac{M}{\bar{r}}\right)^{-2} dt^2 + \left(1 + \frac{M}{\bar{r}}\right)^2 d\bar{r}^2 + \bar{r}^2 \left(1 + \frac{M}{\bar{r}}\right)^2 d\Omega_2^2.$$

From here we are able to define an Harmonic function (satisfying Laplace's equation $\Delta H = 0$) $H = (1 + \frac{M}{\bar{r}})$, giving the metric

$$ds^2 = -H^{-2} dt^2 + H^2 (d\bar{r}^2 + \bar{r}^2 d\Omega_2^2).$$

Looking at long distance, as $\bar{r} \rightarrow \infty$, the metric becomes asymptotically flat, suggesting that gravity is very weak. We can also look at the spacetime around $\bar{r} = 0$, which corresponds to the region around the event horizon of the extremal Reissner-Nordström black hole, and as such is known as *near-horizon*. In this regime, the metric takes the form

$$ds^2 \rightarrow -\left(\frac{\bar{r}}{M}\right)^2 dt^2 + \left(\frac{M}{\bar{r}}\right)^2 d\bar{r}^2 + M^2 d\Omega_2^2,$$

and, by making the substitution $z = \frac{M^2}{\bar{r}}$, it can be brought into a simpler form

$$ds^2|_{\bar{r} \approx 0} = \frac{M^2}{z^2} (-dt^2 + dz^2) + M^2 d\Omega_2^2. \quad (34)$$

The near horizon geometry of the extremal Reissner-Nordström black hole is precisely the product space $AdS_2 \times S^2$, both of radius M . Although innocuous-looking, this is an important result, as it is the first glimpse into the effect that the presence of a gravitational object has on the geometry of spacetime.

6.4 p -branes

We make an extension from the pointlike black hole to p -spatial dimensions. These are known as p -branes, and are solutions of 10-dimension supergravity equations, sourcing the graviton, dilaton and the Ramond-Ramond potentials in the 10-dimensional supergravity spectra. The metric reads [15]

$$ds^2 = Z_p^{-\frac{1}{2}}(r) \left(-K(r) dt^2 + \sum_{i=1}^p dx_i^2 \right) + Z_p^{\frac{1}{2}}(r) \left(\frac{dr^2}{K(r)} + r^2 d\Omega_{8-p}^2 \right), \quad (35)$$

where [15]

- $Z_p(r) = 1 + \alpha_p \left(\frac{r_p}{r}\right)^{7-p}$
- $K(r) = 1 - \left(\frac{r_H}{r}\right)^{7-p}$
- $\alpha_p = \sqrt{1 + \left(\frac{r_H^{7-p}}{2r_p^{7-p}}\right)^2} - \frac{r_H^{7-p}}{2r_p^{7-p}}$

p -branes span p -dimensions, but remain localised in the other $9-p$ dimensions transverse to the brane, admitting spherical symmetry. Because of this, we can parameterise the space in these dimensions with a radial coordinate r and polar coordinates on the $8-p$ sphere. Observing the metric for the p -brane, the analogy to the black holes described above is obvious, and it is natural to ask whether there is an analogous definition for an 'event horizon' and coordinate singularity. The answer is yes: the event horizon, with a coordinate singularity, is positioned at the radius $r = r_H$, and the curvature singularity is once again at $r = 0$. We can take $r_H = 0$, giving the metric

$$ds^2 = H_p^{-\frac{1}{2}}(r) \left(-dt^2 + \sum_{i=1}^p dx_i^2 \right) + H_p^{\frac{1}{2}}(r) (dr^2 + r^2 d\Omega_{8-p}^2),$$

giving the *extremal* p -brane, analogous to the Reissner-Nordström metric. We define the harmonic function $H_p(r) = 1 + (\frac{r_p}{r})^{7-p}$ (as $\alpha_p = 1$). This also generalises to the situation where we have more than one brane, known as the multicentre solution[15]

$$H_p(r) = 1 + \sum_{i=1}^N \frac{r_p^{7-p}}{|\vec{r} - \vec{r}_i|^{7-p}}, \quad (36)$$

which represents N parallel branes situated at some arbitrary position $\vec{r}_i$, each with charge N_i , such that the total charges sum to N . For the purposes of this investigation, the example that we would like to concentrate on is $p = 3$. For all other choices of p , the area of the $8-p$ sphere vanishes for $r = 0$, giving a singularity of zero area. However, for $p = 3$, the factor of r in the harmonic function cancels with the coefficient of the 5-sphere, leaving a finite area. For $p = 3$ we can explore the geometry of this solution near $r = 0$.

$$ds^2|_{r \approx 0} = \frac{r^2}{r_3^2} (-dt^2 + d\vec{x}^2) + \frac{r_3^2}{r^2} (dr^2 + r^2 d\Omega_5^2).$$

Making an analogous substitution to the Reissner-Nordström case, $z = \frac{L^2}{r}$, gives the metric in Poincaré coordinates

$$ds^2|_{r \approx 0} = \frac{L^2}{z^2} (-dt^2 + d\vec{x}^2 + dz^2) + L^2 d\Omega_5^2. \quad (37)$$

The near horizon geometry is the product spacetime $AdS_5 \times S^5$, and will play a very important role within the correspondence!

7 AdS/CFT Correspondence

Finally, with all of the ingredients in place, we come to the correspondence. In the previous sections two objects were studied: the D-brane and the p -brane. In 1995, Polchinski showed that these two objects are in fact the same - supergravity solutions correspond to the dynamical endpoints of strings.[24] This gives two viewpoints of D3-branes

- a configuration of N parallel D3-branes with strings stretched between. When they coincide, the low energy dynamics are described by $\mathcal{N} = 4$ Super Yang-Mills theory with $SU(N)$ gauge symmetry
- a configuration of N extremal D3-branes which are solutions of supergravity. The near-horizon geometry of these solutions is the product of spacetime $AdS_5 \times S^5$.

These viewpoints form the foundation of the conjecture, but we must consider the decoupling limits to make it apparent.

7.1 Decoupling Limit: Gauge Theory

We begin by considering Type IIB String Theory, with closed strings propagating in the bulk and open strings that end on the brane. We can therefore describe the dynamics of the theory with the action

$$S_{IIB} = S_{brane} + S_{bulk} + S_{int},$$

where:

- S_{brane} is the action defined on the $(3+1)$ -dimensional world volume of the brane
- S_{bulk} is the action ten dimensional supergravity in the bulk
- S_{int} is the action describing the interaction between the bulk and the brane modes

In the low energy, $l_s \rightarrow 0$, limit we saw that the effective action on the brane is that of SYM gauge theory. Furthermore, the interaction term depends on the string length, $S_{int} \sim l_s^4$, and thus also vanishes[1]. The brane modes decouple from the bulk modes, leaving a SYM defined on the brane and free supergravity propagating in the bulk.

7.2 Decoupling Limit: Supergravity Theory

Next we consider the D-branes as supergravity solutions. The near horizon geometry can be described by $AdS_5 \times S^5$, and as $r \rightarrow \infty$, the geometry is asymptotically flat. The action of a relativistic massive particle is given by

$$S \sim \int \mathcal{L} d\tau \sim \int (g_{\nu\nu} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau} - m^2)^{\frac{1}{2}} d\tau.$$

Singling out time, the action can be extremised to find the Euler-Lagrange Equations

$$\frac{d}{d\tau} \left(\frac{\partial \mathcal{L}}{\partial \left(\frac{dt}{d\tau} \right)} \right) - \frac{\partial \mathcal{L}}{\partial t} = 0.$$

From Birkhoff's theorem and the extension to p -spatial branes, the supergravity solutions are static, and the second term vanishes, giving

$$\frac{d}{d\tau} \left(-2H_p^{-\frac{1}{4}} \frac{dt}{d\tau} \right) = 0,$$

and thus, as we can determine a constant of motion, we see the proper time interval is related to dt by

$$d\tau = H_p^{-\frac{1}{4}} dt.$$

Suppose an observer sat at constant radius r_1 emits a photon to another observer at $r = \infty$

$$\begin{aligned} H_p(r_1) &= 1 + \left(\frac{r_p}{r_1} \right)^{7-p} \\ H_p(r = \infty) &= 1 + \left(\frac{r_p}{\infty} \right)^{7-p} = 1. \end{aligned}$$

The proper time intervals are $d\tau_1 = H_p^{-\frac{1}{4}}(r_1)dt$ and $d\tau_\infty = dt$. The frequency of the photons are inversely related to the proper time, and the redshift can be calculated by taking the ratio of frequencies between both observers

$$\frac{d\tau_1}{d\tau_\infty} = \frac{\omega_\infty}{\omega_1}$$

$$\begin{aligned} H_p^{-\frac{1}{4}} \omega_1 &= \omega_\infty \\ H_p^{-\frac{1}{4}} E_1 &= E_\infty, \end{aligned}$$

as $E \sim \omega$. Therefore we see that the photon has been redshifted by a factor of $H_p^{-\frac{1}{4}}$. As $r_1 \rightarrow 0$, the more the photon redshifts and the lower the energy to an observer at infinity. This is due to the gravitational potential well, analogous to approaching the event horizon of a black hole. The dynamics in the throat, where the spacetime is $AdS_5 \times S^5$, decouple from the dynamics in the bulk, where we also have a low energy regime (spacetime is asymptotically flat)[22], and the supergravity is free. Both scenarios contain supergravity decoupled in flat space. Therefore, we are led to Malcadena's conjecture[16]:

Type IIB superstring theory on $AdS_5 \times S^5$ is dual to $\mathcal{N} = 4$ $SU(N)$ Super-Yang Mills theory in $(3+1)$ -dimensions.

7.3 Limits and Validity

Now a relation between the gauge and string couplings has been made, the validity of the correspondence can be explored. For field theories, we can use perturbative analysis when the couplings are weak. For Super Yang-Mills, this corresponds to

$$g_{YM}^2 N \sim g_s N \sim \frac{L^4}{l_s^4} \ll 1. \quad (38)$$

On the other hand, supergravity only becomes reliable when the radius of curvature is greater than the string length

$$\frac{L^4}{l_s^4} \sim g_s N \sim g_{YM}^2 N \gg 1. \quad (39)$$

For supergravity to be valid, we typically require N to be large. There seems to be a complication - the two sides of the correspondence are valid in completely different regimes. This is the reason why the correspondence is called a duality - it relates the strong coupling of one theory to the weak coupling of the other, which in turn makes it very difficult to prove.

There are three forms of the correspondence with varying degrees of strength. These are:

- weakest form: valid for large $g_s N$, thus valid for the supergravity approximation. However, may not be valid for the full string theory away from this limit.
- mid-strength form: here we take the 't Hooft limit, keeping $\lambda = g_{YM}^2 N$ fixed, but taking $N \rightarrow \infty$ and $g_s \rightarrow 0$
- strongest form: valid for all g_s and N .

7.4 The Field-Operator Correspondence

For a duality to exist between two theories, there should be a one-to-one correspondence and mapping between their observables: fields, operators, and correlation functions. In the previous section we saw that the topological expansion derived by 't Hooft allowed us to relate the gauge theory coupling constant to that of the strings, $g_{YM}^2 \sim g_s$. In string theory, the string coupling is related to the dilaton field by

$$g_s = e^{\langle \phi \rangle}, \quad (40)$$

where $\langle \phi \rangle$ denotes the expectation value of the dilaton, which is set by the boundary conditions given at infinity[1]. Therefore, if the coupling strength of the gauge theory is changed, this will change the string theory boundary conditions for the dilation at the infinite boundary of space. For string theory in AdS_5 , we see that the conformal boundary is that of a 4-dimensional Minkowski space, and can introduce a coupling of the CFT that lives on the boundary

$$\int d^4x \phi_0(x) \mathcal{O}, \quad (41)$$

where $\phi_0(x)$ is the source for the CFT operator $\mathcal{O}$. Because the field is fully determined by its boundary conditions, we want to ensure that the value of the bulk field tends towards $\phi_0(x)$ as it approaches the boundary

$$\phi(z, \vec{x})|_{z=0} = \phi_0(\vec{x}).$$

On the CFT side of the correspondence, we have a source-operator relation, and would like to compute the correlation functions for distinct points $\langle \mathcal{O}(x_1) \dots \mathcal{O}(x_n) \rangle$. Witten extended this to defining a generating functional $\langle \exp(\int_{S^4} \phi_0 d^4x) \rangle_{CFT}$ and proposing the precise correspondence[31]

$$\langle \exp(\int_{S^4} \phi_0 d^4x) \rangle_{CFT} = \mathcal{Z}_S[\phi(z, \vec{x})|_{z=0} = \phi_0(\vec{x})]. \quad (42)$$

The right hand side is the classical supergravity partition function, which may be written as

$$\mathcal{Z}_S[\phi(z, \vec{x})|_{z=0} = \phi_0(\vec{x})] = \exp(-S_{SUGRA}(\phi)).$$

The Witten prescription will be used in the sections below, where we look at the structure of the propagators connecting the bulk dynamics to the boundary, and from this determine the correlation functions of the Super Yang-Mills theory

7.5 Wave Equation in AdS Space

The string theory side of the correspondence has a bulk-boundary relation, and we must understand how to express the dynamics of the scalar field in the bulk in terms of the boundary operator. We want to solve the wave equation in AdS_5 , i.e. the Klein-Gordon equation for a scalar field

$$(\square - m^2) \phi = 0. \quad (43)$$

To do this, we make use of the metric given in Poincaré coordinates

$$ds^2 = \frac{L^2}{z^2} (dz^2 + \eta_{ij} dx^i dx^j).$$

The boundary in these coordinates is given as $z \rightarrow 0$, which corresponds to the radial coordinate $r \rightarrow \infty$, thus, the gauge theory 'lives at infinity'. The metric in these coordinates is given as

$$g_{\mu\nu} = \frac{L^2}{z^2} (1, -1, 1, \dots, 1) \quad g^{\mu\nu} = \frac{z^2}{L^2} (1, -1, 1, \dots, 1).$$

The d'Alembertian

$$\begin{aligned}
\Box &= \frac{1}{\sqrt{-g}} \partial_\mu (g^{\mu\nu} \sqrt{-g} \partial_\nu) \\
&= \frac{z^n}{L^n} \left[\partial_z \left(\frac{z^2}{L^2} \frac{L^n}{z^n} \partial_z \right) + \frac{L^n}{z^n} \frac{z^2}{L^2} g^{ij} \partial_i \partial_j \right] \\
&= \frac{z^n}{L^n} \left[\partial_z \left(\frac{L^{n-2}}{z^{n-2}} \partial_z \right) + \frac{L^{n-2}}{z^{n-2}} \eta^{ij} \partial_i \partial_j \right],
\end{aligned}$$

where $g = \det g_{\mu\nu} = -\frac{L^{2n}}{z^{2n}}$ and the (i, j) components are just the flat-space metric. Taking the Leibniz rule

$$\begin{aligned}
&= \frac{z^n}{L^n} \left[\frac{L^{n-2}}{z^{n-2}} \partial_z^2 - (n-2) \frac{L^{n-2}}{z^{n-3}} \partial_z + \frac{L^{n-2}}{z^{n-2}} \eta^{ij} \partial_i \partial_j \right] \\
&= \frac{z^2}{L^2} \left[\partial_z^2 - \frac{(n-2)}{z} \partial_z + \eta^{ij} \partial_i \partial_j \right].
\end{aligned}$$

We follow the method of Avery[3] and rotate into Euclidean space to study the behaviour of the field ϕ near the boundary, with d'Alembertian

$$\Box = \frac{z^2}{L^2} \left[\partial_z^2 - \frac{(n-2)}{z} \partial_z + \nabla_{n-1}^2 \right] \quad (44)$$

and consider the ansatz for the solution

$$\begin{aligned}
\phi(z, \vec{x}) &= \frac{z^\Delta}{(z^2 + \vec{x}^2)^\Delta} \\
\phi(z, \vec{x}) &= \frac{z^\Delta}{(z^2 + \vec{x}^2)^\Delta}
\end{aligned} \quad (45)$$

for some parameter Δ . We see

$$\begin{aligned}
\partial_z \phi &= \frac{\Delta z^{\Delta-1} (z^2 + \vec{x}^2)^\Delta - 2\Delta z (z^2 + \vec{x}^2)^{\Delta-1} z^\Delta}{(z^2 + \vec{x}^2)^{2\Delta}} \\
&= \frac{z^\Delta \Delta}{(z^2 + \vec{x}^2)^\Delta} \left[z^{-1} - \frac{2z}{(z^2 + \vec{x}^2)} \right] \\
&= \frac{\Delta \phi}{z} \left[\frac{\vec{x}^2 - z^2}{\vec{x}^2 + z^2} \right]
\end{aligned}$$

and with similar analysis[3]

$$\begin{aligned}
\nabla^2 \phi &= \frac{\phi \Delta}{z^2 (z^2 + \vec{x}^2)} \left[2(2\Delta + 3 - n) z^2 \vec{x}^2 - 2(n-1) z^4 \right] \\
\partial_z^2 \phi &= \frac{\phi \Delta}{z^2 (z^2 + \vec{x}^2)^2} \left[(\Delta-1) \vec{x}^4 - 2(\Delta+2) z^2 \vec{x}^2 + (\Delta+1) z^4 \right].
\end{aligned}$$

Placing this into the wave equation

$$\Box \phi = \frac{\Delta(\Delta+1-n)\phi}{L^2},$$

which, for AdS_5 gives

$$\begin{aligned}\Delta(\Delta - 4) - m^2 L^2 &= 0 \\ \Delta(\Delta - 4) &= m^2 L^2.\end{aligned}\tag{46}$$

This equation is a polynomial in Δ , and for the massive wave equation in general AdS_n , the solutions are

$$\Delta_{\pm} = \frac{(n-1)}{2} \pm \frac{1}{2} \sqrt{(n-1)^2 + 4m^2 L^2}.\tag{47}$$

The aim is to study the dynamics of the fields in the bulk given some boundary conditions, employing the method of the Green's function, which characterises the response of the bulk in the presence of a point source. A simple example of this is within electrostatics, with a point charge at position $\vec{x}$ - what should the charge density be? Well, at any position $\vec{x}' \neq \vec{x}$, the charge density is zero, but at $\vec{x}$ we have a finite charge within an infinitely small point, giving an infinite density. Algebraically, we can write this as

$$\rho(\vec{x}) = \delta(\vec{x}' - \vec{x})Q.$$

Analogously, we want to look at the response of our scalar field in the presence of a point source, and find the Green's function $G(z, \vec{x}; \vec{x}')$ such that

$$(\square - m^2)G(z, \vec{x}; \vec{x}') = \delta(\vec{x} - \vec{x}').\tag{48}$$

Looking at the scalar field

$$\phi(z, \vec{x}) = \frac{z^\Delta}{(z^2 + \vec{x}^2)^\Delta},$$

we see that for $\vec{x} \neq 0$, the field vanishes as we approach the boundary at $z \rightarrow 0$, and at $\vec{x} = 0$, the field diverges. This behaviour is suggestive of a delta function, but to observe whether this holds we must integrate over the $n-1$ directions and approach $z = 0$ [3].

$$\int d^{n-1}x \phi = z^\Delta \int d^{n-1}x \frac{1}{(z^2 + \vec{x}^2)^\Delta} = z^\Delta \Omega_{n-2} \int_0^\infty dr \frac{r^{n-2}}{(z^2 + r^2)^\Delta},$$

Where Ω_{n-2} is the volume of an $(n-2)$ -dimensional ball, with the standard result

$$\begin{aligned}\Omega_{n-2} &= \frac{2\pi^{(n-1)/2}}{\Gamma(\frac{n-1}{2})} \\ \Rightarrow &= z^\Delta \frac{2\pi^{(n-1)/2}}{\Gamma(\frac{n-1}{2})} \int_0^\infty dr \frac{r^{n-2}}{(z^2 + r^2)^\Delta} \\ &= z^\Delta \frac{2\pi^{(n-1)/2}}{\Gamma(\frac{n-1}{2})} \int_0^\infty (2r dr) \frac{\frac{1}{2}r^{n-3}}{(z^2 + r^2)^\Delta} \\ &= z^\Delta \frac{2\pi^{(n-1)/2}}{\Gamma(\frac{n-1}{2})} \frac{1}{z^{2\Delta}} \int_0^\infty (2r dr) \frac{\frac{1}{2}r^{n-3}}{(1 + \frac{r^2}{z^2})^\Delta}.\end{aligned}$$

Making a change of variables $t = r^2/z^2$

$$\begin{aligned}
\Rightarrow &= z^\Delta \frac{2\pi^{(n-1)/2}}{\Gamma(\frac{n-1}{2})} \frac{z^{n-1}}{2z^{2\Delta}} \int_0^\infty \frac{t^{n-1}}{(1+t)^\Delta} dt \\
&= z^\Delta \frac{2\pi^{(n-1)/2}}{\Gamma(\frac{n-1}{2})} \frac{z^{n-1}}{2z^{2\Delta}} B\left(\frac{n-1}{2}, \Delta - \frac{n-1}{2}\right) \\
&= \frac{\pi^{(n-1)/2}}{\Gamma(\frac{n-1}{2})} \frac{z^{n-1}}{z^\Delta} \frac{\Gamma(\frac{n-1}{2})\Gamma(\Delta - \frac{n-1}{2})}{\Gamma(\Delta)} \\
&= \pi^{(n-1)/2} z^{n-1-\Delta} \frac{\Gamma(\Delta - \frac{n-1}{2})}{\Gamma(\Delta)},
\end{aligned}$$

where we used the beta function

$$B(x, y) = \int_0^\infty \frac{t^{x-1}}{t^{x+y}} dt = \frac{B(x)B(y)}{B(x+y)}.$$

We denote the coefficient of z as a normalisation constant C_n , and eventually find that

$$\int dx^{n-1} \phi = C_n z^{n-1-\Delta}. \quad (49)$$

The scalar field doesn't quite behave as a delta-function, but $\phi z^{\Delta+1-n}$ does, as z approaches the boundary. The dynamics of the bulk field can be described in terms of the boundary field and some boundary-to-bulk propagator. We give the bulk field as

$$\phi(z, \vec{x}) = \int d^{n-1}y K_\Delta(z, \vec{x}; \vec{y}) \phi_0(\vec{y}),$$

where

$$K_\Delta(z, \vec{x}; \vec{y}) = \frac{1}{C_n} \frac{z^\Delta}{(z^2 + (\vec{x} - \vec{y})^2)^\Delta}, \quad (50)$$

such that the wave equation is solved and the boundary is approached as $K_\Delta(z, \vec{x}; \vec{y})|_{z \rightarrow 0} = z^{n-1-\Delta} \delta(\vec{x} - \vec{y})$. This is the *bulk-to-boundary* propagator. It is this powerful tool which allows us to write the dynamics of the bulk in terms of the boundary for the two and three point functions. The solution to the wave equation had two roots

$$\Delta_\pm = \frac{(n-1)}{2} \pm \frac{1}{2} \sqrt{(n-1)^2 + 4m^2 L^2},$$

and the asymptotic behaviour near the boundary forces us to take the Δ_+ solution, so as we approach the boundary the field behaves like $\phi|_{z \rightarrow 0} \sim z^{4-\Delta_+} \phi(\vec{y}) = z^{\Delta_-} \phi_0(\vec{y})$. We can isolate the behaviour that satisfies this within the neighbourhood of the boundary, such that

$$\phi(\vec{x}, \epsilon) = \epsilon^{\Delta_-} \phi_0(\vec{y}), \quad (51)$$

where we eventually take the limit $\epsilon \rightarrow 0$ to reach the boundary. Because the dilaton field is dimensionless, this implies that the boundary value has mass dimension Δ_- . This boundary field also acts as a source for the action coupling term $\int d^4x \phi_0 \mathcal{O}$, which must be dimension, implying that the operator $\mathcal{O}$ has a conformal weight $4 - \Delta_- = \Delta_+$. In the next section we will look at the correlation functions for operators of this conformal weight, checking that they conform to the restrictions imposed by the high symmetry of the conformal group.

7.6 Two-Point and Three-Point Functions

7.6.1 Two-Point Functions

In the previous section we saw that a field in the bulk of AdS_5 can be described in terms of its boundary value ϕ_0 via the bulk-to-boundary propagator, $K_{\Delta_+}(z, \vec{x}; \vec{y})$. This gives the bulk-to-boundary relation

$$\phi(z, \vec{x}) = \int d^4y K_{\Delta_+}(z, \vec{x}; \vec{y}) \phi_0(\vec{y}) = \frac{1}{C_n} \int d^4y \frac{z^{\Delta_+}}{(z^2 + |\vec{x} - \vec{y}|^2)^{\Delta_+}} \phi_0(\vec{y}).$$

The dynamics of a free scalar field in AdS_5 are given by the SUGRA action

$$S_{SUGRA} = \frac{1}{2} \int \sqrt{g} (g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - m^2 \phi^2) d^4x dz.$$

This can be integrated by parts, and by imposing the on-shell wave equation $(\square - m^2)\phi = 0$, only the boundary term remains

$$\sqrt{g} g^{\mu\nu} (\partial_\mu \phi \partial_\nu \phi) = \partial_\mu (\sqrt{g} g^{\mu\nu} \phi \partial_\nu \phi) - \phi \partial_\mu (\sqrt{g} g^{\mu\nu} \partial_\nu \phi) = \partial_\mu (\sqrt{g} g^{\mu\nu} \phi \partial_\nu \phi) - \phi \sqrt{g} \square \phi$$

$$\begin{aligned} S_{SUGRA} &= \frac{1}{2} \int \partial_\mu (\sqrt{g} g^{\mu\nu} \phi \partial_\nu \phi) d^4x dz - \int \sqrt{g} \phi (\square - m^2) \phi d^4x dz \\ &= \frac{1}{2} \int \partial_\mu (\sqrt{g} g^{\mu\nu} \phi \partial_\nu \phi) d^4x dz. \end{aligned}$$

Witten noticed a nice trick here [31], using the divergence theorem to express this in terms of a surface integral

$$= \lim_{z \rightarrow 0} \left(-\frac{1}{2} \int_B \sqrt{g} g^{zz} \phi \partial_z \phi d^4x \right), \quad (52)$$

with a normal vector whose components are only non-zero in the z direction, i.e. they're perpendicular to the boundary. Setting the curvature radius $L = 1$ for ease and in Poincaré coordinates, $g^{zz} = z^2$ and $\sqrt{g} = z^{-5}$ for AdS_5

$$\Rightarrow = \lim_{z \rightarrow 0} -\frac{1}{2} \int_B z^{-3} \phi \partial_z \phi d^4x.$$

The behaviour of $\partial_z \phi$ goes as

$$\begin{aligned} \partial_z \phi(\vec{x}, z; \vec{y}) &= \frac{1}{C_n} \int d^4y \partial_z \left[\frac{z^{\Delta_+}}{(z^2 + |\vec{x} - \vec{y}|^2)^{\Delta_+}} \right] \phi_0(\vec{y}) \\ &= \frac{\Delta_+}{C_n} \int d^4y \left[\frac{z^{\Delta_+-1} (z^2 + |\vec{x} - \vec{y}|^2)^{\Delta_+} - 2z^{\Delta_++1} (z^2 + |\vec{x} - \vec{y}|^2)^{\Delta_+-1}}{(z^2 + |\vec{x} - \vec{y}|^2)^{2\Delta_+}} \right] \phi_0(y) \\ &= \frac{\Delta_+}{C_n} \int d^4y \left[\frac{z^{\Delta_++1} + z^{\Delta_+-1} |\vec{x} - \vec{y}|^2 - 2z^{\Delta_++1}}{(z^2 + |\vec{x} - \vec{y}|^2)^{\Delta_++1}} \right] \phi_0(\vec{y}), \end{aligned}$$

and recalling, as we approach the boundary, ϕ behaves as $\phi|_{z \rightarrow 0} = z^{\Delta_-} \phi_0$. We also find that the powers of z cancel, leaving

$$S_{SUGRA} = \frac{1}{2} \frac{\Delta_+}{C_n} \int d^4x d^4y \frac{\phi_0(\vec{x})\phi_0(\vec{y})}{|\vec{x} - \vec{y}|^{2\Delta_+}}. \quad (53)$$

From Witten's postulate, the correlation function is given by

$$\langle \mathcal{O}(\vec{x}_1) \mathcal{O}(\vec{x}_2) \rangle = \left[\left(-i \frac{\delta}{\delta \phi_0(\vec{x}_1)} \right) \left(-i \frac{\delta}{\delta \phi_0(\vec{x}_2)} \right) e^{-S_{SUGRA}} \right] |_{\phi_0=0},$$

which, of course, is the normal definition of the correlation function by taking the functional derivative of the generating functional with sources ϕ_0 .

$$\begin{aligned} &= \left(-\frac{\delta}{\delta \phi_0(\vec{x}_2)} \right) \left[\frac{1}{2} \frac{\Delta_+}{C_n} \left(\frac{\phi_0(\vec{x}_1)\delta(\vec{x}_1 - \vec{x})}{|\vec{x} - \vec{y}|^{2\Delta_+}} + \frac{\phi_0(\vec{x}_1)\delta(\vec{x}_1 - \vec{y})}{|\vec{x} - \vec{y}|^{2\Delta_+}} \right) e^{-S_{SUGRA}} \right] |_{\phi_0=0} \\ &= \left[-\frac{1}{2} \frac{\Delta_+}{C_n} \left(\frac{\delta(\vec{x}_1 - \vec{x})\delta(\vec{x}_2 - \vec{y})}{|\vec{x} - \vec{y}|^{2\Delta_+}} + \frac{\delta(\vec{x}_1 - \vec{y})\delta(\vec{x}_2 - \vec{x})}{|\vec{x} - \vec{y}|^{2\Delta_+}} \right) \right] \\ \langle \mathcal{O}(\vec{x}_1) \mathcal{O}(\vec{x}_2) \rangle &\sim \frac{\Delta_+}{C_n} \frac{1}{|\vec{x}_1 - \vec{x}_2|^{2\Delta_+}}. \end{aligned} \quad (54)$$

7.6.2 Three-Point Functions

For three point functions, we look for a cubic term in the supergravity action

$$S_{SUGRA} = \int d^5x \sqrt{g} \left[\sum_i \frac{1}{2} (\partial \phi_i)^2 + \frac{1}{2} m_i^2 \phi_i^2 + \lambda \phi_1 \phi_2 \phi_3 \right],$$

where for each field there is the bulk-to-boundary relation

$$\phi(z, \vec{x}) = \int d^4x_i K_{\Delta_+}(z, \vec{x}; \vec{x}_i) \phi_0(\vec{x}_i),$$

and using the same methods as the two point function, the three point function is given by

$$\begin{aligned} \langle \mathcal{O}(\vec{x}_1) \mathcal{O}(\vec{x}_2) \mathcal{O}(\vec{x}_3) \rangle &= -\lambda \int d^5x \sqrt{g} K_{\Delta_+}(x; \vec{x}_1) K_{\Delta_+}(x; \vec{x}_2) K_{\Delta_+}(x; \vec{x}_3) \\ &= \frac{\lambda C_{123}}{|\vec{x}_1 - \vec{x}_2|^{\Delta_1 + \Delta_2 - \Delta_3} |\vec{x}_1 - \vec{x}_3|^{\Delta_1 + \Delta_3 - \Delta_2} |\vec{x}_2 - \vec{x}_3|^{\Delta_2 + \Delta_3 - \Delta_1}}, \end{aligned} \quad (55)$$

for some constant C_{123} , whose value is determined by the evaluation of the x integral [1]. Comparing equations (55), (56) with the conformally restricted forms of the correlations functions in (17) and (18), we see that these correlations are exactly what we would expect for operators of conformal dimension Δ_+ .

8 Conclusion

In this paper, we have seen how two seemingly unrelated theories, one with gravity and one strictly without, can be brought together to form the duality that is the AdS/CFT Correspondence. This particularly exciting example of a strong/weak duality allows one to probe the strong coupling of field theories, difficult to access otherwise, through a perturbative expansion in a gravity theory. From this we can learn many lessons in QCD, and indeed use the the correspondence to formulate the idea of holographic QCD, providing a systematic approach in describing the messier sides of the field theory, such as nuclear physics and heavy ion collisions. By bringing string theory back to its original roots, we had to use a number of components from vast expanses of physics, each one crucial in providing a comprehensive understanding of the conjecture. Throughout the paper, there were tantalising hints of a duality between AdS_5/CFT_4 , including

- The isometry group of AdS_5 is the conformal group symmetry in four dimensions, $SO(2, 4)$
- The conformal boundary of AdS_5 space is that of Minkowski in four dimensions, $\mathbb{R}^{1,3}$, suggesting this domain is the home for the conformal field theory
- 't Hooft's large N limit and idea to look at the topological structure of Feynman diagrams uncovered the relation between the Yang-Mills coupling g_{YM} and the string coupling, g_s

There was no safe leap across the chasm without the introduction of D3-branes, extended objects lying at the heart of AdS_5/CFT_4 . By looking at these as endpoints of open strings and solutions to supergravity, we were able to put forward the conjecture that Type IIB string theory on $AdS_5 \times S^5$ is dual to $\mathcal{N} = 4$ $SU(N)$ Super-Yang Mills theory in $(3 + 1)$ -dimensions. The holographic principle allowed us to use the dynamics of the bulk in AdS_5 space to formulate the two- and three-point correlation functions through the boundary value of the field ϕ_0 , which acted as a source for the conformal operators.

This, however, is only one particular example of the gauge/gravity duality and, through the use of holography, we are able to choose a variety of brane configurations from both string theory and M-theory to realise more exotic field theories. In doing so, we illuminate shrouded paths of the labyrinth and, step by step, strive to provide a correspondence describing the nature around us.

Acknowledgements

I would like to thank Professor Dan Waldram for all of his help both throughout this thesis and the last one. I have thoroughly enjoyed working with him, and he has made me want to be a physicist in the future. I would also like to thank my parents for their unending love and support, and all of my friends for being there for me, even in the darkest of times. Finally, thank you to the physics gang, it has been the best.

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From Gravity to Thermal Gauge Theories: The Ads/CFT Correspondence

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ISSN 0075-8450

e-ISSN 1616-6361

ISBN 978-3-642-04863-0

e-ISBN 978-3-642-04864-7

DOI 10.1007/978-3-642-04864-7

Springer Heidelberg Dordrecht London New York

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Cover design: eStudio Calamar, Berlin/Figueres

Printed on acid-free paper

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Preface

This book is an edited version of the review talks given in the Fifth Aegean School on the AdS/CFT correspondence, held in Adamas on Milos Island, Greece, from 21st to 26th of September 2009. The aim of this book is to present an advanced multiauthored textbook which meets the needs of both the postgraduate students and the young researchers, in the fields of Field Theory, String Theory, Gravity and Condensed Matter Physics.

The AdS/CFT correspondence is a powerful tool in studying strongly coupled phenomena in field theory using results from a weakly coupled gravity background. The principle of the AdS/CFT was developed within the string theory and it was proved to be very useful in describing strongly coupled phenomena in gauge theories like quark-gluon plasma and heavy ions collisions. Soon it was realized that its applicability can be extended, in a more phenomenological approach, to condensed matter systems and to systems described by fluid dynamics.

The selected contributions to this volume are aimed to describe the principle of the AdS/CFT correspondence in its field theoretic formulation in string theory, its applicability to holographic QCD and to heavy ions collisions and to give an account of processes in fluid dynamics and of phenomena in condensed matter physics, which can be studied with the use of this principle.

In the introductory part of the book the article by Christos Charmousis after introducing the basic properties of the anti de Sitter spacetime, it discusses the static black holes in this space, their basic properties and the novel topological effects due to the presence of a negative cosmological constant. The anti-de Sitter spacetime is a necessary ingredient to build up the gravity sector of the dual conformal field theory. To extract information about the transport coefficients of the boundary dual theory the properties of the background gravity sector should be known, best described by perturbation theory. The article by George Siopsis reviews the perturbations of black holes in asymptotically anti-de Sitter space and it shows how the quasi-normal modes governing these perturbations can be calculated analytically and it discusses the implications on the hydrodynamics of gauge theory fluids per the AdS/CFT correspondence.

The contribution by Philip Argyres introduces the basic concepts of the correspondence and its foundation in string theory. He first reviews the properties of d -dimensional conformal field theories and describes their relation to quantum gravitational theories on $(d + 1)$ -dimensional anti-de Sitter spacetimes. The 't Hooft limit of $U(N)$ Yang-Mills theory is reviewed and then it is described how an appropriate limit of type IIB superstring theory with D3-branes can be used to motivate a precise and computable correspondence between a 4-dimensional conformal field theory and a quantum gravitational theory on $\text{AdS}_5 \times S^5$. He finally discusses an extension of this construction in which probe branes on the AdS spacetime are included.

The second part of the book deals with the holographic realization of the AdS/CFT correspondence. The first article presented by Elias Kiritsis reviews the applications of the correspondence to QCD. In particular it provides a detail review of holographic models based on Einstein-dilaton gravity with a potential in 5 dimensions. Such theories, for a judicious choice of potential are very close to the physics of large- N Yang-Mills theory both at zero and finite temperature. The model can be used to calculate transport coefficients, like bulk viscosity, the drag force jet quenching parameters, relevant for the physics of the quark-gluon plasma.

The next article by Romuald Janik in a pedagogical way presents the techniques of the AdS/CFT correspondence which can be applied to the study of real time dynamics of a strongly coupled plasma system. These methods are based on solving gravitational Einstein's equations on the string/gravity side of the AdS/CFT correspondence. These AdS/CFT methods provide a fascinating arena interrelating General Relativity phenomena with strongly coupled gauge theory physics.

The contribution by Veronika Hubeny discusses the fluid/gravity correspondence which originates from the AdS/CFT correspondence. This correspondence constitutes a one-to-one map between configurations of a conformal fluid dynamics in d dimensions and solutions to Einstein's equations in $d + 1$ dimensions. In particular, the bulk solutions describe a regular generic, non-uniform and dynamical black hole which at late times settles down to a stationary planar black hole. The iterative construction of such solutions is indicated and the key physical properties are extracted.

The article by Amos Yarom is on heavy ions collisions. This contribution provides a review of two particular applications of the gauge-gravity duality to heavy ion collisions. The first involves a study of the wake of a quark as it travels through the quark gluon plasma and its possible connection to measurements of jet correlations carried out at the relativistic heavy ion collider at Brookhaven. The second section provides, via the gauge/gravity duality, a lower bound on the entropy produced in a collision of two energetic distributions. This is then compared to particle multiplicity in gold-gold collisions.

This part of the book ends with the article by Jiro Soda. In his article, he explains how the AdS/CFT correspondence is related to the Randall-Sundrum braneworld models. There are two different versions of these braneworlds models, namely, the single-brane model and the two-brane model. In the case of the

single-brane model, the relation between the geometrical and the AdS/CFT correspondence approach using the gradient expansion method is revealed. In the case of two-brane system, he shows that the AdS/CFT correspondence play an important role in the sense that the low energy effective field theory can be described by the conformally coupled scalar-tensor theory where the radion plays the role of the scalar field.

The last part of the book consists of four articles and they discuss various aspects of the application of the AdS/CFT correspondence to condensed matter physics. In the first article Subir Sachdev reviews two classes of strong coupling problems in condensed matter physics, and describes insights gained by application of the AdS/CFT correspondence. The first class concerns non-zero temperature dynamics and transport in the vicinity of quantum critical points described by relativistic field theories. The second class concerns symmetry breaking transitions of two-dimensional systems in the presence of gapless electronic excitations at isolated points or along lines (i.e. Fermi surfaces) in the Brillouin zone.

The next article by Gary Horowitz gives an introduction to the theory of holographic superconductors. These are superconductors that have a dual gravitational description using gauge/gravity duality. After introducing a suitable gravitational theory, he discusses its properties in various regimes: the probe limit, the effects of backreaction, the zero temperature limit, and the addition of magnetic fields. Using the gauge/gravity dictionary, these properties reproduce many of the standard features of superconductors.

The string theory realization of holographic superconductors is given in the next contribution by Matthias Kaminski. After reviewing the basic D-brane physics and gauge/gravity methods at finite temperature he constructs the gravity dual of a D3/D7-brane system yielding a superconducting or superfluid vector-condensate. The corresponding gauge theory is $3 + 1$ -dimensional $N = 2$ supersymmetric Yang-Mills theory with $SU(N_c)$ color and $SU(2)$ flavor symmetry and it shows a second order phase transition typical to superconductivity. Condensates of this nature are comparable to those recently found experimentally in p-wave superconductors such as a ruthenate compound.

The final article is by Tassos Petkou who discusses some attempts to construct a Kalb-Ramond superconductor. The article starts by explaining the holographic implications of torsional degrees of freedom in the context of AdS_4/CFT_3 , emphasizing in particular the physical interpretation of the latter as carriers of the non-trivial gravitational magnetic field. It presents a new exact 4-dimensional gravitational background with torsion and argue that it corresponds to the holographic dual of a 3d system undergoing parity symmetry breaking. Finally, it compares the new gravitational background with known wormhole solutions—with and without cosmological constant—and argue that they can all be unified under an intriguing Kalb-Ramond superconductivity framework.

The Fifth Aegean School and consequently this book, became possible with the kind support of many people and organizations. The School was organized by the Physics Department of the National Technical University of Athens, and supported by the Physics Department of the University of Tennessee. We received financial

support from the following sources and organizations and this is gratefully acknowledged: Ministry of National Education and Religious Affairs, Alexander Onassis Foundation, the “S&B Industrial Minerals S.A.”, General Secretariat of Aegean and Islands Policy, Springer Lecture Notes in Physics. We specially thank the staff of the Milos Conference Center “George Iliopoulos” for making available to us all the excellent facilities of the Center and for helping us to run smoothly the school. The administrative support of the Fifth Aegean School was taken up with great care by Fani and Christina Siatra. We acknowledge the help of Vassilis Zamarias who designed and maintained the website of the School. We also thank Kostas Anagnostopoulos for helping us in editing this book.

Last, but not least, we are grateful to the staff of Springer-Verlag, responsible for the Lecture Notes in Physics, whose abilities and help contributed greatly to the appearance of this book.

Athens, July 2010

Lefteris Papantonopoulos

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Part I: Introduction to the AdS/CFT Correspondence

Chapter 1

Introduction to Anti de Sitter Black Holes

Christos Charmousis

Abstract These introductory notes concern basic properties of negative constant curvature spacetimes and their black holes. For comparison purposes we will begin by reviewing flat spacetime, the spacetime diagram and two particular patches, Milne and Rindler. We will then discuss anti de Sitter, its symmetries, basic properties and the construction of the spacetime diagram. We then look into the properties of anti de Sitter spacetime giving some global and local parametrisations. We will study the static black holes and then discuss their basic properties and novel topological effects due to the presence of a negative cosmological constant. We show using the classical Euclidean path integral approach their thermodynamic properties in the canonical ensemble with a heat bath of constant temperature. Finally we discuss, rather briefly, stationary and axially symmetric spacetimes and some properties of the rotating black holes.

1.1 Spacetimes of Constant Curvature

1.1.1 Spaces of Maximal Symmetry and Constant Curvature

Let us consider in d dimensions Einstein's equations with cosmological constant Λ ,

$$G_{AB} + \Lambda g_{AB} = 0. \quad (1.1)$$

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Here, G_{AB} is the Einstein tensor and capital Latin letters denote d dimensional spacetime indices. The simplest of solutions to (1.1) we can consider are spaces locally characterised by the geometric condition,

$$R_{ABCD} = \frac{R}{(d-1)d} (g_{AC} g_{BD} - g_{AD} g_{BC}), \quad (1.2)$$

where d is the spacetime dimension and R , Ricci scalar curvature. Using (1.1) we see that,

$$G_{AB} = -R g_{AB} \frac{d-2}{2d} = -\Lambda g_{AB}.$$

In other words $R = \frac{2d}{d-2} \Lambda$ are the, locally, constant curvature solutions of the Einstein equations with cosmological constant. They are Einstein spaces which are locally homogeneous. In particular for $\Lambda = 0$ we have flat-Minkowski spacetime, for $\Lambda > 0$ positively curved, de Sitter (dS) spacetime and for $\Lambda < 0$ anti de Sitter spacetime (adS).

The above three spacetimes are of maximal symmetry and therefore by definition admit the maximal number of Killing vectors. Flat spacetime for example is isometric under the group of Poincaré transformations, in other words d -dimensional Lorentz coordinate transformations plus d translations. This is obvious in the inertial Minkowskian system of coordinates. Therefore we have $d + d(d-1)/2 = d(d+1)/2$ Killing vectors as generators of these symmetries all together. Since this is the maximal number of Killing vectors (see *Exercise 1*) this spacetime is by definition maximally symmetric. Flat spacetime is defined as the unique space of zero Riemann curvature (and of trivial topology).

Before moving on to non zero constant curvature spacetimes it is useful to classify the maximally symmetric $n = d - 1$ -spacelike sections. They are respectively representing locally Euclidean, spherical and hyperbolic sections and their line element can be written in a compact fashion as

$$dK_n^2 = \frac{d\chi^2}{1 - \kappa\chi^2} + \chi^2 d\Omega_{n-1}^2, \quad \chi \geq 0, \quad (1.3)$$

for normalised curvature $\kappa = 0, 1, -1$ respectively. The spherical line element $d\Omega_{n-1}^2$ ($n > 1$) is given by the iterative relation,

$$\begin{aligned} d\Omega_k^2 &= d\theta_k^2 + \sin^2(\theta_k) d\Omega_{k-1}^2, \dots, \quad d\Omega_1 = d\theta_1, \\ \theta_k &\in [0, \pi[, \dots, \theta_1 \in [0, 2\pi[, \quad k = 2, \dots, n-1. \end{aligned} \quad (1.4)$$

Setting $\chi = \sin \phi$ or $\chi = \sinh \phi$ in (1.3) for $\kappa = 1, -1$ respectively gives us the usual line element for the unit sphere and hyperboloid,

$$d\Omega_n^2 = d\phi^2 + \sin^2 \phi d\Omega_{n-1}^2, \quad \phi \in [0, \pi] \quad (1.5)$$

$$dH_n^2 = d^2 + \sinh^2 d\Omega_{n-1}^2, \quad \in [0, +\infty[. \quad (1.6)$$

We will be making extensive use of these parametrisations later on.

1.1.2 Flat Spacetime

1.1.2.1 Conformal Spacetime Diagram

To understand some of the basic properties of anti de Sitter space it is useful to first make a rapid go through the case of flat space. This way we can introduce relevant definitions in a more intuitive setting and compare properties. We turn our attention to flat spacetime and in particular to its conformal spacetime diagram (see for example [1]). One can write d dimensional Minkowski spacetime (1.3) as

$$ds^2 = -dt^2 + dr^2 + r^2 d\Omega_{n-1}^2, \quad (1.7)$$

in a spherical coordinate system (compare with (1.3)). To obtain the structure of flat spacetime at infinity we will go to a spacetime metric which is conformally equivalent,

$$\tilde{g}_{AB}(x) = \Omega^2(x) g_{AB},$$

to the initial one (1.7). The important point to realise is that the structure we obtain at asymptotic infinity will be common to all asymptotically flat spacetimes. This is due to the fact that a conformal transformation will shrink or stretch spacetime lengths but will not alter the null cones and therefore asymptotic properties. The main idea therefore is to use conformal transformations in order to bring asymptotic infinities to finite values for the conformally transformed metric. This is the central idea of Carter–Penrose diagrams.

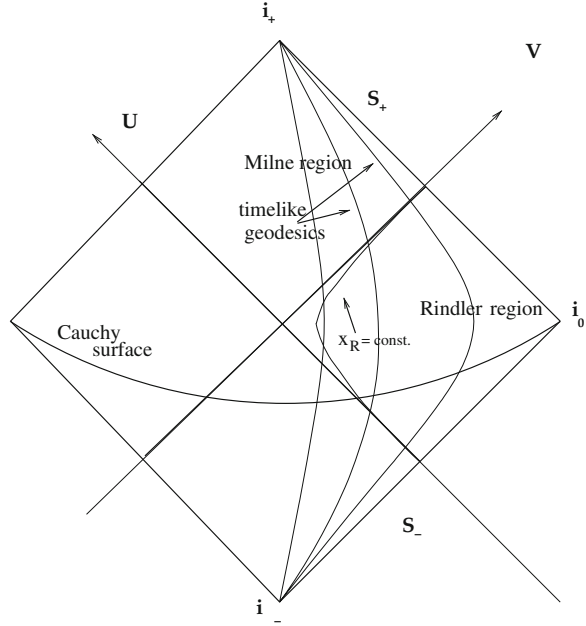
To achieve this for flat spacetime we first pass to null retarded and advanced coordinates, $u = t - r, v = t + r$, where u or $v = \text{constant}$ correspond to null radial geodesics of (1.7). Now we can bring coordinate infinity to finite values via the coordinate transformation, $\tan \tilde{U} = u, \tan \tilde{V} = v$ with $-\frac{1}{2}\pi < \tilde{U}, \tilde{V} < \frac{1}{2}\pi$. Go once again to time-space like coordinates $\tilde{t} = \tilde{U} + \tilde{V}, \tilde{r} = \tilde{V} - \tilde{U}$. Then (1.7) is conformally (and not coordinate) equivalent to

$$d\tilde{s}^2 = -d\tilde{t}^2 + d\tilde{r}^2 + \sin^2 \tilde{r} d\Omega_{n-1}^2$$

with conformal factor, $\Omega = 2 \cos(\tilde{U}) \cos(\tilde{V})$, i.e., $ds^2 = \Omega^2 d\tilde{s}^2$ (compare with (1.3)). Its important to keep track of the coordinate ranges,

$$-\pi < \tilde{t} + \tilde{r} < \pi, \quad -\pi < \tilde{t} - \tilde{r} < \pi, \quad \tilde{r} \geq 0. \quad (1.8)$$

Fig. 1.1 Three dimensional diagram and spacetime diagram (right) of Minkowski spacetime. The quarter diamonds on the *left* are the regions of Rindler and Milne spacetimes



We can now draw the Carter–Penrose spacetime diagram for flat spacetime (see Fig. 1.1). The domain defined by (1.8) is a triangle with its boundary defining asymptotic infinity of flat spacetime. It will correspond to a diamond shaped region of the Einstein cylinder (see [1]). In the diagram $\mathfrak{I}^-$ and $\mathfrak{I}^+$ stand for null past infinity and future null infinity, respectively, whereas i^- , i^+ are the endpoints of timelike geodesics $r = \text{constant}$. i^0 is that of spacelike geodesics $t = \text{constant}$.

A Cauchy surface is a spacelike surface intersecting all null and timelike inextensible geodesics. It will inevitably touch i^0 asymptotically. As its name indicates a Cauchy surface is therefore a valid set of initial data (for our second order Einstein field equations). A non-geodesic curve, such as $x_R = \text{constant}$, can reach null infinity $\mathfrak{I}^+$ if it is uniformly accelerated approaching the speed of light as $t \rightarrow +\infty$. The tangent curve emanating from $x_R = \text{constant}$ at $\mathfrak{I}^+$ is the acceleration horizon of the worldline $x_R = \text{constant}$. No events above this line can be witnessed by the observer whose worldline is $x_R = \text{constant}$. These effects concern Rindler and Milne spacetimes which we briefly turn to now.

1.1.2.2 Rindler and Milne Spacetimes

The line element for (spherical) Rindler spacetime is given by

$$ds^2 = -x_R^2 dt_R^2 + dx_R^2 + x_R^2 \cosh^2 t_R d\Omega_{n-1}^2, \quad x_R > 0, \quad (1.9)$$

and we immediately note that for $x_R = 0$ we have a singularity. Direct calculation of the Riemann tensor for this spacetime gives us identically zero i.e., we have a coordinate (and not curvature) singularity and furthermore Rindler spacetime is just a coordinate patch of flat spacetime. Indeed the coordinate transformation relating it to (1.7) is given by

$$\tanh t_R = \frac{t}{r} = \frac{\sin(\tilde{V} + \tilde{U})}{\sin(\tilde{V} - \tilde{U})}, \quad x_R = \sqrt{r^2 - t^2} = \sqrt{-\tan \tilde{U} \tan \tilde{V}}. \quad (1.10)$$

From the above we see that Rindler spacetime covers only part of the global spacetime diagram since $\tilde{U}\tilde{V} < 0$. The lines $\tilde{V} = 0$ and $\tilde{U} = 0$ corresponding to, $x_R = 0$ and $t_R \rightarrow \pm\infty$, are event horizons for the observer with coordinate time t_R and worldline $x_R = \text{constant}$. An observer with worldline $x_R = \text{constant}$ is in uniform acceleration $a = \frac{1}{x_R}$ since his trajectory in the original inertial Minkowski coordinates are hyperbolas (1.10). The inverse transformation reads,

$$t = x_R \sinh t_R, \quad r = x_R \cosh t_R.$$

To get Milne spacetime we can consider,

$$x_R \rightarrow it_M, \quad t_R \rightarrow x_M + \frac{i\pi}{2}.$$

This transformation, is one involving complex coordinates, however, it maps us to a real metric which is a different portion of flat spacetime ($\tilde{U}\tilde{V} > 0$). It is Milne spacetime,

$$ds^2 = -dt_M^2 + t_M^2 dH_n^2, \quad t_M > 0, \quad (1.11)$$

which is now a cosmological or time dependent type of metric. Notice it has no initial big bang singularity. An observer of Milne spacetime has rather a cosmological horizon at $t_M = 0$ and has Hubble expansion rate $H = \frac{1}{t_M}$ in proper time. Note that (just like in de Sitter spacetime) there is no horizon problem in this cosmological metric due to the fact that Milne spacetime is free of curvature singularities.

1.1.3 Anti de Sitter Spacetime

1.1.3.1 Definition, Boundary and Isometries

Useful and elegant representations of de Sitter and anti de Sitter spacetimes are obtained by embedding d dimensional hypersurfaces in $d + 1$ dimensional flat spacetime. Then de Sitter spacetime of curvature scale a is defined as the hyperboloid,

$$-X_0^2 + \sum_{i=1}^d X_i^2 = a^2, \quad (1.12)$$

embedded in $d + 1$ dimensional Minkowski spacetime. De Sitter space is topologically $\mathfrak{R} \times S^{d-1}$ and thus its spatial sections are compact (see also [1 or 2–4]). We shall focus on adS space from now on.

AdS spacetime in turn is obtained by considering the hyperboloid,

$$X_0^2 + X_d^2 - \sum_{i=1}^{d-1} X_i^2 = l^2 \quad (1.13)$$

of radius of curvature $l > 0$ embedded in the $d + 1$ dimensional spacetime,

$$ds^2 = -(dX_0^2 + dX_d^2) + \sum_{i=1}^{d-1} dX_i^2, \quad (1.14)$$

where note the double time coordinates. Clearly, any element of the Lorentz group $SO(2, d - 1)$ will leave (1.14) and (1.13) unchanged (by construction). Also we see that translation invariance is broken by (1.13). Since $SO(2, d - 1)$ has $d(d + 1)/2$ Killing generators just like flat spacetime, which is of maximal symmetry, $SO(2, d - 1)$ is the precise isometry group of adS.

A second important point is that adS spacetime has a conformal boundary¹ at the hyperboloid infinity. To see this we rescale all coordinates by $X_A \rightarrow X_A \Lambda$ and take $\Lambda \rightarrow \infty$. This limit defines the boundary as

$$X_0^2 + X_d^2 - \sum_{i=1}^{d-1} X_i^2 = 0, \quad (1.15)$$

$$X_A = X_A \Lambda, \Lambda \in \mathfrak{R}. \quad (1.16)$$

Suppose first that $X_0 \neq 0$: we can then divide by X_0 and then rescale. We therefore have that the boundary verifies,

$$-X_d^2 + \sum_{i=1}^{d-1} X_i^2 = 1,$$

which is a hyperboloid in $d - 1$ dimensions. This is just $d - 1$ dimensional de Sitter space according to (1.12). The topology is that of $\mathfrak{R} \times S^{d-2}$. If $X_0 = 0$ on the other hand, from (1.15) we have a sphere in $d - 2$ dimensions (times a point). Adding the two spaces together the boundary is a maximally symmetric space $S^1 \times S^{d-2}$.

¹ The boundary is called conformal for it admits not one but an equivalence class of metrics which are related via a conformal transformation, $g_{\mu\nu}^{\text{boundary}} = \Omega \Gamma_{\mu\nu}^{\text{boundary}}$. This equivalence class is manifest in the arbitrariness of Λ in (1.16).

Since the adS isometry acts on this space the boundary preserves $SO(1, d-1)$ symmetry. Note, however, that additionally we still have d extra transformations (1.16) for the boundary metric, 1 dilatation and $d-1$ special conformal transformations. The symmetry group is the conformal group $\mathcal{C}(1, d-2)$ that possesses, Lorentz and angle preserving transformations. Due to this last property, which has to do with the fact that we are dealing with an equivalence class of metrics, the boundary of adS admits as many symmetries as the bulk adS space. In other words the symmetry group of the boundary $\mathcal{C}(1, d-2)$, is isomorphic to the symmetry group of the bulk adS space, $SO(2, d-1)$.

1.1.3.2 Parametrisations

Let us now construct line elements for adS space in d dimensions. A global parametrisation is constructed as follows. Given the form of (1.13) we consider two spheres $X_0^2 + X_d^2 = r_1^2$, $\sum_{i=1}^{d-1} X_i^2 = r_2^2$ of radii r_1, r_2 such that

$$r_1^2 - r_2^2 = l^2. \quad (1.17)$$

This equation is solved setting $r_1 = l \cosh(u/l)$, $r_2 = l \sinh(u/l)$ where $u \in [0, +\infty[$. Now we just take the relevant parametrisations in polar spherical coordinates, (1.3) for $\kappa = 0$, and replace them in the line element (1.14),

$$\begin{aligned} ds^2 &= -(dr_1^2 + r_1^2 d^2) + dr_2^2 + r_2^2 d\Omega_{d-2}^2 \\ &= -l^2 \cosh^2(u/l) d^2 + du^2 + l^2 \sinh^2(u/l) d\Omega_{d-2}^2. \end{aligned} \quad (1.18)$$

The embedding in flat space is thus defined in coordinates, $(t, u, \theta_1, \dots, \theta_{d-2})$ as,

$$\begin{aligned} X_0 &= l \cosh(u/l) \sin, & X_d &= l \cosh(u/l) \cos, \\ X_1 &= l \sinh(u/l) \cos \theta_1, \dots, & X_{d-1} &= l \sinh(u/l) \sin \theta_1 \dots \cos \theta_{d-2}. \end{aligned} \quad (1.19)$$

The boundary lies at $u \rightarrow \infty$. This is the global parametrisation of adS since all points of the hyperboloid are taken into account exactly once. This metric is solution to the Einstein equations with cosmological constant,

$$2\Lambda = -\frac{(d-1)(d-2)}{l^2}. \quad (1.20)$$

Note that the timelike coordinate is an angular coordinate $\in [-\pi, \pi]$. This signifies that adS is a spacetime with closed timelike curves! We can, however, get around this; since the space is not simply connected (i.e. the time circle cannot be topologically reduced to a point) we can unwrap the circle of the time coordinate and take a new coordinate $t \in [-\infty, +\infty]$ with $t \equiv t + 2\pi$ in each 2π -interval. This means that we are effectively taking infinite copies of the hyperboloid. This is the universal covering of adS space,

$$ds^2 = -l^2 \cosh^2(u/l) dt^2 + du^2 + l^2 \sinh^2(u/l) d\Omega_{d-2}^2. \quad (1.21)$$

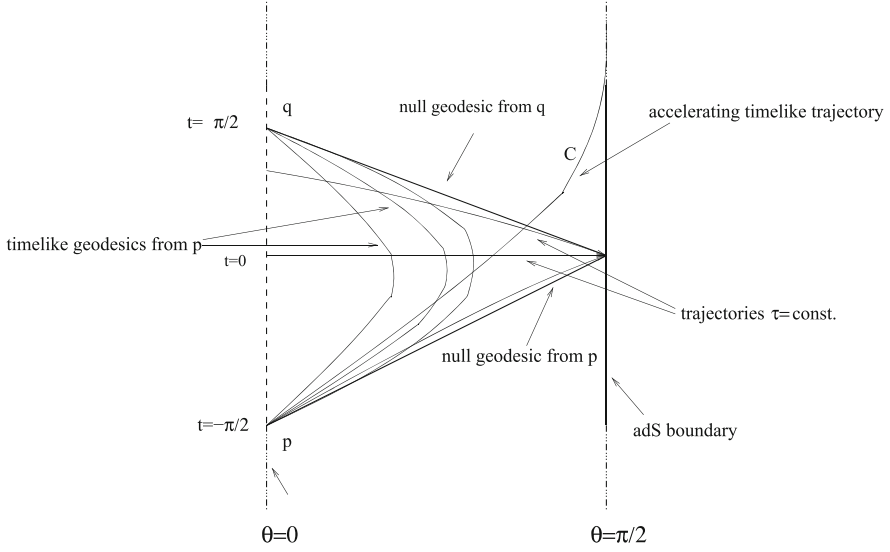


Fig. 1.2 Spacetime diagram with timelike geodesics of adS spacetime (1.29). All past and future timelike geodesics from $\tau = 0$ focus in the past at point p and in the future at point q . The resulting “geodesic” triangular region covers only part of the timelike future of p . Similarly, the Cauchy surface at $t = 0$ covers only the triangular region in between p and q and their null past and future, respectively. Any event beyond this triangle is not causally connected to $\tau = 0$ unless suitable reflective boundary conditions are imposed at the adS boundary. The accelerating timelike curve C “escapes” from this bounded region and gets to see beyond, ending up at the boundary of adS

This space is, however, not a Cauchy space (or in other words globally hyperbolic). We can of course define a Cauchy surface but it represents no longer sufficient initial data to describe the entire space of adS. Data on the boundary of adS have to be also specified (see Fig. 1.2). We will come back to this briefly when studying the spacetime diagram.

Also note that for the normal adapted coordinate u , ∂_u is not a Killing vector for adS. In adS space we miss a global spacelike Killing vector generating translational invariance in u (in de Sitter space we do not have a global timelike Killing vector). Taking $u = u_0$ constant with u_0 large we see that,

$$ds^2 \sim l^2 e^{\frac{2u_0}{l}} (-dt^2 + d\Omega_{d-2}^2),$$

that the geometry of the boundary is indeed topologically $\mathbb{R} \times S^{d-2}$ for the universal covering of adS. Had we considered we would have got, $S^1 \times S^{d-2}$.

Let us now consider a local parametrisation defined by

$$\begin{aligned} y &= l \ln \frac{X_d + X_{d-1}}{l}, & \tilde{t} &= \frac{X_0}{X_d + X_{d-1}}, \\ x_i &= \frac{X_i}{X_d + X_{d-1}}, & i &= 1, \dots, d-2. \end{aligned} \quad (1.22)$$

This parametrisation covers only half of the hyperboloid since we must have $X_d + X_{d-1} > 0$ in order for (1.22) to be well defined. In order to find the relevant line element we must invert (1.22). Using (1.13) it is easy to show that,

$$X_d + X_{d-1} = l e^{\frac{y}{l}}, \quad X_d - X_{d-1} = l e^{\frac{y}{l}} \left(1 + e^{\frac{2y}{l}} \frac{\sum_{j=1}^{d-2} x_j^2 - \tilde{t}^2}{l^2} \right).$$

Taking the sum and the difference we then obtain the inverse transformation,

$$\begin{aligned} X_{d-1} &= l \sinh\left(\frac{y}{l}\right) - \frac{l}{2} e^{\frac{y}{l}} (x^2 - \tilde{t}^2), \\ X_0 &= \tilde{t} e^{\frac{y}{l}}, \quad X_i = l x_i e^{\frac{y}{l}}, \\ X_d &= l \cosh\left(\frac{y}{l}\right) + \frac{l}{2} e^{\frac{y}{l}} (x^2 - \tilde{t}^2), \end{aligned} \quad (1.23)$$

where $x^2 = \sum_{j=1}^{d-2} x_j^2$ (compare with the global parametrisation (1.19)). Inserting into (1.14) we now get the desired line element,

$$ds^2 = l^2 e^{\frac{2y}{l}} (-d\tilde{t}^2 + dx^2) + dy^2, \quad (1.24)$$

with $y \in [-\infty, +\infty]$ measuring proper distance. Note that by rescaling the $(\tilde{t}, x)$ coordinates we can get rid of the l^2 factor. Planar coordinates are obtained by setting $r = l e^{\frac{y}{l}}$. We obtain,

$$ds^2 = r^2 (-d\tilde{t}^2 + dx^2) + \frac{l^2 dr^2}{r^2}, \quad (1.25)$$

with $r > 0$. We will use this chart for planar black holes. Again note from (1.22) that planar coordinates cover only half of the hyperboloid. To cover all of adS we take two portions $r > 0$ and $r < 0$. Note that the boundary is attained at $r \rightarrow \infty$ whereas we have a horizon at $r = 0$ due to the fact that we cannot cover all of adS space. This is a degenerate (i.e. of zero temperature) Killing horizon associated to the Poincaré flat slicing of adS space (we will come back to this in a moment). Finally the Poincaré chart is obtained as the conformally flat version of adS by setting $z = \frac{l^2}{r}$,

$$ds^2 = \frac{l^2}{z^2} (-d\tilde{t}^2 + dx^2 + dz^2), \quad (1.26)$$

where now the boundary is at $z = 0$ for this chart.

Let us now find a proper time parametrisation of adS. We proceed as for the global parametrisation modulo we now start by considering, $X_0^2 = \rho_1^2, X_d^2 - \sum_{i=1}^{d-1} X_i^2 = \rho_2^2$ of radii ρ_1, ρ_2 such that

$$\rho_1^2 + \rho_2^2 = l^2, \quad (1.27)$$

which is solved setting $\rho_1 = l \sin \tau, \rho_2 = l \cos \tau$. We repeat the same procedure for the hyperboloid $X_d^2 - \sum_{i=1}^{d-1} X_i^2 = \rho_2^2$ with $X_d = \rho_2 \cosh$ and then we parametrise $\sum_{i=1}^{d-1} X_i^2$ using (1.3) for $\kappa = 0$. All in all we have,

$$\begin{aligned} X_0 &= \sin \tau, & X_d &= l \cos \tau \cosh, \\ X_1 &= l \cos \tau \sinh \cos \theta_1, \dots, X_{d-1} &= l \cos \tau \sinh \sin \theta_1 \dots \cos \theta_{d-1} \end{aligned} \quad (1.28)$$

and replacing in the line element (1.14) we get,

$$ds^2 = -l^2 d\tau^2 + l^2 \cos^2 \tau dH_{d-1}^2. \quad (1.29)$$

This coordinate system is only defined for $\tau \in]-\frac{1}{2}\pi, \frac{1}{2}\pi[$. Therefore in this patch $-l \leq X_0 \leq l$ (see Fig. 1.2).

1.1.3.3 Spacetime Diagram

Let us turn our attention now to the spacetime diagram for adS. Our starting point is the global coordinate system (1.21). As before we write the metric in a conformally flat form and define novel coordinates so as to bring infinity of the radial coordinate to a new finite coordinate value. Time, however, is either periodic in or infinite in t . This amounts to solving $d\theta = dul \cosh \frac{t}{l}$ and therefore considering the coordinate transformation,

$$\tan\left(\frac{\theta}{2} + \frac{\pi}{4}\right) = e^{\frac{t}{l}}, \quad (1.30)$$

with $\theta \in [0, \frac{\pi}{2}]$. The line element is

$$ds^2 = \frac{l^2}{\cos^2 \theta} (-dt^2 + d\theta^2 + \sin^2 \theta d\Omega_{n-1}^2) \quad (1.31)$$

and it is conformally equivalent to a quarter of the Einstein cylinder (1.7). The difference here is that $\theta \in [0, \frac{\pi}{2}]$ rather than $\theta \in [0, \pi[$ for flat space. Also note that we cannot make a conformal transformation which brings time infinities to finite values for then the conformal factor explodes. It is useful for comparison between different charts to note that,

$$\begin{aligned} X_0 &= l \frac{\sin t}{\cos \theta}, & X_d &= l \frac{\cos t}{\cos \theta}, \\ X_1 &= l \tan \rho \cos \theta_1, \dots, X_{d-1} &= l \tan \rho \sin \theta_1 \dots \cos \theta_{n-1}. \end{aligned} \quad (1.32)$$

Therefore the spacetime diagram consists of an infinite strip of length $\pi/2$ (for the universal covering). The boundary resides at $\theta = \frac{\pi}{2}$, the endpoint of both future

and past null geodesics. For timelike geodesics we use the proper time slicing, (1.29). Indeed comparing X_0, X_d for (1.32) and (1.28) we get

$$\sin t = \sin \tau \cos \theta, \quad \cos t = \frac{1}{\tanh} \sin \theta \quad (1.33)$$

for τ and constant respectively (see Fig. 1.2).

Note that timelike infinity in the past i^- or future i^+ is infinite. As a consequence there is no finite Cauchy spacelike surface at all in this space. This is because although any constant t_0 surface ($t = 0$ or $\tau = 0$ in Fig. 1.2), X say, covers the whole of adS at instant $t = t_0$, however, we can find null surfaces that never intersect X . This means that universal adS is not globally hyperbolic: Cauchy data on arbitrary spacelike surface X , determines the system's evolution only in a region bounded by a null hypersurface called a Cauchy horizon for X (the triangular region in Fig. 1.2). Physics on adS depends also on the boundary conditions imposed at the boundary. Secondly all timelike geodesics emanating from $t = 0$ focus at point q and diverge at p never reaching the boundary of adS. Unlike de Sitter space which inflates spacetime events away from each other, anti-de-Sitter never allows free-falling particles to escape to the boundary. There are actually regions in the future of p which can never be reached by any geodesic as is shown in the figure but only by accelerated observers as the curve C in (1.2). This is a sign that the gravitational potential in adS will be a bounding potential for gravitons or free particles.

Planar (or Poincaré) coordinates cover the region $r \geq 0$ which gives us the triangular region in the spacetime diagram. Indeed we have (1.23) $r = X_d + X_{d-1} = \frac{l}{\cos \theta} (\cos t + \sin \theta \cos \Omega)$ given the global patch (1.19) in coordinates (1.31) (see Fig. 1.3) Note that the $r = \text{const}$ worldlines are now accelerating trajectories and as a result we have the presence of an adS horizon at $r \sim \infty$. The Poincaré patch is in this sense similar to the Rindler patch in flat spacetime.

Exercise 1 Show that for a Killing vector we have

$$\nabla_A \nabla_B \xi_C = -R_{BCA}^D \xi_D.$$

Deduce that any Killing vector can be obtained by its values ξ^A and $\nabla_A \xi_B$ at any point $P \in \mathcal{M}$. From Killing's relation deduce therefore that there are at most $\frac{d(d+1)}{2}$ linearly independent Killing vectors in $\mathcal{M}$ and hence at most $\frac{d(d+1)}{2}$ isometries of the metric (for more details see [5]).

Exercise 2 The constant curvature slicings of adS are given by the following line element,

$$ds^2 = -\left(\kappa + \frac{r^2}{l^2}\right)dt^2 + \frac{dr^2}{\kappa + \frac{r^2}{l^2}} + r^2 dK_{n-1}^2. \quad (1.34)$$

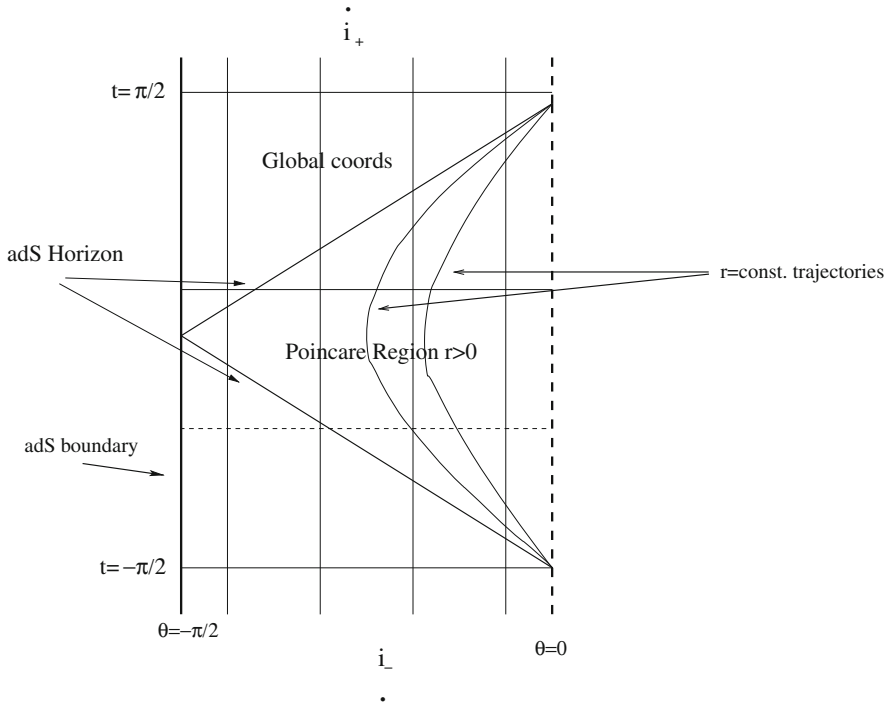


Fig. 1.3 On the right a 3 dimensional representation of (universal) adS space with all but one (symbolically Ω in the figure) of the angular coordinates fixed. The boundary of adS is represented by the infinite cylinder, the bulk of adS space by its interior. The fundamental domain is the cylinder bounded the horizontal cups $t = \pm\pi/2$. The Poincaré region is bounded by the tilted adS horizon caps $r = 0$ and covers half of the fundamental domain for $t \in [-\pi/2, \pi/2]$. On the left a projected 2 dimensional view of the Poincaré region [3]

For $\kappa = 0$ we obtain the flat slicing already encountered (1.25). Show that the spherical slicing is nothing but that of global coordinates (1.21). Find the coordinate transformation relating the flat and spherical parametrisations. Find then the region of validity of the Poincaré coordinates. Show that the hyperbolic slicing gives a time dependent version of adS. Find its domain of validity.

Exercise 3 Show that the global coordinate system for de Sitter space is,

$$ds^2 = -dt^2 + a^2 \cosh^2\left(\frac{t}{a}\right) d\Omega_n^2. \quad (1.35)$$

Therefore there is no global timelike Killing vector in de Sitter space. Construct the Poincaré slicing and the Carter–Penrose diagram for de-Sitter space (see for example [1]).

1.2 Static Black Holes

1.2.1 Basic Properties

Assume a $d = n + 1$ dimensional spacetime such that it has $n - 1$ dimensional sections of constant curvature given by the line element (1.3). Then one can show, for metrics of such symmetry, that the general solution of Einstein's equations with cosmological constant admits a locally timelike Killing vector. This is a slightly generalised version of Birkhoff's theorem for vacuum spacetime (see for example [6]). The solution reads (see [7, 8])

$$ds^2 = -V(r)dt^2 + \frac{dr^2}{V(r)} + \frac{r^2}{l^2}dK_{n-1}^2, \quad n \geq 2, \quad (1.36)$$

where

$$V(r) = \kappa - \frac{\mu}{r^{n-2}} + \frac{r^2}{l^2}, \quad (1.37)$$

where the $(n-1)$ dimensional metric dK_{n-1}^2 is given by (1.3) and the adS curvature length l is related to the cosmological constant (1.20). Parameter μ is an integration constant which, as we will see, is associated to the black hole mass [9, 10]. For the charged version of these black holes and their thermodynamics see [11].

Unlike de Sitter or flat spacetime, black holes in adS [12] will exist for all three values of $\kappa = 0, 1, -1$. In other words black hole horizons do not have only spherical topology in adS, we can have toroidal or even hyperbolic black holes. The horizons have to undergo topological identifications so as to make the horizon surface compact. For flat horizons, $\kappa = 0$, one can simply identify all flat directions in order to get a torus of $n - 1$ dimensions. One can identify flat directions non-trivially, creating non orientable horizon surfaces such as the Klein bottle in 2 dimensions and so forth. For $\kappa = -1$ the construction is more complicated. Here we give a rapid overview but the interested reader should consult [13–16]. Take a locally H^2 horizon (in a 4 dimensional hyperbolic black hole) and consider a quotient space $\Sigma = H^2/\Gamma$. Here, H^2 is modded out by its discrete subgroup Γ made of discrete boosts of $SO(2, 1)$ (which as we saw earlier belongs to the symmetry group of H^2). Then Σ turns out to be a compact space of genus g . The compact space has $4g$ sides and the sum of its angles has to give an overall angle of 2π in order to avoid conical singularities (see Fig. 1.4). The fundamental domain of Σ is a polygon whose edges are geodesics of H^2 . The action of Γ will identify opposing edges. The simplest case free of conical singularities is a regular hyperbolic octagon with opposite edges identified [13]. Often these black holes are referred to as topological black holes [7, 14–17] due to the identifications one has to undergo in order to compactify the horizon geometries. In fact, any Einstein space of dimension $n - 1$ can form a horizon for an adS black hole [7].

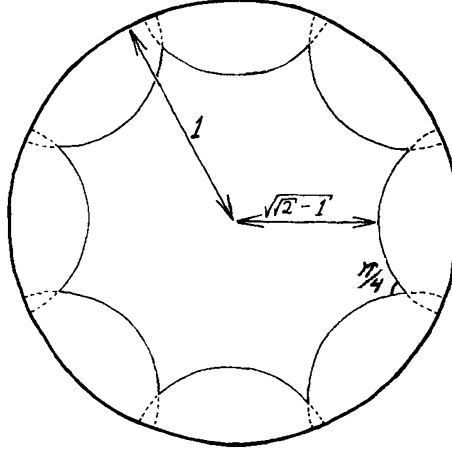


Fig. 1.4 The fundamental region depicting a regular hyperbolic octagon in the Poincaré hyperbolic unit disk, $D = \{x \in \mathbb{R}^2 : |x| < 1\}$ with metric $ds^2 = \frac{dx^2 + dy^2}{(1-x^2-y^2)^2}$. Notice how distances get warped as we approach the boundary. Each depicted arc is orthogonal to the disks boundary. Thus each polygon side is a geodesic segment in this space. The sum of angles gives 2π ensuring the absence of conical singularities. The opposite edges of the polygon are identified via the action of Γ . The thus constructed horizon surface is a compact space which is locally hyperbolic. This figure is taken from [6] where the detailed construction can be found

The solutions for $\kappa = +1$ are sometimes called “Schwarzschild-adS” solutions because they reduce to the standard Schwarzschild solution when the cosmological constant vanishes, $l \rightarrow \infty$, and to adS in global coordinates when $\mu = 0$. In fact, these are the genuine asymptotically adS solutions precisely because asymptotically we recover the global patch of adS space. Moreover, their topology is $\mathbb{R}^2 \times S^{n-1}$, and the horizon is the sphere S^{n-1} , like that of the Schwarzschild solution.

Given that these black holes are static their horizons, in this coordinate system, are the zeros of the potential. We will denote their (largest root) event horizon by $r = r_h$, $V(r_h) = 0$. This event horizon has typically a non-zero temperature which can be calculated the standard Euclidean way: Indeed consider $t \rightarrow i\tau$. We have

$$ds^2 = V(r)d\tau^2 + \frac{dr^2}{V(r)} + r^2 dK_{n-1}^2 \quad (1.38)$$

and the metric is then of Euclidean signature for $r > r_h$. This can be seen by expanding around $r = r_h$,

$$ds^2 \sim \left(\frac{1}{4} V_{r_h}'^2 \right) \rho^2 d\theta^2 + d\rho^2 + \dots, \quad (1.39)$$

with radial isotropic (or cylindrical) coordinate $\rho = \sqrt{\frac{2(r-r_h)}{|V_{r_h}'|}}$. Clearly in order to evade a conical singularity at the origin of the axis, $r = r_h$, we must impose the periodicity,

$$\beta = \frac{4\pi}{|V'(r_h)|}. \quad (1.40)$$

As we will see, the Euclidean quantum field propagator, with the above periodic boundary conditions, describes a canonical ensemble of states in thermal equilibrium with a heat bath of temperature $T = \beta^{-1}$ [18] where

$$\beta = \frac{4\pi l^2 r_h}{nr_h^2 + \kappa(n-2)l^2}. \quad (1.41)$$

This relation can be inverted to find

$$r_h = \frac{2\pi l^2}{n\beta} \left[1 \pm \sqrt{1 - \kappa \frac{n(n-2)\beta^2}{4\pi^2 l^2}} \right], \quad (1.42)$$

which allows us to take β as the parameter that determines the solution. For $\kappa = +1$ the minus branch for r_h exists and corresponds to small black holes in adS. Notice that in the limit where $r_h \gg l$ the $\kappa = \pm 1$ classes of solutions approach the planar black hole class $\kappa = 0$. This admits an interpretation in terms of an “infinite volume” limit, in which the curvature radius of S^{n-1} or H^{n-1} is much larger than the thermal wavelength of the system [19].

Setting $\mu = 0$ we recover differing patches of adS space. For $\kappa = 0$ we obtain the Poincaré patch which covers only part of adS. In this case $r \rightarrow \infty$ is the boundary of adS whereas $r = 0$ is a horizon. This is a degenerate Killing horizon meaning that there is no temperature associated with it (This is also true for the $\kappa = 1$ case). These are then the ground states for their respective class of solutions parametrised by μ . In other words adS space is not associated with some temperature or inversely we can construct any temperature adS state by suitably fixing the period of Euclidean time. This is similar to flat spacetime and in contrast to de Sitter spacetime. For $\kappa = -1$ however, we have a bifurcate Killing horizon at $r_h = l$ with $r > l$. The hyperbolic slicing covers a yet smaller triangular portion of adS but the horizon in question has now temperature, $\beta = 2\pi l$. This patch is very similar to the Rindler patch of flat spacetime. Therefore in this case it is not clear what is the ground state of the system. We stress, however, that the $\mu = 0$ case is the only one which has no curvature singularity. The issue of the ground state is an important question since when calculating the partition function it is important to specify the background solution with which to annihilate divergences. AdS/CFT is capital in resolving this and we will come back to this in a moment [9, 10].

Furthermore, for the $\kappa = -1$ class of black holes [8], and in contrast to the $\kappa = +1, 0$ classes, the zero temperature solution exists and is different from the one that is isometric to adS. In fact, for $\kappa = -1$ there is a range of negative values for μ such that the solutions still possess regular horizons. When we are in such a configuration we also have an inner horizon much like for a RN black hole. The minimum values of μ and r_h that are compatible with cosmic censorship and for which the horizon is degenerate are [7],

$$\mu_{\text{ext}} = -\frac{2}{n-2} \left(\frac{n-2}{n} \right)^{n/2} l^{n-2}, \quad r_{\text{ext}} = \sqrt{\frac{n-2}{n}} l, \quad (1.43)$$

and,

$$\mu_{\text{ext}} = -\frac{l^2}{4}, \quad r_{\text{ext}} = \frac{l}{\sqrt{2}}, \quad \text{for } n = 4. \quad (1.44)$$

For these values of the parameters, the black hole is extremal with coinciding inner and event horizons.

1.2.2 Thermodynamics

The thermodynamics of adS black holes have many interesting properties which are rather different from their asymptotically flat or de Sitter cousins. We will use here the path integral method developed by Hartle and Hawking [20] in order to calculate the partition function and then, using standard thermodynamic formulas, the basic thermodynamic quantities for adS black holes. Already as we mentioned above, the $\kappa = -1$ case, presents subtleties due to the fact that the background adS solution possesses a temperature at $r_h = l$ whereas at the same time there is an extremal black hole of zero temperature. Therefore it is not clear, at least naively, which is the right background one should use. The adS/CFT correspondence cures this ambiguity by providing through the boundary CFT the correct geometric counter-terms. These are used to cancel out the infinities and provide a background independent way to calculate the partition function [9, 10, 21, 22]. For this presentation we will concentrate on pre-adS/CFT method as a motivation and comparison with adS/CFT which will be studied later on. We therefore concentrate here solely on the $\kappa = 1$ case (for the conserved charges see [23, 24]) giving in particular an overview of the Hawking–Page phase transition.

We shall consider a canonical ensemble with a constant temperature heat bath. The system has thus fixed temperature but is allowed to exchange energy with the heat bath. In asymptotically flat space although a black hole can be in equilibrium with thermal radiation at some common constant temperature T_0 , this equilibrium is unstable once gravitational corrections are taken into account. In other words, if the mass of the black hole were to increase due to some infalling matter then the temperature would decrease and the black hole would continue to grow. This means that the canonical ensemble, where the black hole is in thermal equilibrium with a heat bath of constant temperature, is ill defined if gravitational energy flow is not held fixed. In adS space, however, the thermal radiation remains confined close to the black hole since the gravitational potential, $V \sim r^2/l^2$ increases for large r . This is true for any choice of origin

since adS is a homogeneous space. Non-zero rest mass particles are confined and prevented from escaping to infinity and one can consider a canonical ensemble description for any given temperature T . Effectively, though volume of space is infinite, adS provides a confining gravitational box. The canonical ensemble partition function is defined via a path integral [25] that takes us from a given configuration $S, (S, g, \phi) \longrightarrow (S', g', \phi')$ to S' via all possible paths. These paths represent arbitrary matter fields and arbitrary metrics which are asymptotically flowing to zero and adS, respectively, in periodic time τ with period β (1.40). Here ϕ represents collectively matter fields and g Euclidean metrics. Symbolically,

$$\langle g', \phi', S' | g, \phi, S \rangle = \int D(\phi, g) \exp(-I[\phi, g]). \quad (1.45)$$

The path integral is taken in Euclidean signature in order to optimise regularity. Indeed, this permits the amplitude to be an elliptic operator with an exponential *damping* factor rather than an oscillating one. All defined above possible configurations are allowed but one can argue that regular classical solutions, i.e. saddle points of the action, are going to give the dominant contributions to this integral and in particular to the partition function $Z = e^{-I}$ where,

$$I_{sol} = -\frac{1}{16\pi G} \int d^d x \sqrt{g} (R - 2\Lambda) + BT = \frac{d-1}{8\pi G} \int d^d x \sqrt{g} = \frac{d-1}{8\pi G} Vol \quad (1.46)$$

Here BT is the Gibbons–Hawking boundary term yielding a zero contribution for adS (unlike flat space where it is the sole term giving a contribution). We have immediately run into a problem: the volume integral resulting from this calculation is clearly infinite, both for adS space and the black hole in question. We will therefore here, have to regularise by some vacuum (background) solution behaving similarly at asymptotic infinity to the black hole in question, so as to get rid of the infinities. We choose the background as the ground state since it is a regular solution satisfying the same boundary conditions *for any temperature of the heat bath*. We resort to [26] considering an upper cut-off $r < R$, subtracting the two volume integrals and then taking the limit $R \rightarrow \infty$.

Let us take a closer look [19]. Our background in question is pure adS in global coordinates. For adS and the black hole we have, respectively,

$$Vol_1(R) = \int_0^{\beta_0} d\tau \int_0^R dr \int_{S^{d-2}} d\Omega r^{d-2}, \quad (1.47)$$

$$Vol_2(R) = \int_0^{\beta} d\tau \int_{r_+}^R dr \int_{S^{d-2}} d\Omega r^{d-2}. \quad (1.48)$$

and note the differences in the bounds of integration for the two integrals. Although the period β for the black hole is fixed by (1.41), for adS space it is arbitrary, β_0 . We can therefore fix β_0 so that *the temperature of both configurations is the same at $r = R$* . This is consistent with the fact that we are assuming a common heat bath for the canonical ensemble at temperature T . This sets,

$$\beta_0 = \frac{\beta \sqrt{1 + \frac{R^2}{l^2} - \frac{\mu}{R^{d-3}}}}{\sqrt{1 + \frac{R^2}{l^2}}}. \quad (1.49)$$

After evaluating the action difference and taking the limit we finally obtain,

$$-\log(Z) = I = \frac{\text{Vol}(S^{d-2}) r_+^{d-2} (l^2 - r_+^2)}{4G[(d-1)r_+^2 + (d-3)l^2]}. \quad (1.50)$$

The action I we have calculated is essentially the difference in between the saddle points present in the partition fuction. We see that I is positive for *small* r_h compared to l which means that the tunnelling probability from adS to a black hole is exponentially *suppressed*. We have thus semi-classical stability. On the contrary the sign is inverted for larger r_h than l , which points to an instability, physically favouring tunnelling to black holes. In between we can expect to find a phase transition. Let us now pursue to find in a standard way the thermodynamic quantities. The mean energy of thermal radiation is $E = \sum_{\text{states}} p_i E_i$, where $p_i = \frac{1}{Z} e^{-\beta E_i}$ and therefore,

$$E = -\frac{\partial}{\partial \beta} \log Z = \frac{(d-2)\text{Vol}(S^{d-2})(l^{-2}r_+^{d-1} - r_+^{d-3})}{16\pi G} = M, \quad (1.51)$$

where $M = \frac{(d-2)\text{Vol}(S^{d-2})\mu}{16\pi G}$ is the gravitational mass of the black hole ([23, 24] for $\kappa = 1$, with the general conserved charges for locally asymptotically adS spacetimes given in [21]). The entropy $S = \sum_{\text{states}} p_i \log p_i$ is given by

$$S = \beta E - I = \frac{1}{4G} r_h^{d-2} \text{Vol}(S^{d-2}) = \frac{A}{4G} \quad (1.52)$$

where A is the volume of the horizon. Here we see how in adS the entropy-area relation is verified. The heat capacity i.e., the amount of heat energy required to increase the temperature by a unit quantity, is given by

$$\begin{aligned} C &= \frac{\partial E}{\partial T} = \frac{\partial E}{\partial r_h} \frac{\partial r_h}{\partial T} \\ &= \frac{(n-1)\text{Vol}(S^{d-2})r_h^{n-3}}{8n} (nr_h^2 \\ &\quad + (n-2)l^2) \left[1 \pm \sqrt{1 - \kappa \frac{n(n-2)\beta^2}{4\pi^2 l^2}} \right], \end{aligned} \quad (1.53)$$

with C , the second derivative of the action with respect to temperature, gives thermodynamic stability. The free energy is given by $F = E - TS = IT$

We can finally check the possible phases: first we note that r_h is an increasing function of the mass M . Now it is easy to check that $0 \leq \beta < \beta_{\max}$, where $\beta_{\max} = \frac{2\pi l}{\sqrt{(d-3)(d-1)}}$. Hence for temperatures $T < T_{\min} = \frac{1}{\beta_{\max}}$ there is no black hole and we have a pure (adS) phase of thermal radiation with negative free energy. For $T > T_{\min}$ there are two black holes with $M_- < M_+$ (1.42). The black hole with the smaller mass has negative specific heat. It is therefore unstable to decay either to a larger black hole or to pure thermal adS. The bigger black hole has on the contrary positive specific heat and is thus thermodynamically stable. Furthermore for $T_{\min} < T < T_{HP}$ the free energy of the plus branch is less than pure thermal adS which is energetically favoured. However, for $T > T_{HP}$ the situation is inversed and the large black hole state has less free energy than thermal adS space. AdS space can therefore tunnel towards this large black hole which becomes the energetically preferred state.

1.3 Beyond Static Black Holes

Suppose we now reduce the symmetry of spacetime, requiring a stationary and axially symmetric spacetime. As is often the case once symmetry is reduced the field equations are no longer integrable and knowledge of the full spectrum of solutions is lost. The presence of the cosmological constant presents a genuine complication with respect to the vacuum case which remains integrable for the stationary axially symmetric case (see [27–33] and references within).

Our aim in this section is to describe the difficulty associated to the presence of Λ and to discuss briefly this open problem. Let us focus here mostly on the case of $d = 4$ dimensions to simplify the problem as much as possible. Consider a spacetime with two killing vectors one which is locally timelike and one axial which are no longer orthogonal to each other. In other words we have lost $d - 2$ maximal symmetry and we have traded stationarity for staticity. The general metric ansatz reads,

$$ds^2 = e^{2v} \alpha^{-1/2} (dt^2 + dz^2) + \alpha e^{-\frac{9}{2}} d\varphi^2 - \alpha e^{\frac{9}{2}} (dt + Ad\varphi)^2, \quad (1.54)$$

where $\partial_t, \partial_\varphi$ are the Killing vectors in this adapted coordinate system. The metric functions, v, α and Ω depend on r and z . Note that we can still undertake 2 dimensional conformal transformations in the (r, z) plane without loss of generality. The field equations read,

$$\Delta \alpha = -2\Lambda \alpha^{\frac{1}{2}} e^{2v}, \quad (1.55)$$

$$\vec{\nabla} \cdot (e^{\Omega} \alpha \vec{\nabla} A) = 0, \quad (1.56)$$

$$\frac{1}{\alpha} \vec{\nabla} \cdot (\alpha \vec{\nabla} \Omega) + 2e^\Omega (\vec{\nabla} A)^2 = 0, \quad (1.57)$$

$$\Delta v + \frac{1}{4} (\vec{\nabla} \Omega)^2 - e^\Omega (\vec{\nabla} A)^2 = \frac{\Lambda \alpha}{\alpha}, \quad (1.58)$$

$$2v_{,u} \frac{\alpha_{,u}}{\alpha} - \frac{\alpha_{,\zeta\bar{\zeta}}}{\alpha} = \frac{1}{8} (\Omega_{,\zeta})^2 - \frac{1}{2} e^\Omega A_{,\zeta}^2 \quad (\zeta \leftrightarrow \bar{\zeta}), \quad (1.59)$$

and correspond to a 4 dimensional spacetime with cosmological constant [16]. The derivative operators are the usual flat 2 dimensional operators with respect to r and z and $\zeta = r + iz$.

This system is a slight generalisation of the Lewis–Papapetrou system for $\Lambda = 0$ (see for example [27–33]). In particular static and axially symmetric metrics (usually called Weyl metrics) with $\Lambda = 0$ are obtained setting $A = 0$. The vacuum case is long known to be integrable [27–33]. This is no longer true in the presence of the cosmological constant and also in higher dimensions [34]. In higher dimensions this is due to the appearance of novel curvature scales, much like the cosmological constant, associated to the additional Killing vectors. However, when all curvature scales are set to zero (all spatial Killing vectors have $U(1)$ symmetry) the system is integrable for $D > 4$ and has been studied extensively in [35]. To picture how this works for vacuum and fails for $\Lambda \neq 0$ let us switch off in (1.55–1.59) the curvature scale Λ and A (for simplicity). Note then from (1.55) that α is a 2 dimensional harmonic function and we can use the 2 dimensional conformal transformations of the metric (1.54) to fix, without any loss of generality $\alpha = r$. Then (1.57), is just Laplace’s equation, written in 3 dimensional cylindrical coordinates, a linear ODE! Solving for Ω is in fact the same as solving for a Newtonian axial source! Given any Newtonian source we can find (and even superpose) Ω , the corresponding Weyl potential for the static and axisymmetric metric (1.54). In fact using this Newtonian analogy we can construct multi black hole solutions (see for example [34, 36–38]). In essence once we have determined the source corresponding to a black hole (a linear massive rod lying on the $r = 0$ axis) we just superpose such sources on the symmetry axis. For a non-linear theory as GR, however, this is not all. Using the non-linear (1.59) we solve for v obtaining the full metric solution. Typically v picks up conical singularities on the axis $r = 0$ which are interpreted as keeping the static equilibrium between attractive gravitational sources. For a double black hole we have either two infinite cosmic strings lying on $r = 0$ pulling apart the black holes or a rigid strut in between the rods keeping them apart [36–38]. The former corresponds to a conical defect, $v(0, z) > 0$, whereas the latter to a conical excess, $v(0, z) < 0$.

Firstly go back to $\Lambda \neq 0$. Note now that the curvature scale Λ spoils this integrability at step 1 given that we can no longer adequately fix the coordinate system as before (1.55). Furthermore secondly, switching back on A takes us from the static to the stationary case. Integrability for $\Lambda = 0$ is still maintained largely due to the fact that (1.56–1.57) combine together to form a complex equation, the Ernst equation with respect to a complex potential $\mathcal{E}$ that has as its real part the

Weyl potential and as its imaginary part a dual of the rotation A [39, 40]. It is interesting that this mathematical property remains true even when $\Lambda \neq 0$. [41], [42]. The Ernst equation replacing (1.56), (1.57) reads,

$$\frac{1}{\alpha} \vec{\nabla} \cdot (\alpha \vec{\nabla} \mathcal{E}_-) = \frac{(\vec{\nabla} \mathcal{E}_-)^2}{\text{Re}(\mathcal{E}_-)} + \text{Re}(\mathcal{E}_-) \frac{\Delta \alpha}{\alpha}, \quad (1.60)$$

with $\mathcal{E} = e^{\frac{2}{3}\alpha} + i\omega$.

Let us now look at a specific example solution first found by Carter, which generalises the Kerr metric in the presence of a cosmological constant. The solution reads,

$$ds_4^2 = -\frac{\Delta}{\rho^2} \left(dt - \frac{a \sin^2 \theta}{\Xi_a} d\varphi \right)^2 + \frac{\Delta_\theta \sin^2 \theta}{\rho^2} \left(a dt - \frac{r^2 + a^2}{\Xi_a} d\varphi \right)^2 + \rho^2 \left(\frac{dr^2}{\Delta} + \frac{d\theta^2}{\Delta_\theta} \right), \quad (1.61)$$

where k is the adS curvature scale, M is the black hole mass, a the angular momentum parameter and

$$\Delta = (r^2 + a^2)(1 + k^2 r^2) - 2Mr, \quad (1.62)$$

$$\Delta_\theta = 1 - a^2 k^2 \cos^2 \theta, \quad \Xi_a = 1 - a^2 k^2, \quad (1.63)$$

$$\rho^2 = r^2 + a^2 \cos^2 \theta, \quad \Lambda = -3k^2. \quad (1.64)$$

Rotation is bounded since the solution remains valid for $a^2 < k^2$ and is singular at $a^2 = k^2$. The exterior event horizon is the largest zero of $\Delta(r_+) = 0$ and the extremal limit is obtained when the inner and outer horizons coincide. This also sets the lower bound for the mass parameter, $m \geq m_{\text{ext}}$ which is extremised for $a^2 = k^2$. Unlike the static black holes here the zeros of Δ do not correspond to the position of the Killing horizon of ∂_t . In fact the geometric region defined by the null Killing generator of the horizon and the Killing horizon of ∂_t defines the ergoregion as in the case of Kerr spacetime. In this region we have superradiance effects for certain wave modes due to the possible energy extraction of the black hole with the reflected wave mode being of greater energy than the initial one ($E_{\text{scattered}} > E_{\text{initial}}$). Now an intriguing possibility arises [43]. AdS, with its confining gravitational potential, can act as an amplifier to this effect. This is true for massive modes due to the confining potential not allowing them to escape to infinity but shooting them back into the hole, and massless modes alike, due to the reflecting boundary conditions imposed at the adS boundary. One expects through this effect that the black hole will gradually loose its angular momentum [43]. Such an instability was found for small black holes in [44].

In 5 dimensions an interesting difference occurs [43] (for the generalization of Carter's solution to higher dimensions see [45–48]) and this effect, can in some

cases, be washed away. First there is an additional angular momentum parameter due to the extra Killing vector. The null generator of the horizon reads, $\chi = \partial_t + \Omega_\phi \partial_\phi + \Omega \partial$ where Ω_i is the rotational velocity associated to each angular momentum parameter. If the hole is sufficiently slow rotating $\Omega_i < k$ with respect to the curvature scale of adS, the vector χ remains timelike all the way up to infinity. In other words in adS matter can be in a co-rotating frame with the black hole, something impossible for a Kerr spacetime where χ is spacelike and an external observer would have to move at a speed greater than light to keep up with the black hole rotation. This effect also agrees with the rotating black hole thermodynamics in adS [43] where a similar phase transition occurs as for the static case we studied earlier (see also [49] for a more general discussion including the case of adS black rings and other assorted higher dimensional black objects).

Let us finally give the Ernst potential for the rotating black hole solution found by Carter. We can transit in between the coordinate system of (1.61) and (1.54) by setting

$$\frac{dr^2}{\Delta} = dr^2, \quad \frac{d\theta^2}{\Delta_\theta} = dz^2, \quad (1.65)$$

meaning that z and r are implicitly given as functions of θ and r , respectively. Using (1.54), this is all we need to know in order to identify the different components:

$$\alpha = \frac{\sin \theta}{\Xi_a} \sqrt{\Delta \Delta_\theta}, \quad (1.66)$$

$$A = \frac{a \sin^2 \theta (\Delta - \Delta_\theta (r^2 + a^2))}{\Xi_a (a^2 \Delta_\theta \sin^2 \theta - \Delta)}, \quad (1.67)$$

$$e^\Omega = \frac{\Xi_a^2 (\Delta - a^2 \Delta_\theta \sin^2 \theta)^2}{\Delta \Delta_\theta \rho^4 \sin^2 \theta}, \quad (1.68)$$

$$e^{2v} = \rho^2 \alpha^{1/2}. \quad (1.69)$$

The electric Ernst potential for Carter's solution is given by

$$\mathcal{E} = \frac{1}{\rho^2} (\Delta - a^2 \sin^2 \theta \Delta_\theta - 2ia \cos \theta (k^2 \rho^2 r + M)). \quad (1.70)$$

If there is no rotation, $a = 0$, the Ernst potential is real and corresponds to the $\kappa = 1$ black holes we studied in the previous section. For $M = 0$ we have pure adS but the potential is still complex since the metric has non-zero angular momentum.² If $\Lambda = 0$, $\mathcal{E}$ is the usual Ernst potential in the coordinates of (1.61).

² This is quite unlike the situation for Kerr's solution at asymptotic infinity.

Acknowledgements It is a pleasure to thank Bruno Boisseau, Jeff Dufaux, Renaud Parentani and Kostas Skenderis for helpful comments and discussion. I am especially grateful to Roberto Emparan and Bernard Linet for reading through the manuscript and making numerous helpful and critical comments and corrections. I last but not least thank Jihad Mourad for discussion and explanations regarding properties of adS space.

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Chapter 2

Perturbations of Anti de Sitter Black Holes

George Siopsis

Abstract I review perturbations of black holes in asymptotically anti de Sitter space. I show how the quasi-normal modes governing these perturbations can be calculated analytically and discuss the implications on the hydrodynamics of gauge theory fluids per the AdS/CFT correspondence. I also discuss phase transitions of hairy black holes with hyperbolic horizons and the dual superconductors emphasizing the analytical calculation of their properties.

2.1 Introduction

The perturbations of a black hole are governed by quasi-normal modes (QNMs). The latter are typically obtained by solving a wave equation for small fluctuations in the black hole background subject to the conditions that the flux be ingoing at the horizon and outgoing at asymptotic infinity. These boundary conditions in general lead to a discrete spectrum of complex frequencies whose imaginary part determines the decay time of the small fluctuations

$$\Im\omega = \frac{1}{\tau}. \quad (2.1)$$

There is a vast literature on quasi-normal modes and I make no attempt to review it here. Instead, I concentrate on obtaining analytic expressions for QNMs of black hole perturbations in asymptotically AdS space. One can rarely obtain analytic expressions in closed form. Instead, I discuss techniques which allow one to

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calculate the spectrum perturbatively in an asymptotic regime (high or low overtones). For high overtones, the frequencies at leading order are proportional to the radius of the horizon. For low overtones, one in general obtains an additional frequency which is inversely proportional to the horizon radius. Thus for large black holes there is a gap between the lowest frequency and the rest of the spectrum of QNMs. I pay special attention to the lowest frequencies because they govern the behavior of the gauge theory fluid on the boundary according to the AdS/CFT correspondence. The latter may have experimental consequences pertaining to the formation of the quark-gluon plasma in heavy ion collisions. Moreover, I discuss phase transitions to hairy black holes which correspond to a dual superconducting phase. I concentrate on the case of black holes with a hyperbolic horizon because their properties can be understood analytically.

In [Sect. 2.2](#) I discuss scalar, gravitational and electromagnetic perturbations of an AdS Schwarzschild black hole analytically calculating the QNM spectrum in the high frequency regime. In [Sect. 2.3](#) I calculate the QNM spectrum analytically in the low frequency regime and discuss its relevance to the hydrodynamic behavior of the dual gauge theory fluid on the boundary. In [Sect. 2.4](#) I introduce hairy black holes and discuss their phase transition in the case of a hyperbolic horizon which can be understood analytically. The dual gauge theory corresponds to a superconductor whose properties can be calculated via electromagnetic perturbations. Finally, I conclude with [Sect. 2.5](#).

2.2 Perturbations

In this section I discuss scalar, gravitational and electromagnetic perturbations of an AdS Schwarzschild black hole in d dimensions analytically calculating the QNM spectrum in the high frequency regime. Low overtones will be discussed in the next section.

The metric of an AdS Schwarzschild black hole is

$$ds^2 = -\left(\frac{r^2}{l^2} + K - \frac{2\mu}{r^{d-3}}\right)dt^2 + \frac{dr^2}{\frac{r^2}{l^2} + K - \frac{2\mu}{r^{d-3}}} + r^2 d\Sigma_{K,d-2}^2. \quad (2.2)$$

I shall choose units so that the AdS radius $l = 1$. The horizon radius and Hawking temperature are, respectively,

$$2\mu = r_+^{d-1} \left(1 + \frac{K}{r_+^2}\right), \quad T_H = \frac{(d-1)r_+^2 + K(d-3)}{4\pi r_+}. \quad (2.3)$$

The mass and entropy of the hole are, respectively,

$$M = (d-2)(K + r_+^2) \frac{r_+^{d-3}}{16\pi G} \text{Vol}(\Sigma_{K,d-2}), \quad S = \frac{r_+^{d-2}}{4G} \text{Vol}(\Sigma_{K,d-2}). \quad (2.4)$$

The parameter K determines the curvature of the horizon and the boundary of AdS space. For $K = 0, +1, -1$ we have, respectively, a flat ($\mathbb{R}^{d-2}$), spherical ($\mathbb{S}^{d-2}$) and hyperbolic ($\mathbb{H}^{d-2}/\Gamma$, topological black hole, where Γ is a discrete group of isometries) horizon (boundary).

The harmonics on $\Sigma_{K,d-2}$ satisfy

$$(\nabla^2 + k^2)\mathbb{T} = 0. \quad (2.5)$$

For $K = 0$, k is the momentum; for $K = +1$, the eigenvalues are quantized,

$$k^2 = l(l + d - 3) - \delta, \quad (2.6)$$

whereas for $K = -1$,

$$k^2 = \xi^2 + \left(\frac{d-3}{2}\right)^2 + \delta, \quad (2.7)$$

where ξ is discrete for non-trivial Γ . $\delta = 0, 1, 2$ for scalar, vector, or tensor perturbations, respectively.

According to the AdS/CFT correspondence, QNMs of AdS black holes are expected to correspond to perturbations of the dual Conformal Field Theory (CFT) on the boundary. The establishment of such a correspondence is hindered by difficulties in solving the wave equation governing the various types of perturbation. In three dimensions one obtains a hypergeometric equation which leads to explicit analytic expressions for the QNMs [1, 2]. In five dimensions one obtains a Heun equation and a derivation of analytic expressions for QNMs is no longer possible. On the other hand, numerical results exist in four, five and seven dimensions [3–5].

2.2.1 Scalar Perturbations

To find the asymptotic form of QNMs, we need to find an approximation to the wave equation valid in the high frequency regime. In three dimensions the resulting wave equation will be an exact equation (hypergeometric equation). In five dimensions, I shall turn the Heun equation into a hypergeometric equation which will lead to an analytic expression for the asymptotic form of QNM frequencies in agreement with numerical results.

2.2.1.1 AdS₃

In three dimensions the wave equation for a massless scalar field is

$$\frac{1}{r}\partial_r\left(r^3\left(1 - \frac{r_+^2}{r^2}\right)\partial_r\Phi\right) - \frac{1}{r^2 - r_+^2}\partial_t^2\Phi + \frac{1}{r^2}\partial_x^2\Phi = 0. \quad (2.8)$$

Writing the wavefunction in the form

$$\Phi = e^{i(\omega t - p x)} \Psi(y), \quad y = \frac{r_+^2}{r^2}, \quad (2.9)$$

the wave function becomes

$$y^2(y-1)((y-1)\Psi')' + \hat{\omega}^2 y \Psi + \hat{p}^2 y(y-1)\Psi = 0 \quad (2.10)$$

to be solved in the interval $0 < y < 1$, where

$$\hat{\omega} = \frac{\omega}{2r_+} = \frac{\omega}{4\pi T_H}, \quad \hat{p} = \frac{p}{2r_+} = \frac{p}{4\pi T_H}. \quad (2.11)$$

For QNMs, we are interested in the solution

$$\Psi(y) = y(1-y)^{i\hat{\omega}} {}_2F_1(1 + i(\hat{\omega} + \hat{p}), 1 + i(\hat{\omega} - \hat{p}); 2; y), \quad (2.12)$$

which vanishes at the boundary ($y \rightarrow 0$). Near the horizon ($y \rightarrow 1$), we obtain a mixture of ingoing and outgoing waves,

$$\Psi \sim A_+(1-y)^{-i\hat{\omega}} + A_-(1-y)^{+i\hat{\omega}}, \quad A_{\pm} = \frac{\Gamma(\pm 2i\hat{\omega})}{\Gamma(1 \pm i(\hat{\omega} + \hat{p}))\Gamma(1 \pm i(\hat{\omega} - \hat{p}))}.$$

Setting $A_- = 0$, we deduce the quasi-normal frequencies

$$\hat{\omega} = \pm \hat{p} - in, \quad n = 1, 2, \dots \quad (2.13)$$

which form a discrete spectrum of complex frequencies with $\Im \hat{\omega} < 0$.

2.2.1.2 AdS₅

Restricting attention to the case of a large black hole, the massless scalar wave equation reads

$$\frac{1}{r^3} \partial_r (r^5 f(r) \partial_r \Phi) - \frac{1}{r^2 f(r)} \partial_t^2 \Phi - \frac{1}{r^2} \nabla^2 \Phi = 0, \quad f(r) = 1 - \frac{r_+^4}{r^4}. \quad (2.14)$$

Writing the solution in the form

$$\Phi = e^{i(\omega t - p \cdot x)} \Psi(y), \quad y = \frac{r^2}{r_+^2} \quad (2.15)$$

the radial wave equation becomes

$$(y^2 - 1)(y(y^2 - 1)\Psi')' + \left(\frac{\hat{\omega}^2}{4} y^2 - \frac{\hat{p}^2}{4} (y^2 - 1) \right) \Psi = 0. \quad (2.16)$$

For QNMs, we are interested in the analytic solution which vanishes at the boundary and behaves as an ingoing wave at the horizon. The wave equation contains an additional (unphysical) singularity at $y = -1$, at which the

wavefunction behaves as $\Psi \sim (y+1)^{\pm\hat{\omega}/4}$. Isolating the behavior of the wavefunction near the singularities $y = \pm 1$,

$$\Psi(y) = (y-1)^{-i\hat{\omega}/4} (y+1)^{\pm\hat{\omega}/4} F_{\pm}(y), \quad (2.17)$$

we shall obtain two sets of modes with the same $\Im\hat{\omega}$, but opposite $\Re\hat{\omega}$.

$F_{\pm}(y)$ satisfies the Heun equation

$$\begin{aligned} y(y^2-1)F''_{\pm} + \left\{ \left(3 - \frac{i \pm 1}{2} \hat{\omega} \right) y^2 - \frac{i \pm 1}{2} \hat{\omega} y - 1 \right\} F'_{\pm} \\ + \left\{ \frac{\hat{\omega}}{2} \left(\pm \frac{i\hat{\omega}}{4} \mp 1 - i \right) y - (i \mp 1) \frac{\hat{\omega}}{4} - \frac{\hat{p}^2}{4} \right\} F_{\pm} = 0 \end{aligned} \quad (2.18)$$

to be solved in a region in the complex y -plane containing $|y| \geq 1$ which includes the physical regime $r > r_+$.

For large $\hat{\omega}$, the constant terms in the polynomial coefficients of F' and F are small compared with the other terms, therefore they may be dropped. The wave equation may then be approximated by a hypergeometric equation

$$(y^2-1)F''_{\pm} + \left\{ \left(3 - \frac{i \pm 1}{2} \hat{\omega} \right) y - \frac{i \pm 1}{2} \hat{\omega} \right\} F'_{\pm} + \frac{\hat{\omega}}{2} \left(\pm \frac{i\hat{\omega}}{4} \mp 1 - i \right) F_{\pm} = 0, \quad (2.19)$$

in the asymptotic limit of large frequencies $\hat{\omega}$. The acceptable solution is

$$F_0(x) = {}_2F_1(a_+, a_-; c; (y+1)/2), \quad a_{\pm} = 1 - \frac{i \pm 1}{4} \hat{\omega} \pm 1, \quad c = \frac{3}{2} \pm \frac{1}{2} \hat{\omega}. \quad (2.20)$$

For proper behavior at the boundary ($y \rightarrow \infty$), we demand that F be a *polynomial*, which leads to the condition

$$a_+ = -n, \quad n = 1, 2, \dots \quad (2.21)$$

Indeed, it implies that F is a polynomial of order n , so as $y \rightarrow \infty$, $F \sim y^n \sim y^{-a_+}$ and $\Psi \sim y^{-i\hat{\omega}/4} y^{\pm\hat{\omega}/4} y^{-a_+} \sim y^{-2}$, as expected.

We deduce the quasi-normal frequencies [6]

$$\hat{\omega} = \frac{\omega}{4\pi T_H} = 2n(\pm 1 - i) \quad (2.22)$$

in agreement with numerical results.

It is perhaps worth mentioning that these frequencies may also be deduced by a simple monodromy argument [6]. Considering the monodromies around the singularities, if the wavefunction has no singularities other than $y = \pm 1$, the contour around $y = +1$ may be unobstructedly deformed into the contour around $y = -1$, which yields

$$\mathcal{M}(1)\mathcal{M}(-1) = 1. \quad (2.23)$$

Since the respective monodromies are $\mathcal{M}(1) = e^{\pi\hat{\omega}/2}$ and $\mathcal{M}(-1) = e^{\mp i\pi\hat{\omega}/2}$, using $\Im\hat{\omega} < 0$, we deduce $\hat{\omega} = 2n(\pm 1 - i)$, in agreement with our result above.

2.2.2 Gravitational Perturbations

Next I consider gravitational perturbations. For definiteness, I concentrate on the case of spherical black holes ($K = +1$). I shall derive analytic expressions for QNMs [7] including first-order corrections [8]. The results are in good agreement with results of numerical analysis [9]. Extension to other forms of the horizon is straightforward [10].

The radial wave equation for gravitational perturbations in the black-hole background (2.2) can be cast into a Schrödinger-like form,

$$-\frac{d^2\Psi}{dr_*^2} + V[r(r_*)]\Psi = \omega^2\Psi, \quad (2.24)$$

in terms of the tortoise coordinate defined by

$$\frac{dr_*}{dr} = \frac{1}{f(r)}. \quad (2.25)$$

The potential V for the various types of perturbation has been found by Ishibashi and Kodama [11]. For tensor, vector and scalar perturbations, one obtains, respectively,

$$V_T(r) = f(r) \left\{ \frac{\ell(\ell + d - 3)}{r^2} + \frac{(d - 2)(d - 4)f(r)}{4r^2} + \frac{(d - 2)f'(r)}{2r} \right\} \quad (2.26)$$

$$V_V(r) = f(r) \left\{ \frac{\ell(\ell + d - 3)}{r^2} + \frac{(d - 2)(d - 4)f(r)}{4r^2} - \frac{rf'''(r)}{2(d - 3)} \right\} \quad (2.27)$$

$$\begin{aligned} V_S(r) = & \frac{f(r)}{4r^2} \left[\ell(\ell + d - 3) - (d - 2) + \frac{(d - 1)(d - 2)\mu}{r^{d-3}} \right]^{-2} \\ & \times \left\{ \frac{d(d - 1)^2(d - 2)^3\mu^2}{R^2r^{2d-8}} - \frac{6(d - 1)(d - 2)^2(d - 4)[\ell(\ell + d - 3) - (d - 2)]\mu}{R^2r^{d-5}} \right. \\ & + \frac{(d - 4)(d - 6)[\ell(\ell + d - 3) - (d - 2)]^2r^2}{R^2} + \frac{2(d - 1)^2(d - 2)^4\mu^3}{r^{3d-9}} \\ & + \frac{4(d - 1)(d - 2)(2d^2 - 11d + 18)[\ell(\ell + d - 3) - (d - 2)]\mu^2}{r^{2d-6}} \\ & + \frac{(d - 1)^2(d - 2)^2(d - 4)(d - 6)\mu^2}{r^{2d-6}} \\ & - \frac{6(d - 2)(d - 6)[\ell(\ell + d - 3) - (d - 2)]^2\mu}{r^{d-3}} \\ & - \frac{6(d - 1)(d - 2)^2(d - 4)[\ell(\ell + d - 3) - (d - 2)]\mu}{r^{d-3}} \\ & \left. + 4[\ell(\ell + d - 3) - (d - 2)]^3 + d(d - 2)[\ell(\ell + d - 3) - (d - 2)]^2 \right\}, \end{aligned}$$

Near the black hole singularity ($r \sim 0$),

$$V_T = -\frac{1}{4r_*^2} + \frac{\mathcal{A}_T}{[-2(d-2)\mu]^{\frac{1}{d-2}}} r_*^{-\frac{d-1}{d-2}} + \dots, \quad \mathcal{A}_T = \frac{(d-3)^2}{2(2d-5)} + \frac{\ell(\ell+d-3)}{d-2}, \quad (2.28)$$

$$V_V = \frac{3}{4r_*^2} + \frac{\mathcal{A}_V}{[-2(d-2)\mu]^{\frac{1}{d-2}}} r_*^{-\frac{d-1}{d-2}} + \dots, \quad \mathcal{A}_V = \frac{d^2 - 8d + 13}{2(2d-15)} + \frac{\ell(\ell+d-3)}{d-2} \quad (2.29)$$

and

$$V_S = -\frac{1}{4r_*^2} + \frac{\mathcal{A}_S}{[-2(d-2)\mu]^{\frac{1}{d-2}}} r_*^{-\frac{d-1}{d-2}} + \dots, \quad (2.30)$$

where

$$\mathcal{A}_S = \frac{(2d^3 - 24d^2 + 94d - 116)}{4(2d-5)(d-2)} + \frac{(d^2 - 7d + 14)[\ell(\ell+d-3) - (d-2)]}{(d-1)(d-2)^2}. \quad (2.31)$$

I have included only the terms which contribute to the order I am interested in. The behavior of the potential near the origin may be summarized by

$$V = \frac{j^2 - 1}{4r_*^2} + \mathcal{A} r_*^{-\frac{d-1}{d-2}} + \dots \quad (2.32)$$

where $j = 0$ (2) for scalar and tensor (vector) perturbations.

On the other hand, near the boundary (large r),

$$V = \frac{j_\infty^2 - 1}{4(r_* - \bar{r}_*)^2} + \dots, \quad \bar{r}_* = \int_0^\infty \frac{dr}{f(r)}, \quad (2.33)$$

where $j_\infty = d-1$, $d-3$ and $d-5$ for tensor, vector and scalar perturbations, respectively.

After rescaling the tortoise coordinate ($z = \omega r_*$), the wave equation to first order becomes

$$\left(\mathcal{H}_0 + \omega^{-\frac{d-3}{d-2}} \mathcal{H}_1 \right) \Psi = 0, \quad (2.34)$$

where

$$\mathcal{H}_0 = \frac{d^2}{dz^2} - \left[\frac{j^2 - 1}{4z^2} - 1 \right], \quad \mathcal{H}_1 = -\mathcal{A} z^{-\frac{d-1}{d-2}}. \quad (2.35)$$

By treating $\mathcal{H}_1$ as a perturbation, one may expand the wave function

$$\Psi(z) = \Psi_0(z) + \omega^{-\frac{d-3}{d-2}} \Psi_1(z) + \dots \quad (2.36)$$

and solve the wave equation perturbatively.

The zeroth-order wave equation,

$$\mathcal{H}_0 \Psi_0(z) = 0, \quad (2.37)$$

may be solved in terms of Bessel functions,

$$\Psi_0(z) = A_1 \sqrt{z} J_{\frac{1}{2}}(z) + A_2 \sqrt{z} N_{\frac{1}{2}}(z). \quad (2.38)$$

For large z , it behaves as

$$\begin{aligned} \Psi_0(z) &\sim \sqrt{\frac{2}{\pi}} [A_1 \cos(z - \alpha_+) + A_2 \sin(z - \alpha_+)] \\ &= \frac{1}{\sqrt{2\pi}} (A_1 - iA_2) e^{-i\alpha_+} e^{iz} + \frac{1}{\sqrt{2\pi}} (A_1 + iA_2) e^{+i\alpha_+} e^{-iz}, \end{aligned}$$

where $\alpha_{\pm} = \frac{\pi}{4}(1 \pm j)$.

At the boundary ($r \rightarrow \infty$), the wavefunction ought to vanish, therefore the acceptable solution is

$$\Psi_0(r_*) = B \sqrt{\omega(r_* - \bar{r}_*)} J_{\frac{1}{2}}(\omega(r_* - \bar{r}_*)). \quad (2.39)$$

Indeed, $\Psi \rightarrow 0$ as $r_* \rightarrow \bar{r}_*$, as desired.

Asymptotically (large z), it behaves as

$$\Psi(r_*) \sim \sqrt{\frac{2}{\pi}} B \cos[\omega(r_* - \bar{r}_*) + \beta], \quad \beta = \frac{\pi}{4}(1 + j_{\infty}). \quad (2.40)$$

This ought to be matched to the asymptotic form of the wavefunction in the vicinity of the black-hole singularity along the Stokes line $\Im z = \Im(\omega r_*) = 0$. This leads to a constraint on the coefficients A_1, A_2 ,

$$A_1 \tan(\omega \bar{r}_* - \beta - \alpha_+) - A_2 = 0. \quad (2.41)$$

By imposing the boundary condition at the horizon

$$\Psi(z) \sim e^{iz}, \quad z \rightarrow -\infty, \quad (2.42)$$

one obtains a second constraint. To find it, one needs to analytically continue the wavefunction near the black hole singularity ($z = 0$) to negative values of z . A rotation of z by $-\pi$ corresponds to a rotation by $-\frac{\pi}{d-2}$ near the origin in the complex r -plane. Using the known behavior of Bessel functions

$$J_{\nu}(e^{-i\pi} z) = e^{-i\pi\nu} J_{\nu}(z), \quad N_{\nu}(e^{-i\pi} z) = e^{i\pi\nu} N_{\nu}(z) - 2i \cos \pi\nu J_{\nu}(z), \quad (2.43)$$

for $z < 0$ the wavefunction changes to

$$\Psi_0(z) = e^{-i\pi(j+1)/2} \sqrt{-z} \left\{ [A_1 - i(1 + e^{i\pi j})A_2] J_{\frac{j}{2}}(-z) + A_2 e^{i\pi j} N_{\frac{j}{2}}(-z) \right\} \quad (2.44)$$

whose asymptotic behavior is given by

$$\Psi \sim \frac{e^{-i\pi(j+1)/2}}{\sqrt{2\pi}} [A_1 - i(1 + 2e^{i\pi i})A_2] e^{-iz} + \frac{e^{-i\pi(j+1)/2}}{\sqrt{2\pi}} [A_1 - iA_2] e^{iz}. \quad (2.45)$$

Therefore one obtains a second constraint

$$A_1 - i(1 + 2e^{i\pi i})A_2 = 0. \quad (2.46)$$

The two constraints are compatible provided

$$\left| \begin{array}{cc} 1 & -i(1 + 2e^{i\pi i}) \\ \tan(\omega\bar{r}_* - \beta - \alpha_+) & -1 \end{array} \right| = 0, \quad (2.47)$$

which yields the quasi-normal frequencies [7]

$$\omega\bar{r}_* = \frac{\pi}{4} (2 + j + j_\infty) - \tan^{-1} \frac{i}{1 + 2e^{i\pi i}} + n\pi. \quad (2.48)$$

The first-order correction to the above asymptotic expression may be found by standard perturbation theory [8]. To first order, the wave equation becomes

$$\mathcal{H}_0\Psi_1 + \mathcal{H}_1\Psi_0 = 0. \quad (2.49)$$

The solution is

$$\begin{aligned} \Psi_1(z) = & \sqrt{z} N_{\frac{j}{2}}(z) \int_0^z dz' \frac{\sqrt{z'} J_{\frac{j}{2}}(z') \mathcal{H}_1 \Psi_0(z')}{\mathcal{W}} \\ & - \sqrt{z} J_{\frac{j}{2}}(z) \int_0^z dz' \frac{\sqrt{z'} N_{\frac{j}{2}}(z') \mathcal{H}_1 \Psi_0(z')}{\mathcal{W}}. \end{aligned} \quad (2.50)$$

where $\mathcal{W} = 2/\pi$ is the Wronskian.

The wavefunction to first order reads

$$\Psi(z) = \{A_1[1 - b(z)] - A_2 a_2(z)\} \sqrt{z} J_{\frac{j}{2}}(z) + \{A_2[1 + b(z)] + A_1 a_1(z)\} \sqrt{z} N_{\frac{j}{2}}(z), \quad (2.51)$$

where

$$\begin{aligned} a_1(z) &= \frac{\pi\mathcal{A}}{2} \omega^{-\frac{d-3}{d-2}} \int_0^z dz' z'^{-\frac{1}{d-2}} J_{\frac{j}{2}}(z') J_{\frac{j}{2}}(z'), \\ a_2(z) &= \frac{\pi\mathcal{A}}{2} \omega^{-\frac{d-3}{d-2}} \int_0^z dz' z'^{-\frac{1}{d-2}} N_{\frac{j}{2}}(z') N_{\frac{j}{2}}(z'), \\ b(z) &= \frac{\pi\mathcal{A}}{2} \omega^{-\frac{d-3}{d-2}} \int_0^z dz' z'^{-\frac{1}{d-2}} J_{\frac{j}{2}}(z') N_{\frac{j}{2}}(z'), \end{aligned}$$

and $\mathcal{A}$ depends on the type of perturbation.

Asymptotically, it behaves as

$$\Psi(z) \sim \sqrt{\frac{2}{\pi}} [A'_1 \cos(z - \alpha_+) + A'_2 \sin(z - \alpha_+)], \quad (2.52)$$

where

$$A'_1 = [1 - \bar{b}]A_1 - \bar{a}_2 A_2, \quad A'_2 = [1 + \bar{b}]A_2 + \bar{a}_1 A_1 \quad (2.53)$$

and I introduced the notation

$$\bar{a}_1 = a_1(\infty), \quad \bar{a}_2 = a_2(\infty), \quad \bar{b} = b(\infty). \quad (2.54)$$

The first constraint is modified to

$$A'_1 \tan(\omega \bar{r}_* - \beta - \alpha_+) - A'_2 = 0. \quad (2.55)$$

Explicitly,

$$[(1 - \bar{b}) \tan(\omega \bar{r}_* - \beta - \alpha_+) - \bar{a}_1]A_1 - [1 + \bar{b} + \bar{a}_2 \tan(\omega \bar{r}_* - \beta - \alpha_+)]A_2 = 0. \quad (2.56)$$

To find the second constraint to first order, one needs to approach the horizon. This entails a rotation by $-\pi$ in the z -plane. Using

$$\begin{aligned} a_1(e^{-i\pi}z) &= e^{-i\pi\frac{d-3}{d-2}}e^{-i\pi j}a_1(z), \\ a_2(e^{-i\pi}z) &= e^{-i\pi\frac{d-3}{d-2}}\left[e^{i\pi j}a_2(z) - 4\cos^2\frac{\pi j}{2}a_1(z) - 2i(1 + e^{i\pi j})b(z)\right], \\ b(e^{-i\pi}z) &= e^{-i\pi\frac{d-3}{d-2}}[b(z) - i(1 + e^{-i\pi j})a_1(z)], \end{aligned}$$

in the limit $z \rightarrow -\infty$ one obtains

$$\Psi(z) \sim -ie^{-ij\pi/2}B_1 \cos(-z - \alpha_+) - ie^{ij\pi/2}B_2 \sin(-z - \alpha_+), \quad (2.57)$$

where

$$\begin{aligned} B_1 &= A_1 - A_1 e^{-i\pi\frac{d-3}{d-2}}[\bar{b} - i(1 + e^{-i\pi j})\bar{a}_1] \\ &\quad - A_2 e^{-i\pi\frac{d-3}{d-2}}\left[e^{+i\pi j}\bar{a}_2 - 4\cos^2\frac{\pi j}{2}\bar{a}_1 - 2i(1 + e^{+i\pi j})\bar{b}\right] \\ &\quad - i(1 + e^{i\pi j})\left[A_2 + A_2 e^{-i\pi\frac{d-3}{d-2}}[\bar{b} - i(1 + e^{-i\pi j})\bar{a}_1] + A_1 e^{-i\pi\frac{d-3}{d-2}}e^{-i\pi j}\bar{a}_1\right] \\ B_2 &= A_2 + A_2 e^{-i\pi\frac{d-3}{d-2}}[\bar{b} - i(1 + e^{-i\pi j})\bar{a}_1] + A_1 e^{-i\pi\frac{d-3}{d-2}}e^{-i\pi j}\bar{a}_1. \end{aligned}$$

Therefore the second constraint to first order reads

$$[1 - e^{-i\pi\frac{d-3}{d-2}}(i\bar{a}_1 + \bar{b})]A_1 - [i(1 + 2e^{i\pi j}) + e^{-i\pi\frac{d-3}{d-2}}((1 + e^{i\pi j})\bar{a}_1 + e^{i\pi j}\bar{a}_2 - i\bar{b})]A_2 = 0 \quad (2.58)$$

Compatibility of the two first-order constraints yields

$$\begin{vmatrix} 1 + \bar{b} + \bar{a}_2 \tan(\omega\bar{r}_* - \beta - \alpha_+) & i(1 + 2e^{i\pi j}) + e^{-i\pi\frac{d-3}{d-2}}((1 + e^{i\pi j})\bar{a}_1 + e^{i\pi j}\bar{a}_2 - i\bar{b}) \\ (1 - \bar{b}) \tan(\omega\bar{r}_* - \beta - \alpha_+) - \bar{a}_1 & 1 - e^{-i\pi\frac{d-3}{d-2}}(i\bar{a}_1 + \bar{b}) \end{vmatrix} = 0, \quad (2.59)$$

leading to the first-order expression for quasi-normal frequencies,

$$\begin{aligned} \omega\bar{r}_* &= \frac{\pi}{4}(2 + j + j_\infty) + \frac{1}{2i} \ln 2 + n\pi \\ &\quad - \frac{1}{8} \left\{ 6i\bar{b} - 2ie^{-i\pi\frac{d-3}{d-2}}\bar{b} - 9\bar{a}_1 + e^{-i\pi\frac{d-3}{d-2}}\bar{a}_1 + \bar{a}_2 - e^{-i\pi\frac{d-3}{d-2}}\bar{a}_2 \right\}, \end{aligned}$$

where

$$\begin{aligned} \bar{a}_1 &= \frac{\pi\mathcal{A}}{4} \left(\frac{n\pi}{2\bar{r}_*} \right)^{-\frac{d-3}{d-2}} \frac{\Gamma(\frac{1}{d-2})\Gamma(\frac{j}{2} + \frac{d-3}{2(d-2)})}{\Gamma^2(\frac{d-1}{2(d-2)})\Gamma(\frac{j}{2} + \frac{d-1}{2(d-2)})} \\ \bar{a}_2 &= \left[1 + 2 \cot \frac{\pi(d-3)}{2(d-2)} \cot \frac{\pi}{2} \left(-j + \frac{d-3}{d-2} \right) \right] \bar{a}_1 \\ \bar{b} &= -\cot \frac{\pi(d-3)}{2(d-2)} \bar{a}_1. \end{aligned}$$

Thus the first-order correction is $\sim \mathcal{O}(n^{-\frac{d-3}{d-2}})$.

The above analytic results are in good agreement with numerical results [9] (see Ref. [8] for a detailed comparison).

2.2.3 Electromagnetic Perturbations

The electromagnetic potential in four dimensions is

$$V_{\text{EM}} = \frac{\ell(\ell+1)}{r^2} f(r). \quad (2.60)$$

Near the origin,

$$V_{\text{EM}} = \frac{j^2 - 1}{4r_*^2} + \frac{\ell(\ell+1)r_*^{-3/2}}{2\sqrt{-4\mu}} + \dots, \quad (2.61)$$

where $j = 1$. Therefore a vanishing potential to zeroth order is obtained. To calculate the QNM spectrum one needs to include first-order corrections from the outset. Working as with gravitational perturbations, one obtains the QNMs

$$\omega \bar{r}_* = n\pi - \frac{i}{4} \ln n + \frac{1}{2i} \ln(2(1+i)\mathcal{A}\sqrt{\bar{r}_*}), \quad \mathcal{A} = \frac{\ell(\ell+1)}{2\sqrt{-4\mu}}. \quad (2.62)$$

Notice that the first-order correction behaves as $\ln n$, a fact which may be associated with gauge invariance.

As with gravitational perturbations, the above analytic results are in good agreement with numerical results [9] (see Ref. [8] for a detailed comparison).

2.3 Hydrodynamics

There is a correspondence between $\mathcal{N} = 4$ Super Yang–Mills (SYM) theory in the large N limit and type-IIB string theory in $\text{AdS}_5 \times S^5$ (AdS/CFT correspondence). In the low energy limit, string theory is reduced to classical supergravity and the AdS/CFT correspondence allows one to calculate all gauge field-theory correlation functions in the strong coupling limit leading to non-trivial predictions on the behavior of gauge theory fluids. For example, the entropy of $\mathcal{N} = 4$ SYM theory in the limit of large 't Hooft coupling is precisely 3/4 its value in the zero coupling limit.

The long-distance (low-frequency) behavior of any interacting theory at finite temperature must be described by fluid mechanics (hydrodynamics). This leads to a universality in physical properties because hydrodynamics implies very precise constraints on correlation functions of conserved currents and the stress-energy tensor. Their correlators are fixed once a few transport coefficients are known.

2.3.1 Vector Perturbations

I start with vector perturbations and work in the d -dimensional Schwarzschild background (2.2) with $K = +1$ (spherical horizon and boundary). It is convenient to introduce the coordinate [12]

$$u = \left(\frac{r_+}{r}\right)^{d-3}. \quad (2.63)$$

The wave equation becomes

$$-(d-3)^2 u^{\frac{d-4}{d-3}} \hat{f}(u) \left(u^{\frac{d-4}{d-3}} \hat{f}(u) \Psi' \right)' + \hat{V}_V(u) \Psi = \hat{\omega}^2 \Psi, \quad \hat{\omega} = \frac{\omega}{r_+}, \quad (2.64)$$

where prime denotes differentiation with respect to u and I have defined

$$\hat{f}(u) \equiv \frac{f(r)}{r^2} = 1 - u^{\frac{2}{d-3}} \left(u - \frac{1-u}{r_+^2} \right) \quad (2.65)$$

$$\begin{aligned} \hat{V}_V(u) \equiv \frac{V_V}{r_+^2} = \hat{f}(u) & \left\{ \hat{L}^2 + \frac{(d-2)(d-4)}{4} u^{-\frac{2}{d-3}} \hat{f}(u) \right. \\ & \left. - \frac{(d-1)(d-2) \left(1 + \frac{1}{r_+^2} \right)}{2} u \right\}, \end{aligned} \quad (2.66)$$

where $\hat{L}^2 = \frac{\ell(\ell+d-3)}{r_+^2}$.

First I consider the large black hole limit $r_+ \rightarrow \infty$ keeping $\hat{\omega}$ and $\hat{L}$ fixed (small). Factoring out the behavior at the horizon ($u = 1$)

$$\Psi = (1-u)^{-i\frac{\hat{\omega}}{d-1}} F(u), \quad (2.67)$$

the wave equation simplifies to

$$\mathcal{A}F'' + \mathcal{B}_{\hat{\omega}}F' + \mathcal{C}_{\hat{\omega},\hat{L}}F = 0, \quad (2.68)$$

where

$$\begin{aligned} \mathcal{A} &= -(d-3)^2 u^{\frac{2d-8}{d-3}} (1-u^{\frac{d-1}{d-3}}) \\ \mathcal{B}_{\hat{\omega}} &= -(d-3)[d-4-(2d-5)u^{\frac{d-1}{d-3}}]u^{\frac{d-5}{d-3}} - 2(d-3)^2 \frac{i\hat{\omega}}{d-1} \frac{u^{\frac{2d-8}{d-3}}(1-u^{\frac{d-1}{d-3}})}{1-u} \\ \mathcal{C}_{\hat{\omega},\hat{L}} &= \hat{L}^2 + \frac{(d-2)[d-4-3(d-2)u^{\frac{d-1}{d-3}}]}{4} u^{-\frac{2}{d-3}} \\ &\quad - \frac{\hat{\omega}^2}{1-u^{\frac{d-1}{d-3}}} + (d-3)^2 \frac{\hat{\omega}^2}{(d-1)^2} \frac{u^{\frac{2d-8}{d-3}}(1-u^{\frac{d-1}{d-3}})}{(1-u)^2} \\ &\quad - (d-3) \frac{i\hat{\omega}}{d-1} \frac{[d-4-(2d-5)u^{\frac{d-1}{d-3}}]u^{\frac{d-5}{d-3}}}{1-u} - (d-3)^2 \frac{i\hat{\omega}}{d-1} \frac{u^{\frac{2d-8}{d-3}}(1-u^{\frac{d-1}{d-3}})}{(1-u)^2}. \end{aligned}$$

One may solve this equation perturbatively by separating

$$(\mathcal{H}_0 + \mathcal{H}_1)F = 0, \quad (2.69)$$

where

$$\begin{aligned} \mathcal{H}_0 F &\equiv \mathcal{A}F'' + \mathcal{B}_0 F' + \mathcal{C}_{0,0} F \\ \mathcal{H}_1 F &\equiv (\mathcal{B}_{\hat{\omega}} - \mathcal{B}_0)F' + (\mathcal{C}_{\hat{\omega},\hat{L}} - \mathcal{C}_{0,0})F. \end{aligned}$$

Expanding the wavefunction perturbatively,

$$F = F_0 + F_1 + \dots \quad (2.70)$$

at zeroth order the wave equation reads

$$\mathcal{H}_0 F_0 = 0 \quad (2.71)$$

whose acceptable solution is

$$F_0 = u^{\frac{d-2}{2(d-3)}}, \quad (2.72)$$

being regular at both the horizon ($u = 1$) and the boundary ($u = 0$, or $\Psi \sim r^{-\frac{d-2}{2}} \rightarrow 0$ as $r \rightarrow \infty$). The Wronskian is

$$\mathcal{W} = \frac{1}{u^{\frac{d-4}{d-3}}(1 - u^{\frac{d-1}{d-3}})} \quad (2.73)$$

and another linearly independent solution is

$$\check{F}_0 = F_0 \int \frac{\mathcal{W}}{F_0^2}, \quad (2.74)$$

which is unacceptable because it diverges at both the horizon ($\check{F}_0 \sim \ln(1 - u)$ for $u \approx 1$) and the boundary ($\check{F}_0 \sim u^{-\frac{d-4}{2(d-3)}}$ for $u \approx 0$, or $\Psi \sim r^{\frac{d-4}{2}} \rightarrow \infty$ as $r \rightarrow \infty$).

At first order the wave equation reads

$$\mathcal{H}_0 F_1 = -\mathcal{H}_1 F_0 \quad (2.75)$$

whose solution may be written as

$$F_1 = F_0 \int \frac{\mathcal{W}}{F_0^2} \int \frac{F_0 \mathcal{H}_1 F_0}{\mathcal{A} \mathcal{W}}. \quad (2.76)$$

The limits of the inner integral may be adjusted at will because this amounts to adding an arbitrary amount of the unacceptable solution. To ensure regularity at the horizon, choose one of the limits of integration at $u = 1$ rendering the integrand regular at the horizon. Then at the boundary ($u = 0$),

$$F_1 = \check{F}_0 \int_0^1 \frac{F_0 \mathcal{H}_1 F_0}{\mathcal{A} \mathcal{W}} + \text{regular terms}. \quad (2.77)$$

The coefficient of the singularity ought to vanish,

$$\int_0^1 \frac{F_0 \mathcal{H}_1 F_0}{\mathcal{A} \mathcal{W}} = 0, \quad (2.78)$$

which yields a constraint on the parameters (dispersion relation)

$$\mathbf{a}_0 \hat{L}^2 - i \mathbf{a}_1 \hat{\omega} - \mathbf{a}_2 \hat{\omega}^2 = 0. \quad (2.79)$$

After some algebra, one arrives at

$$\mathbf{a}_0 = \frac{d-3}{d-1}, \quad \mathbf{a}_1 = d-3. \quad (2.80)$$

The coefficient $\mathbf{a}_2$ may also be found explicitly for each dimension d , but it cannot be written as a function of d in closed form. It does not contribute to the dispersion relation at lowest order. E.g., for $d = 4, 5$, one obtains, respectively

$$\mathbf{a}_2 = \frac{65}{108} - \frac{1}{3} \ln 3, \quad \frac{5}{6} - \frac{1}{2} \ln 2. \quad (2.81)$$

Equation (2.79) is quadratic in $\hat{\omega}$ and has two solutions,

$$\hat{\omega}_0 \approx -i \frac{\hat{L}^2}{d-1}, \quad \hat{\omega}_1 \approx -i \frac{d-3}{\mathbf{a}_2} + i \frac{\hat{L}^2}{d-1}. \quad (2.82)$$

In terms of the frequency ω and the quantum number ℓ ,

$$\omega_0 \approx -i \frac{\ell(\ell+d-3)}{(d-1)r_+}, \quad \frac{\omega_1}{r_+} \approx -i \frac{d-3}{\mathbf{a}_2} + i \frac{\ell(\ell+d-3)}{(d-1)r_+^2}. \quad (2.83)$$

The smaller of the two, ω_0 , is inversely proportional to the radius of the horizon and is not included in the asymptotic spectrum. The other solution, ω_1 , is a crude estimate of the first overtone in the asymptotic spectrum, nevertheless it shares two important features with the asymptotic spectrum: it is proportional to r_+ and its dependence on ℓ is $\mathcal{O}(1/r_+^2)$. The approximation may be improved by including higher-order terms. This increases the degree of the polynomial in the dispersion relation (2.79) whose roots then yield approximate values of more QNMs. This method reproduces the asymptotic spectrum derived earlier albeit not in an efficient way.

To include finite size effects, I shall use perturbation theory (assuming $1/r_+$ is small) and replace $\mathcal{H}_1$ by

$$\mathcal{H}'_1 = \mathcal{H}_1 + \frac{1}{r_+^2} \mathcal{H}_+ \quad (2.84)$$

where

$$\mathcal{H}_+ F \equiv \mathcal{A}_+ F'' + \mathcal{B}_+ F' + \mathcal{C}_+ F. \quad (2.85)$$

The coefficients may be easily deduced by collecting $\mathcal{O}(1/r_+^2)$ terms in the exact wave equation. One obtains

$$\begin{aligned} \mathcal{A}_+ &= -2(d-3)^2 u^2 (1-u) \\ \mathcal{B}_+ &= -(d-3)u \left[(d-3)(2-3u) - (d-1) \frac{1-u}{1-u^{\frac{d-1}{d-3}}} u^{\frac{d-1}{d-3}} \right] \\ \mathcal{C}_+ &= \frac{d-2}{2} \left[d-4 - (2d-5)u - (d-1) \frac{1-u}{1-u^{\frac{d-1}{d-3}}} u^{\frac{d-1}{d-3}} \right]. \end{aligned}$$

Interestingly, the zeroth order wavefunction F_0 is an eigenfunction of $\mathcal{H}_+$,

$$\mathcal{H}_+ F_0 = -(d-2)F_0, \quad (2.86)$$

therefore the first-order finite-size effect is a simple shift of the angular momentum operator

$$\hat{L}^2 \rightarrow \hat{L}^2 - \frac{d-2}{r_+^2}. \quad (2.87)$$

The QNMs of lowest frequency are modified to

$$\omega_0 = -i \frac{\ell(\ell+d-3) - (d-2)}{(d-1)r_+} + \mathcal{O}(1/r_+^2). \quad (2.88)$$

For $d = 4, 5$, we have respectively,

$$\omega_0 = -i \frac{(\ell-1)(\ell+2)}{3r_+}, \quad -i \frac{(\ell+1)^2 - 4}{4r_+} \quad (2.89)$$

in agreement with numerical results [9, 13].

One deduces from (2.88) the maximum lifetime of the vector modes,

$$\tau_{\max} = \frac{4\pi}{d} T_H. \quad (2.90)$$

In the case of a flat horizon ($K = 0$),

$$\omega_0 = -i \frac{k^2}{(d-1)r_+}, \quad (2.91)$$

which leads to the diffusion constant

$$D = \frac{1}{4\pi T_H}. \quad (2.92)$$

In the case of a hyperbolic horizon ($K = -1$), a similar calculation yields [10]

$$\omega_0 = -i \frac{\xi^2 + \frac{(d-1)^2}{4}}{(d-1)r_+}, \quad \tau = \frac{1}{|\omega_0|} < \frac{16\pi}{(d-1)^2} T_H. \quad (2.93)$$

It follows that for $d = 5$, these modes live longer than their spherical counterparts which is important for plasma behavior.

2.3.2 Scalar Perturbations

Next I consider scalar perturbations which are computationally more involved but phenomenologically more important because their spectrum contains the lowest frequencies and therefore the longest living modes. For a scalar perturbation we ought to replace the potential $\hat{V}_V$ by

$$\begin{aligned} \hat{V}_S(u) = & \frac{\hat{f}(u)}{4} \left[\hat{m} + \left(1 + \frac{1}{r_+^2} \right) u \right]^{-2} \\ & \times \left\{ d(d-2) \left(1 + \frac{1}{r_+^2} \right)^2 u^{\frac{2d-8}{d-3}} - 6(d-2)(d-4)\hat{m} \left(1 + \frac{1}{r_+^2} \right) u^{\frac{d-5}{d-3}} \right. \\ & + (d-4)(d-6)\hat{m}^2 u^{-\frac{2}{d-3}} + (d-2)^2 \left(1 + \frac{1}{r_+^2} \right)^3 u^3 \\ & + 2(2d^2 - 11d + 18)\hat{m} \left(1 + \frac{1}{r_+^2} \right)^2 u^2 \\ & + \frac{(d-4)(d-6) \left(1 + \frac{1}{r_+^2} \right)^2}{r_+^2} u^2 - 3(d-2)(d-6)\hat{m}^2 \left(1 + \frac{1}{r_+^2} \right) u \\ & \left. - \frac{6(d-2)(d-4)\hat{m} \left(1 + \frac{1}{r_+^2} \right)}{r_+^2} u + 2(d-1)(d-2)\hat{m}^3 + d(d-2) \frac{\hat{m}^2}{r_+^2} \right\}, \end{aligned} \quad (2.94)$$

where $\hat{m} = 2 \frac{\ell(\ell+d-3)-(d-2)}{(d-1)(d-2)r_+^2} = \frac{2(\ell+d-2)(\ell-1)}{(d-1)(d-2)r_+^2}$.

In the large black hole limit $r_+ \rightarrow \infty$ with $\hat{m}$ fixed (small), the potential simplifies to

$$\begin{aligned} \hat{V}_S^{(0)}(u) = & \frac{1 - u^{\frac{d-1}{d-3}}}{4(\hat{m} + u)^2} \left\{ d(d-2)u^{\frac{2d-8}{d-3}} - 6(d-2)(d-4)\hat{m}u^{\frac{d-5}{d-3}} \right. \\ & + (d-4)(d-6)\hat{m}^2 u^{-\frac{2}{d-3}} + (d-2)^2 u^3 \\ & \left. + 2(2d^2 - 11d + 18)\hat{m}u^2 - 3(d-2)(d-6)\hat{m}^2 u + 2(d-1)(d-2)\hat{m}^3 \right\}. \end{aligned} \quad (2.95)$$

The wave equation has an additional singularity due to the double pole of the scalar potential at $u = -\hat{m}$. It is desirable to factor out the behavior not only at the horizon, but also at the boundary and the pole of the scalar potential,

$$\Psi = (1 - u)^{-i\frac{\omega}{d-1}} \frac{u^{\frac{d-4}{2(d-3)}}}{\hat{m} + u} F(u). \quad (2.96)$$

Then the wave equation reads

$$\mathcal{A}F'' + \mathcal{B}_{\hat{\omega}}F' + \mathcal{C}_{\hat{\omega}}F = 0, \quad (2.97)$$

where

$$\begin{aligned} \mathcal{A} &= -(d-3)^2 u^{\frac{2d-8}{d-3}} (1 - u^{\frac{d-1}{d-3}}) \\ \mathcal{B}_{\hat{\omega}} &= -(d-3) u^{\frac{2d-8}{d-3}} (1 - u^{\frac{d-1}{d-3}}) \left[\frac{d-4}{u} - \frac{2(d-3)}{\hat{m}+u} \right] \\ &\quad - (d-3) [d-4 - (2d-5) u^{\frac{d-1}{d-3}}] u^{\frac{d-5}{d-3}} - 2(d-3)^2 \frac{i\hat{\omega}}{d-1} \frac{u^{\frac{2d-8}{d-3}} (1 - u^{\frac{d-1}{d-3}})}{1-u} \\ \mathcal{C}_{\hat{\omega}} &= -u^{\frac{2d-8}{d-3}} (1 - u^{\frac{d-1}{d-3}}) \left[-\frac{(d-2)(d-4)}{4u^2} - \frac{(d-3)(d-4)}{u(\hat{m}+u)} + \frac{2(d-3)^2}{(\hat{m}+u)^2} \right] \\ &\quad - \left[\left\{ d-4 - (2d-5) u^{\frac{d-1}{d-3}} \right\} u^{\frac{d-5}{d-3}} + 2(d-3) \frac{i\hat{\omega}}{d-1} \frac{u^{\frac{2d-8}{d-3}} (1 - u^{\frac{d-1}{d-3}})}{1-u} \right] \\ &\quad \times \left[\frac{d-4}{2u} - \frac{d-3}{\hat{m}+u} \right] - (d-3) \frac{i\hat{\omega}}{d-1} \frac{[d-4 - (2d-5) u^{\frac{d-1}{d-3}}] u^{\frac{d-5}{d-3}}}{1-u} \\ &\quad - (d-3)^2 \frac{i\hat{\omega}}{d-1} \frac{u^{\frac{2d-8}{d-3}} (1 - u^{\frac{d-1}{d-3}})}{(1-u)^2} + \frac{\hat{V}_{\mathbf{S}}^{(0)}(u) - \hat{\omega}^2}{1 - u^{\frac{d-1}{d-3}}} \\ &\quad + (d-3)^2 \frac{\hat{\omega}^2}{(d-1)^2} \frac{u^{\frac{2d-8}{d-3}} (1 - u^{\frac{d-1}{d-3}})}{(1-u)^2} \end{aligned}$$

I shall define the zeroth-order wave equation as $\mathcal{H}_0 F_0 = 0$, where

$$\mathcal{H}_0 F \equiv \mathcal{A}F'' + \mathcal{B}_0 F'. \quad (2.98)$$

The acceptable zeroth-order solution is

$$F_0(u) = 1, \quad (2.99)$$

which is plainly regular at all singular points ($u = 0, 1, -\hat{m}$). It corresponds to a wavefunction vanishing at the boundary ($\Psi \sim r^{-\frac{d-4}{2}}$ as $r \rightarrow \infty$).

The Wronskian is

$$\mathcal{W} = \frac{(\hat{m}+u)^2}{u^{\frac{2d-8}{d-3}} (1 - u^{\frac{d-1}{d-3}})} \quad (2.100)$$

and an unacceptable solution is $\check{F}_0 = \int \mathcal{W}$. It can be written in terms of hypergeometric functions. For $d \geq 6$, it has a singularity at the boundary, $\check{F}_0 \sim u^{-\frac{d-5}{d-3}}$ for $u \approx 0$, or $\Psi \sim r^{\frac{d-6}{2}} \rightarrow \infty$ as $r \rightarrow \infty$. For $d = 5$, the acceptable wavefunction behaves as $r^{-1/2}$ whereas the unacceptable one behaves as $r^{-1/2} \ln r$. For $d = 4$, the roles of F_0 and $\check{F}_0$ are reversed, however the results still valid because the correct boundary condition at the boundary is a Robin boundary condition [12, 14]. Finally, note that $\check{F}_0$ is also singular (logarithmically) at the horizon ($u = 1$).

Working as in the case of vector modes, one arrives at the first-order constraint

$$\int_0^1 \frac{\mathcal{C}_{\hat{\omega}}}{\mathcal{AW}} = 0, \quad (2.101)$$

because $\mathcal{H}_1 F_0 \equiv (\mathcal{B}_{\hat{\omega}} - \mathcal{B}_0)F'_0 + \mathcal{C}_{\hat{\omega}}F_0 = \mathcal{C}_{\hat{\omega}}$. This leads to the dispersion relation

$$\mathbf{a}_0 - \mathbf{a}_1 i\hat{\omega} - \mathbf{a}_2 \hat{\omega}^2 = 0, \quad (2.102)$$

After some algebra, one obtains

$$\mathbf{a}_0 = \frac{d-1}{2} \frac{1 + (d-2)\hat{m}}{(1 + \hat{m})^2}, \quad \mathbf{a}_1 = \frac{d-3}{(1 + \hat{m})^2}, \quad \mathbf{a}_2 = \frac{1}{\hat{m}} \{1 + O(\hat{m})\}. \quad (2.103)$$

For small $\hat{m}$, the quadratic equation has solutions

$$\hat{\omega}_0^{\pm} \approx -i \frac{d-3}{2} \hat{m} \pm \sqrt{\frac{d-1}{2}} \hat{m} \quad (2.104)$$

related to each other by $\hat{\omega}_0^+ = -\hat{\omega}_0^{-*}$, which is a general symmetry of the spectrum.

Finite size effects at first order amount to a shift of the coefficient $\mathbf{a}_0$ in the dispersion relation

$$\mathbf{a}_0 \rightarrow \mathbf{a}_0 + \frac{1}{r_+^2} \mathbf{a}_+ \quad (2.105)$$

After some tedious but straightforward algebra, we obtain

$$\mathbf{a}_+ = \frac{1}{\hat{m}} \{1 + O(\hat{m})\}. \quad (2.106)$$

The modified dispersion relation yields the modes

$$\hat{\omega}_0^{\pm} \approx -i \frac{d-3}{2} \hat{m} \pm \sqrt{\frac{d-1}{2}} \hat{m} + 1. \quad (2.107)$$

In terms of the quantum number ℓ ,

$$\omega_0^{\pm} \approx -i(d-3) \frac{\ell(\ell+d-3) - (d-2)}{(d-1)(d-2)r_+} \pm \sqrt{\frac{\ell(\ell+d-3)}{d-2}}, \quad (2.108)$$

in agreement with numerical results [13].

Notice that the imaginary part is inversely proportional to r_+ , as in vector case. In the scalar case, we also obtained a finite real part independent of r_+ .

The maximum lifetime of a gravitational scalar mode is found from (2.108) to be

$$\tau_{\max} = \frac{d-2}{(d-3)d} 4\pi T_H. \quad (2.109)$$

In the case of a flat horizon ($K = 0$), one obtains

$$\omega = \pm \frac{k}{\sqrt{d-2}} - i \frac{d-3}{(d-1)(d-2)r_+} k^2, \quad (2.110)$$

showing that the speed of sound is

$$v = \frac{1}{\sqrt{d-2}} \quad (2.111)$$

as expected for a CFT and the diffusion constant is

$$D = \frac{d-3}{d-2} \frac{1}{4\pi T_H}. \quad (2.112)$$

For a hyperbolic horizon ($K = -1$), a similar calculation yields [10]

$$\omega = \pm \sqrt{\frac{\xi^2 + (\frac{d-3}{2})^2}{d-2}} - i \frac{(d-3)[\xi^2 + \frac{(d-1)^2}{4}]}{(d-1)(d-2)r_+}, \quad \tau < \frac{4(d-2)}{(d-3)(d-1)^2} 4\pi T_H. \quad (2.113)$$

In the physically relevant case $d = 5$, evidently the $K = -1$ scalar modes live longer than any other modes, which is important for plasma behavior.

2.3.3 Tensor Perturbations

Finally, for completeness I discuss the case of tensor perturbations. Unlike the other two cases of gravitational perturbations, the asymptotic spectrum of tensor perturbations is the entire spectrum. To see this, note that in the large black hole limit, the wave equation reads

$$\begin{aligned} & - (d-3)^2 (u^{\frac{2d-8}{d-3}} - u^3) \Psi'' - (d-3) [(d-4)u^{\frac{d-5}{d-3}} - (2d-5)u^2] \Psi' \\ & + \left\{ \hat{L}^2 + \frac{d(d-2)}{4} u^{-\frac{2}{d-3}} + \frac{(d-2)^2}{4} u - \frac{\hat{\omega}^2}{1 - u^{\frac{d-1}{d-3}}} \right\} \Psi = 0. \end{aligned}$$

For the zeroth-order equation, we may set $\hat{L} = 0 = \hat{\omega}$. The resulting equation may be solved exactly. Two linearly independent solutions are ($\Psi = F_0$ at zeroth order)

$$F_0(u) = u^{\frac{d-2}{2(d-3)}}, \quad \check{F}_0(u) = u^{-\frac{d-2}{2(d-3)}} \ln\left(1 - u^{\frac{d-1}{d-3}}\right). \quad (2.114)$$

Neither behaves nicely at both ends ($u = 0, 1$). Therefore both are unacceptable which makes it impossible to build a perturbation theory to calculate small frequencies which are inversely proportional to r_0 . This negative result is in agreement with numerical results [9, 13] and in accordance with the AdS/CFT correspondence. Indeed, there is no ansatz that can be built from tensor spherical harmonics $\mathbb{T}_{ij}$ satisfying the linearized hydrodynamic equations, because of the conservation and tracelessness properties of $\mathbb{T}_{ij}$.

2.3.4 Hydrodynamics on the AdS Boundary

The above results in the bulk dictate the hydrodynamic behavior of the dual gauge theory fluid on the conformal boundary. To see the correspondence, one needs to understand the hydrodynamics in the linearized regime of a $d - 1$ dimensional fluid with dissipative effects. The fluid lives on a space with metric

$$ds_{\mathbb{S}}^2 = -dt^2 + d\Sigma_{K,d-2}^2. \quad (2.115)$$

The hydrodynamic equations are simply the requirement that the stress-energy momentum tensor be conserved,

$$\nabla_\mu T^{\mu\nu} = 0. \quad (2.116)$$

As the duality corresponds to a conformal field theory one must also demand scale invariance which implies

$$T^\mu_\mu = 0, \quad \epsilon = (d - 2)p, \quad \zeta = 0, \quad (2.117)$$

where ϵ , p and ζ are the energy density, pressure and bulk viscosity of the fluid. In the rest frame of the fluid, the velocity field is $u^\mu = (1, 0, 0, 0)$ and the pressure p_0 is constant. Consider a perturbation

$$u^\mu = (1, u^i), \quad p = p_0 + \delta p, \quad (2.118)$$

Applying the hydrodynamic equations, one obtains

$$\begin{aligned} (d - 2)\partial_t \delta p + (d - 1)p_0 \nabla_i u^i &= 0 \\ (d - 1)p_0 \partial_t u^i + \partial^i \delta p - \eta \left[\nabla^j \nabla_j u^i + K(d - 3)u^i + \frac{d - 4}{d - 2} \partial^i (\nabla_j u^j) \right] &= 0, \end{aligned} \quad (2.119)$$

where I used the curvature tensor $R_{ij} = K(d - 3)g_{ij}$.

For *vector perturbations*, consider the *ansatz*

$$\delta p = 0, \quad u^i = C_V e^{-i\omega t} \mathbb{V}^i, \quad (2.120)$$

where $\mathbb{V}^i$ is a vector harmonic.

The hydrodynamic equations imply

$$-i\omega(d-1)p_0 + \eta[k_V^2 - K(d-3)] = 0. \quad (2.121)$$

Using

$$\frac{\eta}{p_0} = (d-2) \frac{\eta}{s} \frac{S}{M} = \frac{4\pi\eta}{s} \frac{r_+}{K + r_+^2}, \quad (2.122)$$

with ω from the gravity dual, one obtains for large r_+ ,

$$\frac{\eta}{s} = \frac{1}{4\pi} \quad (2.123)$$

which is the standard value of the ratio in gauge theory fluids with a gravity dual [15].

For *scalar perturbations*, consider the *ansatz*

$$u^i = \mathcal{A}_S e^{-i\omega t} \partial^i \mathbb{S}, \quad \delta p = \mathcal{B}_S e^{-i\omega t} \mathbb{S}, \quad (2.124)$$

where $\mathbb{S}$ is a scalar harmonic.

The hydrodynamic equations imply the system of equations

$$\begin{aligned} (d-2)i\omega\mathcal{B}_S + (d-1)p_0 k_S^2 \mathcal{A}_S &= 0 \\ \mathcal{B}_S + \mathcal{A}_S \left[-i\omega(d-1)p_0 - 2(d-3)K\eta + 2\eta k_S^2 \frac{d-3}{d-2} \right] &= 0. \end{aligned} \quad (2.125)$$

The determinant must vanish,

$$\begin{vmatrix} (d-2)i\omega & (d-1)p_0 k_S^2 \\ 1 & -i\omega(d-1)p_0 - 2(d-3)K\eta + 2\eta k_S^2 \frac{d-3}{d-2} \end{vmatrix} = 0. \quad (2.126)$$

Arguing along the same lines as for vector perturbations, we arrive at

$$\frac{\eta}{s} = \frac{1}{4\pi} \quad (2.127)$$

which is the same result as the one obtained with vector QNMs.

2.3.5 Conformal Soliton Flow

The above results have been applied to the study of the quark-gluon plasma which forms in heavy ion collisions (at the Relativistic Heavy Ion Collider (RHIC) and elsewhere). In the case of a spherical horizon ($K = +1$), the boundary of space-time is $S^3 \times \mathbb{R}$. This may be conformally mapped onto a flat Minkowski space. Then by holographic renormalization, the AdS_5 -Schwarzschild black hole is dual

to a spherical shell of plasma on the four-dimensional Minkowski space which first contracts and then expands (conformal soliton flow) [13].

Quasi-normal modes govern the properties of this plasma with long-lived modes (i.e., of small $\Im\omega$) having the most influence. For example, one obtains the ratio

$$\frac{v_2}{\delta} = \frac{1}{6\pi} \Re \frac{\omega^4 - 40\omega^2 + 72}{\omega^3 - 4\omega} \sin \frac{\pi\omega}{2}, \quad (2.128)$$

where $v_2 = \langle \cos 2\phi \rangle$ evaluated at $\theta = \frac{\pi}{2}$ (mid-rapidity) and averaged with respect to the energy density at late times; $\delta = \frac{\langle y^2 - x^2 \rangle}{\langle y^2 + x^2 \rangle}$ is the eccentricity at time $t = 0$. Numerically, $\frac{v_2}{\delta} = 0.37$, which compares well with the result from RHIC data, $\frac{v_2}{\delta} \approx 0.323$ [16].

Another observable is the thermalization time which is found to be

$$\tau = \frac{1}{2|\Im\omega|} \approx \frac{1}{8.6T_{\text{peak}}} \approx 0.08 \text{ fm/c}, \quad T_{\text{peak}} = 300 \text{ MeV} \quad (2.129)$$

not in agreement with the RHIC result $\tau \sim 0.6 \text{ fm/c}$ [17], but still encouragingly small. For comparison, the corresponding result from perturbative QCD is $\tau \gtrsim 2.5 \text{ fm/c}$ [18, 19].

In the case of a hyperbolic horizon (topological black hole; $K = -1$), one needs to work with a conformal map from $\mathbb{H}^{d-2}/\Gamma \times \mathbb{R}$ to a $(d-1)$ -dimensional Minkowski space. Finding an explicit form of this map for $d = 5$ involves a considerable amount of numerical work. However, it is important that one consider this case because the modes of hyperbolic black holes live the longest [10].

2.4 Phase Transitions

In this section I discuss hairy black holes in asymptotically AdS space and their duals. At low temperatures, an instability leads to symmetry breaking and the formation of a dual superconductor. Electromagnetic perturbations of the black hole determine the conductivity in the bulk. First I review the case of a flat horizon ($K = 0$) [20] and then I discuss the case of hyperbolic horizon ($K = -1$) where exact analytical results are obtained [21].

2.4.1 $K = 0$

Consider a scalar Ψ of mass $m^2 = -2$, which is above the Breitenlohner-Freedman (BF) bound coupled to an electromagnetic potential A_μ in $3 + 1$ dimensions. The Lagrangian density is

$$\mathcal{L} = -\frac{1}{2}\partial_\mu\Psi\partial^\mu\Psi + \Psi^2 - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{q^2}{2}\Psi^2(\partial_\mu\theta - A_\mu)(\partial^\mu\theta - A^\mu), \quad (2.130)$$

where θ is a Stückelberg field. q is an arbitrary parameter which can be thought of as the electric charge of the scalar field Ψ (one may instead turn Ψ into a complex scalar field of charge q coupled to an electromagnetic potential in a standard fashion).

The Lagrangian density (2.130) is invariant under the $U(1)$ gauge transformation

$$A_\mu \rightarrow A_\mu + \partial_\mu\omega, \quad \theta \rightarrow \theta + \omega. \quad (2.131)$$

To fix the gauge, set

$$\theta = 0. \quad (2.132)$$

Working in the probe limit ($q \rightarrow \infty$) in which there is no back reaction to the metric, assume that the fields propagate in the black hole background (2.2) with $d = 4$ and $K = 0$. The radius of the horizon and Hawking temperature are, respectively,

$$r_+ = (2\mu)^{1/3}, \quad T = \frac{3r_+}{4\pi}. \quad (2.133)$$

Assuming spherical symmetry and an electrostatic potential $A_0 = \Phi(r)$, the field equations yield two coupled non-linear differential equations [20]

$$\begin{aligned} \Psi'' + \left(\frac{f'}{f} + \frac{2}{r}\right)\Psi' + \left(\frac{\Phi}{f}\right)^2\Psi + \frac{2}{f}\Psi &= 0 \\ \Phi'' + \frac{2}{r}\Phi' - \frac{2\Psi^2}{f}\Phi &= 0, \end{aligned} \quad (2.134)$$

where I set $q = 1$ and

$$f(r) = r^2 - \frac{2\mu}{r} \quad (2.135)$$

As $r \rightarrow \infty$, one obtains the boundary behavior

$$\Psi = \frac{\Psi^{(1)}}{r} + \frac{\Psi^{(2)}}{r^2} + \dots, \quad \Phi = \Phi^{(0)} + \frac{\Phi^{(1)}}{r} + \dots \quad (2.136)$$

where one of the $\Psi^{(i)} = 0$ ($i = 1, 2$) for stability, $\Phi^{(0)}$ is the chemical potential and $\Phi^{(1)} = -\rho$ (charge density).

Below a critical temperature T_0 a condensate forms,

$$\langle \mathcal{O}_i \rangle = \sqrt{2}\Psi^{(i)} \quad (2.137)$$

of an operator of dimension $\Delta = i$.

At $T = T_0$, one may set $\Psi = 0$ in the equation for Φ and deduce (using $\Phi(r_+) = 0$)

$$\Phi = \rho \left(\frac{1}{r_+} - \frac{1}{r} \right). \quad (2.138)$$

Then the equation for Ψ turns into an eigenvalue problem which yields

$$T_0 \approx 0.226\sqrt{\rho}, \quad 0.118\sqrt{\rho}$$

depending on the boundary conditions.

To study the properties of the dual CFT, apply an electromagnetic perturbation. It obeys the wave equation

$$A'' + \frac{f'}{f}A' + \left(\frac{\omega^2}{f^2} - \frac{2\Psi^2}{f} \right)A = 0, \quad (2.139)$$

to be solved subject to the boundary conditions that it be ingoing at the horizon, $A \sim f^{-i\omega/(4\pi T)}$, and at the boundary ($r \rightarrow \infty$),

$$A = A^{(0)} + \frac{A^{(1)}}{r} + \dots \quad (2.140)$$

Ohm's law yields the conductivity

$$\sigma(\omega) = \frac{A^{(1)}}{i\omega A^{(0)}}. \quad (2.141)$$

For $T \geq T_0$, $\Psi = 0$, therefore $A \sim e^{i\omega r_*}$ where $r_* = \int dr/f(r)$ is the tortoise coordinate. It follows that

$$\sigma(\omega) = 1. \quad (2.142)$$

At low T , for $\langle \mathcal{O}_1 \rangle \neq 0$, we have

$$\Psi \approx \frac{\langle \mathcal{O}_1 \rangle}{\sqrt{2} r}$$

Since $r_+ \rightarrow 0$, we obtain $A \sim e^{i\omega' r_*}$, where $\omega' = \sqrt{\omega^2 - \langle \mathcal{O}_1 \rangle^2}$. Therefore, for $\omega < \langle \mathcal{O}_1 \rangle$, $\Re \sigma = 0$, i.e., we obtain a superconductor with a gap.

2.4.2 $K = -I$

Turning to the case of a hyperbolic horizon [21], choose a scalar Ψ of mass $m^2 = -2$, as before, but conformally coupled with potential

$$V(\Psi) = \frac{8\pi G}{3} \Psi^4$$

The system has an exact solution (MTZ black hole [22])

$$ds^2 = -f_{MTZ}(r)dt^2 + \frac{dr^2}{f_{MTZ}(r)} + r^2 d\sigma^2, \quad f_{MTZ} = r^2 - \left(1 + \frac{r_0}{r}\right)^2, \quad (2.143)$$

with

$$\Psi(r) = -\sqrt{\frac{3}{4\pi G}} \frac{r_0}{r + r_0}, \quad \Phi = 0. \quad (2.144)$$

The temperature, entropy and mass are, respectively,

$$T = \frac{1}{\pi} \left(r_+ - \frac{1}{2} \right), \quad S_{MTZ} = \frac{\sigma}{4G} (2r_+ - 1), \quad M_{MTZ} = \frac{\sigma r_+}{4\pi G} (r_+ - 1). \quad (2.145)$$

and the law of thermodynamics $dM = TdS$ holds.

At $M = 0$, the MTZ black hole coincides with the topological black hole with no hair (Eq. 2.2) with $d = 4$, $K = -1$),

$$ds_{\text{AdS}}^2 = -(r^2 - 1)dt^2 + \frac{dr^2}{r^2 - 1} + r^2 d\Sigma^2 \quad (2.146)$$

and an enhanced scaling symmetry (pure AdS space) emerges at the critical temperature

$$T_0 = \frac{1}{2\pi} \quad (2.147)$$

At this point there is a phase transition which can be seen by calculating the difference in free energies,

$$\Delta F = F_{TBH} - F_{MTZ} = -\frac{\sigma}{8\pi G} \pi^3 l^3 (T - T_0)^3 + \dots, \quad (2.148)$$

showing that there is a third-order phase transition at T_0 .

Perturbative stability of the MTZ black hole has also been demonstrated for $T < T_0$ ($M < 0$) [21]. Comparing with the flat case, note that here both $\Psi^{(1)}$ and $\Psi^{(2)}$ are non-vanishing, yet the MTZ black hole is stable. However this is true only if the mass is negative which is never the case with a flat horizon. Also, here the condensation of the scalar field has a geometrical origin and is due entirely to its coupling to gravity.

Moreover, the heat capacities in the normal and superconducting (corresponding to the MTZ black hole) phases, respectively, as $T \rightarrow 0$ exhibit a power-law behavior

$$C_n \approx \frac{\pi\sigma}{3\sqrt{3}G} T, \quad C_s \approx \frac{\pi\sigma}{2G} T, \quad (2.149)$$

Since both $\Psi^{(1)}$ and $\Psi^{(2)}$ are non-vanishing, we have a multi-trace deformation of the CFT [23] with a condensate

$$\langle \mathcal{O}_1 \rangle = \sqrt{\frac{3\pi^3}{2G}} (T_0^2 - T^2). \quad (2.150)$$

It should be noted that the deformation does not break the global $U(1)$ symmetry because Ψ is a real field (see Eq. 2.130).

To study the conductivity, apply an electromagnetic perturbation. It obeys the wave equation (2.139) which may be solved using first-order perturbation theory in q^2 ,

$$A = e^{-i\omega r_*} + \frac{q^2}{2i\omega} e^{i_*} \int_{r_+}^r dr' \Psi^2(r') e^{-2i_*} - \frac{q^2}{2i\omega} e^{-i_*} \int_{r_+}^r dr' \Psi^2(r'). \quad (2.151)$$

The conductivity to first order in q^2 is

$$\sigma(\omega) = \frac{A^{(1)}}{i\omega A^{(0)}} = 1 - \frac{q^2}{i\omega} \int_{r_+}^{\infty} dr \Psi^2(r) e^{-2i\omega r_*}. \quad (2.152)$$

The superfluid density is found from

$$\Re[\sigma(\omega)] \sim \pi n_s \delta(\omega), \quad \Im[\sigma(\omega)] \sim \frac{n_s}{\omega}, \quad \omega \rightarrow 0. \quad (2.153)$$

One obtains

$$n_s = q^2 \int_{r_+}^{\infty} dr \Psi^2(r) = \frac{3q^2}{4\pi G} \frac{r_0^2}{r_+ + r_0} = \alpha (T_0 - T)^2, \quad \alpha = \frac{3\pi q^2}{4G}. \quad (2.154)$$

Near $T = 0$,

$$n_s(0) - n_s(T) \approx \frac{\alpha}{\pi} T^\delta, \quad \delta = 1 \quad (2.155)$$

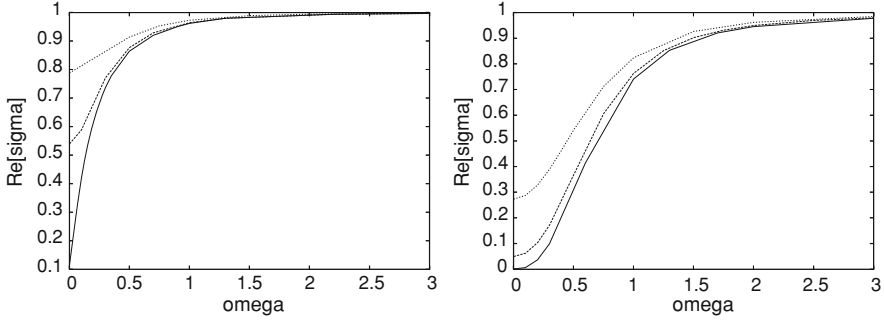
In Table 2.1, this analytic prediction is compared against exact numerical results for various values of the charge q . Naturally, the agreement is best at small values of q .

Table 2.1 The exponent δ characterizing the low-temperature dependence of the superfluid density n_s

$q/\sqrt{G}$	1	3	5
δ	1.025 ± 0.007	1.52 ± 0.03	1.78 ± 0.03

Table 2.2 Numerical vs analytical results for the normal and superfluid densities

$q/\sqrt{G}$	$\gamma_{\text{numerical}}$	$\gamma_{\text{analytical}}$	$\alpha_{\text{numerical}}$	$\alpha_{\text{analytical}}$
0.1	0.0020	0.0024	0.0225	0.024
0.5	0.0538	0.0597	0.552	0.589
1.0	0.187	0.239	2.196	2.356
2.0	0.684	0.955	8.678	9.425
3.0	1.325	2.15	20.35	21.21
5.0	2.522	5.97	52.90	58.90

**Fig. 2.1** The real part of the conductivity vs ω for $q/\sqrt{G} = 2$ (left) and $q/\sqrt{G} = 5$ (right) and $T = 0.0032, 0.032, 0.064$. The *lowest* curve corresponds to the lowest temperature

The normal, non-superconducting, component of the DC conductivity is

$$n_n = \lim_{\omega \rightarrow 0} \Re[\sigma(\omega)]. \quad (2.156)$$

Therefore,

$$\ln n_n = 2q^2 \int_{r_+}^{\infty} dr \Psi^2(r) r_*. \quad (2.157)$$

At low T ,

$$n_n \sim T^\gamma, \quad \gamma = \frac{3q^2}{4\pi G}. \quad (2.158)$$

This analytic result and the prediction for the parameter α determining the critical behavior of the superfluid density are compared against exact numerical results in Table 2.2. Again, the agreement is best at small q .

Figures 2.1 and 2.2 show the frequency dependence of the real and imaginary, respectively, parts of the conductivity. The real part of the conductivity becomes smaller as we increase the charge q . Unfortunately, numerical instabilities also increase and we have not been able to produce reliable numerical results above $q/\sqrt{G} = 5$. The superconductor appears to be *gapless*. However, a gap is likely to

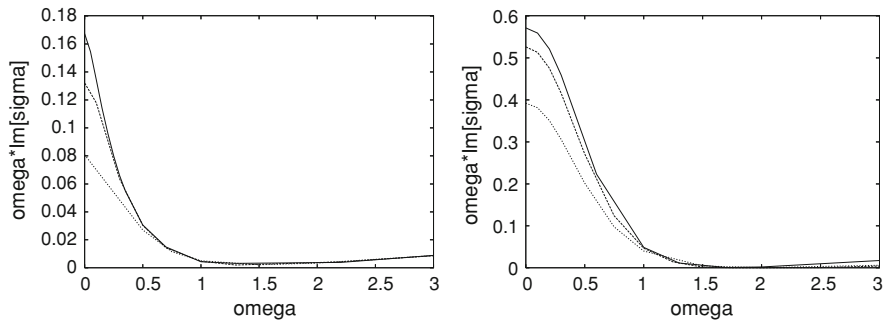


Fig. 2.2 2 The imaginary part of the conductivity multiplied by ω vs ω for $q/\sqrt{G} = 2$ (left) and $q/\sqrt{G} = 5$ (right) and $T = 0.0032, 0.032, 0.064$. The *uppermost* curve corresponds to the lowest temperature

develop above a certain value of the charge q , as indicated by the trend in the graphs as q increases.

2.5 Conclusion

The quasi-normal modes that govern perturbations of black holes in asymptotically AdS space are a powerful tool in understanding the hydrodynamic behavior of a gauge theory fluid at strong coupling. Here I focused on the analytic calculation of QNMs. I discussed both high overtones and low frequencies. I applied the results on gravitational perturbations to the understanding of the quark-gluon plasma produced in heavy ion collisions at RHIC and the LHC. I also considered hairy black holes whose electromagnetic perturbations allow one to analyze the conductivity of the dual conformal field theory and the phase transition to a superconducting state. I reviewed the case of a flat horizon and compared the results with those from black holes with hyperbolic horizon for which exact hairy solutions have been constructed (MTZ black holes [22]). In all these cases, only the low-lying QNMs were needed. It is unclear what physical role high overtones play.

Acknowledgement Research supported in part by the US Department of Energy under grant DE-FG05-91ER40627.

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Chapter 3

Introduction to the AdS/CFT Correspondence

Philip C. Argyres

Abstract We review the basic properties of d -dimensional conformal field theories (CFTs) and describe their relation to quantum gravitational theories on $d + 1$ -dimensional anti-de Sitter (AdS_{d+1}) space-times. We briefly review the 't Hooft limit of $U(N)$ Yang–Mills theory, and then describe how an appropriate limit of type IIB superstring theory with D3-branes can be used to motivate a precise and computable correspondence between a four-dimensional conformal field theory and a quantum gravitational theory on $\text{AdS}_5 \times S^5$. We then discuss an extension of this construction in which probe branes on the AdS space-time are included.

3.1 Introduction

These lectures review the foundations of the AdS/CFT correspondence, emphasizing the logical structure of the arguments that lead to this conjectured equivalence of a (non-gravitational) quantum field theory with a higher-dimensional theory of quantum gravity. Included are summaries of the basic ingredients in the correspondence: d -dimensional conformal field theories, anti-de Sitter (AdS) space-times, and the 't Hooft limit of Yang–Mills (YM) theories. A basic familiarity with string theory [1–3] is assumed. Although the framework for how computations in the AdS/CFT correspondence can be carried out is set up, the description of detailed and efficient computational techniques are left for other lectures in this volume.

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A basic introduction to the AdS/CFT correspondence of a descriptive nature, together with an elementary introduction to string theory can be found in [4], while [5] gives a descriptive survey of extensions and some applications of the correspondence. Some more detailed reviews of the AdS/CFT correspondence and modern developments of the formalism for carrying out computations are [6–8].

3.2 CFTs

Scale-invariant quantum field theories are important as possible end-points of renormalization group flows on the space of cut-off effective field theories. Knowledge of them and their relevant deformations partially organizes the space of quantum field theories; see Fig. 3.1. Scaling a local or quasi-local quantum field theory to its infrared (IR), or long-wavelength, limit results in a local scale-invariant field theory, by definition. Scaling to the ultraviolet (UV), or short distances, need not always lead to a scale-invariant local field theory (*e.g.*, the non-local “little string theories” are a known exception; see [9] for a review), but many such examples are known, including all the asymptotically free and scale invariant Yang–Mills theories in four space–time dimensions.

It is believed [10] that unitary scale-invariant theories are also conformally invariant: the space–time symmetry group Poincaré $\times$ dilatations enlarges to the conformal group of symmetries. In what follows we will concentrate on such conformal field theories (CFTs) for which there exists a local, conserved, traceless energy–momentum tensor $T^{\mu\nu}(x)$

3.2.1 Conformal Algebra

The conformal group in d space–time dimensions is the group of reparameterizations which preserve the (flat) space–time metric, $g^{\mu\nu}$, up to a local scale factor, where $\mu, \nu \in \{1, \dots, d\}$. (We will work in Euclidean signature where convenient.)

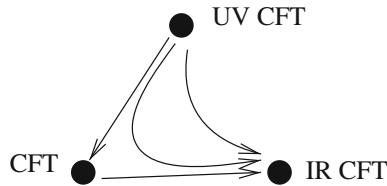


Fig. 3.1 Effective field theory RG flows from UV to IR CFTs, which appear as fixed points of RG flows. CFTs can also govern the behavior of flows at intermediate scales, as shown

For $d > 2$ its generators are (Lorentz) rotations, $M^{\mu\nu}$, plus translations, dilatations, and special conformal transformations which, respectively, act infinitesimally as

$$\begin{aligned} P^\mu : x^\mu &\rightarrow x^\mu + \alpha^\mu, \\ D : x^\mu &\rightarrow (1 + \epsilon)x^\mu, \\ K_\nu : x^\mu &\rightarrow x^\mu + \epsilon_\nu (g^{\mu\nu}x^2 - 2x^\mu x^\nu). \end{aligned} \quad (3.1)$$

They obey the algebra

$$[D, K_\mu] = iK_\mu, \quad [D, P_\mu] = -iP_\mu, \quad [P_\mu, K_\nu] = 2iM_{\mu\nu} - 2ig_{\mu\nu}D, \quad (3.2)$$

where the other commutators either vanish or follow from rotational invariance. By defining $2d + 1$ additional rotation generators from P_μ , D , and K_μ ,

$$M_{\mu d+1} \equiv (K_\mu - P_\mu)/2, \quad M_{\mu d+2} \equiv (K_\mu + P_\mu)/2, \quad M_{d+1 d+2} \equiv D, \quad (3.3)$$

we see that the d -dimensional conformal algebra is equivalent to $SO(d, 2)$ in Minkowski signature or $SO(d + 1, 1)$ in Euclidean signature.

3.2.2 Local Field Operators

As in usual quantum field theory, we classify local field operators $\mathcal{O}(x^\mu)$, by their transformation properties of the little group $SO(d) \times SO(2) \subset SO(d, 2)$ of the conformal group. In radial quantization in Euclidean signature, the $SO(d)$ irreducible representation is the “spin” of the field, while the charge under the (infinite cover of the) $SO(2)$ subgroup is the scaling dimension, Δ , of the field,

$$\mathcal{O}_\Delta(\lambda x^\mu) = \lambda^{-\Delta} \mathcal{O}_\Delta(x^\mu) \quad \Leftrightarrow \quad [D, \mathcal{O}_\Delta(0)] = -i\Delta \mathcal{O}_\Delta(0) \quad (3.4)$$

In radial quantization we foliate Euclidean $\mathbf{R}^d$ by $(d-1)$ -spheres, S^{d-1} , concentric at the origin, and define the Hilbert space of states of the CFT at a given radial slice. The dilatation operator, D , then generates the evolution of states to different radial slices. In this case there is a correspondence between local operators and states in the Hilbert space. First, we denote by $|0\rangle$ the conformal vacuum state, defined to be the state annihilated by all the generators of the conformal algebra. Then each local operator $\mathcal{O}(x^\mu)$ defines a state at the origin (i.e., in the zero radius limit of the slicing) by $\mathcal{O}(0)|0\rangle$. In a CFT, by scale invariance this is a state on any size S^{d-1} around the origin. Conversely any state on an S^{d-1} created by the action of operators at smaller radii can be written as a state created by a local operator insertion at the origin by shrinking the sphere.

Primary operators are those annihilated by the special conformal generators, K^μ , at the origin. By translating such operators to arbitrary position, it follows that they obey the commutation relations

$$\begin{aligned}
[P_\mu, \mathcal{O}_\Delta(x)] &= i\partial_\mu \mathcal{O}_\Delta(x), \\
[M_{\mu\nu}, \mathcal{O}_\Delta(x)] &= \left[i(x_\mu \partial_\nu - x_\nu \partial_\mu) + \Sigma_{\mu\nu}^R \right] \mathcal{O}_\Delta(x), \\
[D, \mathcal{O}_\Delta(x)] &= i(x^\mu \partial_\mu - \Delta) \mathcal{O}_\Delta(x), \\
[K_\mu, \mathcal{O}_\Delta(x)] &= \left[i(x^2 \partial_\mu - 2x_\mu x^\nu \partial_\nu + 2x_\mu \Delta) - 2x^\nu \Sigma_{\mu\nu}^R \right] \mathcal{O}_\Delta(x).
\end{aligned} \tag{3.5}$$

Here $\Delta \in \mathbf{R}$ is the conformal dimension of the primary operator, and $\Sigma_{\mu\nu}^R$ are the representation matrices of the irreducible spin R of the primary which acts on its spin indices (which we have suppressed).

Unitarity puts lower bounds on the conformal dimension, Δ , of primaries depending on their spin, R ; see [11] for a review. In particular, for a scalar primary, $\Delta \geq (d-2)/2$, with equality only for a free field.

Descendant operators are those made by acting on a primary operator with P^μ s. All local operators are found in this way. Note that acting with P^μ (K^μ) increases (decreases) the scaling dimension Δ of the operator by 1.

3.2.3 Conformal Correlators

Correlators of descendant fields can be derived from those of the primary fields by taking derivatives. Conformal invariance determines the form of the correlators of primaries in terms of their scaling dimensions and spins up to undetermined functions of conformal invariants made up of the positions of the insertions of the operators. For example, for scalar primaries, the 2-, 3-, and 4-point functions are determined to have the forms

$$\begin{aligned}
\langle \mathcal{O}_{\Delta_1}(x_1) \mathcal{O}_{\Delta_2}(x_2) \rangle &= \delta_{1,2} \prod_{i < j}^2 |x_{ij}|^{-\Delta}, \\
\langle \mathcal{O}_{\Delta_1}(x_1) \mathcal{O}_{\Delta_2}(x_2) \mathcal{O}_{\Delta_3}(x_3) \rangle &= c_{123} \prod_{i < j}^3 |x_{ij}|^{\Delta - 2\Delta_i - 2\Delta_j}, \\
\langle \mathcal{O}_{\Delta_1}(x_1) \mathcal{O}_{\Delta_2}(x_2) \mathcal{O}_{\Delta_3}(x_3) \mathcal{O}_{\Delta_4}(x_4) \rangle &= c_{1234}(u, v) \prod_{i < j}^4 |x_{ij}|^{\frac{1}{3}\Delta - \Delta_i - \Delta_j}.
\end{aligned} \tag{3.6}$$

Here $x_{ij} \equiv x_i - x_j$ and $\Delta \equiv \sum_i \Delta_i$ are the sum of the dimensions of the operators in the correlator. For the 2-point function the Kronecker δ represents a choice of normalization of the primary fields. For the 3-point function, the c_{ijk} are

undetermined constant coefficients, while for the 4-point function $c_{1234}(u, v)$ is an undetermined function of the two independent conformally invariant cross-ratios

$$u \equiv \frac{|x_{12}||x_{34}|}{|x_{13}||x_{24}|}, \quad v \equiv \frac{|x_{14}||x_{23}|}{|x_{13}||x_{24}|}. \quad (3.7)$$

The number of independent conformal invariants grows with the number of insertion points, so higher-point correlators would appear to be even less constrained by conformal invariance.

However, the state-operator correspondence implies that the product of any two conformal primaries can be rewritten as a linear combination of conformal operators inserted at a nearby point (as long as there are no other insertions in between the two primaries). This expansion is called the operator product expansion (OPE) of the two primaries. For example, for two scalar primaries it has the form

$$\mathcal{O}_{\Delta_i}(x)\mathcal{O}_{\Delta_j}(0) = \sum_k c_{ijk} |x|^{-\Delta_i-\Delta_j+\Delta_k} (\mathcal{O}_{\Delta_k}(0) + \text{descendants}) \quad (3.8)$$

where the sum on the right side is over all primaries and their descendants. The coefficients appearing in front of the descendants of each primary are completely determined by conformal invariance. It is also easy to see that the coefficients c_{ijk} appearing in front of each primary on the right side are the same as the coefficients appearing in the 3-point functions.

In principle all n -point correlators are determined by the OPEs, since any n -point function can be replaced by an infinite sum of $(n-1)$ -point functions by using the OPE for any two adjacent insertions. Thus, the CFT is completely determined by the data $\{\Delta_i, \text{spins}, c_{ijk}\}$ for all the primaries (labelled by i, j, k). However, this data is highly constrained: unitarity and associativity (“crossing symmetry”) of the OPEs imply many non-trivial relations among the CFT data, so arbitrary lists of $\{\Delta_i, \text{spins}, c_{ijk}\}$ will not in general define a consistent CFT.

Since the associativity constraints from the OPEs are notoriously difficult to even state, let alone solve, we look for other approaches to constructing CFTs. The AdS/CFT correspondence is such an alternative approach.

It is based on a different, useful way of encoding the correlation functions of a CFT in terms of its partition function,

$$Z[\phi_{\Delta_i}] \equiv \left\langle \exp \left(\int d^d x \phi_{\Delta_i}(x) \mathcal{O}_{\Delta_i}(x) \right) \right\rangle_{\text{CFT}}. \quad (3.9)$$

This functional of the classical *sources* $\phi_{\Delta_i}(x)$ associated with each field operator in the CFT generates the correlation functions by taking derivatives of Z with respect to the sources:

$$\langle \mathcal{O}_{\Delta_1}(x_1) \mathcal{O}_{\Delta_2}(x_2) \dots \rangle = \frac{\partial^n Z[\phi_{\Delta_i}]}{\partial \phi_{\Delta_1}(x_1) \partial \phi_{\Delta_2}(x_2) \dots} \Big|_{\phi_{\Delta_i}=0}. \quad (3.10)$$

The conformal invariance of the correlators is reflected in the conformal invariance of $Z[\phi_\Delta]$. In particular, under scaling,

$$\int d^d x \phi_\Delta(x) \mathcal{O}_\Delta(x) = \int d^d(\lambda x) \phi_\Delta(\lambda x) \mathcal{O}_\Delta(\lambda x) = \lambda^{d-\Delta} \int d^d x \phi_\Delta(\lambda x) \mathcal{O}_\Delta(x), \quad (3.11)$$

so Z is invariant under the following scaling transformation of the sources:

$$\phi_\Delta(x) \rightarrow \lambda^{d-\Delta} \phi_\Delta(\lambda x). \quad (3.12)$$

In general, the sources transform in field representations of the conformal group and $Z[\phi_\Delta]$ is an invariant combination of these fields.

3.3 AdS/CFT Correspondence

We now introduce a trick to generate conformally invariant $Z[\phi_\Delta]$ s. It is not guaranteed to generate all CFTs, perhaps just a special subset of them. The idea is to copy the way we write actions as invariants of field representations of the Poincaré group, and apply it instead to the conformal group. Local field representations of the Poincaré group are functions on $\mathbf{R}^d$ valued in finite dimensional representations of the Lorentz group. Poincaré invariant actions are formed by taking translationally invariant integrals over $\mathbf{R}^d$ of scalar combinations of fields and their derivatives formed using the Minkowski (Euclidean) metric on $\mathbf{R}^d$. The key point in this familiar construction is that d -dimensional Minkowski (Euclidean) space is the space whose isometry group is the d -dimensional Poincaré group.

3.3.1 AdS Geometry

To apply the same strategy to construct conformal invariant functions we want to find the space whose isometry group is the d -dimensional conformal group $SO(d+1, 1)$. (We will work in Euclidean signature for now.) Since $SO(d+1, 1)$ is the $(d+2)$ -dimensional Lorentz rotation group, flat $\mathbf{R}^{d+2}$ might work. But this space has additional translational symmetries beyond the Lorentz rotations. So we remove them by restricting to a $(d+1)$ -dimensional “sphere” of radius R : this preserves the rotational symmetry, but will leave no translational symmetries.

To realize $SO(d+1, 1)$, we really want a Lorentzian “sphere”, that is, a hyperboloid

$$X^2 + V_+^2 - V_-^2 = R^2, \quad (3.13)$$

where $X := \{X_1, \dots, X_d\}$ and $V_\pm$ are Cartesian coordinates on $\mathbf{R}^{d+1,1}$ with metric

$$ds^2 = dX^2 + dV_+^2 - dV_-^2. \quad (3.14)$$

The surface in $\mathbf{R}^{d+1,1}$ described by (3.13) is called the $(d+1)$ -dimensional (Euclidean) anti-de Sitter space, or AdS_{d+1} , of radius R .

Useful coordinates on this space are $\{x, z\}$ defined by

$$X = \frac{R}{z} \mathbf{x}, \quad V_{\pm} = \frac{1}{2} \left(z + \frac{\mathbf{x}^2 \pm R^2}{z} \right), \quad (3.15)$$

which parametrizes solutions of the hyperboloid constraint (3.13) for $\mathbf{x} \in \mathbf{R}^d$ and $z > 0$. In these coordinates (called Poincaré patch coordinates), the AdS metric reads

$$ds^2 = \frac{R^2}{z^2} (dz^2 + d\mathbf{x}^2). \quad (3.16)$$

Thus AdS_{d+1} is conformal to the upper-half space $z > 0$ of $\mathbf{R}^{d+1}$ see Fig. 3.2.

A closely related set of coordinates are $\mathbf{x}$ and $r = R^2/z$, in which

$$ds^2 = \frac{R^2}{r^2} dr^2 + \frac{r^2}{R^2} d\mathbf{x}^2. \quad (3.17)$$

r is often called the radial coordinate of the AdS. Here $r = \infty$ is the boundary of AdS, and $r = 0$ can be thought of as a horizon. Note that both are infinitely distant from any finite r .

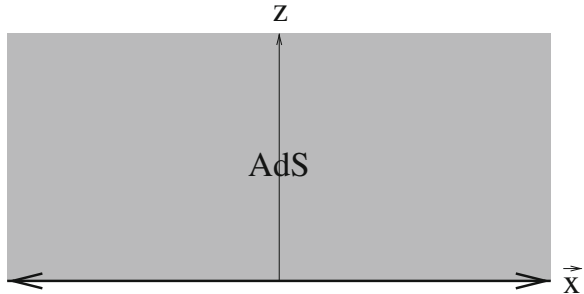
To go to Minkowski signature, one simply changes the d -dimensional Euclidean factor $d\mathbf{x}^2$ in the AdS metric to

$$d\mathbf{x}^2 \rightarrow -dt^2 + d\mathbf{x}^2, \quad (3.18)$$

the d -dimensional Minkowski metric, where now on the right side $\mathbf{x} = (x_1, \dots, x_{d-1})$. Minkowski-AdS has the interesting feature that though the boundary is radially infinitely far from any interior point, it can be reached in finite time by radial light-like signals.

We will work mostly in Euclidean-signature AdS from now on.

Fig. 3.2 AdS_{d+1} is conformal to the upper half-space of $\mathbf{R}^{d+1}$. In these coordinates, the boundary of AdS_{d+1} is conformal to $\mathbf{R}^d$ at $z = 0$



3.3.2 Partition Function

Any generally covariant function of (tensor) fields $\phi(z, \mathbf{x})$ on AdS will be conformally invariant. But these fields live in one more dimension (z) than we want. So we want to restrict to fields on a d -dimensional subspace of AdS_{d+1} while keeping general covariance in the full AdS_{d+1} .

The boundary, $\partial\text{AdS} \simeq \{z=0, \mathbf{x}\}$, is special because the group of conformal isometries acts on it in the same way as the conformal group acts in d -dimensional space-time: translations acting as $\{z, \mathbf{x}\} \rightarrow \{z, \mathbf{x} + \alpha\}$, (Lorentz) rotations acting as $\{z, \mathbf{x}\} \rightarrow \{z, \Lambda \mathbf{x}\}$, and the scaling transformation $\{z, \mathbf{x}\} \rightarrow \{\lambda z, \lambda \mathbf{x}\}$ are all clearly isometries. (The special conformal transformations are more complicated, but not hard to work out.) So boundary values of (tensor) functions on AdS, $\lim_{z \rightarrow 0} \phi(z, \mathbf{x})$, transform as representations of the conformal group on space-time.

We can therefore try to identify any generally covariant (on AdS_{d+1}) function of the boundary values of fields as the partition function of a CFT (as a functional of the boundary values of those fields).

A way of writing a general class of such functions is as a functional integral over fields $\phi(z, \mathbf{x})$ on AdS_{d+1} with a generally covariant measure keeping the boundary values $\phi(0, \mathbf{x}) \sim \bar{\phi}(\mathbf{x})$ fixed:

$$Z[\bar{\phi}(\mathbf{x})] = \int_{\phi|_0 = \bar{\phi}} \mathcal{D}\phi(z, \mathbf{x}) e^{-S[\phi(z, \mathbf{x})]} \quad (3.19)$$

Then, e.g., Z is invariant under scale transformations taking scalar fields $\phi(z, \mathbf{x}) \rightarrow \phi(\lambda z, \lambda \mathbf{x})$. In particular, if ϕ behaves near the boundary like

$$\phi(z, \mathbf{x}) \sim z^{d-\Delta} \bar{\phi}(\mathbf{x}) + \mathcal{O}(z^{d-\Delta+1}) \quad (3.20)$$

it follows that the boundary value $\bar{\phi}(\mathbf{x})$ transforms under scaling as

$$\bar{\phi}(\mathbf{x}) \rightarrow \lambda^{d-\Delta} \bar{\phi}(\lambda \mathbf{x}). \quad (3.21)$$

Comparing to the scaling of sources in the CFT, (3.12), it follows that $\bar{\phi} \equiv \bar{\phi}_\Delta$ is the source of a (scalar) primary operator $\mathcal{O}_\Delta$ of dimension Δ in the CFT. Similar remarks apply to non-scalar fields as well.

Since Z is given as a path integral, it defines a quantum theory if the ϕ fluctuate, i.e., if they have kinetic terms. For example, if ϕ is a free scalar on AdS so that it has action

$$S[\phi] = \frac{1}{2} \int_{\text{AdS}_{d+1}} \sqrt{g} \phi(-D_\mu D^\mu + m^2) \phi, \quad (3.22)$$

then it is not hard to check that its equation of motion has two independent solutions scaling near the boundary as $z^{d-\Delta}$ and z^Δ , so that asymptotically

$$\phi(z, \mathbf{x}) \sim z^{d-\Delta} \bar{\phi}(\mathbf{x}) + \cdots + z^\Delta \phi(\mathbf{x}) + \cdots \quad (3.23)$$

with

$$\Delta = \frac{d}{2} + \sqrt{\left(\frac{d}{2}\right)^2 + m^2 R^2}. \quad (3.24)$$

and $\bar{\phi}$ is the source for $\mathcal{O}_\Delta$ in the CFT. Note that in the limit where the mass of the scalar field goes to infinity—turning off the ϕ -fluctuations—the dimension $\Delta_{\text{energy-momentum}}$ of the associated CFT operator also goes to infinity.

Every CFT has a local tensor, $T^{\mu\nu}(\mathbf{x})$, as a primary operator of dimension $\Delta = d$. It is sourced by $\int h_{\mu\nu} T^{\mu\nu}$, so $h_{\mu\nu}$ is the boundary value of a spin-2 field on AdS. An argument similar to the one for the scalar field shows that $\Delta = d$ for $T^{\mu\nu}$ implies that $m^2 = 0$ for $h_{\mu\nu}$. So the AdS theory must have a dynamical, massless spin-2 field: a graviton. This gives the basic AdS/CFT correspondence [12, 13].

The partition function of a quantum gravity theory on an asymptotically AdS_{d+1} space-time as a function of the boundary values of its fields is the partition function of a CFT_d with the boundary values acting as sources for the primary operators:

$$Z_{\text{qu-grav}}[\bar{\phi}] = Z_{\text{CFT}}[\bar{\phi}] \quad (3.25)$$

It is worth pointing out a few simple but important properties or modifications of the AdS/CFT correspondence as described so far.

- If we replace AdS_{d+1} by $\text{AdS}_{d+1} \times X$ with X any space without boundary, the same procedure still works. The isometry group of X becomes a global internal symmetry of the CFT.
- If we define a partition function by the same procedure but cut off the AdS at $z = \epsilon > 0$ (i.e., make $z = \epsilon$ the boundary), we then keep Poincaré invariance, but break conformal invariance and locality on length scales smaller than ϵ .
- For negative mass-squared in the range $0 > m^2 > -d^2/4R^2$, the associated scaling dimension of the CFT primary is $d > \Delta > d/2$, which is real and above the unitarity bound. This corresponds to the fact [14, 15] that scalars with negative mass-squared in this range are stable in AdS.
- For $(d+2)/2 > \Delta > d/2$, both the $z^{d-\Delta}$ as well as the z^Δ asymptotic solution are normalizable, so we can use either as a source of an operator $\mathcal{O}_\Delta$ or $\mathcal{O}_{d-\Delta}$, respectively [16]. This allows us to describe operators with dimensions down to the unitarity bound $\Delta = (d-2)/2$.

3.3.3 Semi-classical Gravity Limit

The result for the AdS/CFT correspondence summarized by (3.25) immediately raises two questions: (1) Are there consistent quantum gravity theories in which we can compute the left side of (3.25) and (2) What class of CFTs do they correspond to?

The only examples of quantum gravity theories whose consistency we have confidence in are string theories. It is difficult to compute string theory partition functions on AdS backgrounds, except in the weak-coupling, low-energy limit, in which case it reduces to classical Einstein gravity coupled to other massless string fields. So we now describe the correspondence in more detail in this limit.

A gravitational theory on AdS_{d+1} has the low-energy effective action

$$\frac{1}{\kappa^2} \int d^{d+1}x \sqrt{g} (\mathcal{R} + \alpha' \mathcal{R}^2 + \dots) \quad (3.26)$$

where $\mathcal{R}$ is the Ricci scalar curvature and the dots denote an infinite series of higher-derivative terms built from curvature tensors and covariant derivatives (as well as other possible fields). The relative sizes of these terms define the basic length scales governing strength of gravity and size of higher-derivative terms, respectively:

$$\text{Planck length: } \ell_p \sim \kappa^{\frac{2}{d-1}}, \quad \text{String length: } \ell_s \sim \sqrt{\alpha'}. \quad (3.27)$$

In an AdS background $\mathcal{R} \sim R^{-2}$, so gravity is weak and the higher-derivative terms are small when

$$\ell_p \ll R, \quad \text{and} \quad \ell_s \ll R. \quad (3.28)$$

In this limit the semi-classical (saddle-point) approximation to the gravitational partition function is then

$$Z_{\text{qu-grav}}[\bar{\phi}] \sim \sum_{\{\phi_{cl}\}} e^{-S_{\text{Einstein}}[\phi_{cl}]} \quad (3.29)$$

where $\{\phi_{cl}\}$ are the classical field values (including the space-time metric) found by extremizing the action subject to the $\phi|_{\partial} = \bar{\phi}$ boundary conditions (including the boundary condition that the space-time is asymptotically AdS), and the sum is over all such extrema. S_{Einstein} is the Einstein action, i.e., (3.26) with all the α' and higher-derivative terms dropped.

In this semi-classical gravity limit, there is now a concrete calculational procedure for extracting correlation functions of the associated CFT via the AdS/CFT correspondence.

1. Find the dominant saddle point.
2. Each $(d+1)$ -dimensional (or “bulk”) field, ϕ , obeys a second order partial differential equation of motion on the (asymptotically) AdS space, with near-boundary asymptotic expansion

$$\phi(z, \mathbf{x}) \sim z^{d-\Delta} \bar{\phi}(\mathbf{x}) + \dots + z^{\Delta} \varphi(\mathbf{x}) + \dots \quad (3.30)$$

3. $\phi \equiv \phi_{\Delta}$ is associated with the source for an $\mathcal{O}_{\Delta}$ CFT primary.
4. With boundary conditions fixing $\bar{\phi}(\mathbf{x})$, then $\varphi(\mathbf{x})$ is determined by the equations of motion, so $\phi = \phi[\bar{\phi}]$.

5. Evaluate S_{Einstein} on these solutions to get, by the AdS/CFT correspondence (3.25),

$$S_{\text{Einstein}}[\bar{\phi}] = -\ln(Z_{\text{CFT}}[\bar{\phi}]) \equiv -W_{\text{CFT}}[\bar{\phi}]. \quad (3.31)$$

6. Then the “connected” correlators in the CFT are

$$\langle \mathcal{O}_{\Delta_1}(\mathbf{x}_1) \cdots \mathcal{O}_{\Delta_n}(\mathbf{x}_n) \rangle_{\text{CFT-conn}} = - \frac{\partial^n S_{\text{Einstein}}[\{\bar{\phi}\}]}{\partial \bar{\phi}_{\Delta_1}(\mathbf{x}_1) \cdots \partial \bar{\phi}_{\Delta_n}(\mathbf{x}_n)} \Big|_{\bar{\phi}=0}. \quad (3.32)$$

To actually compute $S_{\text{Einstein}}[\bar{\phi}]$ and its derivatives, one must regulate (by cutting off consistently at $z = \epsilon > 0$), renormalize (by adding local counterterms on the $z = \epsilon$ boundary) to preserve conformal invariance, then take the $\epsilon \rightarrow 0$ limit to extract finite answers. See for example [8]. As an example of the kind of results that one finds, for the simplest case of a free massive scalar plus $-(g/4)\phi^4$ interaction term, with some work one finds [8]

$$\begin{aligned} \langle \mathcal{O}_{\Delta}(\mathbf{x}) e^{\int \bar{\phi}_{\Delta} \mathcal{O}_{\Delta}} \rangle_{\text{CFT}} &= (2\Delta - d) \varphi(\mathbf{x}) \\ \langle \mathcal{O}_{\Delta}(\mathbf{x}_1) \mathcal{O}_{\Delta}(\mathbf{x}_2) \rangle_{\text{CFT}} &= (2\Delta - d) \frac{\Gamma(\Delta)}{\pi^{d/2} \Gamma(\Delta - \frac{d}{2})} \frac{1}{|\mathbf{x}_{12}|^{2\Delta}} \\ \langle \mathcal{O}_{\Delta}(\mathbf{x}_1) \mathcal{O}_{\Delta}(\mathbf{x}_2) \mathcal{O}_{\Delta}(\mathbf{x}_3) \rangle &= 0 \\ \langle \mathcal{O}_{\Delta}(\mathbf{x}_1) \mathcal{O}_{\Delta}(\mathbf{x}_2) \mathcal{O}_{\Delta}(\mathbf{x}_3) \mathcal{O}_{\Delta}(\mathbf{x}_4) \rangle &= \dots \text{explicit but complicated.} \dots \end{aligned}$$

Many other more difficult examples have been worked out in the literature, notably correlators of the energy momentum tensor and of global conserved currents in the CFT.

3.4 Large N

We now briefly review a few salient points about the large- N expansion of gauge theories. These will be important in the precise form of the AdS/CFT correspondence that will be discussed in the next section.

Consider a $U(N)$ Yang–Mills theory in the ’t Hooft limit, i.e., take $N \rightarrow \infty$ keeping $\lambda \equiv g_{\text{ym}}^2 N$ fixed. The $U(N)$ YM action has the general form

$$\mathcal{L} \sim \frac{N}{\lambda} \text{tr}(d\phi d\phi + \phi^2 d\phi + \phi^4), \quad (3.33)$$

for fields ϕ in the adjoint representation of the gauge group. ϕ here represents either the vector boson or an adjoint scalar superpartner of the vector boson in a super–Yang–Mills (SYM) theory. In the ’t Hooft limit the growth of the action

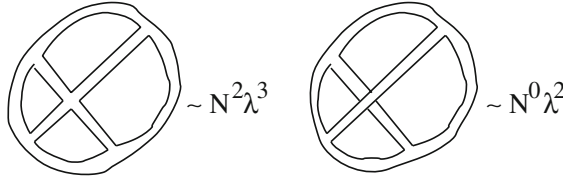


Fig. 3.3 Two sample Feynman diagrams and their associated N and λ weights. Note that the left diagram is planar, and can be interpreted as a triangulation of a disk; while the right one is not planar: it is the triangulation of a torus minus a disk

with N makes the theory look semi-classical, i.e., dominated by the saddle point contributions.

We can write the adjoint indices as ϕ_j^i with $i, j = 1, \dots, N$, corresponding to the $N \otimes \bar{N}$ decomposition of the adjoint representation. Then $\langle \phi_j^i \phi_\ell^k \rangle \propto \delta_\ell^i \delta_k^j$, so we can notate propagators in Feynman diagrams as oppositely oriented double lines with each line tracking the fundamental or antifundamental indices, see Fig. 3.3. Associate to each interaction a vertex, to each propagator an edge, and to each loop a face, to form a triangulation of some oriented surface. Denote the number of vertices, edges, and faces, by E , V , and F , respectively.

Since each vertex has a factor of N/λ , each propagator λ/N , and each face N (from the associated trace over the fundamental indices), a given diagram has weight $N^{V-E+F} \lambda^{E-V}$. Since $V - E + F = 2 - 2g$, where g is the genus of the surface, and extending to diagrams with n external propagators (with appropriate normalization), we find that connected correlators have the general form

$$\langle \mathcal{O}_1 \cdots \mathcal{O}_n \rangle \sim \sum_{g=0}^{\infty} N^{2-2g-n} F_{g,n}(\lambda). \quad (3.34)$$

So correlators are dominated by the $g = 0$, or “planar”, diagrams in the ’t Hooft limit.

In this limit, 2-point functions $\sim N^0$, 3-point functions $\sim N^{-1}$, etc., so $1/N$ acts like an effective coupling constant in large- N YM (in addition to the ’t Hooft coupling λ).

3.5 D3-Branes and $\text{AdS}_5 \times S^5$

Now we will use a specific quantum gravity theory—string theory—to derive a specific example of the AdS/CFT correspondence. In particular we will derive the *Maldacena conjecture* [17]: type IIB string on $\text{AdS}_5 \times S^5$ is equivalent to four-dimensional $\mathcal{N} = 4$ supersymmetric $\text{SU}(N)$ YM. The argument can be organized into three basic steps.

1. Identify $SU(N)$ SYM as a sector of the low energy limit of IIB strings in the presence of N parallel D3-branes at weak coupling.
2. Identify the space–time geometry sourced by the D3-branes as approximately an $\text{AdS}_5 \times S^5$ “throat” glued into an asymptotically flat region whose gravitational modes decouple from the string modes on the throat.
3. Compare these two low energy descriptions to identify the SYM sector with the $\text{AdS}_5 \times S^5$ sector with a specific mapping of parameters.

The same type of argument can be used to find many other examples.

3.5.1 SYM from D-Branes

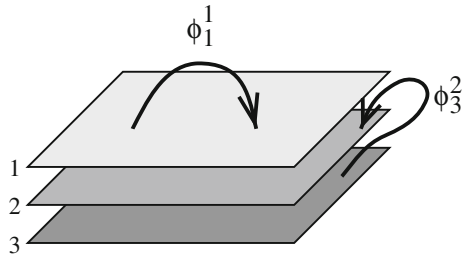
In IIB string theory place a stack of N parallel D3-branes extended along the $\mathbf{x} = x_{0,1,2,3}$ directions, and take the limit in which they become coincident. At weak enough string coupling, $g_s \ll 1$, this stack is well-described as a 4-manifold where oriented open strings may end, see Fig. 3.4. Label the D3-branes by indices $i, j, \dots = 1, \dots, N$. The lightest modes of open strings connecting the branes then carry adjoint $U(N)$ labels. They fill out a massless four-dimensional $\mathcal{N} = 4$ vector supermultiplet of the $U(N)$ SYM theory. The SYM coupling is the open string coupling, $g_{\text{ym}} \sim \sqrt{g_s}$.

For small enough g_s at fixed N , the gravitational back-reaction of the D3-branes is small, since even though the D3-brane tension τ_3 is proportional to $N g_s^{-1}$, the gravitational coupling κ^2 is proportional to g_s^2 , so the net strength of the D3-brane source goes as $\tau_3 \kappa^2 \propto g_s N$.

In the low energy limit, $E \ll \ell_s^{-1}$ or $E^2 \alpha' \rightarrow 0$, only the massless string modes can be excited. These modes are the bulk (10-dimensional) supergravity modes of the closed strings and the (four-dimensional) $U(N)$ SYM modes of the open strings. The couplings between the four-d SYM and ten-d supergravity sectors are higher-derivative, and so vanish in the low energy limit.

Since the gravitational coupling $\kappa \sim g_s \alpha'^2$, the dimensionless coupling strength is $E^4 \kappa$ which vanishes as the energy scale $E \rightarrow 0$. Thus the supergravity sector is IR-free. On the other hand, the SYM sector stays at fixed coupling, g_{ym}^2 , since

Fig. 3.4 A stack of nearly coincident D3-branes labelled by indices 1, 2, 3, and two sample open string modes labelled by the ordered pair of indices associated with the brane the string starts on and ends on



$\mathcal{N} = 4$ SYM is a CFT for all g_{ym} . In other words, the gauge coupling is an exactly marginal coupling so does not run with scale. (More precisely, the diagonal $U(1) \subset U(N)$ decouples and is free, corresponding to the overall translational degrees of freedom of the N D3-branes. So only the $SU(N)$ SYM factor stays at fixed non-zero coupling.) The net result of this analysis is that:

In the small g_s limit at fixed N , the low energy effective action of IIB strings in the presence of N coincident D3-branes consists of three decoupled sectors:

$$\begin{aligned} & (\text{free 10-d supergravity}) \times \\ & (\text{free 4-d } U(1) \text{ SYM}) \times \\ & (4\text{-d } SU(N) \text{ SYM CFT with } g_{\text{ym}}^2 \sim g_s). \end{aligned} \quad (3.35)$$

3.5.2 D3-Brane Near Horizon Geometry

Now look at the same system in the $g_s \rightarrow 0$, $N \rightarrow \infty$ limit, keeping $g_s N \equiv \lambda$ fixed. In this limit the gravitational back-reaction of the D3-branes cannot be neglected, since the D3-brane tension times the gravitational coupling is proportional to $N g_s^{-1} \times g_s^2 = \lambda$. In the low energy and $g_s \rightarrow 0$ limit, the N D3-branes source a classical supergravity solution with metric and RR 5-form field strength given by

$$\begin{aligned} ds^2 &= f^{-1/2}(-dt^2 + dx^2) + f^{1/2}(dr^2 + r^2 d\Omega_5^2) \\ F_5 &= (1 + *)dt \wedge dx_1 \wedge dx_2 \wedge dx_3 \wedge df^{-1} \end{aligned} \quad (3.36)$$

where

$$f \equiv 1 + \frac{R^4}{r^4}, \quad \text{and} \quad R^4 \equiv 4\pi g_s N \alpha'^2. \quad (3.37)$$

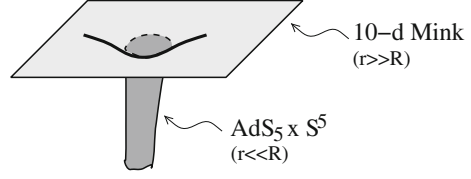
For $r \gg R$, $f \rightarrow 1$, so the solution asymptotes to flat $\mathbf{R}^{9,1}$, while for $r \ll R$, $f^{1/2} \rightarrow R^2/r^2$, so the metric becomes

$$ds^2 \sim \frac{R^2}{r^2} dr^2 + \frac{r^2}{R^2}(-dt^2 + dx^2) + d\Omega_5^2 \quad (3.38)$$

which we recognize as $\text{AdS}_5 \times S^5$ in Poincaré patch coordinates; compare to (3.17). The AdS boundary at $r = \infty$ is replaced by a transition to flat $\mathbf{R}^{9,1}$; see Fig. 3.5.

We now want to consider the low energy effective description of physics in this geometry. There are low energy scattering states of the massless supergravity modes in the $\mathbf{R}^{9,1}$ asymptotic region. There are also arbitrarily low-energy modes far down the AdS “throat”. To see this, consider an object of fixed energy E_* as measured in the frame of a co-moving observer. From the point of view of an asymptotic observer at $r = \infty$ (where we are measuring all our observables) it has

Fig. 3.5 A cartoon illustrating how the $\text{AdS}_5 \times S^5$ “throat” geometry for $r \ll R$ is attached to the flat $\mathbf{R}^{9,1}$ geometry for $r \gg R$



a red-shifted energy $E = \sqrt{g_{tt}}E_* = f^{-1/4}E_*$. So $E \sim rE_*/R \rightarrow 0$ as $r \rightarrow 0$. Therefore states of *arbitrary* finite energy E_* in the $\text{AdS}_5 \times S^5$ throat are low-energy excitations.

Finally there are also a few massless modes corresponding (from the asymptotic $\mathbf{R}^{9,1}$ perspective) to the translational zero-modes of the whole throat. (They are the “singleton” boundary modes on the AdS.) Since they are the translational modes of the whole stack of D3-branes, they are equivalent to a free four-dimensional $U(1)$ SYM theory. In summary,

In the small g_s , fixed $\lambda = g_s N$ limit, the low energy effective action of IIB strings in the presence of N coincident D3-branes has three sectors:

$$\begin{aligned}
 &(\text{free 10-d supergravity}) \times \\
 &(\text{free 4-d } U(1) \text{ SYM}) \times \\
 &(\text{IIB string theory on } \text{AdS}_5 \times S^5 \text{ with } R^4 = 4\pi\lambda\alpha'^2 \text{ and } \kappa = g_s\alpha'^2).
 \end{aligned} \tag{3.39}$$

Furthermore, these three sectors decouple in the low-energy limit, since the cross-section for an $\mathbf{R}^{9,1}$ supergravity wave of frequency ω to scatter off the throat ($r < R$) region is $\sigma \sim \omega^3 R^8$, so vanishes in the low-energy $\omega \rightarrow 0$ limit. Likewise, as the low-energy throat modes are localized closer to $r = 0$, escape to the asymptotically flat region is energetically suppressed. Finally, low energy $\mathbf{R}^{9,1}$ gravitational waves can not excite the massive throat translational modes, while the associated singleton modes on the boundary of AdS decouple from bulk AdS modes. Thus all three sectors decouple.

3.5.3 Strong/Weak Duality

Comparing these two low energy descriptions, (3.35) and (3.39), leads to the Maldacena conjecture:

$$\begin{aligned}
 &\text{Type IIB string theory on } \text{AdS}_5 \times S^5 \\
 &\quad \text{is equivalent to} \\
 &4\text{-dimensional } \mathcal{N} = 4 \text{ supersymmetric } SU(N) \text{ YM} \\
 &\quad \text{with parameters identified as} \\
 &4\pi g_s = g_{ym}^2, \text{ and } R^4/\alpha'^2 = \lambda \equiv g_{ym}^2 N.
 \end{aligned}$$

It is just a conjecture because the low energy descriptions of (3.35) and (3.39) were in different limits (the first at fixed N , the second as $N \rightarrow \infty$). Note that $\mathcal{O}(\alpha')$ string worldsheet corrections are $\mathcal{O}(1/\sqrt{\lambda})$ corrections in the Yang–Mills theory, and at fixed λ , the $\mathcal{O}(g_s)$ string loop corrections are $\mathcal{O}(1/N)$ corrections on the Yang–Mills side. The N D3-brane near horizon AdS geometry came from a classical supergravity solution, i.e., it did not contain g_s or α' corrections, so it is not *a priori* clear that the AdS solution should still be valid in the regime where the SYM description was derived.

Thus there are different possible versions of the conjecture that one can imagine:

Weak: valid only for $g_s N \rightarrow \infty$: neither α' nor g_s corrections agree.

Medium: valid for all $g_s N$ but only for $N \rightarrow \infty$: only α' corrections agree.

Strong: valid for all $g_s N$ and all N : an exact equivalence.

The strong version of the conjecture allows *all* kinds of finite energy interior processes and objects in the AdS space–time, including space–times with different topologies (e.g., black holes). So in this version,

$$Z_{\text{CFT}} = \sum \forall \text{ asymptotic AdS geometries} \quad (3.40)$$

Most tests of the conjecture are by computing at large N quantities whose λ -dependence is determined by supersymmetry, though a few are also checks that $1/N$ corrections match as well.

In any case, the string theory on $\text{AdS}_5 \times S^5$ is currently only really calculable in the classical supergravity limit where $g_s \ll 1$ (so no string loops) and $\ell_s \ll R$ (so no α' corrections). In terms of YM parameters this means that $N \gg \lambda \gg 1$, which is the planar 't Hooft limit, but at *strong* 't Hooft coupling. On the other hand, the YM theory is only under perturbative control at small λ and finite N . A great deal of the power of Maldacena's conjecture comes not just from the fact that it is an explicit realization of the AdS/CFT conjecture, but also that weak coupling on one side of the equivalence is strong coupling on the other.

3.6 Extensions

There are many examples, refinements, extensions and deformations of the AdS/CFT correspondence; see the other lectures in this volume for a flavor. Two extensions which are basic and play an important role in many other applications involve

- turning on a finite temperature, and
- adding brane probes.

The finite temperature extension can be derived by the same argument as in the last section, but keeping a finite energy density on the D3-branes. In this case the

supergravity solution become a black 3-brane, and the near-horizon limit is the AdS_5 -black hole $\times S^5$ geometry. The AdS black hole Hawking temperature is the same as the temperature of the SYM theory. This extension of the AdS/CFT correspondence is straightforward in the sense that it does not change the underlying equivalence, but involves identifying certain excited states on the two sides of the correspondence. Nevertheless, this extension leads to rich and beautiful physics, such as the correspondence between the deconfinement transition in finite volume SYM and the instability of small black holes in AdS space [18].

The brane probe approximation in the AdS/CFT correspondence can be more subtle, so I will describe an example of it in more detail. This particular example can be motivated by the following question: How can one add fundamental (as opposed to adjoint) matter to the SYM theory in the AdS/CFT correspondence? In the weak coupling limit, we need string states with one end on the D3-brane stack (carrying a fundamental color index) and the other end elsewhere. Since fundamental strings end on D-branes, we should add other kinds of D-branes to the initial setup. We call these other branes “flavor branes” since they will label different flavors of fundamental matter.

In general it is hard to find supergravity solutions for the gravitational back-reaction of adding flavor branes. However, if the number, N_f , of flavor branes is much smaller than the number, N , of (color) D3-branes, this back-reaction can be ignored in the large N limit. To see this, recall that Newton’s constant $\kappa^2 \sim g_s^2 \sim N^{-2}$ in the ’t Hooft limit (where $\lambda = g_s N$ is kept fixed), and the tension of N_f Dp-branes is $N_f T_{\mu\nu}^{\text{Dp}} \sim N_f g_s^{-1} \sim N_f N$, so the gravitational back-reaction of the flavor branes is $\kappa^2 N_f T_{\mu\nu}^{\text{Dp}} \sim N_f/N$ which vanishes as $N \rightarrow \infty$ with N_f fixed. Thus such probe branes need only satisfy their classical equations of motion in the unperturbed background space–time generated by the N color branes. These equations come from extremizing the probe branes’ world volume [or the DBI action if the brane’s $U(1)$ field is turned on].

It was shown in [19, 20] that probe flavor D7-branes can be added in such a way as to keep half of the $\mathcal{N} = 4$ supersymmetry of the SYM theory. The required embedding of the flavor brane in $\text{AdS}_5 \times S^5$ is shown in Fig. 3.6. The dual theory is then interpreted to be $\mathcal{N} = 2 SU(N)$ SYM with N_f massive quark hypermultiplets. Since the probe D7-branes only added new open string states—identified with the fundamental hypermultiplets on the YM side—the dual theory can no longer be a CFT, but must in fact have non-zero beta function. The AdS/CFT correspondence then implies that the dual string background should not be asymptotically AdS. This is reflected in the fact that a D7 brane is co-dimension 2 in 10 dimensions, and so no matter how light, its tension will deform the asymptotic structure of the space (by causing a deficit angle). Thus, strictly speaking, D7-branes can never satisfy the probe approximation, and the $\text{AdS}/\mathcal{N} = 2$ SYM correspondence is not exact, but is broken by N_f/N effects.

Higher co-dimension probes are less problematic. For example, fundamental string and D1-brane probes of the AdS_5 geometry have proven very useful, either as strings extending to the boundary of AdS whose worldsheets are dual to

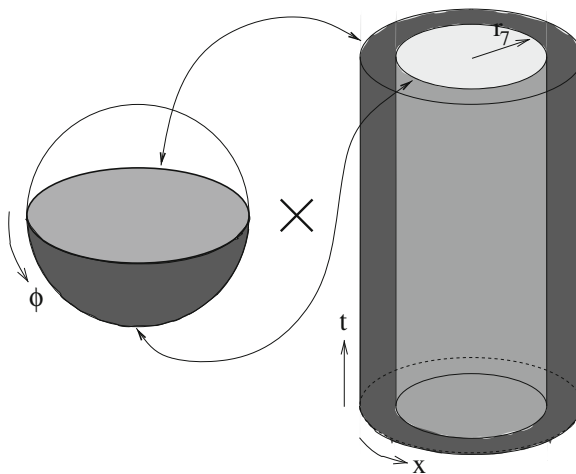


Fig. 3.6 The topology of the embedding of a probe D7-brane in $\text{AdS}_5 \times S^5$. The *solid cylinder* is the AdS factor, with t its axial direction, r its radial direction, and $\mathbf{x}$ its angular direction (s). The D7-brane fills the AdS cylinder from $r = r_7$ out to the boundary ($r = \infty$). The *sphere* is the S^5 factor, and the D7-brane fills an S^4 hemisphere. The D7-brane is embedded in such a way that constant r slices of the *cylinder* are associated with constant latitude ϕ S^3 s, with the equatorial S^3 corresponding to $r = \infty$, and the degenerate point S^3 at the south pole of the S^5 to $r = r_7$

insertions of the non-local Wilson and 't Hooft loop operators in the CFT, or as strings ending on probe D7-branes dual to massive quark states in the $\mathcal{N} = 2$ SYM. Both of these scenarios have led to physically rich and active areas of development of the AdS/CFT correspondence following on the work of [21, 22] and [23, 24], respectively.

Acknowledgments It is a pleasure to thank the organizers for inviting me to lecture at the summer school, and to thank K. Skenderis and M. Rangamani for helpful comments.

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Part II: Holography and the AdS/CFT Correspondence

Chapter 4

Improved Holographic QCD

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Abstract We provide a review to holographic models based on Einstein-dilaton gravity with a potential in five dimensions. Such theories, for a judicious choice of potential are very close to the physics of large- N YM theory both at zero and finite temperature. The zero temperature glueball spectra as well as their finite temperature thermodynamic functions compare well with lattice data. The model can be used to calculate transport coefficients, like bulk viscosity, the drag force and jet quenching parameters, relevant for the physics of the Quark–Gluon Plasma.

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4.1 Introduction

The experimental efforts at RHIC [1–4] have provided a novel window in the physics of the strong interactions. The consensus on the existing data is that shortly after the collision, a ball of quark–gluon plasma (QGP) forms that is at thermal equilibrium, and subsequently expands until its temperature falls below the QCD transition (or crossover) where it finally hadronizes. Relativistic hydrodynamics describes very well the QGP [5], with a shear-viscosity to entropy density ratio close to that of $\mathcal{N} = 4$ SYM [6, 7]. The QGP is at strong coupling, and it necessitates a treatment beyond perturbative QCD approaches, [8–10]. Moreover, although the shear viscosity from $\mathcal{N} = 4$ seems to be close to that “measured” by experiment, lattice data indicate that in the relevant RHIC range $1 \leq \frac{T}{T_c} \leq 3$ the QGP seems not to be a fully conformal fluid. Therefore the bulk viscosity may play a role near the phase transition [11–13]. The lattice techniques have been successfully used to study the thermal behavior of QCD, however they are not easily extended to the computation of hydrodynamic quantities. They can be used however, together with parametrizations of correlators in order to pin down parameters [12, 13]. On the other hand, approaches based on holography have the potential to address directly the real-time strong coupling physics relevant for experiment.

In the bottom-up holographic model of AdS/QCD [14, 15], the bulk viscosity is zero as conformal invariance is essentially not broken (the stress tensor is traceless). In the soft-wall model [16], no reliable calculation can be done for glue correlators and therefore transport coefficients are ill-defined. Similar remarks hold for other phenomenologically interesting observables as the drag force and the jet quenching parameter [17–21].

Top-down holographic models of QCD displaying all relevant features of the theory have been difficult to obtain. Bottom-up models based on AdS slices [22] have given some insights mostly in the meson sector [14, 15] but necessarily lack many important holographic features of QCD. A hybrid approach has been advocated [23–25] combining features of bottom-up and top-down models. An similar approach was proposed independently in [26]. Such an approach, called Improved Holographic QCD (or IHQCD for short) is essentially a five-dimensional dilaton-gravity system with a non-trivial dilaton potential. Flavor can be eventually added in the form of N_f space-time filling $D4 - \overline{D4}$ brane pairs, supporting $U(N_f)_L \times U(N_f)_R$ gauge fields and a bi-fundamental scalar [27]. The UV asymptotics of the potential are fixed by QCD perturbation theory, while the IR asymptotics of the potential can be fixed by confinement and linear glueball asymptotics. An analysis of the finite temperature behavior [28, 29] has shown that the phase structure is exactly what one would expect from YM. A potential with a single free parameter tuned to match the zero temperature glueball spectrum was able to agree with the thermodynamic behavior of glue to a good degree [28]. Similar results, but with somewhat different potentials were also obtained in [26, 30].

In [26, 28, 29] it was shown that Einstein-dilaton gravity with a strictly monotonic dilaton potential that grows sufficiently fast, generically shares the same phase

structure and thermodynamics of finite-temperature pure Yang–Mills theory at large N_c . There is a deconfinement phase transition (dual to a Hawking–Page phase transition between a black hole and thermal gas background on the gravity side), which is generically first order. The latent heat scales as N_c^2 . In the deconfined gluon-plasma phase, the free energy slowly approaches that of a free gluon gas at high temperature, and the speed of sound starts from a small value at T_c and approaches the conformal value $c_s^2 = 1/3$ as the temperature increases. The deviation from conformal invariance is strongest at T_c , and is signaled by the presence of a non-trivial gluon condensate, which on the gravity side emerges as a deviation of the scalar solution that behaves asymptotically as r^4 close to the UV boundary. In the CP-violating sector, the topological vacuum density $\text{Tr}F\tilde{F}$ has zero expectation value in the deconfined phase, in agreement with lattice results [31] and large- N_c expectations.

The analysis performed in [29] was completely general and did not rely on any specific form of the dilaton potential $V(\lambda)$. A detailed analysis of an explicit model in [32] shows that the thermodynamics matches *quantitatively* the thermodynamics of pure Yang–Mills theory. The (dimensionless) free energy, entropy density, latent heat and speed of sound, obtained on the gravity side by numerical integration of the 5D field equations, can be compared with the corresponding quantities, calculated on the lattice for pure Yang–Mills at finite- T , resulting in excellent agreement, for the temperature range that is accessible by lattice techniques. The same model also shows a good agreement with the lattice calculation of glueball mass ratios at zero temperature, and the value of the deconfining critical temperature (in units of the lowest glueball mass) is also in good agreement with the lattice results.

In short, the model we present gives a good phenomenological holographic description of most static properties¹ (spectrum and equilibrium thermodynamics) of large- N_c pure Yang–Mills, as computed on the lattice, for energies up to several times T_c . Thus it constitutes a good starting point for the computation of dynamical observables in a *realistic* holographic dual to QCD (as opposed to e.g. $\mathcal{N} = 4$ SYM), such as transport coefficients and other hydrodynamic properties that are not easily accessible by lattice techniques, at energies and temperatures relevant for relativistic heavy-ion collision experiments. We will report on such a calculation in the near future.

The vacuum solution in this model is described in terms of two basic bulk fields, the metric and the dilaton. These are not the only bulk fields however, as the bulk theory is expected to have an a priori infinite number of fields, dual to all possible YM operators. In particular we know from the string theory side that there are a few other low mass fields, namely the RR axion (dual to the QCD θ -angle) the NSNS and RR two forms B_2 and C_2 as well as other higher-level fields. With the exception of the RR axion, such fields are dual to higher-dimension and/or higher-spin operators of YM. Again, with the exception of the RR axion, they are not expected to play an important role into the structure of the vacuum and this is

¹ There are very few observables also that are not in agreement with YM. They are discussed in detail in [25].

why we neglect them when we solve the equations of motion. However, they are going to generate several new towers of glueball states beyond those that we discuss in this paper (namely the 0^{++} glueballs associated to dilaton fluctuations, 2^{++} glueballs associated to graviton fluctuations and 0^{-+} glueballs associated to RR axion fluctuations). Such fields can be included in the effective action and the associated glueball spectra calculated. Since we do not know the detailed structure of the associated string theory, their effective action will depend on more semi-phenomenological functions like $Z(\lambda)$ in (4.45). These functions can again be determined in a way similar to $Z(\lambda)$. In particular including the B_2 and C_2 field will provide 1^{+-} glueballs among others. Fields with spin greater than 2 are necessarily stringy in origin. We will not deal further with extra fields, like B_2 and C_2 and other as they are not particularly relevant for the purposes of this model, namely the study of finite temperature physics in the deconfined case. We will only consider the axion, as its physics is related to the CP-odd sector of YM with an obvious phenomenological importance.

It is well documented that string theory duals of YM must have strong curvatures in the UV regime. This has been explained in detail in [25] where it was also argued, that although the asymptotic AdS boundary geometry is due to the curvature non-linearities of the associated string theory, the inwards geometry is perturbative around AdS, with logarithmic corrections, generating the YM perturbation theory. The present model is constructed so that it takes the asymptotic AdS geometry for granted, by introducing the associated vacuum energy by hand, and simulates the perturbative YM expansion by an appropriate dilaton potential. In the IR, we do not expect strong curvatures in the string frame, and indeed the preferred backgrounds have this property. In this sense the model contains in itself the relevant expected effects that should arise from strong curvatures in all regimes. These issues have been explained in [23, 24] and in more detail in [25].

A different and interesting direction is the use of such models to study the expansion of the plasma and the associated dynamics. Such a context is similar to what happens on cosmology, especially the one related to the Randall–Sundrum setup. Indeed in this case the expansion can be found by following the geodesic motion of probe branes in the relevant background [33–36]. This generalizes to more complicated backgrounds [36] like the ones studied here.

Once we have a holographic model we trust, we should calculate observables, like transport coefficients that are hard to calculate on the lattice. A first class of transport coefficients are viscosity coefficients.² A general fluid is characterized by two viscosity coefficients, the shear η and the bulk viscosity ζ . The shear viscosity in strongly coupled theories described by gravity duals was shown to be universal [6, 7]. In particular, the ratio η/s , with s the entropy density, is equal to $\frac{1}{4\pi}$. This is correlated to the universality of low-energy scattering of gravitons from black

² These are the leading transport coefficients in the derivative expansion. There are subleading coefficients that have been calculated recently for $\mathcal{N} = 4$ SYM [37, 38]. However, at the present level of accuracy, they cannot affect substantially the comparison to experimental data [5].

holes. It is also known that deviations from this value can only be generated by higher curvature terms that contain the Riemann tensor (as opposed to the Ricci tensor of the scalar curvature). In QCD, as the theory is strongly coupled in the temperature range $T_c \leq T \leq 3T_c$, we would expect that $\eta/s \simeq \frac{1}{4\pi}$. Recent lattice calculations [39] agree with this expectations although potential systematic errors in lattice calculations of transport coefficients can be large.

Conformal invariance forces the bulk viscosity to vanish. Therefore the $\mathcal{N} = 4$ SYM plasma, being a conformal fluid, has vanishing bulk viscosity. QCD on the other hand is not a conformal theory. The classical theory is however conformally invariant and asymptotic freedom implies that conformal invariance is a good approximation in the UV. This would suggest that the bulk viscosity to entropy ratio is negligible at large temperatures. However it is not expected to be so in the IR: as mentioned earlier lattice data indicate that in the relevant RHIC range $1 \leq \frac{T}{T_c} \leq 3$ the QGP seems not to be a fully conformal fluid. Therefore the bulk viscosity may play a role near the phase transition.

So far there have been two approaches that have calculated the bulk viscosity in YM/QCD [12, 40–43] and have both indicated that the bulk viscosity rises near the phase transition as naive expectation would suggest. The first used the method of sum rules in conjunction with input from Lattice thermodynamics [40–43]. It suggested a dramatic rise of the bulk viscosity near T_c although the absolute normalization of the result is uncertain. The reason is that this method relies on an ansatz for the density associated with stress-tensor two point functions that are otherwise unknown.

The second method [12] relies on a direct computation of the density at low frequency of the appropriate stress-tensor two-point function. As this computation is necessarily Euclidean, an analytic continuation is necessary. The values at a finite number of discrete Matsubara frequencies are not enough to analytically continue. An ansatz for the continuous density is also used here, which presents again a potentially large systematic uncertainty.

Calculations in IHQCD support a rise of the bulk viscosity near T_c , but the values are much smaller than previously expected. Studies of how this affects hydrodynamics at RHIC [44] suggest that this implies a fall in radial and elliptic flow.

Another class of interesting experimental observables is associated with quarks, and comes under the label of “jet quenching”. Central to this is the expectation that an energetic quark will loose energy very fast in the quark–gluon plasma because of strong coupling. This has as a side effect that back-to back jets are suppressed. Moreover if a pair of energetic quarks is generated near the plasma boundary then one will exit fast the plasma and register as an energetic jet, while the other will thermalize and its identity will disappear. This has been clearly observed at RHIC and used to study the energy loss of quarks in the quark–gluon plasma.

Heavy quarks are of extra importance, as their mass masks some low-energy strong interaction effects, and can be therefore cleaner probes of plasma energy loss. There are important electron observables at RHIC [45] that can probe heavy-quark energy loss in the strongly coupled quark–gluon plasma. Such observables are also expected to play an important role in LHC [46].

A perturbative QCD approach to calculate the energy loss of a heavy quark in the plasma has been pursued by calculating radiative energy loss [47–49]. However its application to the RHIC plasma has recently raised problems, based on comparison with data. A phenomenological coefficient used in such cases is known as the jet quenching coefficient $\hat{q}$, and is defined as the rate of change of the average value of transverse momentum square of a probe. Current fits [45, 50] indicate that a value of order 10 GeV²/fm or more is needed to describe the data while perturbative approaches are trustworthy at much lower values.

Several attempts were made to compute quark energy loss in the holographic context, relevant for $\mathcal{N} = 4$ SYM.³ In some of them [18, 52] the jet-quenching coefficient $\hat{q}$ was calculated via its relationship to a light-like Wilson loop. Holography was then used to calculate the appropriate Wilson loop. The $\hat{q}$ obtained scales as $\sqrt{\lambda}$ and as the third power of the temperature,

$$\hat{q}_{\text{conformal}} = \frac{\Gamma[\frac{3}{4}]}{\Gamma[\frac{5}{4}]} \sqrt{2\lambda} \pi^{\frac{3}{2}} T^3. \quad (4.1)$$

A different approach chooses to compute the drag force acting a string whose UV end-point (representing an infinitely heavy quark) is forced to move with constant velocity v [17, 19–21] in the context of $\mathcal{N} = 4$ SYM plasma. The result for the drag force is

$$F_{\text{conformal}} = \frac{\pi}{2} \sqrt{\lambda} T^2 \frac{v}{\sqrt{1-v^2}} \quad (4.2)$$

and is calculated by first studying the equilibrium configuration of the appropriate string world-sheet string and then calculating the momentum flowing down the string. This can be the starting point of a Langevin evolution system, as the process of energy loss has a stochastic character, as was first pointed out in [53] and more recently pursued in [54–60].

Such a system involves a classical force, that in this case is the drag force, and a stochastic noise that is taken to be Gaussian and which is characterized by a diffusion coefficient. There are two ingredients here that are novel. The first is that the Langevin evolution must be relativistic, as the quarks can be very energetic. Such relativistic systems have been described in the mathematical physics literature [61, 62] and have been used in phenomenological analyses of heavy-ion data [50]. They are known however to have peculiar behavior, since demanding an equilibrium relativistic Boltzmann distribution, provides an Einstein relation that is pathological at large temperatures. Second, the transverse and longitudinal diffusion coefficients are not the same [57]. A first derivation of such Langevin dynamics from holography was given in [57]. This has been extended in [60] where the thermal-like noise was associated and interpreted in terms of the world-sheet horizon that develops on the probe string.

³ Most are reviewed in [51].

Most of the transport properties mentioned above have been successfully computed in $\mathcal{N} = 4$ SYM and a lot of debate is still waged as to how they can be applied to QCD in the appropriate temperature range [19, 20, 63–65]. A holographic description of QCD has been elusive, and the best we have so far have been simple bottom up models.

In the simplest bottom-up holographic model known as AdS/QCD [14, 15], the bulk viscosity is zero as conformal invariance is essentially not broken (the stress tensor is traceless), and the drag force and jet quenching essentially retain their conformal values.

In the soft-wall model [16], no reliable calculation can be done for glue correlators and therefore transport coefficients are ill-defined, as bulk equations of motion are not respected. Similar remarks hold for other phenomenologically interesting observables as the drag force and the jet quenching parameter.

The shear viscosity of IHQCD is the same as that of $\mathcal{N} = 4$ SYM, as the model is a two derivative model. Although this is not a good approximation in the UV of QCD, it is expected to be a good approximation in the energy range $T_c \leq T \leq 5T_c$. The bulk viscosity in IHQCD rises near the phase transition but ultimately stays slightly below the shear viscosity. There is a general holographic argument that any (large- N) gauge theory that confines color at zero temperature should have an increase in the bulk viscosity-to-entropy density ratio close to T_c .

The drag force on heavy quarks, and the associated diffusion times, can be calculated and found to be momentum depended as anticipated from asymptotic freedom. Numerical values of diffusion times are in the region dictated by phenomenological analysis of heavy-ion data. The medium-induced corrections to the quark mass (needed for the diffusion time calculation) can be calculated, and they result in a mildly decreasing effective quark mass as a function of temperature. This is consistent with lattice results. Finally, the jet-quenching parameter can be calculated and found to be comparable at T_c to the one obtained by extrapolation from $\mathcal{N} = 4$ SYM. Its temperature dependence is however different and again reflects the effects of asymptotic freedom.

4.2 The 5D Model

The holographic dual of large N_c Yang Mills theory, proposed in [23, 24], is based on a five-dimensional Einstein-dilaton model, with the action⁴:

$$S_5 = -M_p^3 N_c^2 \int d^5x \sqrt{g} \left[R - \frac{4}{3} (\partial\Phi)^2 + V(\Phi) \right] + 2M_p^3 N_c^2 \int_{\partial M} d^4x \sqrt{h} K. \quad (4.3)$$

⁴ Similar models of Einstein-dilaton gravity were proposed independently in [26] to describe the finite temperature physics of large N_c YM. They differ in the UV as the dilaton corresponds to a relevant operator instead of the marginal case we study here. The gauge coupling e^Φ also asymptotes to a constant instead of zero in such models.

Here, M_p is the five-dimensional Planck scale and N_c is the number of colors. The last term is the Gibbons–Hawking term, with K being the extrinsic curvature of the boundary. The effective five-dimensional Newton constant is $G_5 = 1/(16\pi M_p^3 N_c^2)$, and it is small in the large- N_c limit.

Of the 5D coordinates $\{x_i, r\}_{i=0\dots 3}$, x_i are identified with the 4D space-time coordinates, whereas the radial coordinate r roughly corresponds to the 4D RG scale. We identify $\lambda \equiv e^\Phi$ with the running 't Hooft coupling $\lambda_t \equiv N_c g_{YM}^2$, up to an *a priori* unknown multiplicative factor,⁵ $\lambda = \kappa \lambda_t$.

The dynamics is encoded in the dilaton potential,⁶ $V(\lambda)$. The small- λ and large- λ asymptotics of $V(\lambda)$ determine the solution in the UV and the IR of the geometry respectively. For a detailed but concise description of the UV and IR properties of the solutions the reader is referred to Sect. 2 of [29]. Here we will only mention the most relevant information:

1. For small λ , $V(\lambda)$ is required to have a power-law expansion of the form:

$$V(\lambda) \sim \frac{12}{\ell^2} (1 + v_0 \lambda + v_1 \lambda^2 + \dots), \quad \lambda \rightarrow 0. \quad (4.4)$$

The value at $\lambda = 0$ is constrained to be finite and positive, and sets the UV AdS scale ℓ . The coefficients of the other terms in the expansion fix the β -function coefficients for the running coupling $\lambda(E)$. If we identify the energy scale with the metric scale factor in the Einstein frame, as in [23, 24], we obtain:

$$\begin{aligned} \beta(\lambda) &\equiv \frac{d\lambda}{d \log E} = -b_0 \lambda^2 - b_1 \lambda^3 + \dots \\ b_0 &= \frac{9}{8} v_0, \quad b_1 = \frac{9}{4} v_1 - \frac{207}{256} v_0^2. \end{aligned} \quad (4.5)$$

2. For large λ , confinement and the absence of bad singularities⁷ require:

$$V(\lambda) \sim \lambda^{2Q} (\log \lambda)^P \quad \lambda \rightarrow \infty, \quad \begin{cases} 2/3 < Q < 2\sqrt{2}/3, & P \text{ arbitrary} \\ Q = 2/3, & P \geq 0 \end{cases}. \quad (4.6)$$

In particular, the values $Q = 2/3, P = 1/2$ reproduce an asymptotically-linear glueball spectrum, $m_n^2 \sim n$, besides confinement. We will restrict ourselves to this case in what follows.

⁵ This relation is well motivated in the UV, although it may be modified at strong coupling (see Sect. 4.3). The quantities we will calculate do not depend on the explicit relation between λ and λ_t .

⁶ With a slight abuse of notation we will denote $V(\lambda)$ the function $V(\Phi)$ expressed as a function of $\lambda \equiv e^\Phi$.

⁷ We call “bad singularities” those that do not have a well defined spectral problem for the fluctuations without imposing extra boundary conditions.

In the large N_c limit, the canonical ensemble partition function of the model just described, can be approximated by a sum over saddle points, each given by a classical solution of the Einstein-dilaton field equations:

$$\mathcal{Z}(\beta) \simeq e^{-S_1(\beta)} + e^{-S_2(\beta)} + \dots \quad (4.7)$$

where S_i are the euclidean actions evaluated on each classical solution with a fixed temperature $T = 1/\beta$, i.e. with euclidean time compactified on a circle of length β . There are two possible types of Euclidean solutions which preserve three-dimensional rotational invariance. In conformal coordinates these are:

1. *Thermal gas solution*,

$$ds^2 = b_o^2(r)(dr^2 + dt^2 + dx_m dx^m), \quad \Phi = \Phi_o(r), \quad (4.8)$$

with $r \in (0, \infty)$ for the values of P and Q we are using;

2. *Black-hole solutions*,

$$ds^2 = b(r)^2 \left[\frac{dr^2}{f(r)} + f(r) dt^2 + dx_m dx^m \right], \quad \Phi = \Phi(r), \quad (4.9)$$

with $r \in (0, r_h)$, such that $f(0) = 1$, and $f(r_h) = 0$.

In both cases Euclidean time is periodic with period β_o and β respectively for the thermal gas and black-hole solution, and three-space is taken to be a torus with volume V_{3o} and V_3 respectively, so that the black-hole mass and entropy are finite.⁸

The black holes are dual to a deconfined phase, since the string tension vanishes at the horizon, and the Polyakov loop has non-vanishing expectation value [66, 67]. On the other hand, the thermal gas background is confining.

The thermodynamics of the deconfined phase is dual to the 5D black-hole thermodynamics. The free energy, defined as

$$\mathcal{F} = E - TS, \quad (4.10)$$

is identified with the black-hole on-shell action; as usual, the energy E and entropy S are identified with the black-hole mass, and one fourth of the horizon area in Planck units, respectively.

The thermal gas and black-hole solutions with the same temperature differ at $O(r^4)$:

$$b(r) = b_o(r) \left[1 + \mathcal{G} \frac{r^4}{\ell^3} + \dots \right], \quad f(r) = 1 - \frac{C}{4} \frac{r^4}{\ell^3} + \dots \quad r \rightarrow 0, \quad (4.11)$$

⁸ The periods and three-space volumes of the thermal gas solution are related to the black-hole solution values by requiring that the geometry of the two solutions are the same on the (regulated) boundary. See [29] for details.

where $\mathcal{G}$ and C are constants with units of energy. As shown in [29] they are related to the enthalpy TS and the gluon condensate $\langle \text{Tr} F^2 \rangle$:

$$C = \frac{TS}{M_p^3 N_c^2 V_3}, \quad \mathcal{G} = \frac{22}{3(4\pi)^2} \frac{\langle \text{Tr} F^2 \rangle_T - \langle \text{Tr} F^2 \rangle_o}{240 M_p^3 N_c^2}. \quad (4.12)$$

Although they appear as coefficients in the UV expansion, C and $\mathcal{G}$ are determined by regularity at the black-hole horizon. For T and S the relation is the usual one,

$$T = -\frac{\dot{f}(r_h)}{4\pi}, \quad S = \frac{\text{Area}}{4G_5} = 4\pi (M_p^3 N_c^2 V_3) b^3(r_h). \quad (4.13)$$

For $\mathcal{G}$ the relation with the horizon quantities is more complicated and cannot be put in a simple analytic form. However, as discussed in [29], for each temperature there exist only specific values of $\mathcal{G}$ (each corresponding to a different black hole) such that the horizon is regular.

At any given temperature there can be one or more solutions: the thermal gas is always present, and there can be different black holes with the same temperature. The solution that dominates the partition function at a certain T is the one with smallest free energy. The free energy difference between the black hole and thermal gas was calculated in [29] to be:

$$\frac{\mathcal{F}}{M_p^3 N_c^2 V_3} = \frac{\mathcal{F}_{BH} - \mathcal{F}_{th}}{M_p^3 N_c^2 V_3} = 15\mathcal{G} - \frac{C}{4}. \quad (4.14)$$

For a dilaton potential corresponding to a confining theory, like the one we will assume, the phase structure is the following [29]:

1. There exists a minimum temperature $T_{\min}$ below which the only solution is the thermal gas.
2. Two branches of black holes (“large” and “small”) appear for $T \geq T_{\min}$, but the ensemble is still dominated by the confined phase up to a temperature $T_c > T_{\min}$.
3. At $T = T_c$ there is a first order phase transition to the *large* black-hole phase. The system remains in the black-hole (deconfined) phase for all $T > T_c$.

In principle there could be more than two black-hole branches, but this will not happen with the specific potential we will use.

4.3 Scheme Dependence

There are several sources of scheme dependence in any attempt to solve a QFT. Different parametrizations of the coupling constant (here λ) give different descriptions. However, physical statements must be invariant under such a change.

In our case, reparametrizations of the coupling constant are equivalent to radial diffeomorphisms as we could use λ as the radial coordinate.

In the holographic context, scheme dependence related to coupling redefinitions translates into field redefinitions for the bulk fields. As the bulk theory is on-shell, all on-shell observables (that are evaluated at the single boundary of space-time) are independent of the field redefinitions showing that scheme-independence is expected. Invariance under radial reparametrizations of scalar bulk invariants is equivalent to RG invariance. Because of renormalization effects, the boundary is typically shifted and in this case field redefinitions must be combined with appropriate radial diffeomorphisms that amount to RG-transformations.

Another source of scheme dependence in our setup comes from the choice of the energy function. Again we may also consider this as a radial coordinate and therefore it is subject to coordinate transformations. A relation between λ and E is the β -function,

$$\frac{d\lambda}{d \log E} = \beta(\lambda). \quad (4.15)$$

β by definition transforms as a vector under λ reparametrizations and as a form under E reparametrizations. $\beta(\lambda)$ can therefore be thought of as a vector field implementing the change of coordinates from λ to E and vice-versa.

Physical quantities should be independent of scheme. They are quantities that are fully diffeomorphism invariant. If the gravitational theory had no boundary there would be no diffeomorphism invariant quantities, except for possible topological invariants. Since we have a boundary, diffeomorphism invariant quantities are defined at the boundary.

Note that scalar quantities are not invariant. To be invariant they must be scalar and constant. We therefore need to construct scalar functions that are invariant under changes of radial coordinates.

We can fix this reparametrization invariance by picking a very special frame. For example choosing the (string) metric in the conformal frame

$$ds^2 = e^{2A} [dr^2 + dx^\mu dx_\mu], \quad \lambda(r) \quad (4.16)$$

or in the domain-wall frame

$$ds^2 = du^2 + e^{2A} dx^\mu dx_\mu, \quad \lambda(u) \quad (4.17)$$

fixes the radial reparametrizations almost completely. In conformal frame, common scalings of r, x^μ are allowed, corresponding to constant shifts of $A(r)$.

Eventually we are led to calculate and compare our results to other ways of calculating (like the lattice). Some outputs are easier to compare (for example correlators). Others are much harder as they are not invariant (like the value of the coupling at a given energy scale).

In the UV such questions are well understood. The asymptotic energy scale is fixed by comparison to conformal field theory examples. This is possible because the space is asymptotically AdS_5 .⁹

The coupling constant is also fixed to leading order from the coupling of the dilaton to D_3 branes (up to an overall multiplicative factor). Subleading (in perturbation theory) redefinitions of the coupling constant and the energy lead to changes in the β -function beyond two loops.

More in detail, as it has been described in [23–25], the general form of the kinetic term for the gauge fields on a D_3 brane is expected to be:

$$S_{F^2} = e^{-\Phi} Z(R, \xi) \text{Tr}[F^2], \quad \xi \equiv -e^{2\Phi} \frac{F_5^2}{5!} \quad (4.18)$$

where $Z(R, \xi)$ is an (unknown) function of curvature R and the five-form field strength, ξ . At weak background fields, $Z \simeq -\frac{1}{4} + \dots$. In the UV regime, expanding near the boundary in powers of the coupling $\lambda \equiv N_c e^\Phi$ we obtain, [25]

$$S_{F^2} = N_c \text{Tr}[F^2] \frac{1}{\lambda} \left[Z(R_*, \xi_*) - \frac{Z_\xi(R_*, \xi_*)}{F_{\xi\xi}(R_*, \xi_*) \sqrt{\xi_*}} \frac{\lambda}{\ell} + \mathcal{O}(\lambda^2) \right] \quad (4.19)$$

where $F(R, \xi)$ is the bulk effective action and R_*, ξ_* are the boundary values for these parameters. Therefore the true 't Hooft coupling of QCD is

$$\lambda_{\text{tHooft}} = -\frac{\lambda}{Z(R_*, \xi_*)} \left[1 + \frac{Z_\xi(R_*, \xi_*)}{Z(R_*, \xi_*) F_{\xi\xi}(R_*, \xi_*) \sqrt{\xi_*}} \frac{\lambda}{\ell} + \mathcal{O}(\lambda^2) \right]. \quad (4.20)$$

In the IR, more important changes can appear between our λ and other definitions as for example in lattice calculations.

In the region of strong coupling we know much less in order to be guided concerning the correct definition of the energy. We can obtain some hints however by comparing with lattice results.¹⁰ In particular, based on lattice calculations using the Schrödinger functional approach [68], it is argued that at long distance L the 't Hooft coupling constant scales as

$$\lambda_{\text{lat}} \sim e^{mL}, \quad m \simeq \frac{3}{4} m_{0^{++}}. \quad (4.21)$$

⁹ As the dilaton is now not constant there is a non-trivial question: in which frame is the metric AdS . In [25] it was argued that this should be the case in the string frame. The difference of course between the string and Einstein frame is subleading in the UV as the coupling constant vanishes logarithmically. But this may not be the case in the IR where we have very few criteria to check. In the model we are using we impose that the space is asymptotically AdS in the Einstein frame as this is the only choice consistent with the whole framework.

¹⁰ We would like to thank K. Kajantie for asking the question, suggesting to compare with lattice data, and providing the appropriate references.

This was based on a specific definition of the coupling constant, and length scale on the lattice as well as on numerical data, and some general expectations on the fall-off of correlations in a massive theory. This suggests an IR β function of the form

$$L \frac{d\lambda}{dL} = \lambda \log \frac{\lambda}{\lambda_0}, \quad \lambda = \lambda_0 e^{mL}. \quad (4.22)$$

On the other hand our β -function at strong coupling uses the UV definition of energy, $\log E = A_E$ (the scale factor in the Einstein frame), $E \sim 1/L$ and is

$$L \frac{d\lambda}{dL} = \frac{3}{2} \lambda \left[1 + \frac{3}{4} \frac{a-1}{a} \frac{1}{\log \lambda} + \dots \right], \quad \lambda \simeq \left(\frac{L}{L_0} \right)^{\frac{3}{2}}. \quad (4.23)$$

where a is a parameter in the IR asymptotics of the potential. The case we consider as best fitting YM is $a = 2$ as then the asymptotic glueball trajectories are linear.

Consider now taking as length scale the string scale factor e^{A_s} in the IR.¹¹ Since it increases, it is consistent to consider it as a monotonic function of length. From its relation to the Einstein scale factor $A_s = A_E + \frac{2}{3} \log \lambda$ and (4.23) we obtain

$$\frac{d\lambda}{dA_s} = \frac{2a}{a-1} \lambda \log \lambda + \dots \quad (4.24)$$

Therefore if we define as length scale in the IR

$$\log L = \frac{2a}{a-1} A_s \rightarrow L = (e^{A_s})^{\frac{2a}{a-1}} \quad (4.25)$$

we obtain a running of the coupling compatible with the given lattice scheme. Note however that $L = (e^{A_s})^{\frac{2a}{a-1}}$ cannot be a global choice but should be only valid in the IR. The reason is that this function is not globally monotonic.

We conclude this section by restating that physical observables are independent of scheme. But observables like the 't Hooft coupling constant do depend on schemes, and it is obvious that our scheme is very different from lattice schemes in the IR.

4.4 The Potential and the Parameters of the Model

We will make the following ansatz for the potential,¹²

$$V(\lambda) = \frac{12}{\ell^2} \left\{ 1 + V_0 \lambda + V_1 \lambda^{4/3} \left[\log \left(1 + V_2 \lambda^{4/3} + V_3 \lambda^2 \right) \right]^{1/2} \right\}, \quad (4.26)$$

¹¹ The string scale factor is not a monotonic function on the whole manifold [23, 24], and this is the reason that it was not taken as a global energy scale. In particular in the UV, e^{A_s} decreases until it reaches a minimum. The existence of the minimum is crucial for confinement. After this minimum e^{A_s} increases and diverges at the IR singularity.

¹² Further studies of IHQCD with different potentials can be found in [69].

which interpolates between the two asymptotic behaviors (4.4) for small λ and (4.6) for large λ , with $Q = 2/3$ and $P = 1/2$. Not all the parameters entering this potential have physical relevance. Below we will discuss the independent parameters of the model, and their physical meaning.

4.4.1 The Normalization of the Coupling Constant λ

As discussed in the previous section, the relation between the bulk field $\lambda(r)$ and the physical QCD 't Hooft coupling $\lambda_t = g_{YM}^2 N_c$ is a priori unknown. In the UV, the identification of the $D3$ -brane coupling to the dilaton implies that the relation is linear, and depends on an a priori unknown coefficient κ , defined as:

$$\lambda = \kappa \lambda_t. \quad (4.27)$$

The coefficient κ can in principle be identified by relating the perturbative UV expansion of the Yang–Mills β -function, to the holographic β -function for the bulk field λ :

$$\beta(\lambda_t) = -\beta_0 \lambda_t^2 - \beta_1 \lambda_t^3 + \dots, \quad \beta_0 = \frac{22}{3(4\pi)^2}, \quad \beta_1 = \frac{51}{121} \beta_0^2, \dots \quad (4.28)$$

$$\beta(\lambda) = -b_0 \lambda^2 - b_1 \lambda^3 + \dots, \quad b_0 = \frac{9}{8} v_0, \quad b_1 = \frac{9}{4} v_1 - \frac{207}{256} v_0^2 \dots \quad (4.29)$$

The two expressions (4.28) and (4.29) are consistent with a linear relation as in (4.27), and expanding the identity $\kappa \beta_t(\lambda_t) = \beta(\kappa \lambda_t)$ to lowest order leads to:

$$\kappa = \beta_0 / b_0. \quad (4.30)$$

Therefore, to relate the bulk field λ to the true coupling λ_t one looks at the linear term in the expansion of the potential. More generally, the other β function coefficients are related by $\beta_n = \kappa^{n+1} b_n$, and the combinations $b_n / b_0^{n+1} = \beta_n / \beta_0^{n+1}$ are κ -independent (however they are scheme-dependent for $n \geq 2$).

As discussed in Sect. 4.3, the introduction of the coefficient κ amounts to a field redefinition and therefore its precise value does not affect physical (scheme-independent) quantities. In this sense, κ is not a parameter that can be fixed by matching some observable computed in the theory. Assuming the validity of the relation (4.27), we could eventually fix κ by matching a RG-invariant (but scheme-dependent) quantity, e.g. λ at a given energy scale.

However, as we discuss later in this section, rescaling λ in the potential (thus changing κ) affects other parameters in the models, that are defined in the string frame, e.g. the fundamental string length ℓ_s : if we hold the physical QCD string tension fixed, the ratio (ℓ_s / ℓ) scales with degree $-2/3$ under a rescaling of κ .

An important point to keep in mind, is that the simple linear relation (4.27) may be modified at strong coupling, but again this does not have any effect on physical

observables. *As long as we compute RG-invariant and scheme-independent quantities, knowledge of the exact relationship $\lambda = F(\lambda_t)$ is unnecessary.*

4.4.2 The AdS Scale ℓ

This is set by the overall normalization of the potential, and its choice is equivalent to fixing the unit of energy. It does not enter dimensionless physical quantities. As usual the AdS length at large N_c is much larger than the Planck length ($\ell_p \sim 1/(M_p N_c^{2/3})$), independently of the 't Hooft coupling.

4.4.3 The UV Expansion Coefficients of $V(\lambda)$

They can be fixed order by order by matching the Yang–Mills β -function. We impose this matching up to two-loops in the perturbative expansion, i.e. $O(\lambda^3)$ in (λ) . One could go to higher orders by adding additional powers of λ inside the logarithm, but since our purpose is not to give an accurate description of the theory in the UV, we choose not to introduce extra parameters.¹³

Identifying the energy scale with the Einstein frame scale factor, $\log E \equiv \log b(r)$, we have the relation (4.29) between the β -function coefficients and the expansion parameters of $V(\lambda)$, with

$$v_0 = V_0, \quad v_1 = V_1 \sqrt{V_2}. \quad (4.31)$$

The term proportional to V_2 in (4.26) is needed to reproduce the correct value of the quantity $b_1/b_0^2 = \beta_1/\beta_0^2 = 51/121$, which is invariant under rescaling of λ . Thus, V_2 is not a free parameter, but is fixed in terms of V_0 and V_1 by:

$$V_2 = b_0^4 \left(\frac{23 + 36 b_1/b_0^2}{81 V_1} \right)^2, \quad b_0 = \frac{9}{8} V_0, \quad \frac{b_1}{b_0^2} = \frac{51}{121}. \quad (4.32)$$

As explained earlier in this section, when discussing the normalization of the coupling, fixing the coefficient V_0 is the same as fixing the normalization κ through (4.30). As we argued, the actual value of κ should not have any physical consequences, so it is tempting to set $V_0 = 1$ by a field redefinition, $\lambda \rightarrow \lambda/V_0$ and eliminate this parameter altogether.

In fact, most of the quantities we will compute are not sensitive to the value of V_0 , but for certain quantities, such as the string tension, some extra care is needed. In general, we can ask whether two models of the same form (4.3), but with

¹³ Moreover, higher order β -function coefficients are known to be scheme-dependent.

different potentials $V(\lambda)$ and $\tilde{V}(\lambda)$, such that $\tilde{V}(\lambda) = V(\alpha\lambda)$ for some constant α , lead to different physical predictions. As we can change from one model to the other simply by a field redefinition $\lambda \rightarrow \alpha\lambda$ (this has no effect on the other terms in the action in the Einstein frame, (4.3), clearly the two potentials lead to the same result for any physical quantity that can be computed unambiguously from the Einstein frame action, e.g. dimensionless ratios between glueball masses, critical temperature, latent heat etc.

However a rescaling of λ does affect the string frame metric, since the latter explicitly contains factors of λ : $b_s(r) = b(r)\lambda^{2/3}$ [23, 24] thus, under the rescaling $\lambda \rightarrow \alpha\lambda$, $b_s(r) \rightarrow \alpha^{2/3}b_s(r)$. This means that any dimensionless ratio of two quantities, such that one of them remains fixed in the string frame and the other in the Einstein frame, will depend on α . An example of this is the ratio ℓ_s/ℓ , where ℓ_s is the string length, that we will discuss shortly.

Therefore, we can safely perform a field redefinition and set V_0 to a given value, as long as we are careful when computing quantities that depend explicitly on the fundamental string length.

Bearing this caveat in mind, we will choose a normalization such that $b_0 = \beta_0$, i.e.

$$V_0 = \frac{8}{9}\beta_0, \quad (4.33)$$

so that the normalization of λ in the UV matches the physical Yang–Mills coupling. With this choice, out of the four free parameters V_i appearing in (4.26) only V_1 and V_3 play a non-trivial role (V_2 being fixed by (4.32)).

4.4.4 The 5D Planck Scale M_p

M_p appears in the overall normalization of the 5D action (4.3). Therefore it enters the overall scale of quantities derived by evaluating the on-shell action, e.g. the free energy and the black-hole mass. It also sets the conversion factor between the entropy and horizon area. M_p cannot be fixed directly as we lack a detailed underlining string theory for YM. To obtain quantitative predictions, M_p must be fixed in terms of the other dimension-full quantity of the model, namely the AdS scale ℓ . As shown in [29] this can be done by imposing that the high-temperature limit of the black-hole free energy be that of a *free* gluon gas with the correct number of degrees of freedom.¹⁴ This requires:

$$(M_p\ell)^3 = \frac{1}{45\pi^2}. \quad (4.34)$$

¹⁴ Note that this is conceptually different from the $\mathcal{N} = 4$ case. There, near the boundary, the theory is strongly coupled and this number must be calculated in string theory. It is different by a factor of 3/4 from the free sYM answer. Here near the boundary the theory is free. Therefore the number of degrees of freedom can be directly inferred.

4.4.5 The String Length

In the non-critical approach the relation between the string length ℓ_s and the 5D Planck length (or the AdS length ℓ) is not known from first principles. The string length does not appear explicitly in the two-derivative action (4.3), but it enters quantities like the static quark–antiquark potential. The ratio ℓ_s/ℓ can be fixed phenomenologically to match the lattice results for the confining string tension.

More in detail, the relation between the fundamental and the confining string tensions T_f and σ is given by:

$$\sigma = T_f b^2(r_*) \lambda^{4/3}(r_*), \quad (4.35)$$

where r_* is the point where the string frame scale factor, $b_s(r) \equiv b(r) \lambda^{2/3}(r)$, has its minimum. Fixing the confining string tension by comparison with the lattice result we can find T_f (more precisely, the dimensionless quantity $T_f \ell^2$, since the overall scale of the metric depends on ℓ). The string length is in turn given by $\ell_s/\ell = 1/\sqrt{2\pi T_f \ell^2}$.

As is clear from (4.35), rescaling $\lambda \rightarrow \alpha \lambda$, keeping the value of the QCD string tension σ and of the AdS scale ℓ fixed, affects the fundamental string length in AdS units as $\ell_s/\ell \rightarrow \alpha^{-2/3}(\ell_s/\ell)$. Therefore two models a and b , defined in the Einstein frame by (4.3), but with potentials related by $V_b(\lambda) = V_a(\alpha \lambda)$, must have different fundamental string tensions in order to reproduce the same result for the QCD string tension. The quantity ℓ_s/ℓ therefore depends on the value of V_0 .

4.4.6 Integration Constants

Besides the parameters appearing directly in the gravitational action, there are also other physically relevant quantities that label different solutions to the 5th order system of field equations. Any solution is characterized by a scale Λ , the temperature T and a value for the gluon condensate $\mathcal{G}$, that correspond to three of the five independent integration constants.¹⁵

Regularity at the horizon fixes $\mathcal{G}$ as a function of T , so that effectively the gluon condensate is a temperature-dependent quantity.

The quantity Λ controls the asymptotic form of the solution, as it enters the dilaton running in the UV: $\lambda \simeq -(b_0 \log r \Lambda)^{-1}$. It can be defined in a reparametrization invariant way as:

¹⁵ The remaining two are the value $f(0)$ which should be set to one for the solution (4.9) to obey the right UV asymptotics, and an unphysical degree of freedom in the reparametrization of the radial coordinate.

$$\Lambda = \ell^{-1} \lim_{\lambda \rightarrow 0} \left\{ b(\lambda) \frac{\exp\left[-\frac{1}{b_0 \lambda}\right]}{\lambda^{b_1/b_0^2}} \right\}, \quad (4.36)$$

and it is fixed once we specify the value of the scale factor $b(\lambda)$ at a given λ_0 .

Every choice of Λ corresponds to an inequivalent class of solutions, that differ by UV boundary conditions. Each class is thermodynamically isolated, since solutions with different Λ 's have infinite action difference. Thus, in the canonical partition sum we need to consider only solutions with a fixed value of Λ . However, this choice is merely a choice of scale, as solutions with different Λ 's will give the same predictions for any dimensionless quantity. In short, Λ is the holographic dual to the QCD strong coupling scale: it is defined by the initial condition to the holographic RG equations, and does not affect dimensionless quantities such as mass ratios, etc. Therefore, as long as all solutions we consider obey the same UV asymptotics, the actual value of Λ is immaterial, since the physical units of the system can always be set by fixing ℓ .

To summarize, the only *nontrivial phenomenological* parameters we have at our disposal are V_1 and V_3 appearing in (4.26). The other quantities that enter our model are either fixed by the arguments presented in this section, or they only affect trivially (e.g. by overall rescaling that can be absorbed in the definition of the fundamental string scale) the physical quantities.

In the next section we present a numerical analysis of the solutions and thermodynamics of the model defined by (4.26), and show that for an appropriate choice of the parameters it reproduces the lattice results for the Yang–Mills deconfinement transition and high-temperature phase as well as the zero temperature glueball data.

4.5 Matching the Thermodynamics of Large- N_c YM

Assuming a potential of the form (4.26), we look for values of the parameters such that the thermodynamics of the 5D model match the lattice results for the thermodynamics of 4D YM. As explained in Sect. 4.3, we set V_0 and V_2 as in (4.33 and 4.32), respectively, with $b_0 = \beta_0 = 22/3(4\pi)^{-2}$.

We then vary V_1 and V_3 only. We fix these parameters by looking at thermodynamic quantities corresponding to the latent heat per unit volume, and the pressure at one value of the temperature above the transition, which we take as $2T_c$.

It is worth remarking that V_1 and V_3 are phenomenological parameters that we use to fit *dimensionless* QCD quantities. The single (dimension-full) parameter of pure Yang–Mills, the strong coupling scale, is an extra input that fixes the overall energy scale of our solution.

Using the numerical method explained in [32], for each set of parameters (V_1, V_3) we numerically generate black-hole solutions for a range of values of λ_h ,

then from the metric at the horizon and its derivative we extract the temperature and entropy functions $T(\lambda_h)$ and $S(\lambda_h)$, and the function $\mathcal{F}(\lambda_h)$ from the integrated form of the first law,

$$\mathcal{F}(\lambda_h) = \int_{\lambda_h}^{+\infty} d\bar{\lambda}_h S(\bar{\lambda}_h) \frac{dT(\bar{\lambda}_h)}{d\bar{\lambda}_h}. \quad (4.37)$$

Here $S(\lambda_h)$ is given by (4.13) and both the large black hole and small black-hole branches are needed in order to get the full result for the free energy. This is because the integral in (4.37) extends to $+\infty$, entering deeply in the small black-hole branch.

The behavior of the thermodynamic functions is shown in Figs. 4.1, 4.2 and 4.3, for the best fit parameter values that we discuss below. One can see the existence of a minimal temperature $T_{\min} = T(\lambda_{\min})$, and a critical value λ_c where $\mathcal{F}$ changes sign. The resulting function $\mathcal{F}(T)$ is shown in Fig. 4.1.

The phase transition is first order, and the latent heat per unit volume L_h , normalized by $N_c^2 T_c^4$, is given by the derivative of the curve in Fig. 4.4 at $T/T_c = 1$.

Fig. 4.1 The free energy density (in units of T_c) as a function of T/T_c , for $V_1 = 14$ and $V_3 = 170$. The *vertical lines* correspond to the critical temperature (*solid*) and the minimum black-hole temperature (*dashed*)

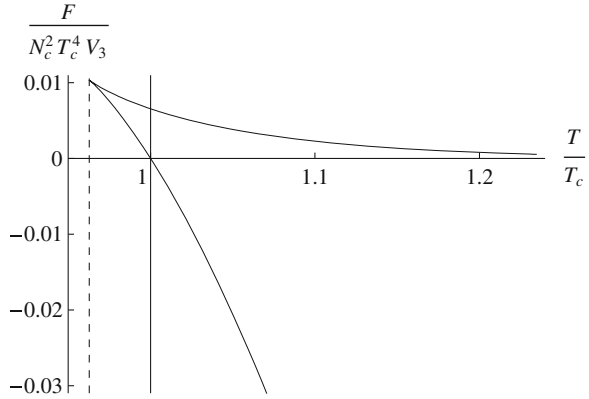


Fig. 4.2 Temperature in units of $T_{\min}$, as a function of λ_h , for $V_1 = 14$ and $V_3 = 170$. The *dashed horizontal and vertical lines* indicate the critical temperature and the critical value of the dilaton field at the horizon

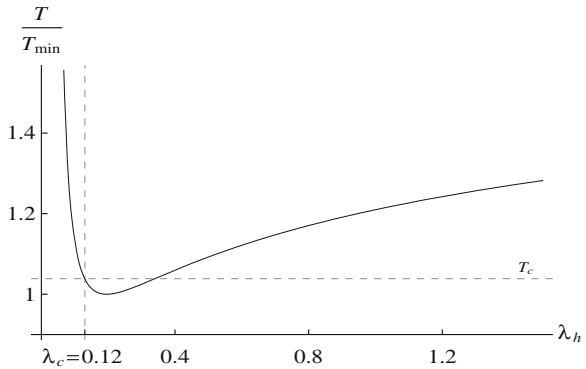


Fig. 4.3 The free energy density in units of $T_{\min}$, as a function of λ_h

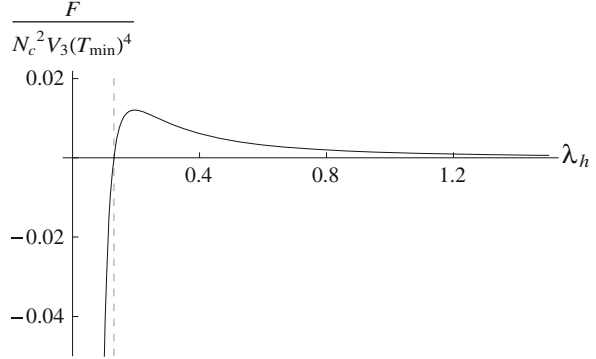
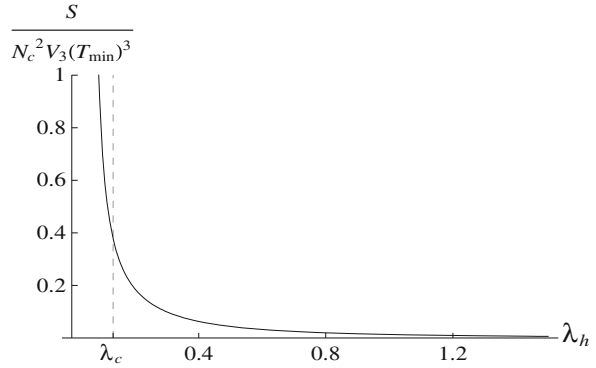


Fig. 4.4 Entropy density in units of $T_{\min}$, as a function of λ_h



Equivalently, L_h is proportional to the jump in the entropy density $s = S/V_3$ at the phase transition from the thermal gas (whose entropy is of $O(1)$, in the limit $N_c \rightarrow \infty$) to the black hole (whose entropy scales as N_c^2 in the same limit): thus, in the large N_c limit,

$$L_h \equiv T \Delta s \simeq T_c s(\lambda_c) \quad (4.38)$$

up to terms of $O(1/N_c^2)$.

To fix V_1 and V_3 we compare our results to the data of G. Boyd et al. [70]. The relevant quantities to compare are the dimensionless ratios $p(T)/T^4$, $e(T)/T^4$ and $s(T)/T^3$, where $p = \mathcal{F}/V_3$ is the pressure, and $e = p + Ts$ is the energy density. Lattice results for these functions are available in the range $T = T_c \sim 5T_c$, and can be seen in Fig. 7 of [70]. The analysis of [70] correspond to $N_c = 3$, but one expects that the thermodynamic functions do not change to much for large N_c .¹⁶

¹⁶ See e.g. [71–73], in which results for $N_c = 8$ do not differ significantly from those for $N_c = 3$ as well as the recent high-precision data by Panero [74, 75].

An additional quantity of relevance is the value for the “dimensionless” latent heat per unit volume, L_h/T_c^4 which for large N_c was found in [76] to be $(L_h/T_c^4)_{lat} = 0.31N_c^2$. The result for $N_c = 3$ is slightly lower ($\simeq 0.28N_c^2$).

As already noted in [28, 29], the qualitative features of the thermodynamic functions are generically reproduced in our setup: the curves $3p(T)/T^4$, $e(T)/T^4$ and $3s(T)/4T^3$ increase starting at T_c , then (very slowly) approach the constant free field value $\pi^2 N_c^2/15$ (given by the Stefan–Boltzmann law) as T increases. By computing the thermodynamic functions for various sets of values of V_1 and V_3 we obtain that:

- (1) V_1 roughly controls the height reached by the curves $p(T)/T^4$, $e(T)/T^4$ and $s(T)/T^3$ at large T/T_c ($\sim$ a few): for larger V_1 the curves approach the free field limit faster;
- (2) V_3 does not affect much the height of the curves at large T/T_c , but on the other hand it changes the latent heat, which is increasing as V_3 decreases.

The best fit corresponds to the values

$$V_1 = 14 \quad V_3 = 170. \quad (4.39)$$

Below we discuss the values of various physical quantities (both related to thermodynamics, and to zero-temperature properties) obtained with this choice of parameters.

4.5.1 Latent Heat and Equation of State

The comparison between the curves $p(T)/T^4$, $e(T)/T^4$ and $s(T)/T^3$ obtained in our models with (4.39), and the lattice results [70] is shown in Fig. 4.5. The match

Fig. 4.5 Temperature dependence of the dimensionless thermodynamic densities, normalized such that they reach the common limiting value $\pi^2/15$ (dashed horizontal line) as $T \rightarrow \infty$. The dots correspond to the lattice data for $N_c = 3$ [70]

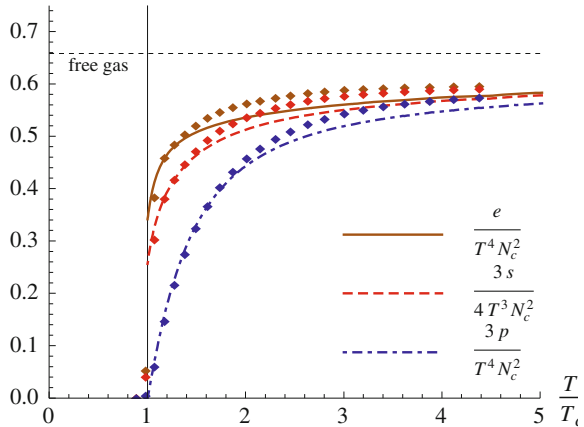
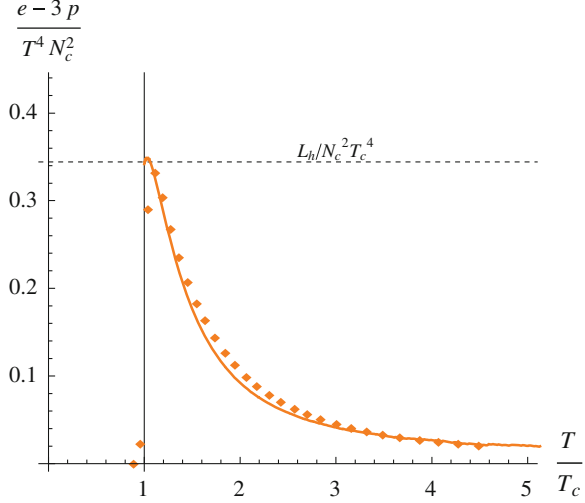


Fig. 4.6 The trace anomaly as a function of temperature in the deconfined phase of the holographic model (*solid line*) and the corresponding lattice data [70] for $N_c = 3$ (*dots*). The peak in the lattice data slightly above T_c is expected to be an artifact of the finite lattice volume. In the infinite volume limit the maximum value of the curve is at T_c , and it equals $L_h/N_c^2 T_c^4$



is remarkably good for $T_c < T < 2T_c$, and deviates slightly from the lattice data in the range up to $5T_c$.

The latent heat we obtain is:

$$L_h/T_c^4 = 0.31N_c^2, \quad (4.40)$$

which matches the lattice result for $N_c \rightarrow \infty$ [76].

An interesting quantity is the *trace anomaly* $(e - 3p)/T^4$, (also known as *interaction measure*), that indicates the deviation from conformality, and it is proportional to the thermal gluon condensate. The trace anomaly in our setup is shown, together with the corresponding lattice data, in Fig. 4.6, and the agreement is again very good. Our results agree even better with recent high-precision lattice calculations of the thermodynamics functions done by Panero at different values of N_c up to $N_c = 8$, [74, 75]. In Fig. 4.7 a comparison (taken from [74, 75]) of the normalized interaction measure with lattice results for different N_c is shown.

We also compute the specific heat per unit volume c_v , and the speed of sound c_s in the deconfined phase, by the relations

$$c_v = -T \frac{\partial^2 F}{\partial T^2}, \quad c_s^2 = \frac{s}{c_v}. \quad (4.41)$$

These are shown in Figs. 4.8 and 4.9 respectively. The speed of sound is shown together with the lattice data, and the agreement is remarkable.

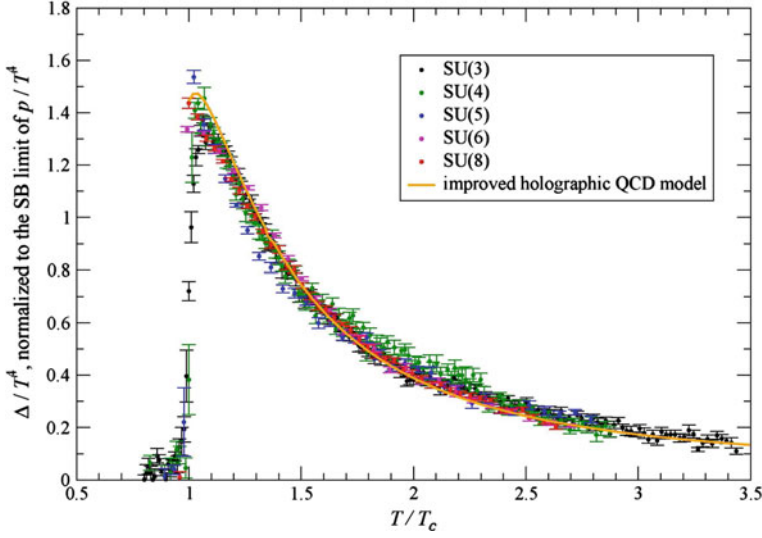
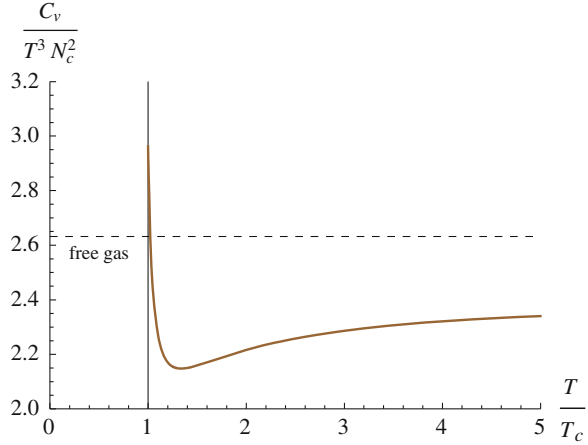


Fig. 4.7 The rescaled trace anomaly (so that it is N_c -independent) as a function of temperature in the deconfined phase of the holographic model (*solid line*) and the corresponding recent high precision lattice data taken from [74, 75] for different N_c . The errors shown are statistical only

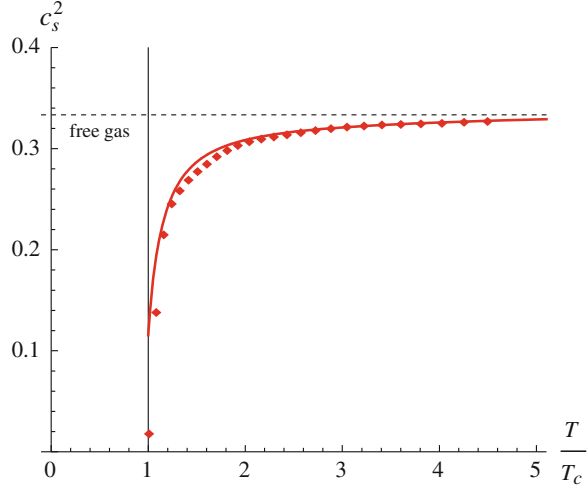
Fig. 4.8 The specific heat (divided by T^3), as a function of temperature, in the deconfined phase of the holographic model



4.5.2 Glueball Spectrum

In [23, 24], the single phenomenological parameters of the potential was fixed by looking at the zero-temperature spectrum, i.e. by computing various glueball mass ratios and comparing them to the corresponding lattice results. The masses are computed by deriving the effective action for the quadratic fluctuations around the background [77] and subsequently reducing the dynamics to four dimensions.

Fig. 4.9 The speed of sound in the deconfined phase, as a function of temperature, for the holographic model (*solid line*) and the corresponding lattice data [70] for $N_c = 3$ (*dots*). The *dashed horizontal line* indicates the conformal limit $c_s^2 = 1/3$



The associated thermodynamics for this potential was studied in [28] which was in qualitative agreement with lattice QCD results, but not in full quantitative agreement. This is due to the fact that the thermodynamics depends more on the details of the potential than the glueball spectrum for the main Regge trajectories. Here we use the potential (4.26), but with the two phenomenological parameters V_1 and V_3 already determined by the thermodynamics (4.39).

The glueball spectrum is obtained holographically as the spectrum of normalizable fluctuations around the zero-temperature background. As explained in the introduction, and motivated in [23–25], here we consider explicitly the 5D metric, one scalar field (the dilaton), and one pseudoscalar field (the axion). As a consequence, the only normalizable fluctuations above the vacuum correspond to spin 0 and spin 2 glueballs¹⁷ (more precisely, states with $J^{PC} = 0^{++}, 0^{-+}, 2^{++}$), each species containing an infinite discrete tower of excited states.

In 4D YM there are many more operators generating glueballs, corresponding to different values of J^{PC} , that are not considered here. These are expected to correspond holographically to other fields in the noncritical string spectrum (e.g. form fields, which may yield spin 1 and CP-odd spin 2 states) and to higher string states that provide higher-spin glueballs. As the main focus is in reproducing the YM thermodynamics in detail rather than the entire glueball spectrum, we choose not to include these states.¹⁸ Therefore we only compare the mass spectrum obtained in our model to the lattice results for the lowest $0^{++}, 0^{-+}, 2^{++}$ glueballs

¹⁷ Spin 1 excitations of the metric can be shown to be non-normalizable.

¹⁸ A further reason is that, unlike the scalar and (to some extent) the pseudoscalar sector that we are considering, the action governing the higher Regge slopes is less and less universal as one goes to higher masses. Only a precise knowledge of the underlying string theory is expected to provide detailed information for such states.

Table 4.1 Glueball masses

	HQCD	$N_c = 3$ [78, 79]	$N_c = \infty$ [80]
$m_{0^{++}}/m_{0^{++}}$	1.61	1.56(11)	1.90(17)
$m_{2^{++}}/m_{0^{++}}$	1.36	1.40(4)	1.46(11)

and their available excited states. These are limited to one for each spin 0 species, and none for the spin 2, in the study of [78, 79], which is the one we use for our comparison. This provides two mass ratios in the CP-even sector and two in the CP-odd sector.

The glueball masses are computed by first solving numerically the zero-temperature Einstein's equations, by setting $f(r) = 1$, and using the resulting metric and dilaton to setup an analogous Schrödinger problem for the fluctuations [23, 24]. The results for the parity-conserving sector are shown in Table 4.1, and are in good agreement with those reported by [78, 79] for $N_c = 3$, whereas the results reported by [80] for large N_c are somewhat larger. The CP-violating sector (axial glueballs) will be discussed separately.

We should add that there are other lattice studies (see e.g. [81]) that report additional excited states. Our mass ratios offer a somewhat worse fit of the mass ratios found in [81] (whose results are not entirely compatible with those of [78, 79] for the states the two studies have in common). We should stress however that reproducing the detailed glueball spectrum is secondary here since the main focus is thermodynamics. However, the comparison of our spectrum to the existing lattice results shows that our model provides a good global fit to 4D YM also with respect to quantities beyond thermodynamics.

Unlike the various mass ratios, the value of any given mass in AdS-length units (e.g. $m_{0^{++}}\ell$) *does depend* on the choice of integration constants in the UV, i.e. on the value of b_{UV} and λ_{UV} . Therefore its numerical value does not have an intrinsic meaning. However it can be used as a benchmark against which all other dimension-full quantities can be measured (provided one always uses the same UV b.c.). On the other hand, given a fixed set of initial conditions, asking that $m_{0^{++}}$ matches the physical value (in MeV) obtained on the lattice, fixes the value of ℓ hence the energy unit.

4.5.3 Critical Temperature

The thermodynamic quantities we have discussed so far, are dimensionless ratios, in units of the critical temperature. To compute T_c , we need an extra dimension-full quantity which can be used independently to set the unit of energy. In lattice studies this is typically the confining string tension σ in the $T = 0$ vacuum, with a value of around $(440 \text{ MeV})^2$, and results are given in terms of the dimensionless ratio $T_c/\sqrt{\sigma}$. In our case we cannot compute σ directly, since it depends on the *fundamental* string tension, which is a priori unknown. Instead, we take the mass $m_{0^{++}}$ of the lowest-lying glueball state as a reference.

We compute m_{0++} with the potential (4.26), with V_1 and V_3 fixed as in (4.39), then compare T_c/m_{0++} to the same quantity obtained on the lattice. For the lattice result, we take the large N_c result of [76], $T_c/\sqrt{\sigma} = 0.5970(38)$, and combine it with the large N_c result for the lowest-lying glueball mass [80], $m_{0++}/\sqrt{\sigma} = 3.37(15)$. The two results are in fair agreement, without need to adjust any extra parameter:

$$\left(\frac{T_c}{m_0}\right)_{\text{hQCD}} = 0.167, \quad \left(\frac{T_c}{m_0}\right)_{\text{lattice}} = 0.177(7). \quad (4.42)$$

In physical units, the critical temperature we obtain is given by

$$T_c = 0.56 \sqrt{\sigma} = 247 \text{ MeV}. \quad (4.43)$$

4.5.4 String Tension

The fundamental string tension $T_f = \frac{1}{2\pi\ell_s^2}$ cannot be computed from first principles in our model, but can be obtained using as extra input the lattice value of the confining string tension σ , at $T = 0$. The fundamental and confining string tensions are related by (4.35).

As for the critical temperature, we can relate T_f to the value of the lowest-lying glueball mass, by using the lattice relation $\sqrt{\sigma} = \frac{m_{0++}}{3.37}$ [80]. Since what we actually compute numerically is $m_{0++}\ell$, this allows us to obtain the string tension T_f (and fundamental string length $\ell_s = 1/\sqrt{2\pi T_f}$ in AdS units:

$$T_f \ell^2 = 0.19, \quad \ell_s/\ell = 0.15. \quad (4.44)$$

This shows that the fundamental string length in our model is about an order of magnitude smaller than the AdS length. The meaning of this fact is a little more complicated conceptually, as the discussion in [25] indicates. Also, we should stress that, as discussed in Sect. 4.4, this result depends on our choice of the overall normalization of λ : changing the potential by $\lambda \rightarrow \kappa\lambda$ will yield different numerical values in (4.44) without affecting the other physical quantities.

Another related observable is the spatial string tension. It is calculated from the expectation value of the rectangular Wilson loop which stretches in spatial dimensions only. This has been calculated on the lattice [82], as well as using the high-temperature (resummed) perturbative expansion plus a zero-temperature calculation of the string tension in three-dimensional YM theory [83]. The two calculation agree reasonably well.

The spatial string tension at finite temperature can be calculated in IHQCD [84] by calculating the relevant Wilson loop. Very good agreement was found with the lattice calculations, especially at temperatures not far from the phase transition.

Finally, several calculations of quark–antiquark potentials exist. At zero temperature the long distance asymptotics of the quark potential was calculated in [23, 24] and used to classify the dilaton potentials as a function of the confinement property. The full quark potential including the short distance behavior was computed in [85]. There a comparison to the Cornell potential was done as well as with quarkonium spectra finding excellent agreement with data. The issue of quarkonium potentials from IHQCD-like theories was also recently discussed in [86].

Finally the Polyakov loop was recently computed [87, 88] in similar Einstein dilaton models that were studied first in [26].

4.5.5 CP-Odd Sector

The CP-odd sector of pure Yang–Mills is described holographically by the addition of a bulk pseudoscalar field $a(r)$ (the *axion*) with action¹⁹:

$$S_{\text{axion}} = \frac{M_p^3}{2} \int d^5x Z(\lambda) \sqrt{-g} (\partial^\mu a) (\partial_\mu a). \quad (4.45)$$

The field $a(r)$ is dual to the topological density operator $\text{Tr} F \tilde{F}$. The prefactor $Z(\lambda)$ is a dilaton-dependent normalization. The axion action is suppressed by a factor $1/N_c^2$ with respect to the action (4.3) for the dilaton and the metric, meaning that in the large- N_c limit one can neglect the back-reaction of the axion on the background.

As shown in [23, 24], requiring the correct scaling of $a(r)$ in the UV, and phenomenologically consistent axial glueball masses, constrain the asymptotics of $Z(\lambda)$ as follows:

$$Z(\lambda) \sim Z_0, \lambda \rightarrow 0; \quad Z(\lambda) \sim \lambda^4, \lambda \rightarrow \infty, \quad (4.46)$$

where Z_0 is a constant. As a simple interpolating function between these large- and small- λ asymptotics we can take the following:

$$Z(\lambda) = Z_0(1 + c_a \lambda^4). \quad (4.47)$$

The parameter Z_0 can be fixed by matching the topological susceptibility of pure Yang–Mills theory, whereas c_a can be fixed by looking at the axial glueball mass spectrum.

¹⁹ This action was justified in [23–25]. The dilaton dependent coefficient $Z(\lambda)$ is encoding both the dilaton dependence as well as the UV curvature dependence of the axion kinetic terms in the associated string theory. We cannot determine it directly from the string theory, but we pin it down by a combination of first principles and lattice input, as we explain further below.

4.5.5.1 Axial Glueballs

As in [23, 24], we can fix c_a by matching to the lattice results the mass ratio m_{0-+}/m_{0++} between the lowest-lying axial and scalar glueball states. This is independent of the overall coefficient Z_0 in (4.47). The lattice value $m_{0-+}/m_{0++} = 1.49$ [78, 79] is obtained for:

$$c_a = 0.26. \quad (4.48)$$

With this choice, the mass of the first excited axial glueball state is in good agreement with the corresponding lattice result [78, 79]:

$$\left(\frac{m_{0-+*}}{m_{0++}}\right)_{\text{hQCD}} = 2.10 \quad \left(\frac{m_{0-+*}}{m_{0++}}\right)_{\text{lattice}} = 2.12(10). \quad (4.49)$$

4.5.5.2 Topological Susceptibility

In pure Yang–Mills, the topological χ susceptibility is defined by:

$$E(\theta) = \frac{1}{2}\chi\theta^2, \quad (4.50)$$

where $E(\theta)$ is the vacuum energy density in presence of a θ -parameter. $E(\theta)$ can be computed holographically by solving for the axion profile $a(r)$ on a given background, and evaluating the action (4.45) on-shell.

In the deconfined phase, the axion profile is trivial, implying a vanishing topological susceptibility [29]. This is in agreement with large- N_c arguments and lattice results [31].

In the low-temperature phase, the axion acquires a non-trivial profile,

$$a(r) = a_{\text{UV}} \frac{F(r)}{F(0)}, \quad F(r) \equiv \int_r^\infty \frac{dr}{Z(\lambda(r))e^{3A(r)}}. \quad (4.51)$$

This profile is shown, for the case at hand, in Fig. 4.10, where the axion is normalized to its UV value.

The topological susceptibility is given by [23, 24]:

$$\chi = M_p^3 F(0)^{-1} = M_p^3 \left[\int_0^\infty \frac{dr}{e^{3A(r)} Z(r)} \right]^{-1}, \quad (4.52)$$

where $Z(r) \equiv Z(\lambda(r))$. Evaluating this expression numerically with $Z(\lambda)$ as in (4.47), and $c_a = 0.26$ (to match the axial glueball spectrum), we can determine the coefficient Z_0 by looking at the lattice result for χ . For $N_c = 3$ [89] obtained $\chi = (191 \text{ MeV})^4$, which requires $Z_0 = 133$.

Fig. 4.10 Axion profile in the radial direction. The x -axis is taken to be the energy scale, $E(r) = E_0 b(r)$, where the unit E_0 is fixed to match the lowest glueball mass

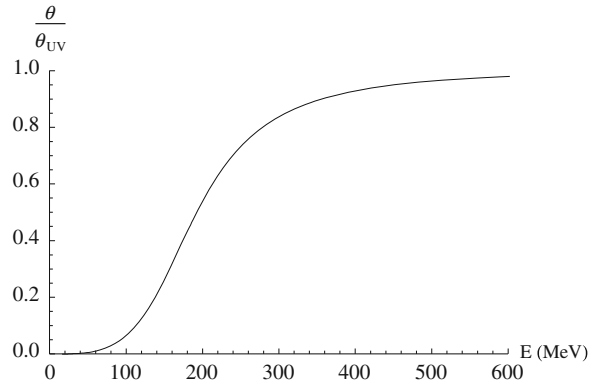
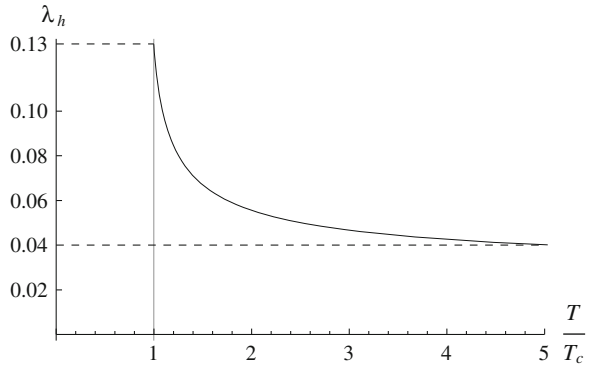


Fig. 4.11 The coupling at the horizon as a function of temperature in the range $T_c - 5T_c$



In Table 4.4 we present a summary of the various physical quantities discussed in this section, as obtained in our holographic model, and their comparison with the lattice results for large N_c (when available) and for $N_c = 3$. The quantities shown in the upper half of the table are the ones that were used to fix the free parameters (reported in the last column) of the holographic model.

4.5.6 Coupling Normalization

Finally, we can relate the field $\lambda(r)$ to the running 't Hooft coupling. All other quantities we have discussed so far are scheme-independent and RG-invariant. This is not the case for the identification of the physical YM 't Hooft coupling, which is scheme dependent.

In the black-hole phase we can take $\lambda_h \equiv \lambda(r_h)$ as a measure of the temperature-dependent coupling. In Fig. 4.11 we show λ_h as a function of the temperature in the range T_c to $5T_c$.

As a reference, we may take the result of [70], that found $g^2(5T_c) \simeq 1.5$ for $N_c = 3$, which translates to $\lambda_t(5T_c) \simeq 5$. On the other hand, if we make the assumption that the identification $\lambda = \lambda_t$ is valid at all scales (not only in the UV), we find in our model $\lambda_t(5T_c) \simeq 0.04$ (see Fig. 4.11), i.e. a factor of 100 smaller than the lattice result.

This discrepancy is almost certainly due to the identification (4.27) being very different from lattice at strong coupling.

4.6 Bulk Viscosity

The bulk viscosity ζ is an important probe of the quark–gluon plasma. Its profile as a function of T reveals information regarding the dynamics of the phase transition. In particular, both from the low-energy theorems and lattice studies [12, 40, 41], there is evidence that ζ increases near T_c .

For a viscous fluid the shear η and bulk ζ viscosities are defined via the rate of entropy production as

$$\frac{\partial s}{\partial t} = \frac{\eta}{T} \left[\partial_i v_j + \partial_j v_i - \frac{2}{3} (\partial \cdot v) \delta_{ij} \right]^2 + \frac{\zeta}{T} (\partial \cdot v)^2. \quad (4.53)$$

Therefore, in a holographic setup, the bulk viscosity can be defined as the response of the diagonal spatial components of the stress–energy tensor to a small fluctuation of the metric. It can be directly related to the retarded Green’s function of the stress–energy tensor by Kubo’s linear response theory (Table 4.2):

Table 4.2 Collected in this table is the complete set of physical quantities that we computed in our model and compared with data

	HQCD	Lattice $N_c = 3$	Lattice $N_c \rightarrow \infty$	Parameter
$[p/(N_c^2 T^4)]_{T=2T_c}$	1.2	1.2	–	$V1 = 14$
$L_h/(N_c^2 T_c^4)$	0.31	0.28 [70]	0.31 [76]	$V3 = 170$
$[p/(N_c^2 T^4)]_{T \rightarrow +\infty}$	$\pi^2/45$	$\pi^2/45$	$\pi^2/45$	$M_p \ell = [45\pi^2]^{-1/3}$
$m_{0^{++}}/\sqrt{\sigma}$	3.37	3.56 [78, 79]	3.37 [80]	$\ell_s/\ell = 0.15$
$m_{0^{++}}/m_{0^{++}}$	1.49	1.49 [78, 79]	–	$c_a = 0.26$
χ	$(191 \text{ MeV})^4$	$(191 \text{ MeV})^4$ [89]	–	$Z_0 = 133$
$T_c/m_{0^{++}}$	0.167	–	0.177(7)	
$m_{0^{++}}/m_{0^{++}}$	1.61	1.56(11)	1.90(17)	
$m_{2^{++}}/m_{0^{++}}$	1.36	1.40(4)	1.46(11)	
$m_{0^{+-}}/m_{0^{++}}$	2.10	2.12(10)	–	

The upper half of the table contains the quantities that we used as input (shown in boldface) for the holographic QCD model (HQCD). Each quantity can be roughly associated to one parameter of the model (last column). The lower half of the table contains our “postdictions” (i.e. quantities that we computed after all the parameters were fixed) and the comparison with the corresponding lattice results. The value we find for the critical temperature corresponds to $T_c = 247 \text{ MeV}$

$$\zeta = -\frac{1}{9} \lim_{\omega \rightarrow 0} \frac{1}{\omega} \text{Im} G_R(\omega, 0), \quad (4.54)$$

where $G_R(w, \mathbf{p})$ is the Fourier transform of retarded Green's function of the stress-energy tensor:

$$G_R(w, \mathbf{p}) = -i \int d^3x dt e^{i\omega t - i\mathbf{p} \cdot \mathbf{x}} \theta(t) \sum_{i,j=1}^3 \langle [T_{ii}(t, \mathbf{x}), T_{jj}(0, 0)] \rangle. \quad (4.55)$$

A direct computation of the RHS on the lattice is non-trivial as it requires analytic continuation to Lorentzian space-time. In Refs. [40, 41] the low energy theorems of QCD, as well as (equilibrium) lattice data at finite temperature were used in order to evaluate a particular moment of the spectral density of the relevant correlator. using a parametrization of the spectral density via two time-dependent constants, one of which is the bulk viscosity a relation for their product was obtained as a function of temperature. This can be converted to a relation for ζ , assuming the other constant varies slowly with temperature.

The conclusion was that ζ/s increases near T_c . Another conclusion is that the fermionic contributions to ζ are small compared to the glue contributions.

The weak point of the approach of [41], is that it requires an ansatz on the spectrum of energy fluctuations, and further assumptions on the other parameters. which are not derived from first principles.

A direct lattice study of the bulk viscosity was also made in [12]. Here, the result is also qualitatively similar 4.12. However, the systematic errors in this computation are large especially near T_c , mostly due to the analytic continuation that one has to perform after computing the Euclidean correlator on the lattice.

The results of references [40, 41] and the assumptions of the lattice calculation have been recently challenged in [90].

4.6.1 The Holographic Computation

The holographic approach offers a new way of computing the bulk viscosity. In the holographic set-up, ζ is obtained from (4.54). Using the standard AdS/CFT prescription, the two point-function of the energy-momentum tensor can be read off from the asymptotic behavior of the metric perturbations $\delta g_{\mu\nu}$. This is similar in spirit to the holographic computation of the shear viscosity [6], but it is technically more involved. A recent treatment of the fluctuation equation governing the scalar mode of a general Einstein-Dilaton system can be found in [91]. Here, we follow the method proposed by [92].

As explained in [92], one only needs to examine the equations of motion in the gauge $r = \Phi$, where the radial coordinate is equal to the dilaton. In our type of metrics, the applicability of this method requires some clarifications, that we provide in [93]. Using $SO(3)$ invariance and the five remaining gauge degrees of freedom the metric perturbations can be diagonalized as

$$\delta g = \text{diag}(g_{00}, g_{11}, g_{11}, g_{11}, g_{55}), \quad (4.56)$$

where

$$g_{00} = -e^{2A}f[1 + h_{00}(\Phi)e^{-i\omega t}], \quad g_{11} = e^{2A}[1 + h_{11}(\Phi)e^{-i\omega t}], \quad (4.57)$$

$$g_{55} = \frac{e^{2B}}{f}[1 + h_{55}(\Phi)e^{-i\omega t}],$$

where the functions A and B emerge from the metric

$$ds^2 = e^{2A(\Phi)}(-fdt^2 + dx_m dx^m) + e^{2B(\Phi)} \frac{d\Phi^2}{f}. \quad (4.58)$$

Here, the fluctuations are taken to be harmonic functions of t while having an arbitrary dependence on Φ .

The bulk viscosity depends only on the correlator of the diagonal components of the metric and so it suffices to look for the asymptotics of h_{11} . Interestingly, in the $r = \Phi$ gauge this decouples from the other components of the metric and satisfies the following equation²⁰

$$h''_{11} - \left(-\frac{8}{9A'} - 4A' + 3B' - \frac{f'}{f}\right)h'_{11} - \left(-\frac{e^{2B-2A}}{f^2}\omega^2 + \frac{4f'}{9fA'} - \frac{f'B'}{f}\right)h_{11} = 0. \quad (4.59)$$

One needs to impose two boundary conditions. First, we require that only the infalling condition survives at the horizon:

$$h_{11} \rightarrow c_b(\Phi_h - \Phi)^{-\frac{i\omega}{4\pi t}}, \quad \Phi \rightarrow \Phi_h, \quad (4.60)$$

where c_b is a normalization factor. The second boundary condition is that h_{11} has unit normalization on the boundary:

$$h_{11} \rightarrow 1, \quad \Phi \rightarrow -\infty. \quad (4.61)$$

Having solved for $h_{11}(\Phi)$, Kubo's formula (4.54) and a wise use of the AdS/CFT prescription to compute the stress–energy correlation function [92] determines the ratio of bulk viscosity as follows.

The AdS/CFT prescription relates the imaginary part of the retarded T_{ii} Green's function to the number flux of the h_{11} gravitons $\mathcal{F}$ [92]:

$$\text{Im } G_R(\omega, 0) = -\frac{\mathcal{F}}{16\pi G_5} \quad (4.62)$$

²⁰ Difference in the various numerical factors in this equation w.r.t [92] is due to our different normalization of the dilaton kinetic term.

where the flux can be calculated as the Noether current associated to the $U(1)$ symmetry $h_{11} \rightarrow e^{i\theta} h_{11}$ in the gravitational action for fluctuations. One finds,

$$\mathcal{F} = i \frac{e^{4A-B} f}{3A'^2} [h_{11}^* h'_{11} - h_{11} h_{11}^*]. \quad (4.63)$$

As $\mathcal{F}$ is independent of the radial variable, one can compute it at any Φ , most easily near the horizon, where h_{11} takes the form (4.60). Using also the fact that $(dA/d\Phi)(\Phi_h) = -8V(\Phi_h)/9V'(\Phi_h)$, one finds

$$\mathcal{F}(\phi) = \frac{27}{32} \phi |c_b(\phi)|^2 e^{3A(\Phi_h)} \frac{V'(\Phi_h)^2}{V(\Phi_h)}. \quad (4.64)$$

Then, (4.54 and 4.62) determine the ratio of bulk viscosity and the entropy density as,

$$\frac{\zeta}{s} = \frac{3}{32\pi} \left(\frac{V'(\Phi_h)}{V(\Phi_h)} \right)^2 |c_b|^2. \quad (4.65)$$

In the derivation we use the Bekenstein–Hawking formula for the entropy density, $s = \exp 3A(\Phi_h)/4G_5$.

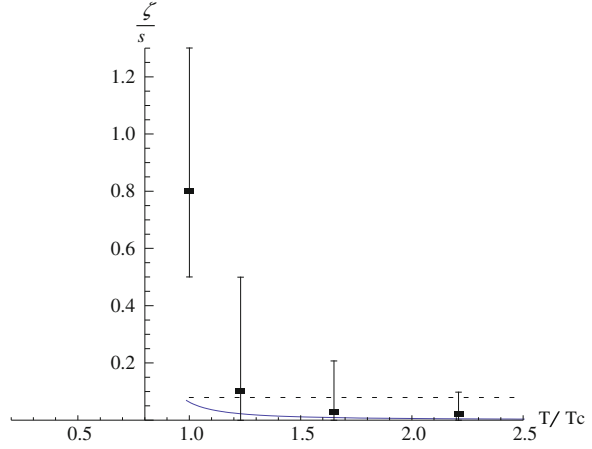
To find ζ we need to find c_b only in the limit $\omega \rightarrow 0$. The computation is performed by numerically solving equation (4.59) with the appropriate boundary conditions. There are two separate methods that one can employ to determine the quantity c_b :

1. One can solve (4.59) numerically with a fixed ω/T , but small enough so that c_b reaches a fixed value. The method is valid also for finite values of ω . From a practical point of view, it is easier to solve (4.59) with the boundary condition (4.60) with a unit normalization factor, read off the value on the boundary $h_{11}(-\infty)$ from the solution and finally use the symmetry of (4.59) under constant scalings of h_{11} to determine $|c_b| = 1/|h_{11}(-\infty)|$.
2. An alternative method of computation that directly extracts the information at $\omega = 0$ follows from the following trick [92]. Instead of solving (4.59) for small but finite ω , one can instead solve it for $\omega = 0$. This is a simpler equation, yet complicated enough to still evade analytic solution. Let us call this solution h_{11}^0 . One numerically solves it by fixing the boundary conditions on the boundary: $h_{11}^0(-\infty) = 1$ and the derivative $dh_{11}^0/d\Phi(-\infty)$ is chosen such that h_{11} is regular at the horizon. Matching this solution to the expansion of (4.60) for small ω then yields $|c_b| = h_{11}^0(\Phi_h)$.

Both methods were used to obtain ζ/s as a function of T and checked that they yield the same result. As explained in [29], most of the thermodynamic observables are easily computed using the method of scalar variables [29, 93].

The results are presented in Fig. 4.12. This figure gives a comparison of the curve obtained by the holographic calculation sketched above by solving (4.59) and the lattice data of [12]. We also show $\eta/s = 1/4\pi$ in this figure for comparison.

Fig. 4.12 Plot of ζ/s (continuous line) calculated in Improved Holographic QCD model. This is compared with the lattice data of [12] that are shown as boxes. The horizontal dashed line is indicating the (universal) value of $\frac{\eta}{s}$ for comparison



The result is qualitatively similar to the lattice result where ζ/s increases as T approaches T_c , however the rate of increase is slower than the lattice. As a result, we obtain a value $\zeta/s(T_c) \approx 0.06$ that is an order of magnitude smaller than the lattice result [12] which is 0.8. Note however that the error bars in the lattice evaluation are large near T_c and do not include all possible systematic errors from the analytic continuation.

We should note the fact that the holographic calculation gives a smaller value for the bulk viscosity near T_c than the lattice calculation is generic and has been found for other potentials with similar IR asymptotics [92]. The fact that the value of ζ/s near T_c is correlated with the IR asymptotics of the potential will be shown further below.

Another fact that one observes from Fig. 4.12 is that ζ/s vanishes in the high T limit. This reflects the conformal invariance in the UV and can be shown analytically as follows. ζ/s is determined by formula (4.65). In the high T limit, (corresponding to $\lambda_h \rightarrow 0$, near the boundary), the fluctuation coefficient $|c_b| \rightarrow 1$. This is because of the boundary condition $h_{11}(\lambda = 0) = 1$. We use the relation between T and λ_h in the high T limit [29],

$$\lambda_h \rightarrow (\beta_0 \log(\pi T/\Lambda))^{-1}. \quad (4.66)$$

Substitution in (4.65) leads to the result,

$$\left. \frac{\zeta}{s} \right|_{\text{large}} \rightarrow \frac{2}{27\pi} \frac{1}{\log^2(\pi T/\Lambda)}, \quad \text{as } T \rightarrow \infty. \quad (4.67)$$

As s itself diverges as T^3 in this limit—it corresponds to an ideal gas—we learn that ζ also diverges as $T^3/\log^2(T)$. Divergence at high T is expected from the bulk-viscosity of an *ideal gas*. We do not expect however the details of the asymptotic result to match with the pQCD result, for the same reasons that the shear-viscosity-to-entropy ratio does not, [25]. We note however, that the asymptotic T -dependence is very similar to the pQCD result [94]:

$$\zeta/s \propto \log^{-2}(\pi T/\Lambda) \log^{-1} \log(\pi T/\Lambda) . \quad (4.68)$$

4.6.2 Holographic Explanation for the Rise of ζ/s Near T_c

With the same numerical methods, one can also compute the ratio ζ/s on the small black-hole branch. As this solution has a smaller value of the action than the large black-hole solution, it is a subleading saddle point in the phase space of the theory, hence bears no direct significance for an holographic investigation of the quark-gluon plasma. However, as we show below, the existence of this branch provides a holographic explanation for the peak in ζ/s in the quark-gluon plasma, near T_c .

From the practical point of view, we find the second numerical method above (solving the fluctuation equation at $\omega = 0$) easier in the range of λ_h that corresponds to the small black hole. The result is shown in Fig. 4.13a. The presence of two branches for $T > T_{\min}$, is made clear in this figure. See also Fig. 4.13b for the respective ranges of λ_h that correspond to small and large BHs. In Fig. 4.13a, ζ/s on the large BH branch is depicted with a solid curve and the small BH branch is depicted with a dashed curve. We observe that ζ/s keeps increasing on the large-BH branch as T is lowered, up to the temperature $T_{\min}$ where the small and large BH branches merge.²¹ On the other hand, on the small BH branch ζ/s keeps increasing as the T is increased, up to a certain $T_{\max}$ that lies between $T_{\min}$ and T_c , see Fig. 4.14. From this point onwards, ζ/s decreases with increasing T .

A simple fact that can be proved analytically is that the derivative of ζ/s diverges at $T_{\min}$. This is also clear from Fig. 4.14. This is shown by inspecting (4.65). The T derivative is determined as $d/dT = (dT/d\lambda_h)d/d\lambda_h$. Whereas the derivative w.r.t λ_h is everywhere smooth,²² the factor $dT/d\lambda_h$ diverges at $T_{\min}$ by definition, see Fig. 4.13b.

Therefore, the presence of a $T_{\min}$ where the large and the small black holes meet, in other words, the presence of a small black-hole branch is responsible for the increase of ζ/s near $T_{\min}$. As in most of the holographic constructions that we analyzed, and specifically in the example we present here, T_c and $T_{\min}$ are close to one another, this fact implies a rise in the bulk viscosity near T_c . This proposal, combined with the fact that *the existence of a small BH branch and color confinement in the dual gauge theory at zero T are in one-to-one correspondence* [29], suggests that *in confining large- N gauge theories, there will be a peak in the ratio ζ/s close to T_c .*

Another fact that can be shown analytically is that ζ/s asymptotes to a finite value as $T \rightarrow \infty$ in the small black-hole branch.²³ We find that,

²¹ As far as the thermodynamics of the gluon plasma is concerned, the temperatures below T_c (on the large BH branch) has little importance, because for $T < T_c$ the plasma is in the confined phase.

²² Note that c_b is also a function of λ_h . As both the fluctuation (4.59) and the boundary conditions are smooth at $\lambda_h = \lambda_{\min}$, one concludes that c_b also is smooth at this point.

²³ See the discussion at appendix B of [93].

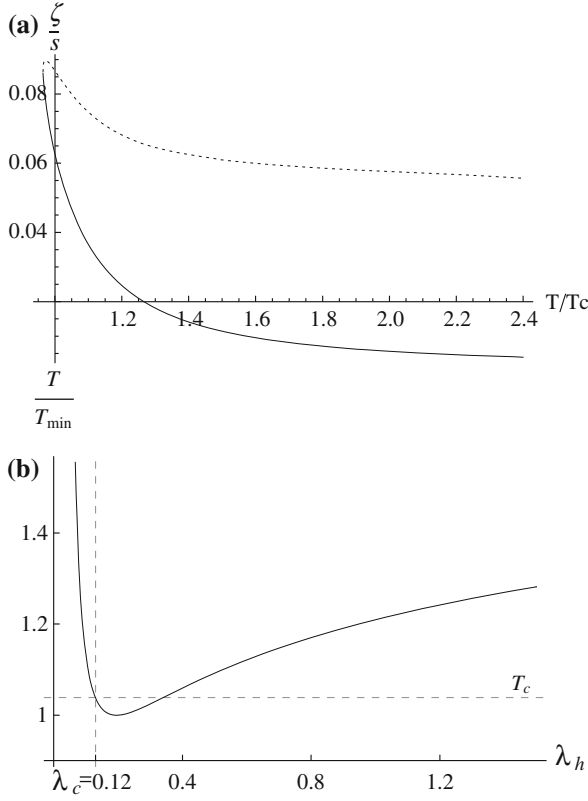


Fig. 4.13 **a** Numerical evaluation of ζ/η both on the large-BH branch (the *solid curve*) and on the small BH branch (the *dashed curve*). T_m denotes $T_{\min}$. **b** The two branches of black-hole solutions, that correspond to different ranges of λ_h . The large BH corresponds to $\lambda_h < \lambda_{\min}$ and the small BH corresponds to $\lambda_h > \lambda_{\min}$

$$\left. \frac{\zeta}{s} \right|_{\text{small}} \rightarrow \frac{1}{6\pi}, \quad \text{as } T \rightarrow \infty. \quad (4.69)$$

As the entropy density vanishes in this limit [29], we conclude that ζ should vanish with the same rate.

For a general potential with strong coupling asymptotics

$$V(\lambda) \sim \lambda^Q \quad \text{as } \lambda \rightarrow \infty, \quad (4.70)$$

taking into account (4.65), (4.69) is modified to

$$\left. \frac{\zeta}{s} \right|_{\text{small}} \rightarrow \frac{3Q^2}{32\pi}, \quad \text{as } r_h \rightarrow r_0. \quad (4.71)$$

where r_0 is the position of the singularity in the zero temperature solution.

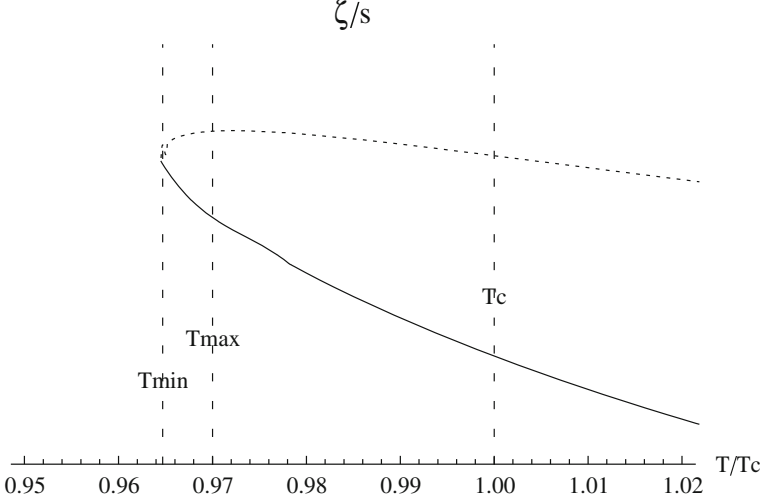


Fig. 4.14 An inset from the Fig. 4.13 around the maximum of ζ/s

For confining theories, the limit $r_h \rightarrow r_0$ corresponds to $T \rightarrow \infty$ on the small BH branch. However, one can show that the result (4.71) holds quite generally, regardless of whether the zero T theory confines or not. In particular, for the non-confining theories—that is either when $Q < 4/3$ or when $Q = 4/3$ but the sub-leading term in the potential vanishes at the singularity—there is only the large black-hole branch and the limit $r_h \rightarrow r_0$ corresponds to the zero T limit of this BH. Thus, we also learn that there exist holographic models that correspond to non-confining gauge theories whose zero T limit yield a constant ζ/s . This constant approaches zero as $Q \rightarrow 0$, i.e. in the limiting AdS case.

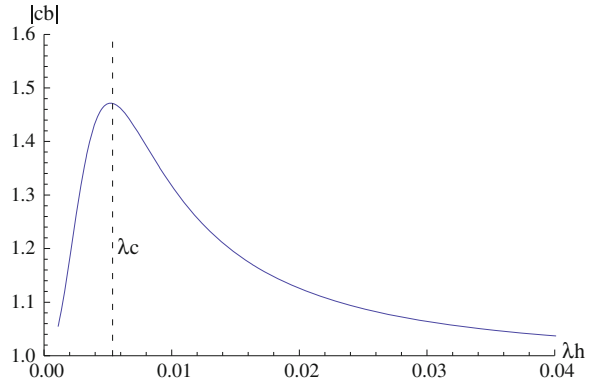
We also see that the asymptotic value of ζ/s in the small BH branch is close to the value of ζ/s near T_c . We shall give an explanation of this fact in the next subsection. Using the asymptotic formula (4.71), the fact that $Q > \frac{4}{3}$ for confinement and $Q \leq \frac{4\sqrt{2}}{3}$ for the IR singularity to be good and repulsive we may obtain a range of values where we expect ζ/s to vary, namely

$$\frac{1}{6\pi} \leq \frac{\zeta}{s} \Big|_{\text{small, asymptotic}} \leq \frac{1}{3\pi}. \quad (4.72)$$

A final observation concerns the coefficient $c_b(\lambda_h)$ in (4.65). This part is the only input from the solution of the fluctuation equation, the rest of (4.65) is fixed by the dilaton potential entirely. We plot the numerical result for c_b in Fig. 4.15 as a function of the coupling at the horizon λ_h .

First of all, Fig. 4.15 provides a check that, the approximate bound of [92] $|c_b| \geq 1$, is satisfied in the entire range. One also observes c_b approaches to 1 in the IR and UV asymptotics. These facts can be understood analytically: In the UV

Fig. 4.15 The coefficient $|c_b|$ of (4.65) as a function of λ_h



(near the boundary) it is because of the boundary condition $c_b = 1$. In the IR, it is more subtle, and it is explained in appendix B of [93].

Finally, we observe that the deviation of c_b from the asymptotic value 1 is maximum around the phase transition point λ_c . In fact, we numerically observed that the top of the curve in Fig. 4.15 coincides with λ_c to a very high accuracy. Whether this is just a coincidence or not, remains to be clarified.

4.6.3 The Adiabatic Approximation

Motivated by the Chamblin-Reall solutions [95], Gubser et al. [26] proposed an approximate adiabatic formula for the speed of sound. In the case when V'/V is a slowly varying function of Φ [26] proposes the following formulae for the entropy density and the temperature:

$$\log s = -\frac{8}{3} \int^{\Phi_h} d\Phi \frac{V}{V'} + \dots, \quad (4.73)$$

$$\log T = \int^{\Phi_h} d\Phi \left(\frac{1}{2} \frac{V'}{V} - \frac{8}{9} \frac{V}{V'} \right) \dots, \quad (4.74)$$

where the ellipsis denote contributions slowly varying in Φ_h .²⁴

It is very useful to reformulate this approximation using the method of scalar variables, which in turn allows us to extract the general T dependence of most of the thermodynamic observables in an approximate form. Here, we apply this formalism to the computation of ζ/s . The method of scalar variables and the details of the adiabatic approximation in the scalar variables are given in [93].

For the scalar variable $X = \frac{\Phi'}{3A'}$ the adiabatic approximation means

²⁴ Various coefficients in these equations differ from [26] due to our different normalization of the dilaton kinetic term.

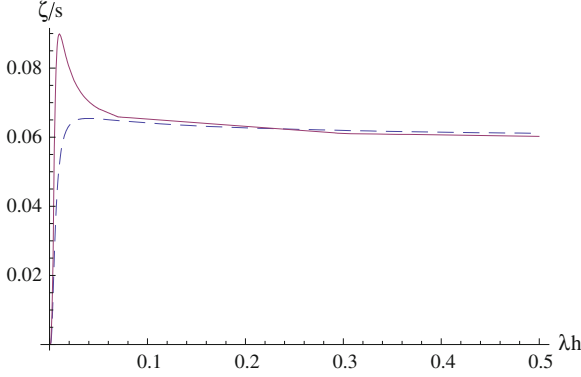
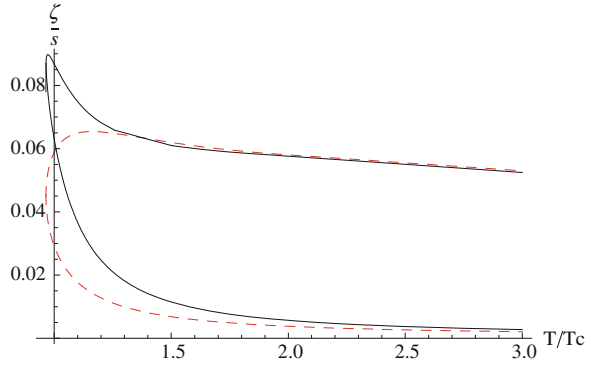


Fig. 4.16 Comparison of the exact ζ/s with the adiabatic approximation in the variable λ_h . *Solid* curve is the full numerical result and the *dashed* curve follows from (4.76)

Fig. 4.17 Comparison of the exact ζ/s with the adiabatic approximation in variable T . *Solid* curve is the full numerical result and the *dashed* curve follows from (4.76)



$$X(\Phi) \approx -\frac{3}{8} \frac{V'(\Phi)}{V(\Phi)}. \quad (4.75)$$

The fluctuation (4.59) greatly simplifies with (4.75). In fact, as shown in [93], the solution becomes independent of Φ . With unit normalization on boundary, the adiabatic solution in the entire range of $\Phi \in (-\infty, \Phi_h)$ becomes $h_{adb}(\Phi) = 1$. Consequently, the coefficient c_b in (4.65) becomes unity, hence:

$$\left. \frac{\zeta}{s} \right|_{adb} = \frac{3}{32\pi} \left(\frac{V'(\Phi_h)}{V(\Phi_h)} \right)^2. \quad (4.76)$$

We plot this function in λ_h in Fig. 4.16, where we also provide the exact numerical result for comparison. Note that in Fig. 4.16 the whole large black-hole branch has been compressed at the left of the figure for $\lambda_h \lesssim 0.04$. The same functions in the variable T/T_c are plotted in Fig. 4.17.

The validity of the adiabatic approximation (4.75), is determined by the rate at which V'/V varies with Φ . In particular, the approximation becomes exact in the

limits where V'/V becomes constant. This happens for a constant potential or a potential that is a single power of λ (exponential in Φ). This is the case in the UV ($\Phi \rightarrow -\infty$, where the potential becomes a constant) and the IR ($\Phi \rightarrow +\infty$ where the potential becomes a power law.). Therefore (4.76) allows us to extract the analytic behavior of ζ/s in the limits $\Phi_h \rightarrow \pm\infty$.

The numerical values one obtains from (4.76) in the intermediate region may differ from the exact result (4.65) considerably, especially near T_c . However, we expect that the general shape will be similar.

Finally, the adiabatic approximation hints at why, in the particular background that we study, ζ/s at T_c is close to the limit value (4.69): In order to see this we rewrite (4.76) as

$$\left. \frac{\zeta}{s} \right|_{adb} = \frac{2}{3\pi} X^2. \quad (4.77)$$

In the limit (4.69) we have $X \rightarrow -1/2$. The only other point where $X = -1/2$, is at the minimum of the string frame scale factor Φ_* . This is the point where the confining string saturates [23, 24]. On the other hand, we expect on general physical grounds that the de-confinement phase transition happens near this point, i.e. $\Phi_c \approx \Phi_*$. Thus, the adiabatic formula predicts that $\zeta/s(\Phi_c)$ be close to the limit value $1/6\pi$.²⁵

4.6.4 Buchel's Bound

In [96], Buchel proposed a bound for the ratio of the bulk and shear viscosities, motivated by certain well-understood holographic examples. In four space-time dimensions the Buchel bound reads,

$$\frac{\zeta}{\eta} \geq 2 \left(\frac{1}{3} - c_s^2 \right). \quad (4.78)$$

We note that the bound is proposed to hold in the entire range of temperature from T_c to ∞ . This bound is trivially satisfied for exact conformal theories such as $\mathcal{N} = 4$ YM, and saturated in theories on Dp branes [96, 97]. With the numerical evaluation at hand, we can check (4.78) in our case. In Fig. 4.18a we plot the LHS and RHS of the bound.²⁶ We clearly see that the bound is satisfied for all

²⁵ This argument may break down for two (dependent) reasons: First of all the adiabatic approximation becomes less good near Φ_c . This is because, in this region V'/V varies relatively more rapidly as a function of Φ . Secondly, precisely because of this, even though Φ_c is not far away from Φ_* the difference can result in a considerable change in the value of ζ/s through (4.76).

²⁶ Since this theory contains two derivatives only, $\frac{\eta}{s}$ has the universal value $1/4\pi$.

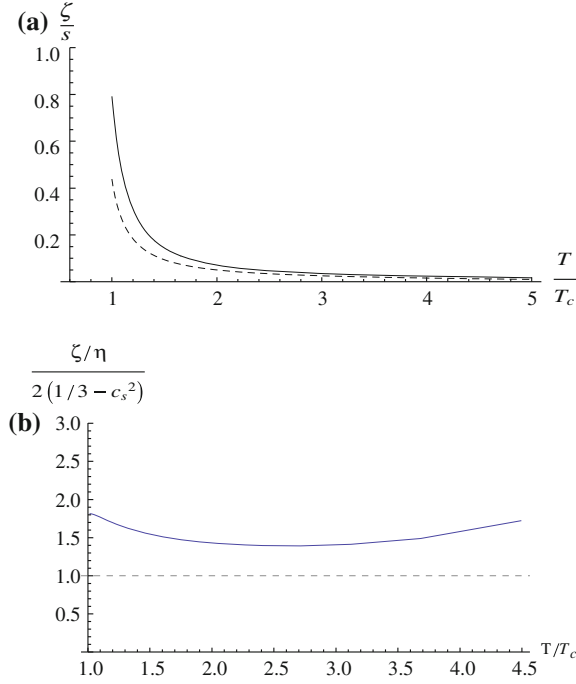


Fig. 4.18 **a** Comparison of ζ/η (solid line) and the RHS of (4.78) (dashed line), obtained using the speed of sound of the IHQCD model [22]. **b** Plot of the function $C(T)$ defined in (4.79) as a function of temperature. The horizontal dashed line indicates where Buchel's bound is saturated. We see that the bound is satisfied in the entire range of temperatures

temperatures. As expected, both the LHS and the RHS of (4.78) vanishes in the high T conformal limit.

A clear picture of Buchel's bound is obtained by defining the function:

$$C(T) = \frac{\zeta/\eta}{2(1/3 - c_s^2)}, \quad (4.79)$$

in terms of which the bound is simply $C > 1$. In Fig. 4.18b we show the function $C(T)$ obtained numerically in our IHQCD model, between T_c and $5T_c$. The values of this function are mildly dependent on temperature, and are between 1.5 and 2, the same range of values that were recently considered in the hydrodynamic codes by Heinz and Song [98].

We may also investigate the fate of the bound at large T , using the asymptotics of ζ/s in (4.67)

$$\left. \frac{\zeta}{s} \right|_{\text{large}} = \frac{2}{27\pi} \frac{1}{\log^2(\pi T/\Lambda)} + \dots, \quad \text{as } T \rightarrow \infty. \quad (4.80)$$

and

$$\frac{1}{c_s^2} - 3 = \frac{4}{3} \frac{1}{\log^2\left(\frac{T}{T_c}\right)} + \frac{32b}{9} \frac{\log\left(\log\left(\frac{T}{T_c}\right)\right)}{\log^3\left(\frac{T}{T_c}\right)} + \dots \quad (4.81)$$

from [29], that can be rewritten as

$$\frac{1}{3} - c_s^2 = \frac{4}{27} \frac{1}{\log^2\left(\frac{T}{T_c}\right)} + \frac{32b}{81} \frac{\log\left(\log\left(\frac{T}{T_c}\right)\right)}{\log^3\left(\frac{T}{T_c}\right)} + \dots \quad (4.82)$$

where $b = \frac{b_1}{b_0^2} = \frac{3 \cdot 34}{2 \cdot 121}$ is the ratio of the two-loop to the one-loop squared β -function coefficients in large- N_c YM.

Since in this class of models $\eta/s = 1/4\pi$ exactly we obtain

$$\lim_{T \rightarrow \infty} \frac{\zeta/\eta}{2(1/3 - c_s^2)} = 1 \quad (4.83)$$

in agreement with a recently derived general formula, in Einstein dilaton gravity [99]

$$\lim_{T \rightarrow \infty} \frac{\zeta/\eta}{2(1/3 - c_s^2)} = 2\pi \frac{4 - \Delta}{4 - 2\Delta} \cot\left(\frac{\pi\Delta}{4}\right) \quad (4.84)$$

where Δ is the scaling dimensions of the scalar operator in the UV, that is marginal in our case.

It has also been suggested recently [100–102] that the speed of sound squared, in Einstein dilaton gravity asymptotes to $1/3$ at high temperatures from below. This is evident in our asymptotic formula (4.82), although the formulae in [100–102] fail to capture correctly the marginal case that is relevant here.

4.7 The Drag Force on Strings and Heavy Quarks

We will now consider an (external) heavy quark moving through an infinite volume of gluon plasma with a fixed velocity v at a finite temperature T [17, 19, 20]. The quark feels a drag force coming from its interaction with the plasma and an external force has to be applied in order for it to keep a constant velocity. In a more realistic set up one would like to describe the deceleration caused by the drag.

The heavy external quark can be described by a string whose endpoint is at the boundary. One can accommodate flavor by introducing D-branes, but we will not do this here. A first step is to describe the classical string “trailing” the quark.

We consider the Nambu–Goto action on the world-sheet of the string.

$$S_{NG} = -\frac{1}{2\pi\ell_s^2} \int d\sigma d\tau \sqrt{\det(-g_{MN}\partial_\alpha X^M \partial_\beta X^N)}, \quad (4.85)$$

where the metric is the string frame metric. The ansatz we are going to use to describe the trailing string is [19, 20]

$$X^1 = vt + \xi(r), \quad X^2 = X^3 = 0, \quad (4.86)$$

along with the gauge choice

$$\sigma = r, \quad \tau = t, \quad (4.87)$$

where r is the (radial) holographic coordinate. The string is moving in the X^1 direction.

This is a “steady-state” description of the moving quark as acceleration and deceleration are not taken into account. For a generic background the action of the string becomes

$$S = -\frac{1}{2\pi\ell_s^2} \int dt dr \sqrt{-g_{00}g_{rr} - g_{00}g_{11}\xi'^2 - g_{11}g_{rr}v^2}. \quad (4.88)$$

Note that g_{00} is negative, and we should check whether our solution produces a real action. For example a straight string stretching from the quark to the horizon is a solution to the equations of motion but has imaginary action.

We note that the action does not depend on ξ but only its derivative, therefore the corresponding “momentum” is conserved

$$\pi_\xi = -\frac{1}{2\pi\ell_s^2} \frac{g_{00}g_{11}\xi'}{\sqrt{-g_{00}g_{rr} - g_{00}g_{11}\xi'^2 - g_{11}g_{rr}v^2}}. \quad (4.89)$$

We solve for ξ' to obtain

$$\xi' = \frac{\sqrt{-g_{00}g_{rr} - g_{11}g_{rr}v^2}}{\sqrt{g_{00}g_{11} \left(1 + g_{00}g_{11}/(2\pi\ell_s^2\pi_\xi)^2\right)}}. \quad (4.90)$$

The numerator changes sign at some finite value of the fifth coordinate r_s . For the solution to be real, the denominator has to change sign at the same point. We therefore determine r_s via the equation

$$g_{00}(r_s) + g_{11}(r_s)v^2 = 0, \quad (4.91)$$

and the constant momentum

$$\pi_\xi^2 = -\frac{g_{00}(r_s)g_{11}(r_s)}{(2\pi\ell_s^2)^2} \quad (4.92)$$

Writing the string-frame metric as

$$ds^2 = e^{2A_s} \left[\frac{dr^2}{f} - f dt^2 + dx \cdot dx \right] \quad (4.93)$$

(4.91) becomes

$$v^2 = f(r_s) \quad (4.94)$$

The induced world-sheet metric is therefore

$$g_{\alpha\beta} = e^{2A_s(r)} \begin{pmatrix} -(f(r) - v^2) & \frac{e^{2A_s(r_s)} v^2}{f(r)} \sqrt{\frac{f(r) - v^2}{e^{4A_s(r)} f(r) - e^{4A_s(r_s)} v^2}} \\ \frac{e^{2A_s(r_s)} v^2}{f(r)} \sqrt{\frac{f(r) - v^2}{e^{4A_s(r)} f(r) - e^{4A_s(r_s)} v^2}} & \frac{e^{4A_s(r)} f^2(r) - v^4 e^{4A_s(r_s)}}{f^2(r) [e^{4A_s(r)} f(r) - v^2 e^{4A_s(r_s)}]} \end{pmatrix} \quad (4.95)$$

We can change the time coordinate to obtain a diagonal induced metric $t = \tau + \zeta(r)$ with

$$\zeta' = \frac{e^{2A_s(r_s)} v^2}{f(r) \sqrt{(f(r) - v^2)(e^{4A_s(r)} f(r) - e^{4A_s(r_s)} v^2)}}$$

The new metric is

$$ds^2 = e^{2A_s(r)} \left[-(f(r) - v^2) d\tau^2 + \frac{e^{4A_s(r)}}{(e^{4A_s(r)} f(r) - e^{4A_s(r_s)} v^2)} dr^2 \right] \quad (4.96)$$

and near $r = r_s$ it has the expansion

$$ds^2 = \left[-f'(r_s) e^{2A_s(r_s)} (r - r_s) + \mathcal{O}((r - r_s)^2) \right] d\tau^2 + \left[\frac{e^{2A_s(r_s)}}{(4v^2 A'_s(r_s) + f'(r_s))(r - r_s)} + \mathcal{O}(1) \right] dr^2 \quad (4.97)$$

This is a world-sheet black-hole metric with horizon at the turning point $r = r_s$.

4.7.1 The Drag Force

The drag force on the quark can be determined by calculating the momentum that is lost by flowing along the string into the horizon:

$$F_{\text{drag}} = \frac{dp_1}{dt} = -\frac{1}{2\pi\ell_s^2} \frac{g_{00}g_{11}\zeta'}{\sqrt{-g}} = \pi\tilde{\xi}. \quad (4.98)$$

This can be obtained by considering the world-sheet Noether currents Π_M^x and expressing the loss of momentum as $\Delta P_{x_1}^z = \int \Pi_1^z$. This may be evaluated at any value of r , but it is more convenient to evaluate it at $r = r_s$.

We finally find that

$$F_{\text{drag}} = -\frac{1}{2\pi\ell_s^2} \sqrt{-g_{00}(r_s)g_{11}(r_s)}. \quad (4.99)$$

Using the form (4.93) of our finite-temperature metric in the string frame we finally obtain

$$F_{\text{drag}} = -\frac{e^{2A_s(r_s)} \sqrt{f(r_s)}}{2\pi\ell_s^2} = -\frac{e^{2A(r_s)} \sqrt{f(r_s)} \lambda(r_s)^{4/3}}{2\pi\ell_s^2}, \quad (4.100)$$

where in the second equality we expressed the force in terms of the Einstein-frame scale factor and the “running” dilaton. Substituting from (4.94) we obtain

$$F_{\text{drag}} = -\frac{v e^{2A_s(r_s)}}{2\pi\ell_s^2} = -\frac{v e^{2A(r_s)} \lambda(r_s)^{4/3}}{2\pi\ell_s^2}, \quad (4.101)$$

Before proceeding further, we will evaluate the drag force for the conformal case of $\mathcal{N} = 4$ SYM where

$$e^{A_s} = \frac{\ell}{r}, \quad v^2 = f(r_s) = 1 - (\pi T r_s)^4, \quad \frac{\ell^2}{\ell_s^2} = \sqrt{\lambda}. \quad (4.102)$$

Substituting in (4.101) we obtain [17–20]

$$F_{\text{conf}} = \frac{\pi}{2} \sqrt{\lambda} T^2 \frac{v}{\sqrt{1 - v^2}}. \quad (4.103)$$

Moving on to YM, to compute the drag force from (4.101) we must first determine ℓ_s in the IHQCD model. In this setup there is no analog of the $\mathcal{N} = 4$ SYM relation (4.102) between ℓ , ℓ_s and λ . Rather, the fundamental string length ℓ_s is determined in a bottom-up fashion, by matching the effective string tension to the QCD string tension σ_c derived from the lattice calculations. We obtain

$$\sigma_c = \frac{1}{2\pi\ell_s^2} e^{2A_{s,o}(r_*)} = \frac{1}{2\pi\ell_s^2} e^{2A_o(r_*)} \lambda_o(r_*)^{4/3}, \quad (4.104)$$

where r_* is the point where the zero-temperature string scale factor (at $T = 0$) $A_{s,o}(r)$ has a minimum. For a typical value of $\sigma_c \sim (440 \text{ MeV})^2$ [78, 79] we find

$$\ell_s = 6.4 \ell, \quad (4.105)$$

where ℓ is the radius of the asymptotic AdS space.

On the other hand, unlike in $\mathcal{N} = 4$ SYM, in the IHQCD model the value of the coupling $\lambda(r_s)$ in (4.101) is not an extra parameter to be fixed by hand, but rather

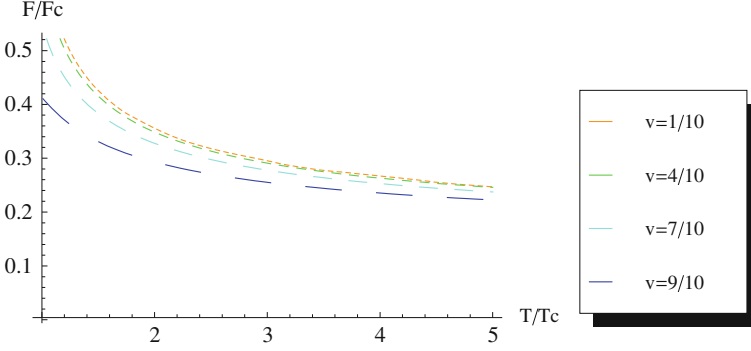


Fig. 4.19 The ratio of the drag force in improved holographic QCD to the drag force in $\mathcal{N} = 4$ SYM is shown. The ratio is computed for different velocities as a function of temperature. The 't Hooft coupling for the $\mathcal{N} = 4$ SYM theory is taken to be 5.5. We chose this value as it is considered in the central region of possible values for the 't Hooft coupling. It is seen that as the velocity increases the value of the ratio decreases

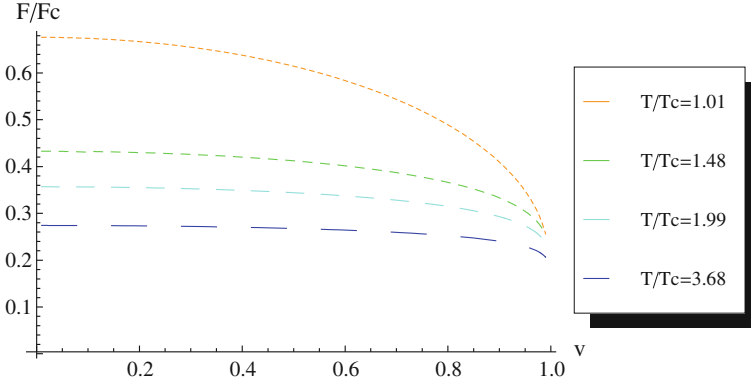


Fig. 4.20 The ratio of the drag force in improved holographic QCD to the drag force in $\mathcal{N} = 4$ SYM is shown. The ratio is computed for different temperatures as a function of velocity. The 't Hooft coupling for the $\mathcal{N} = 4$ SYM theory is taken to be 5.5. As temperature increases the value of the ratio decreases

it is determined dynamically together with the background metric (Figs. 4.19 and 4.20).

4.7.2 The Relativistic Asymptotics

When $v \rightarrow 1$ then $r_s \rightarrow 0$ and we approach the boundary. Near the boundary ($r \rightarrow 0$) we have the following asymptotics of the scale factor and the 't Hooft coupling [23, 24]

$$\begin{aligned}
f(r) &\simeq 1 - \frac{\pi T e^{3A(r_h)}}{\ell^3} r^4 \left[1 + \mathcal{O}\left(\frac{1}{\log(\Lambda r)}\right) \right] + \mathcal{O}(r^8), \\
e^{A(r)} &= \frac{\ell}{r} \left[1 + \mathcal{O}\left(\frac{1}{\log(\Lambda r)}\right) \right] + \dots
\end{aligned} \tag{4.106}$$

and

$$\lambda(r) = -\frac{1}{\beta_0 \log(r\Lambda)} + \mathcal{O}(\log(r\Lambda)^{-2}) \tag{4.107}$$

where r_h is the position of the horizon.

We therefore obtain for the turning point

$$r_s \simeq \left[\frac{\ell^3(1-v^2)}{\pi T e^{3A(r_h)}} \right]^{\frac{1}{4}} \left[1 + \mathcal{O}\left(\frac{1}{\log(1-v^2)}\right) \right], \quad \lambda(r_s) \simeq -\frac{4}{\beta_0 \log[1-v^2]} + \dots \tag{4.108}$$

and the drag force

$$F_{\text{drag}} \simeq -\frac{\sqrt{\pi T \ell b^3(r_h) \lambda^{\frac{8}{3}}(r_s)}}{2\pi \ell_s^2} \frac{v}{\sqrt{1-v^2}} + \dots \tag{4.109}$$

We also use

$$e^{3A(r_h)} = \frac{s(T)}{4\pi M_p^3 N_c^2} = \frac{45\pi \ell^3 s(T)}{N_c^2} \tag{4.110}$$

where $s(T)$ the entropy per unit three-volume, and we write the relativistic asymptotics of the drag force as,

$$\begin{aligned}
F_{\text{drag}} &\simeq -\frac{\sqrt{\pi T \ell b^3(r_h)}}{2\pi \ell_s^2} \frac{v}{\sqrt{1-v^2} \left(-\frac{\beta_0}{4} \log[1-v^2] \right)^{\frac{4}{3}}} + \dots \\
&= -\frac{\ell^2}{\ell_s^2} \sqrt{\frac{45 T s(T)}{4 N_c^2}} \frac{v}{\sqrt{1-v^2} \left(-\frac{\beta_0}{4} \log[1-v^2] \right)^{\frac{4}{3}}} + \dots
\end{aligned} \tag{4.111}$$

The force is proportional to the relativistic momentum combination $v/\sqrt{1-v^2}$ modulo a power of $\log[1-v^2]$. This factor is present because, as argued in [25] the asymptotic metric is AdS in the Einstein frame instead of the string frame. Its effects are not important phenomenologically.

4.7.3 The Non-relativistic Asymptotics

We now consider the opposite limit, $v \rightarrow 0$. In this case the turning point asymptotes to the horizon, $r_s \rightarrow r_h$ and we have the expansion

$$f(r) \simeq 4\pi T(r_h - r) + \mathcal{O}((r_h - r)^2), \quad r_s = r_h - \frac{v^2}{4\pi T} + \mathcal{O}(v^4) \quad (4.112)$$

and

$$\begin{aligned} F_{\text{drag}} &\simeq -\frac{e^{2A(r_h)}\lambda(r_h)^{\frac{4}{3}}}{2\pi\ell_s^2} v \left[1 - \frac{v^2}{2\pi T} A'(r_h) - \frac{v^2}{3\pi T} \frac{\lambda'(r_h)}{\lambda(r_h)} + \mathcal{O}(v^4) \right] \\ &\simeq -\frac{\ell^2}{\ell_s^2} \left(\frac{45\pi s(T)}{N_c^2} \right)^{\frac{2}{3}} \frac{\lambda(r_h)^{\frac{4}{3}}}{2\pi} v + \mathcal{O}(v^3) \end{aligned} \quad (4.113)$$

where primes are derivatives with respect to the conformal coordinate r .

4.7.4 The Diffusion Time

For a heavy quark with mass M_q we may rewrite (4.103) as

$$F_{\text{conf}} \equiv \frac{dp}{dt} = -\frac{1}{\tau} p, \quad p = \frac{M_q v}{\sqrt{1-v^2}} \quad (4.114)$$

where the first equation defines the diffusion time τ . In the conformal case, the diffusion time is constant,

$$\tau_{\text{conf}} = \frac{2M_q}{\pi\sqrt{\lambda} T^2}. \quad (4.115)$$

This is not anymore the case in QCD, where τ defined as above is momentum dependent. We may still define it as in (4.114) in which case we obtain the following limits (Fig. 4.21):

$$\lim_{p \rightarrow \infty} \tau = M_q \frac{\ell_s^2}{\ell^2} \sqrt{\frac{4N_c^2}{45 T s(T)}} \left(\frac{\beta_0}{4} \log \frac{p^2}{M_q^2} \right)^{\frac{4}{3}} + \dots \quad (4.116)$$

$$\lim_{p \rightarrow 0} \tau = M_q \frac{\ell_s^2}{\ell^2} \left(\frac{N_c^2}{45\pi s(T)} \right)^{\frac{2}{3}} \frac{2\pi}{\lambda(r_h)^{\frac{4}{3}}} + \dots \quad (4.117)$$

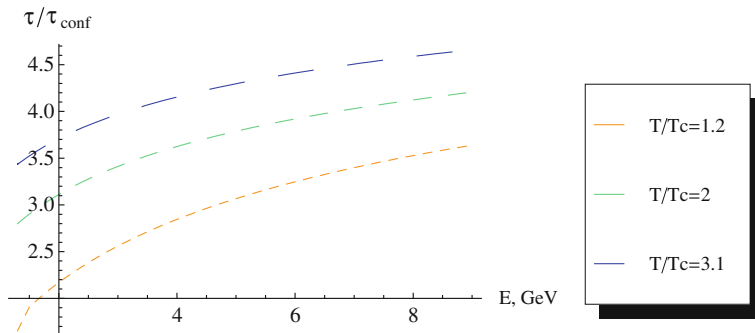


Fig. 4.21 The ratio of the diffusion time in the Improved Holographic QCD model to the diffusion time in $\mathcal{N} = 4$ SYM is shown. The 't Hooft coupling for $\mathcal{N} = 4$ SYM is taken to be $\lambda = 5.5$. The heavy quark has a mass of $M_q = 1.3$ GeV. Note that with the definition of the diffusion time in (4.114) the ratio is the inverse of the ratio of the forces. A similar plot is valid for the bottom quark as well, as the mass drops out of the ratio, although the energy scales are different. In this plot the x-axis is taken to be in MeV units. As temperature increases the ratio also increases

4.7.5 Including the Correction to the Quark Mass

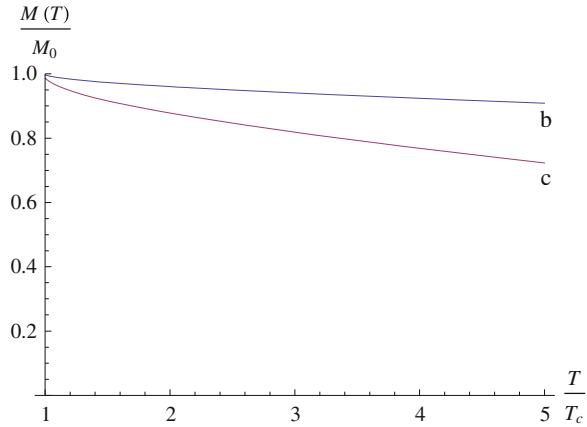
In order to estimate the diffusion time of a quark of finite rest mass, we must take into account the fact that the mass of the quark receives medium-induced corrections. In other words, the mass appearing in (4.114) is a temperature-dependent quantity, $M_q(T) \neq M_q(T=0)$. The ratio $M_q(T)/M_q(0)$ can be estimated holographically by representing a static quark of finite mass by a static, straight string²⁷ stretched along the radial direction starting at a point $r = r_q \neq 0$. At zero temperature, the IR endpoint of the string can be taken as the “confinement” radius, r_* , where the string frame metric reaches its minimum value; At finite temperature, the string ends in the IR at the BH horizon.²⁸ The masses of the quark at zero and finite T are related to the world-sheet action evaluated on the static solution ($\tau = t, \sigma = r$) :

$$M_q(0) = \frac{\ell}{2\pi\ell_s^2} \int_{r_q}^{r_*} dr e^{2A_o(r)} \lambda_o^{4/3}(r), \quad M_q(T) = \frac{\ell}{2\pi\ell_s^2} \int_{r_q}^{r_h} dr e^{2A(r)} \lambda^{4/3}(r). \quad (4.118)$$

²⁷ This representation ignores the fact that the *kinetic* mass of a moving quark may be different from the static mass [17].

²⁸ It would stop at the confinement radius if the latter were closer to the boundary than the horizon, i.e. if $r_*(T) < r_h(T)$. However, in the model we are considering, in the large BH branch we find that the relation $r_h < r_*$ is always satisfied.

Fig. 4.22 Ratios between the thermal mass and the rest mass of the Charm (curve labelled “c”) and Bottom (curve labelled “b”) quarks, as a function of temperature



The value r_q can be fixed numerically by matching $M_q(0)$ to the physical quark mass, and translating the fundamental string tension in physical units by using the relation (4.104), with $\sigma_c = (440 \text{ MeV})^2$. This makes $M_q(T)$ a function of $M_q(0)$. The ratios $M_q(T)/M_q(0)$ we found numerically in the model under consideration is shown in Fig. 4.22 for the Charm ($M(0) = 1.5 \text{ GeV}$) and Bottom ($M(0) = 4.5 \text{ GeV}$) quarks. The fact that, in the deconfined plasma, the quark mass decreases with increasing temperature is a direct consequence of the holographic framework,²⁹ since for higher temperature, the distance to the horizon is smaller. An indication that this result may be in the right direction comes from the lattice computation of the shift in the position of the quarkonium resonance peak at finite temperature [104]: in the deconfined phase the charmonium peak moves to lower mass at higher temperature. Our result for the medium-induced shift in the constituent quark mass is consistent with these observations.

We can now write the diffusion time from (4.101) and (4.114) as:

$$\tau(T, v) = \frac{M_q(T)}{\sigma_c \sqrt{1 - v^2}} \left(\frac{\lambda_o(r_*)}{\lambda(r_s)} \right)^{4/3} e^{2A_o(r_*) - 2A(r_s)}, \quad (4.119)$$

where once again we have eliminated the fundamental string length using (4.104). Given a set of zero- and finite-temperature solutions, (4.119) can be evaluated numerically for different values of the velocity and different quark masses. The results for the Charm ($M_q(0) = 1.5 \text{ GeV}$) and Bottom ($M = 4.5 \text{ GeV}$) quarks are displayed in Fig. 4.23.

²⁹ For a possible field theoretical explanation of this phenomenon, see [103].

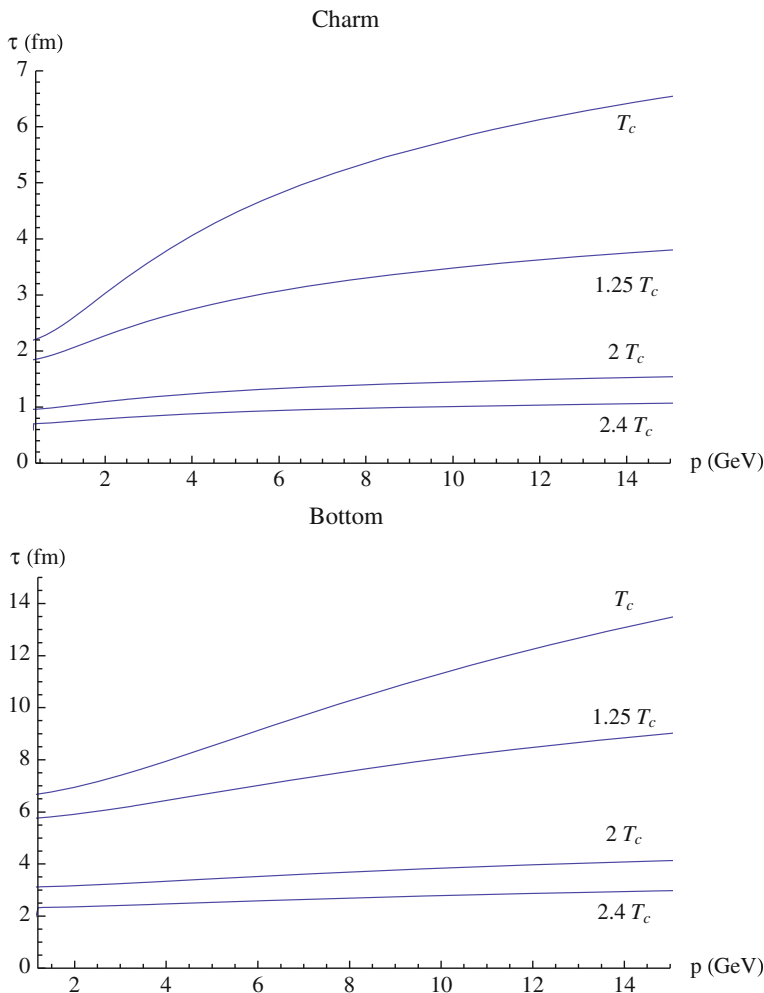


Fig. 4.23 Diffusion time for the Charm and Bottom quarks, as a function of energy, for different ratios of the temperature to the IHQCD transition temperature T_c

4.7.6 Temperature Matching and Diffusion Time Estimates

An important question is how we should choose the temperature in our holographic model in order to compare our results with heavy-ion collision experiments. This is nontrivial, since our setup is designed to describe pure $SU(N_c)$ Yang–Mills, whereas at RHIC temperatures there are three light quark flavors that become relevant. As a consequence, the critical temperatures and the number of degrees of freedom of the two theories are not the same: for pure $SU(N_c)$ Yang Mills we have $N_c^2 - 1$ degrees of freedom and a critical temperature around

260 MeV; For $SU(N_c)$ QCD with N_f flavors the number of degrees of freedom is $N_c^2 - 1 + N_c N_f$, and the transition temperature is lower, around 180 MeV.

In IHQCD, the transition temperature in physical units was calculated to be $T_c = 247$ MeV [32], i.e. close to the lattice result for the pure YM deconfining temperature. From now on, this is the value we will mean when we refer to T_c . This is also close to the temperature of QGP at RHIC, which we will denote T_{QGP} , and is estimated to be around 250 MeV. Since this value is uncertain, below we give our results for a range of temperatures between 200 and 400 MeV. The higher temperatures will be relevant for the LHC heavy-ion collision experiments (see e.g. [105]).

Based on these considerations, there are different ways of fixing the temperature (see e.g. the recent review [51]): one *direct* and two *alternative* schemes (that we call the *energy* and *entropy* scheme).

- *Direct scheme*: The temperature of the holographic model is identified with the temperature of the QGP in the experimental situation (at RHIC or LHC), $T_{\text{ihqcd}}^{(\text{dir})} = T_{\text{QGP}}$.
- *Energy scheme*: One matches the energy densities, rather than the temperatures. The energy density at RHIC is approximately (treating the QCD plasma as a free gas.³⁰) $\epsilon_{\text{QGP}} \simeq (\pi^2/15)(N_c^2 - 1 + N_c N_f)(T_{\text{QGP}})^4$. For $N_c = N_f = 3$, asking that our energy density matches this value requires us to consider the holographic model at temperature $T_{\text{ihqcd}}^{(\epsilon)}$ given by

$$\epsilon_{\text{ihqcd}}(T_{\text{ihqcd}}^{(\epsilon)}) \simeq 11.2(T_{\text{QGP}})^4 \quad (4.120)$$

- *Entropy scheme*: Instead of matching the energy densities, alternatively one can match the entropy density s , which for the QGP, in the free gas approximation, $s_{\text{QGP}} \simeq 4\pi^2/45(N_c^2 - 1 + N_c N_f)(T_{\text{QGP}})^4$. This leads to the identification:

$$s_{\text{ihqcd}}(T_{\text{ihqcd}}^{(s)}) = 14.9(T_{\text{QGP}})^3 \quad (4.121)$$

The temperature translation table between the various schemes is shown in Table 4.3. In that table, $T_c = 247$ MeV is the deconfining temperature of the holographic model.

In Fig. 4.24 we show the comparison between the diffusion times, as a function of initial quark momentum, in the different schemes for the Charm and Bottom quarks, at the temperature $T_{\text{QGP}} = 250$ MeV.

The results for the diffusion times at different temperatures, computed at a reference heavy quark initial momentum $p \approx 10$ GeV, are displayed in Tables 4.4 and 4.5. We see that there is little practical difference between the *entropy* and

³⁰ This is itself an approximation, since as we know both from experiment and in our holographic model, the plasma is strongly coupled up to temperatures of a few T_c .

Table 4.3 Translation table between different temperature identification schemes

T_{QGP} (MeV)	T_{QGP}/T_c	$T_{\text{ihqcd}}^{(\epsilon)}$ (MeV)	$T_{\text{ihqcd}}^{(\epsilon)}/T_c$	$T_{\text{ihqcd}}^{(s)}$ (MeV)	$T_{\text{ihqcd}}^{(s)}/T_c$
190	0.77	259	1.05	274	1.11
220	0.89	290	1.18	302	1.23
250	1.01	325	1.31	335	1.35
280	1.13	361	1.46	368	1.49
310	1.26	398	1.61	402	1.63
340	1.38	434	1.76	437	1.77
370	1.50	471	1.90	472	1.91
400	1.62	508	2.06	507	2.05

The first two columns display temperatures in the direct scheme, (in which the temperature of the holographic model matches the physical QGP temperature) and the corresponding ratio to the IHQCD critical temperature, that was fixed by YM lattice results at $T_c = 247$ MeV [32]. The third and fourth columns display the corresponding temperatures (and respective ratios to T_c) in the energy scheme, and the last two in the entropy scheme

Fig. 4.24 Diffusion times for the Charm and Bottom quarks, as a function of initial momentum, at $T_{\text{QGP}} = 250$ MeV. The different lines represent the in the *direct* scheme (*solid*), *energy* scheme (*dashed*) and *entropy* scheme (*dash-dotted*), all corresponding to the same temperature $T_{\text{QGP}} = 250$ MeV

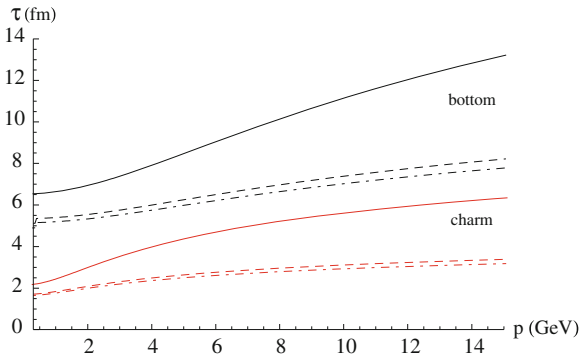


Table 4.4 The diffusion times for the charm quark are shown for different temperatures, in the three different schemes

T_{QGP} , MeV	τ_{charm} (fm/c) (direct)	τ_{charm} (fm/c) (energy)	τ_{charm} (fm/c) (entropy)
220	—	4.0	3.6
250	5.7	3.1	3.0
280	4.3	2.6	2.5
310	3.5	2.1	2.1
340	2.9	1.8	1.8
370	2.5	1.5	1.5
400	2.1	1.3	1.3

Diffusion times have been evaluated with a quark initial momentum fixed at $p \approx 10$ GeV

Table 4.5 Diffusion times for the bottom quark are shown for different temperatures, in the three different schemes

T_{QGP} (MeV)	τ_{bottom} (fm/c) (direct)	τ_{bottom} (fm/c) (energy)	τ_{bottom} (fm/c) (entropy)
220	—	8.9	8.4
250	11.4	7.5	7.1
280	10.1	6.3	6.1
310	8.6	5.4	5.3
340	7.5	4.7	4.7
370	6.6	4.1	4.1
400	5.8	3.6	3.6

Diffusion times have been evaluated with a quark initial momentum fixed at $p \approx 10 \text{ GeV}$

energy schemes; on the other hand the difference between the *direct* scheme and the two alternative schemes can be quite substantial.

4.8 Jet Quenching Parameter

In this section we discuss the jet quenching parameter in the class of holographic models under consideration, and we estimate its numerical value for the concrete model with potential (4.26) and parameters fixed as in [32]. For the holographic computation, we will follow [18, 52]. There is another method available [57], but we will not use it here.

The jet-quenching parameter $\hat{q}$ provides a measure of the dissipation of the plasma and it has been associated to the behavior of a Wilson loop joining two light-like lines. We consider two light-like lines which extend for a distance L^- and are situated distance L apart in a transverse coordinate. Then $\hat{q}$ is given by the large L^+ behavior of the Wilson loop

$$W \sim e^{-\frac{1}{4\sqrt{2}}\hat{q}L^-L^2} . \quad (4.122)$$

We consider the bulk string frame metric

$$ds^2 = e^{2A_s(r)} \left(-f(r)dt^2 + d\vec{x}^2 + \frac{dr^2}{f(r)} \right). \quad (4.123)$$

To address the problem of the Wilson loop we make a change of coordinates to light cone coordinates for the boundary theory

$$x^+ = x_1 + t \quad x^- = x_1 - t \quad (4.124)$$

for which the metric becomes

$$ds^2 = e^{2A_s} \left(dx_2^2 + dx_3^2 + \frac{1}{2}(1-f)(dx_+^2 + dx_-^2) + (1+f)dx_+dx_- + \frac{dr^2}{f} \right). \quad (4.125)$$

The Wilson loop in question stretches across x_2 , and lies at a constant x_+ , x_3 . It is convenient to choose a world-sheet gauge in which

$$x_- = \tau, \quad x_2 = \sigma. \quad (4.126)$$

Then the action of the string stretching between the two lines is given by

$$S = \frac{1}{2\pi\ell_s^2} \int d\sigma d\tau \sqrt{-\det(g_{MN} \partial_x X^M \partial_\beta X^N)} \quad (4.127)$$

and assuming a profile of $r = r(\sigma)$ we obtain

$$S = \frac{L^-}{2\pi\ell_s^2} \int dx_2 e^{2A_s} \sqrt{\frac{(1-f)}{2} \left(1 + \frac{r'^2}{f} \right)}. \quad (4.128)$$

The integrand does not depend explicitly on x_2 , so there is a conserved quantity, c :

$$r' \frac{\partial \mathcal{L}}{\partial r'} - \mathcal{L} = \frac{c}{\sqrt{2}}, \quad (4.129)$$

which leads to

$$r'^2 = f \left(\frac{e^{4A_s}(1-f)}{c^2} - 1 \right). \quad (4.130)$$

A first assessment of this relation involves determining the zeros and the region of positivity of the right-hand side. f is always positive and vanishes at the horizon. For the second factor we need the asymptotics of $e^{4A_s}(1-f)$. This factor remains positive and bounded from below in the interior and up to the horizon. It vanishes however logarithmically near the boundary as

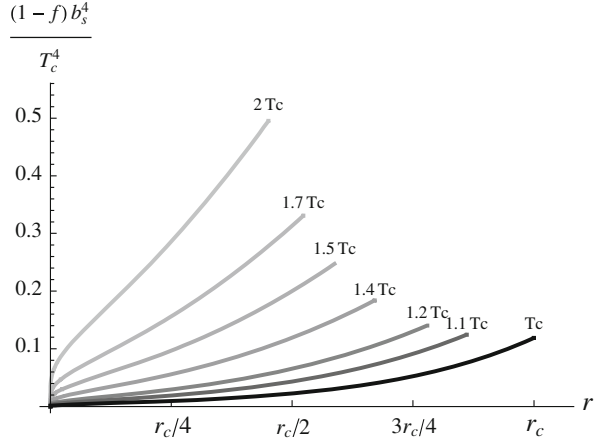
$$e^{4A_s}(1-f) = \pi T \ell e^{3A(r_h)} \left(-\frac{1}{\beta_0 \log(\Lambda r)} \right)^{\frac{8}{3}} \left[1 + \mathcal{O}\left(\frac{1}{\log(\Lambda r)} \right) \right]. \quad (4.131)$$

This is unlike the conformal case where we obtain a constant

$$e^{4A_s}(1-f)|_{\text{conformal}} = (\pi T \ell)^4. \quad (4.132)$$

Because of this, for fixed c , there is a region near the boundary where r'^2 becomes negative. At this stage we will avoid this region, by using a modified boundary at $r = \epsilon$. We will later show that this gymnastics will be irrelevant for

Fig. 4.25 The combination $(1-f)e^{4A_s}$ is plotted as a function of the radial distance, for several temperatures. The radial distance is given in units of the horizon position r_c for the black hole at the critical temperature T_c . All curves stop at the corresponding horizon position



the computation of the jet quenching parameter, as it involves effectively the limit $c \rightarrow 0$ (Fig. 4.25).

We will place the modified boundary $r = \epsilon$ a bit inward from the place $r = r_{\min}$ where the factor $\frac{e^{4A_s}(1-f)}{c^2} - 1$ vanishes:

$$e^{4A_s(r_{\min})}(1-f(r_{\min})) = c^2. \quad (4.133)$$

Therefore we choose $r_{\min} < \epsilon$.

Then, in the range $\epsilon < r < r_h$ the factor $\frac{e^{4A_s}(1-f)}{c^2} - 1$ is positive for sufficiently small c . In this same range, r' vanishes only at $r = r_h$. This is the true turning point of the string world-sheet. This is also what happens in the conformal case. It is also intuitively obvious that the relevant Wilson loop must sample also the region near the horizon.

The constant c is determined by the fact that the two light-like Wilson loops are a $x_2 = L$ distance apart.

$$\frac{L}{2} = \int_{\epsilon}^{r_h} \frac{c dr}{\sqrt{f(e^{4A_s}(1-f) - c^2)}}. \quad (4.134)$$

The denominator vanishes at the turning point. The singularity is integrable.³¹ Therefore, as we are interested in the small L region, it is obvious from the expression above that that c must also be small in the same limit.

This relation can then be expanded in powers of c as

$$\frac{L}{2c} = \int_{\epsilon}^{r_h} \frac{e^{-2A_s} dr}{\sqrt{f(1-f)}} + \frac{c^2}{2} \int_{\epsilon}^{r_h} \frac{e^{-6A_s} dr}{\sqrt{f(1-f)^3}} + \mathcal{O}(c^4). \quad (4.135)$$

³¹ Even if we choose $\epsilon = r_{\min}$, the new singularity at $r = r_{\min}$ is also integrable as suggested from (4.131).

Therefore to leading order in L

$$c = \frac{L}{2 \int_{\epsilon}^{r_h} \frac{e^{-2A_s} dr}{\sqrt{f(1-f)}}} + \mathcal{O}(L^3).. \quad (4.136)$$

We are now ready to evaluate the Nambu–Goto action of the extremal configuration we have found. Starting from (4.128), we substitute r' from (4.130), and change integral variable from $x_2 \rightarrow r$ to obtain

$$S = \frac{2L^-}{2\pi\ell_s^2} \int_{\epsilon}^{r_h} dr \frac{e^{4A_s}(1-f)}{\sqrt{2f(e^{4A_s}(1-f) - c^2)}}. \quad (4.137)$$

As in [18, 52], we subtract from (4.137) the action of two free string straight world-sheets that hang down to the horizon. To compute this action a convenient choice of gauge is $x_- = \tau, r = \sigma$. The action of each sheet is

$$S_0 = \frac{L^-}{2\pi\ell_s^2} \int_{\epsilon}^{r_h} dr \sqrt{g_{-rr}} = \frac{L^-}{2\pi\ell_s^2} \int_{\epsilon}^{r_h} dr e^{2A_s} \sqrt{\frac{1-f}{2f}}. \quad (4.138)$$

The subtracted action is therefore:

$$S_r = S - 2S_0 = \frac{L^- c^2}{2\pi\ell_s^2} \int_{\epsilon}^{r_h} \frac{dr}{e^{2A_s} \sqrt{f(1-f)}} + \mathcal{O}(c^4), \quad (4.139)$$

Using now (4.136) to substitute c we finally obtain

$$S_r = \frac{L^- L^2}{8\pi\ell_s^2} \frac{1}{\int_{\epsilon}^{r_h} \frac{dr}{e^{2A_s} \sqrt{f(1-f)}}} + \mathcal{O}(L^4). \quad (4.140)$$

So far we have evaluated the relevant Wilson loop in the fundamental representation (by using probe quarks). On the other hand, the Wilson loop that defines the jet-quenching parameter is an adjoint one. We can obtain it in the large- N_c limit from the fundamental using $tr_{\text{Adjoint}} = tr_{\text{Fundamental}}^2$. We finally extract the jet-quenching parameter as

$$\hat{q} = \frac{\sqrt{2}}{\pi\ell_s^2} \frac{1}{\int_{\epsilon}^{r_h} \frac{dr}{e^{2A_s} \sqrt{f(1-f)}}}. \quad (4.141)$$

We are now ready to remove the cutoff. As the integral appearing is now well-defined up to the real boundary $r = 0$ we may rewrite it as

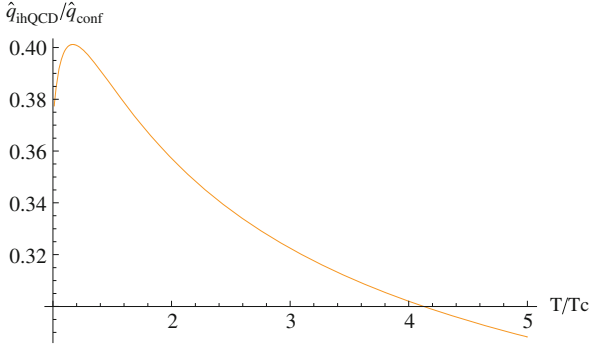


Fig. 4.26 The ratio of the jet quenching parameter in our model to the jet quenching parameter in $\mathcal{N} = 4$ is shown. The integral present in (4.141) has been numerically calculated from an effective cutoff at $r = r_h/1000$. The jet quenching parameter in $\mathcal{N} = 4$ SYM has been calculated with $\lambda_{\text{t Hooft}} = 5.5$

$$\int_{\epsilon}^{r_h} \frac{e^{-2A_s} dr}{\sqrt{f(1-f)}} = \int_0^{r_h} \frac{e^{-2A_s} dr}{\sqrt{f(1-f)}} - I(\epsilon), \quad I(\epsilon) = \int_0^{\epsilon} \frac{e^{-2A_s} dr}{\sqrt{f(1-f)}}. \quad (4.142)$$

In [93] we obtain the small ϵ estimate of $I(\epsilon)$ that vanishes as $\sim \epsilon(\log \epsilon)^{\frac{4}{3}}$. $\sim \epsilon(\log \epsilon)^{\frac{4}{3}}$ We may finally write³²

$$\hat{q} = \frac{\sqrt{2}}{\pi \ell_s^2} \int_0^{r_h} \frac{dr}{e^{2A_s} \sqrt{f(1-f)}}. \quad (4.143)$$

From (4.143) we obtain, in the conformal case:

$$\hat{q}_{\text{conformal}} = \frac{\Gamma\left[\frac{3}{4}\right]}{\Gamma\left[\frac{5}{4}\right]} \sqrt{2\lambda} \pi^{\frac{3}{2}} T^3. \quad (4.144)$$

The conformal value, for the median value of $\lambda = 5.5$ and $T \simeq 250 \text{ MeV}$ gives $\hat{q}_{\text{conformal}} \simeq 1.95 \text{ GeV}^2/\text{fm}$ where we used the conversion $1 \text{ GeV} \simeq 5 \text{ fm}^{-1}$.

Numerical evaluation of (4.143) in the non-conformal IHQCD setup³³ gives us a value of $\hat{q}$ which is lower (at a given temperature) than the conformal value, as shown in Figs. 4.26, 4.27 and 4.28. Tables 4.6, 4.7, 4.8, and 4.9 display the numerical values of the jet quenching parameter at different temperatures in the experimentally relevant range, in different temperature matching schemes.

³² In practise, the previous discussion including regularizing the UV is academic. The numerical calculation is done with a finite cutoff where the boundary conditions for the couplings are imposed.

³³ In this case, the value of ℓ_s appearing in (4.143) is fixed as explained in Sect. 4.4.

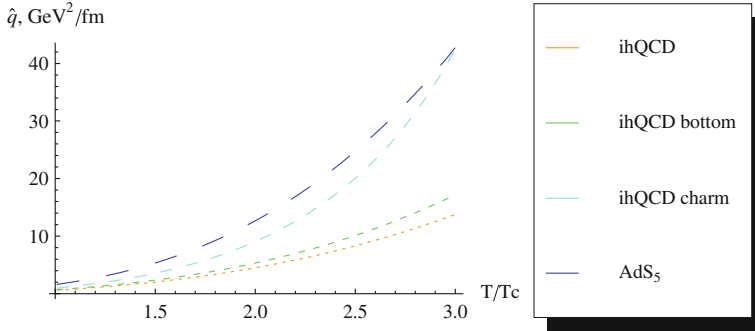


Fig. 4.27 The jet quenching parameter $\hat{q}$ for the Improved Holographic QCD model and $\mathcal{N} = 4$ SYM is shown in units of GeV^2/fm for a region close to $T = T_c$. The *smallest dashed curve* is the ihQCD result with an effective cutoff of $r_{\text{cutoff}} = r_h/1000$. The *small dashed curve* is the ihQCD result with the cutoff from the mass of the Bottom quark. The *medium dashed curve* has a cutoff coming from the Charm mass and *largest dashed curve* is the $\mathcal{N} = 4$ SYM result

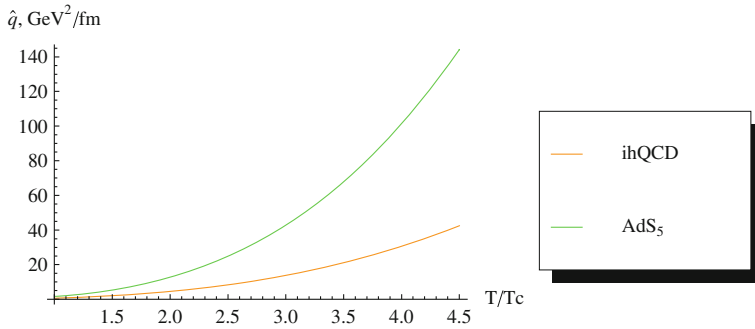


Fig. 4.28 The jet quenching parameter $\hat{q}$ for the Improved Holographic QCD model (*lower curve*) and $\mathcal{N} = 4$ SYM (*upper curve*) are shown in units of GeV^2/fm for temperatures up to $T = 4T_c$

Table 4.6 The jet quenching parameter $\hat{q}$ computed with different cutoffs for the different temperatures shown in the first column

$T_{\text{QGP}}, \text{MeV}$	$\hat{q}(\text{GeV}^2/\text{fm})$ (direct)	$\hat{q}_1(\text{GeV}^2/\text{fm})$ (direct)
220	—	—
250	0.5	0.6
280	0.8	0.8
310	1.1	1.1
340	1.4	1.4
370	1.8	1.8
400	2.2	2.2

The computation is done in the direct scheme. The second column shows $\hat{q}$ with a cutoff at $r_{\text{cutoff}} = r_h/1000$, where r_h is the location of the horizon. In accordance with the conclusions of appendix $\hat{q}$ does not change significantly as we vary the cutoff from $r_h/1000$ to $r_h/100$

Table 4.7 The jet quenching parameter $\hat{q}$ using the three different comparison schemes

$T_{\text{QGP}}, \text{MeV}$	$\hat{q}(\text{GeV}^2/\text{fm})$ (direct)	$\hat{q}(\text{GeV}^2/\text{fm})$ (energy)	$\hat{q}(\text{GeV}^2/\text{fm})$ (entropy)
220	–	0.9	1.0
250	0.5	1.2	1.3
280	0.8	1.6	1.7
310	1.1	2.1	2.2
340	1.4	2.7	2.8
370	1.8	3.4	3.4
400	2.2	4.2	4.2

For lower temperatures the “entropy scheme” gives higher values. As energy is increased the energy and entropy schemes temperatures start to coincide and there is little difference in the jet quenching parameter as well

Table 4.8 The jet quenching parameter $\hat{q}$ using the three different comparison schemes with an effective cutoff provided by the mass of the Charm quark

$T_{\text{QGP}}, \text{MeV}$	$\hat{q}_{\text{charm}}(\text{GeV}^2/\text{fm})$ (direct)	$\hat{q}_{\text{charm}}(\text{GeV}^2/\text{fm})$ (energy)	$\hat{q}_{\text{charm}}(\text{GeV}^2/\text{fm})$ (entropy)
220	–	1.3	1.5
250	0.8	1.8	2.0
280	1.2	2.6	2.8
310	1.7	3.5	3.6
340	2.2	4.6	4.7
370	2.8	5.9	6.0
400	3.6	7.6	7.5

Again, for lower temperatures the “entropy scheme” gives higher values. As energy is increased the energy and entropy schemes temperatures start to coincide and there is little difference in the jet quenching parameter as well. Also when the temperature approaches the quark mass the picture of the heavy quark as a hanging string collapses and results are not reliable

Table 4.9 The jet quenching parameter $\hat{q}$ using the three different comparison schemes with an effective cutoff provided by the mass of the Bottom quark

$T_{\text{QGP}}, \text{MeV}$	$\hat{q}_{\text{bottom}}(\text{GeV}^2/\text{fm})$ (direct)	$\hat{q}_{\text{bottom}}(\text{GeV}^2/\text{fm})$ (energy)	$\hat{q}_{\text{bottom}}(\text{GeV}^2/\text{fm})$ (entropy)
220	–	1.0	1.1
250	0.6	1.4	1.5
280	0.9	1.9	2.0
310	1.2	2.5	2.6
340	1.6	3.2	3.2
370	2.0	4.0	4.0
400	2.5	5.0	4.9

The results are close to the $\hat{q}$ results computed in Table 4.7 since the mass of the Bottom quark is much larger than the temperatures we examine

4.9 Discussion and Outlook

The construction presented in this paper offers a holographic description of large- N_c Yang–Mills theory that is both realistic and calculable, and in quite good agreement with a large number of lattice results both at zero and finite temperature.

It is a phenomenological model; as such it is not directly associated to an explicit string theory construction. In this respect it is in the same class as the models based on pure AdS backgrounds (with hard or soft walls) [14–16, 106, 107], or on IR deformations of the AdS metric [108, 109]. In comparison to them however, the IHQCD approach has the advantage that the dynamics responsible for strong coupling phenomena (such as confinement and phase transitions) is made explicit in the bulk description, and it is tied to the fact that the coupling constant depends on the energy scale and becomes large in the IR. This makes the model consistent and calculable, once the 5D effective action is specified: the dynamics can be entirely derived from the bulk Einstein’s equation. The emergence of an IR mass scale and the finite temperature phase structure are built-in: they need not be imposed by hand and do not suffer from ambiguities related to IR boundary conditions (as in hard wall models) or from inconsistencies in the laws of thermodynamics (as in non-dynamical soft wall models based on a fixed dilaton profile [16] or on a fixed metric [108, 109]). More specifically, in this approach it is guaranteed that the Bekenstein–Hawking temperature of the black hole matches the entropy computed as the derivative of the free energy with respect to temperature.

With an appropriate choice of the potential, a realistic and quantitatively accurate description of essentially all the static properties (spectrum and equilibrium thermodynamics) of the dynamics of pure Yang–Mills can be provided. The main ingredient responsible for the dynamics (the dilaton potential) is fixed through comparison with both perturbative QCD and lattice results. It is worth stressing that such a matching on the quantitative level was only possible because the class of holographic models we discuss *generically* provides a *qualitatively accurate* description of the strong Yang–Mills dynamics. This is a highly non-trivial fact, that strongly indicates that a realistic holographic description of real-world QCD might be ultimately possible.

Although the asymptotics of our potential is dictated by general principles, we base our choice of parameters by comparing with the lattice results for the thermodynamics. There are other physical parameters in the 5D description that do not appear in the potential: the 5D Planck scale, that was fixed by matching the free field thermodynamics in the limit $T \rightarrow \infty$; the coefficients in the axion kinetic term, that were set by matching the axial glueball spectrum and the topological susceptibility (from the lattice). The quantities that we use as input in our fit, as well as the corresponding parameters in the 5D model, are shown in the upper half of Table 4.2.

The fact that our potential has effectively two free parameters depends on our choice of the functional form. This functional form contains some degree of

arbitrariness, in that only the UV and IR asymptotics of $V(\lambda)$ are fixed by general considerations (matching the perturbative β -function in the UV, and a discrete linear glueball spectrum for the IR). Therefore the results presented in this paper offer more a *description*, rather than a *prediction* of the thermodynamics.

Nevertheless, there are several quantities that we successfully “postdict” (i.e. they agree with the lattice results) once the potential is fixed: apart from the good agreement of the thermodynamic functions over the whole range of temperature explored by the lattice studies (see Figs. 4.5, 4.7 and 4.9), they are the lowest glueball mass ratios and the value of the critical temperature. The comparison of these quantities with the lattice results is shown in the lower half of Table 4.2, and one can see that the agreement is overall very good. Moreover the model predicts the masses of the full towers of glueball states in the $0^{+-}, 0^{++}, 2^{++}$ families.

The fact that IHQCD is consistent with a large number of lattice results is clearly not the end of the story: its added value, and one of the main reasons for its interest lies in its immediate applicability beyond equilibrium thermodynamics, i.e. in the dynamic regimes tested in heavy-ion collision experiments. This is a generic feature of the holographic approach, in which there are no obstructions (as opposed to the lattice) to perform real-time computations and to calculate hydrodynamics and transport coefficients. IHQCD provides a framework to compute these quantities in a case where the static properties agree with the real-world QCD at the quantitative level. Therefore the bulk viscosity, drag force and jet quenching parameters were computed in IHQCD.

4.9.1 Bulk Viscosity

The bulk viscosity was computed by calculating the low frequency asymptotics of the appropriate stress tensor correlator holographically. The result is that the bulk viscosity rises near the phase transition but stays always below the shear viscosity. It floats somewhat above the Buchel bound, with a coefficient of proportionality varying between 1 and 2. Therefore it is expected to affect the elliptic flow at the small percentage level [44, 98]. Knowledge of the bulk viscosity is important in extracting the shear viscosity from the data. This result is not in agreement with the lattice result near T_c . In particular the lattice result gives a value for the viscosity that is ten times larger.

The bulk viscosity keeps increasing in the black-hole branch below the transition point until the large BH turns into the small BH at a temperature $T_{\min}$. The bulk viscosity on the small BH background is always larger than the respective one in the large BH background. In particular, it can be shown that the T derivative of the quantity ζ/s diverges at $T_{\min}$. This is the holographic reason for the presence of a peak in ζ/s near T_c . On the other hand, as it is shown in [29], the presence of $T_{\min}$ (i.e. a small BH branch) is in one-to-one correspondence with color confinement at zero T. We arrive thus at the suggestion that in a (large N) gauge theory that confines color at zero T, there shall be a rise in ζ/s near T_c .

An important ingredient here is the value of the viscosity asymptotically in the small BH branch. There the asymptotic value correlates to the IR behavior of the potential. Taking also into account the fact that this asymptotic value is very close to the value of the bulk viscosity near T_c , we can derive bounds that suggest that the bulk viscosity cannot increase a lot near T_c .

4.9.2 Drag Force

The drag force calculated from IHQCD has the expected behavior. Although it increases with temperature, it does so slower than in $\mathcal{N} = 4$ SYM, signaling the effects of asymptotic freedom.

4.9.3 Diffusion Time

Based on the drag force calculation the diffusion times can be computed for a heavy external quark. The numerical values obtained are in agreement with phenomenological models [50]. To accommodate for the fact that IHQCD exhibits a phase transition around $T = 247$ MeV (i.e. about 30% higher than in QCD), the results are compared using alternative schemes, as proposed in [19, 20]. For example, for an external Charm quark of momentum $p = 10$ GeV (in the alternative scheme) a diffusion time of $\tau = 2.6$ fm at temperature $T = 280$ MeV is found. Similarly, for a Bottom quark of the same momentum and at the same temperature, $\tau = 6.3$ fm. Generally the numbers obtained are close to those obtained by [50] and [54].

4.9.4 Jet Quenching

The jet quenching parameter of this model, has been also calculated, based on the formalism of [18, 52] by computing the appropriate light-like Wilson loop. $\hat{q}$ grows with temperature, but slower than the T^3 growth of $\mathcal{N} = 4$ SYM result. Again this can be attributed to the incorporation of asymptotic freedom in IHQCD. Using the alternative scheme to compare with experiment, the results are close to the lower quoted values of $\hat{q}$. For example, for a temperature of $T = 290$ MeV, which in the alternative “energy scheme” corresponds to a temperature of $T = 395$ MeV in our model, we find that $\hat{q} \approx 2 \text{ GeV}^2/\text{fm}$.

However, the numbers obtained for this particular definition of jet quenching parameter seem rather low and indicate that this may not be the most appropriate definition in the holographic context. There are other ways to define $\hat{q}$, in particular

using the fluctuations of the trailing string solution. This gives a direct and more detailed input in the associated Langevin dynamics and captures the asymmetry between longitudinal and transverse fluctuations. It would be interesting to compute this, along the lines set in [57, 59, 60].

Acknowledgements We would like to thank the numerous colleagues that have shared their insights with us via discussions and correspondence since 2007 that this line of research has started: O. Aharony, L. Alvarez-Gaumé, B. Bringoltz, R. Brower, M. Cacciari, J. Casalderrey-Solana, R. Emparan, F. Ferrari, B. Fiol, P. de Forcrand, R. Granier de Cassagnac, L. Giusti, S. Gubser, K. Hashimoto, T. Hertog, U. Heinz, D. K. Hong, G. Horowitz, E. Iancu, K. Intriligator, K. Kajantie, F. Karsch, D. Kharzeev, D. Kutasov, H. Liu, B. Lucini, M. Luscher, J. Mas, D. Mateos, H. B. Meyer, C. Morningstar, V. Niarchos, C. Nunez, Y. Oz, H. Panagopoulos, S. Pal, M. Panero, I. Papadimitriou, A. Paredes, A. Parnachev, G. Policastro, S. Pufu, K. Rajagopal, F. Rocha, P. Romatchke, C. Salgado, F. Sannino, M. Shifman, E. Shuryak, S. J. Sin, C. Skenderis, D. T. Son, J. Sonnenschein, S. Sugimoto, M. Taylor, M. Teper, J. Troost, A. Tseytlin, A. Vainshtein, G. Veneziano, A. Vladikas, L. Yaffe, A. Yarom and U. Wiedemann. This work was partially supported by a European Union grant FP7-REGPOT-2008-1-CreteHEPCosmo-228644, and ANR grant NT05-1-41861. Work of LB has been partly funded by INFN, Ecole Polytechnique (UMR du CNRS 7644), MEC and FEDER under grant FPA2008-01838, by the Spanish Consolider-Ingenio 2010 Programme CPAN (CSD2007-00042) and by Xunta de Galicia (Consellería de Educación and grant PGIDIT06PXIB206185PR). E. K. thanks the organizers and especially E. Papantonopoulos for organizing a very interesting and stimulating school.

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Chapter 5

The Dynamics of Quark-Gluon Plasma and AdS/CFT

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Abstract In these pedagogical lectures, we present the techniques of the AdS/CFT correspondence which can be applied to the study of real time dynamics of a strongly coupled plasma system. These methods are based on solving gravitational Einstein's equations on the string/gravity side of the AdS/CFT correspondence. We illustrate these techniques with applications to the boost-invariant expansion of a plasma system. We emphasize the common underlying AdS/CFT description both in the large proper time regime where hydrodynamic dynamics dominates, and in the small proper time regime where the dynamics is far from equilibrium. These AdS/CFT methods provide a fascinating arena interrelating General Relativity phenomena with strongly coupled gauge theory physics.

5.1 Introduction

The current experimental program of heavy-ion collisions at RHIC and the forthcoming experiments at LHC open an interesting window onto properties of QCD matter at high temperatures, where it appears in the guise of a new phase—the quark-gluon plasma. At asymptotically high temperatures it should be a free gas of quarks and gluons, however, at the experimentally accessible energies there are strong indications that the quark-gluon plasma is indeed a strongly coupled system (see e.g. [1]).

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This poses numerous problems for its theoretical description, yet at the same time makes its study theoretically interesting. One can roughly differentiate the physical properties of the quark-gluon plasma system into two broad categories—static and dynamic (real-time) properties.

The first of these, the static properties, typically involve the study of equilibrium thermodynamics, the entropy, energy density as a function of temperature and more generally properties which can be directly deduced from the Euclidean formulation of finite temperature gauge theory. In this case lattice QCD is an effective tool for accessing these properties in the nonperturbative, strongly coupled regime. It deals directly with QCD and yields quantitative results directly applicable for the QCD quark-gluon plasma.

The second class, the real time dynamic properties of strongly coupled plasma are much more difficult to access. They have to be formulated directly in Minkowski space and since lattice QCD methods are inherently Euclidean, it is very difficult to extrapolate numerical results to Minkowski signature. Moreover, it is exactly these kind of properties which are particularly relevant for the quark-gluon plasma produced in heavy-ion collisions.

To this end let us recall schematically the basic stages of a heavy-ion collision. First the two ultrarelativistic nuclei collide and the plasma is produced in a state very far from equilibrium. Then in a relatively very short time, it becomes thermalized (or at least the pressure becomes isotropic with the residual anisotropy wholly due to flow). From that point on, hydrodynamic phenomenological models seem to describe the properties of the expanding plasma quite well [2, 3]. In particular the plasma expands and cools, and when the temperature reaches the confinement/deconfinement phase transition one expects hadronization to occur.

It would be interesting to understand these various stages of the plasma dynamics directly from first principles. For example, one would like to *derive* the hydrodynamic behaviour from some theoretical framework and not only use it as a phenomenological model. But what is even more interesting is the understanding of the process of thermalization and what governs the short thermalization time necessary for the applicability of hydrodynamic models.

Unfortunately, for the case of QCD we lack appropriate theoretical methods which would be applicable to these kinds of problems at strong coupling. A possible route that one may take is to consider an analogous set of problems in a different theory for which appropriate real-time nonperturbative tools exist.

The new method for studying nonperturbatively various gauge theories (although not directly QCD) is the AdS/CFT correspondence [4–7], which translates dynamical problems in strongly coupled gauge theory into (usually) gravitational ones in higher number of dimensions. Its main advantage is that it works equally well in Minkowski as well as in Euclidean signature. In these lectures we will consider the AdS/CFT correspondence in its simplest original setting for the maximally supersymmetric $\mathcal{N} = 4$ Super-Yang–Mills theory. Of course, one has to be aware that the cost of switching the theory of interest from QCD to $\mathcal{N} = 4$

SYM is that we may most probably lose direct quantitative applicability of our results to realistic heavy-ion collisions. Moreover, there are certain marked differences between the theories which will cause a huge difference for certain physical phenomena, at the same time being unimportant for other questions. We will discuss some of these points in these lectures.

Nevertheless, we would like to point out that currently we do not have *any* gauge theory in which we would have a theoretical understanding of the issues described earlier. Therefore it is very interesting to study these issues for the case of $\mathcal{N} = 4$ SYM and use the results as a point of reference for analyzing the situation in QCD. Later, one could try to generalize these results to more complicated versions of the AdS/CFT correspondence for theories closer to QCD. In essence, this motivation is in line with the statement that *$\mathcal{N} = 4$ SYM is the harmonic oscillator of four dimensional gauge theories*. If one tries to develop some theoretical tools, one should better first apply them to the ‘harmonic oscillator’.

We have tried to make these lectures very pedagogical and self-contained. Our main emphasis in the presentation is to show how one can use the AdS/CFT correspondence as a tool even for far from equilibrium configurations without presupposing any kind of dynamics (which are in fact not known for nonlinear far from equilibrium systems). Therefore our presentation of hydrodynamics is subordinate to this goal, especially as a very general discussion focused on hydrodynamics *per se* is contained in the lectures by Hubeny at this school [8].

The plan of these lectures is as follows. In [Sect. 5.2](#), we introduce the AdS/CFT correspondence, in [Sect. 5.3](#) we compare some properties of plasma in the $\mathcal{N} = 4$ theory and in QCD. Then we proceed to present the AdS/CFT setup specialized to the study of time-dependent dynamics of strongly coupled plasma. We then illustrate these methods in [Sect. 5.5](#) by discussing two important examples—the appearance from this setup of the standard planar AdS black hole, and a planar shock wave. In this section we also discuss some subtleties arising with different choices of coordinate systems which will be relevant later. Then in [Sect. 5.6](#), we introduce the main physical example of a time-dependent plasma configuration—the boost invariant flow. In [Sect. 5.7](#), we analyze its large proper time asymptotics and show how nonlinear perfect fluid dynamics arises from the AdS/CFT methods. In [Sect. 5.8](#), we show how one can see first corrections coming from shear viscosity, and in [Sect. 5.9](#), for completeness, we will summarize briefly the current status of hydrodynamics in AdS/CFT. Then in [Sect. 5.10](#), we introduce physical situations, where the plasma dynamics is not describable by hydrodynamics, and finally, in [Sect. 5.11](#), we apply the AdS/CFT methods to study boost invariant flow in the far from equilibrium small proper time regime. We close the lectures with conclusions and an appendix with a short guide to the literature regarding work done on related topics which were not covered here.

5.2 The AdS/CFT Correspondence

The AdS/CFT correspondence [4–7], in its original form states the equivalence of two apparently completely different theories: the $\mathcal{N} = 4$ supersymmetric Yang–Mills theory (SYM) in four dimensions and type IIB superstrings in a ten-dimensional curved $AdS_5 \times S^5$ background. Since then, it has been generalized in various directions, extending it to a wider class of gauge theories, at the cost of making the dual string backgrounds more complicated. Throughout these lectures we will stay within the context of the AdS/CFT correspondence for $\mathcal{N} = 4$ SYM theory.

The reason why the AdS/CFT correspondence is so interesting is that the nonperturbative strong coupling regime of the $\mathcal{N} = 4$ gauge theory is mapped to the (semi-)classical strings or just (super)gravity which, in contrast to the gauge theory side, is at least theoretically tractable. Therefore one can use the AdS/CFT correspondence as a new method for accessing the very difficult nonperturbative gauge theory physics.

The AdS/CFT correspondence is an equivalence, so in principle any state/phenomenon on the gauge theory side should have its direct counterpart on the string side and vice-versa. However one should keep in mind that the correspondence is an equivalence of gauge and *string* theory, so the dual counterpart does not have to be in the well understood (super)gravity sector. Fortunately, it will turn out that for the questions considered in these lectures namely the study of the dynamics of plasma expansion, the dual description will be purely gravitational.

Apart from the direct physical interest, the AdS/CFT correspondence is also theoretically very interesting as it translates various dynamical gauge theory questions into a geometrical language described by higher-dimensional General Relativity (GR). This leads to quite fascinating links between the two fields, providing a whole range of physically motivated interesting questions which could be addressed by GR methods. In the other direction, various notions introduced by the GR community like dynamical apparent horizons find their application and new interpretation on the gauge theory side.

5.2.1 Effective Degrees of Freedom at Strong Coupling

As an illustration of the use of the AdS/CFT correspondence, and as a justification for the gravitational methods let us consider the question of finding effective degrees of freedom for strongly coupled $\mathcal{N} = 4$ SYM. By the AdS/CFT equivalence, it amounts to asking the same question for superstrings in $AdS_5 \times S^5$.

Let us first recall the case of closed strings in flat space. The string worldsheet action is characterized by a dimensionfull parameter α' (related to the string tension). The various vibrational modes of the string correspond to particles (fields) with distinct masses

$$m_n^2 = \frac{n^2}{\alpha'} \quad (5.1)$$

The massless modes correspond to the graviton (and its whole supergravity multiplet). There is also an infinite tower of massive modes. In the limit of $\alpha' \rightarrow 0$, the massive modes become very heavy and effectively decouple at fixed energies leaving as the governing dynamics just (super)gravity.

In the case of strings in $AdS_5 \times S^5$, the α' parameter becomes proportional to $1/\sqrt{\lambda}$ where $\lambda = g_{YM}^2 N_c$ is the 't Hooft coupling of the dual $\mathcal{N} = 4$ gauge theory. The vibrational modes again split into a massless (super)graviton multiplet and a set of massive modes. The formula (5.1) is no longer exact, but the parametric behaviour with α' still holds. So in the strong coupling limit, those massive string modes become very heavy and effectively decouple leaving essentially supergravity modes as the effective degrees of freedom. Since the AdS/CFT correspondence postulates an equivalence with gauge theory, these should also correspond to the effective degrees of freedom of the gauge theory at strong coupling.

Once we lower the coupling, the massive modes become lighter and their effects will no longer be negligible. Initially, their effects may be absorbed into corrections to the gravitational Einstein–Hilbert action (so-called α' corrections), but at low coupling corresponding to the perturbative regime the spacetime description is not known.

Finally, let us note that the $\mathcal{N} = 4$ SYM theory is quite special in that it allows for such a clean separation between gravity modes and massive string modes. Presumably a dual description of real QCD (or even of large N_c pure YM) would not have such a property.

5.3 Why Study $\mathcal{N} = 4$ Plasma?

Since we will be using the AdS/CFT correspondence for $\mathcal{N} = 4$ SYM as a calculational tool for analyzing strongly coupled dynamics of gauge theory plasma, we will be essentially considering plasma in the supersymmetric $\mathcal{N} = 4$ gauge theory. This theory is completely different from QCD at zero temperature. It is supersymmetric, exactly conformal, does not have confinement. However once we turn on some nonzero temperature (or consider not the vacuum but some appropriate state), supersymmetry is broken and temperature (or energy density) introduces a scale. So qualitatively, we may expect to have similarities with QCD plasma in a certain window of temperatures where it is strongly coupled, approximately conformal and (by definition) deconfined.

However we have to keep in mind some definite differences w.r.t. QCD plasma. Firstly, the $\mathcal{N} = 4$ theory has no running coupling so, in contrast to QCD, even at very high temperatures/energy densities the coupling may remain large. Secondly,

the equation of state of the $\mathcal{N} = 4$ plasma is exactly conformal ($E = 3p$) which is only an approximation for a certain range of temperatures for QCD plasma. Indeed we know, from lattice QCD, that deviations from a conformal equation of state are important close to T_c . In addition, we have other consequences of conformality like that the bulk viscosity for the $\mathcal{N} = 4$ theory is exactly zero. Thirdly, for the $\mathcal{N} = 4$ theory (on Minkowski spacetime) there is no phase transition—no analog of the confinement/deconfinement phase transition of QCD. Therefore as the plasma expands and cools, in the $\mathcal{N} = 4$ theory it will expand indefinitely, while in QCD it will cool down to the phase transition temperature and hadronize.

From the above discussion we see that the applicability of using $\mathcal{N} = 4$ plasma to model real world phenomena depends on the questions asked. It may give a good qualitative picture for the range of temperatures where we have similarities. However, let us note that in this theory we may compute the dynamics from ‘first principles’ (using the AdS/CFT correspondence). For QCD, unfortunately, we do not have any similar calculational technique, even numerical, which would enable us to study real-time dynamics of the strongly coupled quark-gluon plasma. Therefore it is interesting to build up results on strong coupling properties of $\mathcal{N} = 4$ plasma and use them as a point of departure for analyzing or describing QCD plasma. Eventually, one might consider more realistic theories with AdS/CFT duals which are closer to QCD. In those cases generically the dual gravitational backgrounds are much more complicated so it is advantageous to start from the simplest setting for the $\mathcal{N} = 4$ SYM theory.

Another motivation for studying the dynamics of $\mathcal{N} = 4$ plasma is that the natural language of the AdS/CFT correspondence is quite new w.r.t. conventional gauge theory methods. So by studying relatively simple examples we may build up some new physical intuitions within this novel language. Also the interrelations with General Relativity physics are fascinating from the purely theoretical point of view. Last but not least, there may be some unexpected discoveries like the celebrated universality of the shear viscosity to entropy ratio η/s [9], which, at strong coupling, remains equal to $1/4\pi$ for *any* theory with a gravitational dual [10, 11] (see the lectures by Starinets at this school [12]).

5.4 The AdS/CFT Setup for Studying Real-Time Dynamics of Plasma

Let us now briefly review the AdS/CFT correspondence on a more technical level, concentrating on the features relevant to the study of the time evolution of a plasma system.

The S^5 factor in the $AdS_5 \times S^5$ background is associated with a global $SO(6) = SU(4)$ symmetry of the $\mathcal{N} = 4$ theory. In the following, we will only consider systems which do not break this symmetry, so the S^5 factor will be irrelevant and the whole dynamics will be concentrated in the AdS_5 factor.

The five-dimensional Anti-de-Sitter spacetime AdS_5 can be given by the following metric

$$ds^2 = \frac{\eta_{\mu\nu} dx^\mu dx^\nu + dz^2}{z^2} \quad (5.2)$$

with $z \geq 0$. $z = 0$ is the boundary of AdS_5 , while the region $z > 0$ is often called ‘the bulk’. This choice of coordinates covers the Poincare patch of global AdS_5 and is relevant for the case when the dual gauge theory lives in $\mathbb{R}^{1,3}$ Minkowski spacetime. The above geometry can be understood to correspond to the gauge theory vacuum state. In particular gauge theory operators like the energy-momentum tensor $T_{\mu\nu}$ all have vanishing expectation values in this state

$$\langle T_{\mu\nu} \rangle = 0 \quad (5.3)$$

Let us recall that the AdS/CFT correspondence states the equivalence with superstrings in $AdS_5(\times S^5)$. So, on the string side, we may excite any normalizable mode, in particular we may excite gravitons. This will correspond to some states in $\mathcal{N} = 4$ SYM with $\langle T_{\mu\nu} \rangle \neq 0$. We expect a configuration of $\mathcal{N} = 4$ plasma to be a very complicated state which would correspond to exciting very many gravitons. Then it is better to interpret it instead as a change of the background:

$$ds^2 = g_{\alpha\beta}^{5D} dx^\alpha dx^\beta = \frac{g_{\mu\nu}(x^\rho, z) dx^\mu dx^\nu + dz^2}{z^2} \quad (5.4)$$

with the metric coefficients being now generic functions of all the five coordinates. We therefore seek to describe a plasma configuration in terms of the geometry $g_{\mu\nu}(x^\rho, z)$. Let us note that the above choice of the metric (5.4) is always possible after a suitable change of coordinates. Such coordinates, in which the metric has the form (5.4) are called Fefferman–Graham coordinates.

The geometry (5.4) cannot be, however, completely arbitrary. It must form a consistent background for strings, so it must satisfy five-dimensional Einstein’s equations with a negative cosmological constant¹:

$$R_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta}^{5D} R - 6 g_{\alpha\beta}^{5D} = 0 \quad (5.5)$$

Furthermore, for a physical state in the gauge theory this geometry should not have a naked singularity. This turns out to be a crucial requirement with far reaching consequences for the resulting dynamics of the $\mathcal{N} = 4$ plasma, as we will see in the following.

¹ These equations are equivalent to the original ten-dimensional type IIB supergravity equations when we preserve full $SO(6)$ symmetry of S^5 and no other fields are turned on. The negative cosmological constant is a remnant of the RR five-form under this dimensional reduction.

5.4.1 The Gravity $\longrightarrow \langle T_{\mu\nu} \rangle$ Dictionary

Once one has the geometry (5.4) corresponding to some plasma configuration, the key question is what is the energy momentum tensor of that gauge theory system. The way to derive the answer, *holographic renormalization*, has been explained in the lectures by Skenderis [13]. Here we just summarize the outcome derived in [14], which in the Fefferman–Graham coordinates defined by (5.4) takes a particularly simple form.

Suppose that the metric coefficients $g_{\mu\nu}(x^\rho, z)$ have the following Taylor expansion near the boundary²

$$g_{\mu\nu}(x^\rho, z) = \eta_{\mu\nu} + z^4 g_{\mu\nu}^{(4)}(x^\rho) + \dots \quad (5.6)$$

Then the expectation value of the energy-momentum tensor is

$$\langle T_{\mu\nu}(x^\rho) \rangle = \frac{N_c^2}{2\pi^2} \cdot g_{\mu\nu}^{(4)}(x^\rho) \quad (5.7)$$

The spacetime dependence of the energy-momentum tensor carries a lot of information about the dynamics of the plasma—its energy density, momentum flow, stress tensor. Indeed, it is just this information which is exactly the direct outcome of hydrodynamic simulations of realistic heavy-ion collisions. Finally, let us emphasize, to avoid any chance of confusion, that the $T_{\mu\nu}$ is the energy-momentum tensor of the dual gauge theory. On the gravity side we are always dealing with *vacuum* Einstein’s equations.

The construction outlined above leads to the following scenario of investigating a plasma system in strongly coupled $\mathcal{N} = 4$ SYM. One starts from some initial conditions for the five-dimensional Einstein’s equations. Then the geometry is evolved forward in time by solving Einstein’s equations. Finally using the above formula (5.7), one extracts the $\langle T_{\mu\nu} \rangle$ of the corresponding plasma system. The details of the evolution of $T_{\mu\nu}$ are very interesting from the point of view of physics, especially in the far from equilibrium region, where very little is known about thermalization/isotropisation and transition to a hydrodynamic expansion. We will follow this route in Sect. 5.11.

5.4.2 The $\langle T_{\mu\nu} \rangle \longrightarrow$ Gravity Dictionary

It turns out to be very fruitful to ask also the opposite question. Suppose that we are given a certain spacetime profile of the energy momentum tensor $\langle T_{\mu\nu} \rangle$ —how

² Here we always assume that the gauge theory lives in flat Minkowski space, hence the leading $\eta_{\mu\nu}$ and the absence of a z^2 term (see [14] for details).

to construct the dual five-dimensional geometry? The prescription is really just running the preceding recipe backwards. One has to solve Einstein's equations

$$R_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta}^{5D} R - 6 g_{\alpha\beta}^{5D} = 0 \quad (5.8)$$

with the boundary condition

$$g_{\mu\nu}(x^\rho, z) = \eta_{\mu\nu} + z^4 g_{\mu\nu}^{(4)}(x^\rho) + \dots \quad (5.9)$$

where $g_{\mu\nu}^{(4)}(x^\rho)$ is related to $\langle T_{\mu\nu}(x^\rho) \rangle$ through

$$g_{\mu\nu}^{(4)}(x^\rho) = \frac{2\pi^2}{N_c^2} \langle T_{\mu\nu}(x^\rho) \rangle \quad (5.10)$$

It turns out that for a solution to exist, $g_{\mu\nu}^{(4)}(x^\rho)$ has to be traceless and conserved, which is of course physically expected for an energy-momentum tensor in a conformal theory. However here it is just an independent consequence of five-dimensional Einstein's equations.

Once such a $g_{\mu\nu}^{(4)}(x^\rho)$ is chosen, Einstein's equations determine uniquely the solution in the bulk (at least locally i.e. all higher coefficients of the Taylor expansion of $g_{\mu\nu}(x^\rho, z)$ around $z = 0$ are uniquely determined).

In this way we see that for every energy momentum profile which does not violate the standard requirements of energy-momentum conservation and tracelessness we may construct a dual gravity background. However generically, such a geometry will be highly singular with naked singularities in the bulk. The requirement of the absence of naked singularities will very strongly constrain the admissible bulk geometries and hence also the possible spacetime profiles of the energy momentum tensor. This is a nontrivial constraint on the dynamics of the gauge theory as the spacetime profile includes of course the time evolution of $T_{\mu\nu}$.

This line of reasoning was introduced in [15] as a way of determining the possible evolution of the energy momentum tensor. One first picks a family of profiles $T_{\mu\nu}(x^\rho)$, then one constructs for each of them a dual geometry by solving Einstein's equations with appropriate boundary conditions. Finally, one picks the allowed dynamics by requiring that the corresponding dual geometry is nonsingular. We will describe this procedure in detail in the first part of these lectures.

5.5 Exact Analytical Examples

In this section we will illustrate the AdS/CFT methods by analyzing two simple examples of plasma configurations for which the dual gravitational background can be computed in closed form [15]. Both of these examples have also a clear physical interpretation.

5.5.1 A Case Study: Static Uniform Plasma

Let us first consider the simplest configuration of plasma, namely with a uniform and static distribution of energy density filling up the whole spacetime. The energy momentum tensor is just a constant diagonal one:

$$T_{\mu\nu} = \begin{pmatrix} E & 0 & 0 & 0 \\ 0 & p & 0 & 0 \\ 0 & 0 & p & 0 \\ 0 & 0 & 0 & p \end{pmatrix} \quad (5.11)$$

with $E = 3p$. In order to find the dual gravity background, we have to solve Einstein's equations with the boundary conditions given by (5.9) and (5.10). Due to the fact that the energy momentum tensor is constant, the metric will only depend on the z coordinate and the Einstein's equations reduce to ordinary differential equations which can be solved explicitly. The result is

$$ds^2 = -\frac{(1 - z^4/z_0^4)^2}{(1 + z^4/z_0^4)z^2} dt^2 + (1 + z^4/z_0^4) \frac{dx_i^2}{z^2} + \frac{dz^2}{z^2} \quad (5.12)$$

where the parameter z_0 is related to E through

$$E = \frac{3N_c^2}{2\pi^2 z_0^4} \quad (5.13)$$

Although it is not evident at first glance, the geometry (5.12) is exactly the standard *AdS planar black hole* [16], but written in the Fefferman–Graham system of coordinates. We will give the explicit form of the coordinate transformation to the standard AdS Schwarzschild form shortly.

The fact that the dual geometry turns out to be a black hole has significant implications for the physics. Let us note that we did not *assume* that a black hole would appear. It came, in a unique way, from solving Einstein's equations with appropriate boundary conditions.

The parameter z_0 appearing in (5.12) is the location of the black hole horizon. The Hawking temperature T_H which is given by

$$T_H = \frac{\sqrt{2}}{\pi z_0} \quad (5.14)$$

is identified with the gauge theory temperature. This may be most easily seen by computing the Hawking temperature through a Wick rotation of the metric (5.12) and requiring the absence of a conical singularity at $z = z_0$. This requirement leads to a specific periodicity condition for the Euclidean time coordinate which is inversely proportional to the Hawking temperature. But, according to the AdS/CFT correspondence the metric induced on the boundary $z = 0$ (up to an overall rescaling by z^2) is exactly the metric of the (now Euclidean) gauge theory.

Thus the gauge theory also has a compactified Euclidean time with the radius given by the same temperature [16].

Another gravitational concept which carries over to the dual gauge theory is the Bekenstein–Hawking entropy which is identified with the entropy of the dual gauge theory plasma system. In this case, the entropy per spatial three-volume is

$$s = \frac{1}{4G_N} \text{Area} = \frac{N_c^2}{2\pi} \left(\frac{\sqrt{2}}{z_0} \right)^3 \quad (5.15)$$

Now we can use the relation between the horizon parameter z_0 and temperature to express the result completely in terms of gauge theoretical quantities

$$s = \frac{1}{2} N_c^2 \pi^2 T^3 \quad (5.16)$$

Finally, we may use the relation between the energy density E and z_0 , and the link with temperature to express the energy density as a function of T . We get

$$E = \frac{3}{8} N_c^2 \pi^2 T^4 \quad (5.17)$$

The nontrivial factor here is the numerical coefficient, which is *different* from the corresponding one for the free massless gas (Stefan–Boltzmann). Similarly, the entropy density derived earlier is $3/4$ of the free gas answer [17]. This mismatch is quite natural since here we are dealing with a strongly coupled plasma. In fact similar deviations from the Stefan–Boltzmann answer have been observed in lattice studies of QCD thermodynamics above the confinement/deconfinement phase transition.

Before we move on to discuss various systems of coordinates for this geometry, let us note that it is exactly this geometry which is used to describe $\mathcal{N} = 4$ SYM at fixed nonzero temperature T . This interpretation is obvious from the above mentioned Euclidean continuation, but can also be understood directly in Minkowski signature, where a link with the real-time formalism of finite-temperature QFT appears [18–23].

5.5.1.1 Various Coordinate Systems

The geometry (5.12) has been presented in the Fefferman–Graham coordinates, in which the connection to the gauge theory energy-momentum tensor is simplest. However these coordinates have also some significant drawbacks, of which one has to be aware.

Let us first perform a coordinate transformation to bring the metric (5.12) into the standard AdS Schwarzschild form. To this end set

$$z_{\text{std}} = \frac{z}{\sqrt{1 + z^4/z_0^4}} \quad (5.18)$$

Then, the metric becomes

$$ds^2 = -\frac{1 - z_{\text{std}}^4/\tilde{z}_0^4}{z_{\text{std}}^2} dt^2 + \frac{dx_i^2}{z_{\text{std}}^2} + \frac{1}{1 - z_{\text{std}}^4/\tilde{z}_0^4} \frac{dz^2}{z_{\text{std}}^2} \quad (5.19)$$

with $\tilde{z}_0 = z_0/\sqrt{2}$. Looking at the transformation of coordinates (5.18), we see that the Fefferman–Graham coordinates cover only the region between the boundary and the horizon. Even if one would analytically continue the metric for $z > z_0$ one does not go beyond the horizon but rather returns back to the boundary.

The standard Schwarzschild coordinates also break down at the horizon and, in order to have explicit regularity at the horizon, it is convenient to introduce yet another system of coordinates—the (ingoing) Eddington–Finkelstein coordinates.

These coordinates may be obtained from the standard Schwarzschild ones by redefining the time coordinate:

$$t_{EF} = t - \frac{1}{4}\tilde{z}_0 \left(2 \arctan \frac{z_{\text{std}}}{\tilde{z}_0} + \log \frac{\tilde{z}_0 + z_{\text{std}}}{\tilde{z}_0 - z_{\text{std}}} \right) \quad (5.20)$$

The metric becomes then

$$ds^2 = -\frac{1 - z_{\text{std}}^4/\tilde{z}_0^4}{z_{\text{std}}^2} dt_{EF}^2 + 2 \frac{dt_{EF} dz_{\text{std}}}{z_{\text{std}}^2} + \frac{dx_i^2}{z_{\text{std}}^2} \quad (5.21)$$

The crucial advantage of these coordinates is that the horizon is a perfectly regular point and one can enter the region inside the horizon. The lines $x^\mu = \text{const}^\mu$ are null geodesics falling into the black hole and reaching the singularity at $z_{\text{std}} = \infty$. These coordinates were used extensively in V. Hubeny’s lectures at this school with z_{std} substituted by $r = 1/z_{\text{std}}$, which brings them to the canonical form.

Finally let us note that in the formula (5.20), the time coordinate gets an infinite shift at the horizon. In the case of the static black hole geometry this is completely harmless as the metric is time-independent, however for the time dependent geometries which will be the focus of these lectures this shift will give rise to some spurious singularities in the Fefferman–Graham treatment (fortunately appearing only at NNNLO in the large proper time expansion of the geometry).

Some comments are in order here. Of course in General Relativity nothing depends on the choice of coordinate system. This is true if we are dealing with an exact solution of Einstein’s equations—we may analyze it in any coordinate system we like. However if we perform an expansion in time in some coordinate system and deal with approximate solutions truncated at some order, we may get spurious singularities like in Fefferman–Graham at third order.

In these lectures we will nevertheless present the analysis in Fefferman–Graham coordinates (apart from a brief review of the Eddington–Finkelstein approach of Bhattacharyya et al. [24] in Sect. 5.9) The general formulation in Eddington–Finkelstein is considered in detail in the lectures of V. Hubeny, and has been applied to the boost invariant setting in [25–27]. One motivation for this choice of presentation is that our main focus is in reaching the small proper time regime,

where we deal with *exact* solutions of the Einstein's equations and hence do not need to worry about these subtleties. Also there, the analysis of initial conditions in Fefferman–Graham coordinates is simpler.

5.5.2 A Case Study: A Planar Shock Wave

Another case of a gauge theory energy-momentum tensor for which the dual geometry is exactly solvable is a planar shockwave concentrated on the boundary. The only nonvanishing component of $T_{\mu\nu}$ is

$$T_{--} = \mu f(x^-) \quad (5.22)$$

Such a configuration, for $f(x^-) = \mu\delta(x^-)$ represents a planar shock wave of gauge theory matter moving at the speed of light along one light cone direction. It may be understood to represent an analog of an ultrarelativistic nuclei. Then the dual metric is found to be

$$ds^2 = \frac{dx^+ dx^- + f(x^-) z^4 dx^{-2} + dx_\perp^2 + dz^2}{z^2} \quad (5.23)$$

This configuration was proposed in [15], being the simplest member of a family of shock wave solutions with $x_\perp$ dependence derived in [28] (see also [29]). A natural question to consider is a collision of two such shock waves, one propagating along the x^- light cone direction, the other along x^+ . There have been some preliminary investigation along these lines in [30–34]. However, a complete analysis remains to be done. Finally, let us note that this kind of shock wave is *sourceless* in the bulk, in contrast to the shock waves considered recently in [35] and in the lectures of Yarom [36]. For references on work done on these other kinds of shock waves consult [36].

5.6 Boost-Invariant Flow

Let us now concentrate on a concrete evolving plasma system. Since eventually we would like to solve exactly Einstein's equations, one has to introduce as much symmetries as possible to reduce the complexity of the task, at the same time allowing for nontrivial physics to intervene. A natural choice in the context of heavy ion collisions is the requirement of longitudinal boost invariance. This assumption was introduced by Bjorken [37] back in 1983 to model ultrarelativistic collisions. Basically, the motivation is that at infinite energy, a finite boost along the collision axis would not modify the physics. This is certainly not an ideal approximation, however it is used in basically all hydrodynamic codes for modelling relativistic heavy-ion collisions at RHIC [2, 3]. We will make here an

additional assumption that there is no dependence on the transverse coordinates, which corresponds to the limit of infinitely large nuclei. This is not really necessary for discussing the hydrodynamic limit (see [8, 24]) but will be essential in the far from equilibrium regime of small proper times.

When assuming boost invariance, it is natural to pass to proper-time/spacetime rapidity coordinates (τ, y, x_1, x_2)

$$t = \tau \cosh y \quad x_3 = \tau \sinh y \quad (5.24)$$

Then it turns out that the only non-vanishing components of the energy-momentum tensor are $T_{\tau\tau}$, T_{yy} and $T_{xx} \equiv T_{x_1x_1} = T_{x_2x_2}$. Moreover, these components become functions of τ alone.

We should now impose tracelessness $T^\mu_\mu = 0$ and conservation of energy momentum $T^\mu_{;\nu} = 0$ conditions, which take the form

$$\begin{aligned} -T_{\tau\tau} + \frac{1}{\tau^2} T_{yy} + 2T_{xx} &= 0 \\ \tau \frac{d}{d\tau} T_{\tau\tau} + T_{\tau\tau} + \frac{1}{\tau^2} T_{yy} &= 0 \end{aligned}$$

These equations determine $T_{\mu\nu}$ uniquely in terms of a single function $\varepsilon(\tau)$

$$T_{\mu\nu} = \begin{pmatrix} \varepsilon(\tau) & 0 & 0 & 0 \\ 0 & -\tau^3 \frac{d}{d\tau} \varepsilon(\tau) - \tau^2 \varepsilon(\tau) & 0 & 0 \\ 0 & 0 & \varepsilon(\tau) + \frac{1}{2} \tau \frac{d}{d\tau} \varepsilon(\tau) & 0 \\ 0 & 0 & 0 & \varepsilon(\tau) + \frac{1}{2} \tau \frac{d}{d\tau} \varepsilon(\tau) \end{pmatrix} \quad (5.25)$$

The remaining function $\varepsilon(\tau)$ can be interpreted as the energy density of the plasma at mid-rapidity (i.e. at $x_3 = 0$) as a function of (proper-) time.

Let us note that the above decomposition was purely ‘kinematical’—valid in *any* conformal 4D theory at any coupling. The determination of $\varepsilon(\tau)$ will be an issue of understanding the dynamics of the theory of interest—here $\mathcal{N} = 4$ SYM. In particular, weak coupling perturbative considerations [38] lead to the free streaming behaviour

$$\varepsilon(\tau) \sim \frac{1}{\tau} \quad (5.26)$$

If, on the other hand, we would suppose that the plasma system behaves as a perfect fluid, then on top of the decomposition (5.25) we would impose

$$T^{\mu\nu} = (\varepsilon + p)u^\mu u^\nu - p\eta^{\mu\nu} \quad (5.27)$$

with $\varepsilon = 3p$. By our symmetry assumptions $u^\mu = (1, 0, 0, 0)$ and we get in particular $p = \frac{1}{\tau^2} T_{yy} = T_{xx}$, which gives a differential equation for $\varepsilon(\tau)$

$$-\tau \frac{d}{d\tau} \varepsilon(\tau) - \varepsilon(\tau) = \varepsilon(\tau) + \frac{1}{2} \tau \frac{d}{d\tau} \varepsilon(\tau) \quad (5.28)$$

with the celebrated Bjorken solution

$$\varepsilon(\tau) = \frac{\text{const.}}{\tau^{\frac{4}{3}}} \quad (5.29)$$

Other dynamical assumptions would modify the functional form of $\varepsilon(\tau)$. E.g. if the fluid would not be a perfect fluid but would have a nonzero viscosity (proportional to $T^3 \sim \varepsilon^{3/4}$ as should be the case for a conformal theory), then (5.29) would no longer be exact but would have corrections starting with

$$\varepsilon(\tau) = \frac{1}{\tau^{\frac{4}{3}}} \left(1 - \frac{2\eta_0}{\tau^{\frac{4}{3}}} + \dots \right) \quad (5.30)$$

with η_0 related to the shear viscosity. Here we set a single dimensional scale to unity. It can be easily reinstated in all terms by dimensional analysis. Further $1/\tau^{4/3}$ corrections are uniquely determined in terms of η_0 . If the dynamics would follow 2nd order viscous hydrodynamics, these $1/\tau^{4/3}$ corrections would be *different* and would involve additional, second order transport coefficients.

So we see that the knowledge of $\varepsilon(\tau)$ contains a lot of information on the dynamics of plasma. In the rest of these lectures our goal will be to deduce what $\varepsilon(\tau)$ is singled out by the AdS/CFT correspondence. Initially we will concentrate on the large τ asymptotics of $\varepsilon(\tau)$ and then, in Sect. 5.11, move to consider the behaviour of $\varepsilon(\tau)$ for small τ .

5.7 Large Proper Time Behaviour

Let us first concentrate on the large τ asymptotics of $\varepsilon(\tau)$ and consider determining the exponent s in

$$\varepsilon(\tau) \sim \frac{1}{\tau^s} \quad \text{for } \tau \rightarrow \infty \quad (5.31)$$

We will follow here the strategy outlined in Sect. 5.4, and construct, for each s , the dual geometry. Then we will check for which s the dual geometry is nonsingular. This condition will determine s . This approach was proposed in [15], where more details may be found. But first, let us narrow down the range of possible s . We will demand that the energy density in any reference frame is non-negative i.e.

$$T_{\mu\nu} t^\mu t^\nu \geq 0 \quad (5.32)$$

for any time like four-vector t^μ . This leads to the inequalities

$$\varepsilon(\tau) \geq 0 \quad \varepsilon'(\tau) \leq 0 \quad \tau \varepsilon'(\tau) \geq -4\varepsilon(\tau) \quad (5.33)$$

In particular, for (5.31), we obtain that $0 \leq s \leq 4$.

5.7.1 The AdS/CFT Analysis

Now we have to construct the dual geometry to a plasma configuration with the energy density behaving like (5.31). Since the geometry will have the same symmetries as assumed for the plasma system, we are led to the following ansatz

$$ds^2 = \frac{1}{z^2} \left(-e^{a(z,\tau)} d\tau^2 + e^{b(z,\tau)} \tau^2 dy^2 + e^{c(z,\tau)} dx_\perp^2 \right) + \frac{dz^2}{z^2} \quad (5.34)$$

Again, let us reiterate that the above ansatz is completely general. At this stage the choice of the Fefferman–Graham system of coordinates is perfectly legitimate.

According to the approach explained in Sect. 5.4, we have to solve Einstein's equations

$$R_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta}^{5D} R - 6 g_{\alpha\beta}^{5D} = 0 \quad (5.35)$$

with the boundary conditions

$$a(z, \tau) = -z^4 \varepsilon(\tau) + z^6 a_6(\tau) + z^8 a_8(\tau) + \dots \quad (5.36)$$

It is instructive to find the explicit form of the first few coefficients in the above Taylor series.³ Using a computer algebra system we obtain

$$\begin{aligned} a(\tau, z) = & -\varepsilon(\tau) \cdot z^4 + \left\{ -\frac{\varepsilon'(\tau)}{4\tau} - \frac{\varepsilon''(\tau)}{12} \right\} \cdot z^6 + \left\{ \frac{1}{6} \varepsilon(\tau)^2 + \frac{1}{6} \tau \varepsilon'(\tau) \varepsilon(\tau) + \right. \\ & \left. + \frac{1}{16} \tau^2 \varepsilon'(\tau)^2 + \frac{\varepsilon'(\tau)}{128\tau^3} - \frac{\varepsilon''(\tau)}{128\tau^2} - \frac{\varepsilon^{(3)}(\tau)}{64\tau} - \frac{1}{384} \varepsilon^{(4)}(\tau) \right\} \cdot z^8 + \dots \end{aligned} \quad (5.37)$$

Let us now specialize to the case of interest $\varepsilon(\tau) = 1/\tau^s$. We get

$$\begin{aligned} & -z^4 \tau^{-s} + z^6 \left(\frac{1}{6} \tau^{-s-2} s - \frac{1}{12} \tau^{-s-2} s^2 \right) \\ & + z^8 \left(-\frac{1}{16} \tau^{-2s} s^2 - \frac{1}{6} \tau^{-2s} + \frac{1}{6} \tau^{-2s} s + \frac{1}{96} \tau^{-s-4} s^2 - \frac{1}{384} \tau^{-s-4} s^4 \right) + \dots \end{aligned}$$

where the terms dominant for large τ are outlined in bold. Looking at the above formula and analyzing a couple of higher order terms we may convince ourselves that the dominant terms in $a_n(\tau)$ for large τ will be of the form

³ This is an *exact* result without any approximation.

$$z^n a_n(\tau) \sim \frac{z^n}{\tau^{\frac{n}{4}}} = \left(\frac{z}{\tau^{\frac{1}{4}}}\right)^n \quad \text{for large } \tau \quad (5.38)$$

This shows that it is natural to introduce a scaling variable

$$v \equiv \frac{z}{\tau^{\frac{1}{4}}} \quad (5.39)$$

Consequently, the metric coefficients will have an expansion of the form

$$a(z, \tau) = a_0(v) + \frac{1}{\tau^{\#}} a_1(v) + \cdots \quad (5.40)$$

Several comments are in order here. Firstly, the appearance of the scaling variable at late times is a dynamical consequence of the structure of Einstein's equations. We will find later, that for small proper times a similar structure will *not* appear.⁴ Secondly, the separation of dynamics into a scaling variable and an expansion in inverse powers of τ corresponds to a gradient expansion (cf. [24] and the lecture by V. Hubeny [8]). Finally, the appearance of a scaling variable reduces the very complicated Einstein's equations to a system of nonlinear *ordinary* differential equations:

$$\begin{aligned} v(2a'(v)c'(v) + a'(v)b'(v) + 2b'(v)c'(v)) - 6a'(v) - 6b'(v) - 12c'(v) + vc'(v)^2 &= 0 \\ 3vc'(v)^2 + vb'(v)^2 + 2vb''(v) + 4vc''(v) - 6b'(v) - 12c'(v) + 2vb'(v)c'(v) &= 0 \\ 2vsa''(v) + 2sb'(v) + 8a'(v) - vsa'(v)b'(v) - 8b'(v) + vsb'(v)^2 &= 0 \\ + 4vsc''(v) + 4sc'(v) - 2vsa'(v)c'(v) + 2vsc'(v)^2 &= 0 \end{aligned}$$

which can be solved exactly. The solution is

$$\begin{aligned} a(v) &= A(v) - 2m(v) \\ b(v) &= A(v) + (2s - 2)m(v) \\ c(v) &= A(v) + (2 - s)m(v) \end{aligned}$$

where

$$\begin{aligned} A(v) &= \frac{1}{2} (\log(1 + \Delta(s) v^4) + \log(1 - \Delta(s) v^4)) \\ m(v) &= \frac{1}{4\Delta(s)} (\log(1 + \Delta(s) v^4) - \log(1 - \Delta(s) v^4)) \end{aligned}$$

with

$$\Delta(s) = \sqrt{\frac{3s^2 - 8s + 8}{24}}$$

⁴ For the case $\varepsilon(\tau) \rightarrow \text{const.}$ as $\tau \rightarrow 0$.

Now we can analyze the singularities of the above geometry. We see that there is a potential singularity where the argument of the logarithm vanishes. Of course, it may well be a coordinate singularity, so we have to evaluate a curvature invariant like

$$\mathfrak{R} = R^{\mu\nu\alpha\beta} R_{\mu\nu\alpha\beta} \quad (5.41)$$

Moreover, since our geometry is only exact in the scaling limit, we have to evaluate $\mathfrak{R}$ in the same limit i.e. $\tau \rightarrow \infty$, $z \rightarrow \infty$, keeping the ratio $v = \frac{z}{\tau^{1/4}}$ fixed. The resulting expression is quite complicated and can be found in [15]. Its general structure is

$$\mathfrak{R} = \frac{\text{Numerator}(v; s)}{\left(1 - \Delta(s)^2 v^8\right)^4} \quad (5.42)$$

We see that for generic s , there is a 4th order pole singularity in the curvature. It turns out that this singularity gets cancelled by the numerator *only* for a single value of s :

$$s = \frac{4}{3} \quad (5.43)$$

which is just the asymptotic scaling characteristic of perfect fluid hydrodynamics. In this way we see that nonlinear perfect fluid hydrodynamics arises at late stages of plasma expansion as a consequence of the AdS/CFT correspondence.

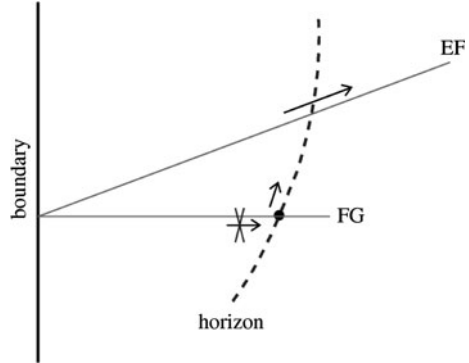
Note that in this way we do not approach the ‘horizon’ directly, but approach it asymptotically along a ‘parallel’ trajectory. An analogous analysis using Eddington–Finkelstein coordinates enables us to pass through the ‘horizon’ and require directly the nonsingularity of the metric there. The difference between the two procedures is summarized in Fig. 5.1. Both methods give equivalent results up to two subleading orders in the large proper-time expansion. In order to go beyond that, however, one has to use Eddington–Finkelstein coordinates (currently the only result in this direction beyond 2nd order is in [39] in the boost-invariant setting).

5.7.2 Perfect Fluid Geometry

Let us now examine more closely the dual geometry corresponding to the perfect fluid value of $s = 4/3$. Then the complicated formulas for the metric coefficients involving generically irrational powers and square roots obtained above simplify drastically and we obtain⁵

⁵ We reinstated here a trivial dimensional scale e_0 .

Fig. 5.1 The difference between using Fefferman–Graham and Eddington–Finkelstein coordinates for checking nonsingularity represented by arrows



$$ds^2 = \frac{1}{z^2} \left[-\frac{\left(1 - \frac{e_0}{3} \frac{z^4}{\tau^{4/3}}\right)^2}{1 + \frac{e_0}{3} \frac{z^4}{\tau^{4/3}}} d\tau^2 + \left(1 + \frac{e_0}{3} \frac{z^4}{\tau^{4/3}}\right) (\tau^2 dy^2 + dx_\perp^2) \right] + \frac{dz^2}{z^2} \quad (5.44)$$

This geometry is analogous to a black hole (cf. (5.12)) with the position of the horizon moving into the bulk as

$$z_0 = \sqrt[4]{\frac{3}{e_0}} \cdot \tau^{\frac{1}{3}} \quad (5.45)$$

This has a clear physical interpretation. Recall that for a static black hole, the position of the horizon in the bulk is inversely proportional to the temperature. Thus here we have a dual counterpart of the plasma undergoing cooling during expansion.⁶ Indeed naively generalizing the static formulas leads to

$$T = \frac{\sqrt{2}}{\pi z_0} = \frac{2^{\frac{1}{2}} e_0^{\frac{1}{4}}}{\pi 3^{\frac{1}{4}}} \tau^{-\frac{1}{3}} \quad (5.46)$$

A more indepth analysis of these phenomenae using the framework of event [41] or dynamical [26, 39, 42] horizons has been made, although a complete understanding of the notions of temperature and entropy in the fully dynamical case is still lacking.

5.8 Plasma Dynamics Beyond Perfect Fluid

One of the key discoveries of the AdS/CFT correspondence was the derivation of the universal value of shear viscosity for plasma at finite nonzero temperature.

⁶ The dual counterpart of cooling was first suggested qualitatively in [40].

This was done using linear response theory in [9]. It is thus interesting to examine how do viscous effects manifest themselves in the current nonlinear setting.

Let us first examine the question whether we can see if the perfect fluid dynamics is violated from the dual gravitational point of view. Suppose that it is not and that consequently

$$\varepsilon(\tau) = \frac{1}{\tau^{\frac{4}{3}}} \quad (5.47)$$

is valid at all proper times without any corrections. Then one can find the next orders in the large τ scaling expansion of the metric coefficients

$$a(z, \tau) = a_0(v) + \frac{1}{\tau^{\frac{4}{3}}} a_2(v) + \dots \quad (5.48)$$

This can be done explicitly since, fortunately, the equations for the corrections are linear albeit still quite complicated. Performing this calculation, and computing the curvature $\mathfrak{R}$ up to this order yields

$$\mathfrak{R} = R_{\alpha\beta\gamma\delta} R^{\alpha\beta\gamma\delta} = \underbrace{R_0(v)}_{\text{nonsingular}} + \frac{1}{\tau^{\frac{4}{3}}} \underbrace{R_2(v)}_{\text{singular!}} + \dots \quad (5.49)$$

where the indicated singularity is of the very strong 4th order pole type. This means that the perfect fluid behaviour (5.47) has to be violated.

Let us now be completely agnostic about viscous hydrodynamics and assume a completely generic type of corrections:

$$\varepsilon(\tau) = \frac{1}{\tau^{\frac{4}{3}}} \left(1 - \frac{2A}{\tau^r} \right) \quad (5.50)$$

Solving the Einstein's equations with the appropriate boundary conditions set by (5.50), computing the curvature⁷ yields

$$\mathfrak{R} = R_{\alpha\beta\gamma\delta} R^{\alpha\beta\gamma\delta} = \underbrace{R_0(v)}_{\text{nonsingular}} + \frac{1}{\tau^r} \underbrace{R_1(v)}_{\text{nonsingular}} + \frac{1}{\tau^{2r}} \underbrace{\tilde{R}_2(v)}_{\text{singular!}} + \frac{1}{\tau^{\frac{4}{3}}} \underbrace{R_2(v)}_{\text{singular!}} + \dots \quad (5.51)$$

The last two terms are always singular. The only way that we may obtain bounded curvature is to make those two terms cancel between themselves. This requires

$$r = \frac{2}{3} \quad (5.52)$$

which is exactly the correct scaling for a correction coming from shear viscosity. Moreover, we have to fine tune the coefficient A to [44]

⁷ Various steps of this calculation were done in [43, 44] and [45].

$$A = 2^{-\frac{1}{2}} 3^{-\frac{3}{4}} \quad (5.53)$$

which is the value corresponding to the value of the shear viscosity to entropy ratio

$$\frac{\eta}{s} = \frac{1}{4\pi} \quad (5.54)$$

In this way we reconfirmed, in a fully nonlinear setting [44], the value of shear viscosity computed at fixed temperature [9].

The above analysis can be repeated for the NNLO correction [46] with the final result for $\varepsilon(\tau)$:

$$\varepsilon(\tau) = \frac{1}{\tau^{\frac{4}{3}}} - \frac{2}{2^{\frac{1}{2}} 3^{\frac{3}{4}}} \frac{1}{\tau^2} + \frac{1 + 2 \log 2}{12\sqrt{3}} \frac{1}{\tau^{\frac{8}{3}}} + \dots \quad (5.55)$$

The coefficient of the NNLO term involves 2nd order transport coefficients. It is at this stage that the pathologies of Fefferman–Graham coordinates surface, leading to a leftover logarithmic singularity in the scaling limit of the curvature (appearing at NNNLO in the metric, which is necessary for obtaining the coefficients of $\varepsilon(\tau)$ at one order lower). The singularity was found to be persistent and not associated with truncating other fields of ten dimensional supergravity [47]. Its origin has been explained in detail in [25] (see also [26, 27]) and can be associated with the singular transformation between coordinates regular at the horizon (Eddington–Finkelstein) and the Fefferman–Graham ones, coupled with performing an expansion of the geometry w.r.t. those coordinates. In order to proceed further, which is however rather impractical analytically, one would have to perform the analysis in Eddington–Finkelstein coordinates (a result in this direction, the NNNLO term in $\varepsilon(\tau)$ is given in [39]).

5.9 Interlude: Hydrodynamics Redux

In the above, we have adopted a very agnostic attitude towards the expected dynamics ruling the time evolution of the energy-momentum tensor of the $\mathcal{N} = 4$ plasma system. We did not assume that one could parametrize the $T_{\mu\nu}$ in terms of such quantities as flow velocity and energy/pressure. We started off from the most general $T_{\mu\nu}$ consistent with the assumed symmetries. By proceeding in this way we have an option of describing dynamics which does not fit at all into a hydrodynamical language. This is in fact the main reason for presenting such an approach here, as in the remaining part of the lectures we would like to address the dynamics of boost-invariant plasma at small proper times where we do not expect hydrodynamic description to be a good starting point.

On the other hand such flexibility has also significant drawbacks. The determination of the transport coefficients of hydrodynamics presented above followed by first deriving from AdS/CFT the explicit form of $\varepsilon(\tau)$. Given that $\varepsilon(\tau)$, one

could ascertain that the leading term is a solution of perfect fluid equations of motion, and together with the subleading term is a solution of viscous hydrodynamic equations with a specific value of the shear viscosity (we leave this as an exercise for the reader).

It is thus very interesting to obtain directly the hydrodynamic equations from AdS/CFT without making the passage through explicit solutions. This task was performed in [24] and is presented in detail in the lectures of V. Hubeny at this school [8]. For completeness, let us just summarize here the main idea.

The static black hole geometry presented in Sect. 5.4 is dual to a plasma at rest, which can be described by a flow vector $u^\mu = (1, 0, 0, 0)$ and an energy density E (or equivalently temperature T). By performing a boost one can obtain a dual geometry to a uniformly moving plasma with four-velocity u^μ .

In Eddington–Finkelstein coordinates it is given explicitly as

$$ds^2 = -2u_\mu dx^\mu dr - r^2 \left(1 - \frac{T^4}{\pi^4 r^4} \right) u_\mu u_\nu dx^\mu dx^\nu + r^2 (\eta_{\mu\nu} + u_\mu u_\nu) dx^\mu dx^\nu \quad (5.56)$$

where $r = \infty$ corresponds to the boundary, $r = T/\pi$ is the horizon while $r = 0$ is the position of the singularity ($r = 1/z_{EF}$ cf. Sect. 5.4). This is an exact solution of Einstein's equations. Now promote T and u^μ to slowly varying functions of the boundary Minkowski coordinates. The geometry (5.56) ceases to be a solution of Einstein's equations and has to be corrected by terms proportional to gradients. These correction terms can be determined with the integration constants fixed by the requirement of nonsingularity at the horizon.

Now from the corrected geometry one can read off the $T_{\mu\nu}$ which is explicitly expressed (similarly to the metric) in terms of T , u^μ and the gradients of u^μ . The numerical constants coming from nonsingularity become exactly the transport coefficients. In this way one obtains

$$\begin{aligned} T_{\text{rescaled}}^{\mu\nu} = & \underbrace{(\pi T)^4 (\eta^{\mu\nu} + 4u^\mu u^\nu)}_{\text{perfect fluid}} - \underbrace{2(\pi T)^3 \sigma^{\mu\nu}}_{\text{viscosity}} + \\ & + \underbrace{(\pi T^2) \left(\log 2 T_{2a}^{\mu\nu} + 2T_{2b}^{\mu\nu} + (2 - \log 2) \left(\frac{1}{3} T_{2c}^{\mu\nu} + T_{2d}^{\mu\nu} + T_{2e}^{\mu\nu} \right) \right)}_{\text{second order hydrodynamics}} \end{aligned}$$

The energy-momentum conservation $\partial_\mu T^{\mu\nu} = 0$ of such a $T^{\mu\nu}$ is by definition the hydrodynamic relativistic Navier–Stokes equation.

From the above construction we see that the appearance of hydrodynamics in the AdS/CFT correspondence is now completely understood. For any solution of the hydrodynamic equations $T(x^\rho)$, $u^\mu(x^\rho)$, the formula (5.56) and its correction terms give an explicit dual metric valid to the same order of the derivative expansion. A similar analysis was performed later in the Fefferman–Graham coordinates [48].

One final thing to note is that the starting point in the above construction is the boosted black hole, which means that one assumes that approximately one is

dealing with an energy-momentum tensor of a hydrodynamic type (i.e. parametrizable by a flow velocity and energy density). This does not always need to be the case, as we shall see shortly, and then one has to return to an ab-initio analysis of Einstein's equations of the type presented in [Sect. 5.7](#).

5.10 Plasma Dynamics Beyond Hydrodynamics

The appearance of hydrodynamic behaviour of strongly coupled plasma as a consequence of the AdS/CFT correspondence is certainly very interesting and satisfying theoretically, however perhaps the most fascinating feature of AdS/CFT is its ability to address the behaviour of a plasma system very far from equilibrium, where in QCD we do not even have a well motivated phenomenological model.

As an example of a configuration which cannot be described, even in any approximation, by hydrodynamic methods consider the problem of plasma isotropisation, extensively studied in various variations within QCD [\[49–52\]](#)

Suppose we have a plasma system uniform in space with anisotropic pressures. In weakly coupled gauge theory one could consider a gas of gluons with non isotropic momentum distributions, like gaussians with different widths for the different momentum components. Then one expects that the pressures would isotropise in time. The energy-momentum tensor of such a system would have the form

$$T_{\mu\nu} = \begin{pmatrix} \varepsilon & 0 & 0 & 0 \\ 0 & p_{\parallel}(t) & 0 & 0 \\ 0 & 0 & p_{\perp}(t) & 0 \\ 0 & 0 & 0 & p_{\perp}(t) \end{pmatrix} \quad (5.57)$$

Note that such a system cannot be described by any form of (even all-order resummed) viscous hydrodynamics, since by symmetry $u^{\mu} = (1, 0, 0, 0)$ and thus has vanishing derivatives. So all viscous terms vanish, while the leading term is clearly of a different form. However nothing stops us from applying the AdS/CFT analysis using Einstein's equations to such a system. This has been first proposed in [\[53\]](#). Subsequently, a numerical study of this system was performed in [\[54\]](#).

Another interesting problem, which we will discuss in the remaining part of these lectures, is the behaviour of the boost-invariant plasma system considered before but now at small proper times. Since the hydrodynamic expansion [\(5.55\)](#)

$$\varepsilon(\tau) = \frac{1}{\tau^4} - \frac{2}{2^{\frac{1}{2}} 3^{\frac{1}{4}}} \frac{1}{\tau^2} + \frac{1 + 2 \log 2}{12\sqrt{3}} \frac{1}{\tau^{\frac{8}{3}}} + \dots \quad (5.58)$$

is an expansion in inverse powers of τ , it completely breaks down as we approach $\tau = 0$. Here the situation is more complicated than in the case of uniform isotropisation [\(5.57\)](#) mentioned above as we expect a mixture of non-equilibrium and hydrodynamic behaviour. In fact it is exactly the question of the transition to

hydrodynamics, and what factors are important in setting the scale of this transition, that is very interesting in the context of heavy-ion collisions. A related more general issue is the observation of thermalization (proposed in [55] to be related to a formation of a black hole in the bulk) and an analysis of the concrete way in which this scenario is realised.

5.11 Dynamics at Small Proper Time

For the reasons described above, we will have to deal with the full Einstein's equations. From the point of view of hydrodynamics treated as a gradient expansion these encompass *all orders* of viscous hydrodynamics together with an infinite set of higher transport coefficients. But apart from these there is additional information contained in the Einstein's equations. Taking the case of a planar black hole as an example, all order hydrodynamics may be identified, on the linearized level, with the lowest quasinormal mode and its exact dependence on spatial momentum. But apart from this lowest mode there is an infinite set of higher quasinormal modes which decay exponentially (in the AdS/CFT context, see in particular [56]). And all of these become important in a far from equilibrium situation such as the early time dynamics of the boost-invariant flow.

Here we will again go to the boost invariant setting and repeat the analysis of Sect. 5.7, but now concentrating on the small τ regime. We follow the analysis of [57]. We will adopt the same ansatz for the metric (5.34) and solve Einstein's equations with the boundary conditions (5.36).

Before going into the details of this construction, let us comment why we are using the Fefferman–Graham system of coordinates. In contrast to the late time expansion, here we will aim at solving the Einstein's equations to an arbitrary accuracy—without performing any kind of scaling limit. Therefore any choice of coordinates works equally well. Moreover the constraint equations for initial data in the Fefferman–Graham system of coordinates are particularly transparent.

Let us first determine the qualitative behaviour of $\varepsilon(\tau)$ at small τ . We will do it in two ways.

5.11.1 The Absence of a Scaling Variable

In the large proper time regime, the structure of Einstein's equations naturally led to the introduction of a scaling variable, which reduced the problem to solving ordinary differential equations and a subsequent expansion in inverse powers of τ . Let us now analyze the solution of Einstein's equations at small τ from this point of view.

We again start from the *exact* power series solution

$$\begin{aligned}
 a(\tau, z) = & -\varepsilon(\tau) \cdot z^4 + \left\{ -\frac{\varepsilon'(\tau)}{4\tau} - \frac{\varepsilon''(\tau)}{12} \right\} \cdot z^6 + \left\{ \frac{1}{6}\varepsilon(\tau)^2 + \frac{1}{6}\tau\varepsilon'(\tau)\varepsilon(\tau) \right. \\
 & \left. + \frac{1}{16}\tau^2\varepsilon'(\tau)^2 + \frac{\varepsilon'(\tau)}{128\tau^3} - \frac{\varepsilon''(\tau)}{128\tau^2} - \frac{\varepsilon^{(3)}(\tau)}{64\tau} - \frac{1}{384}\varepsilon^{(4)}(\tau) \right\} \cdot z^8 + \dots
 \end{aligned} \tag{5.59}$$

and substitute the asymptotics

$$\varepsilon(\tau) \sim \frac{1}{\tau^s} \quad \text{for } \tau \rightarrow 0 \tag{5.60}$$

In this way we obtain

$$\begin{aligned}
 & -z^4 \tau^{-s} + z^6 \left(\frac{1}{6} \tau^{-s-2} s - \frac{1}{12} \tau^{-s-2} s^2 \right) \\
 & + z^8 \left(-\frac{1}{16} \tau^{-2s} s^2 - \frac{1}{6} \tau^{-2s} + 1/6 \tau^{-2s} s + \frac{1}{96} \tau^{-s-4} s^2 - \frac{1}{384} \tau^{-s-4} s^4 \right) + \dots
 \end{aligned} \tag{5.61}$$

where the terms dominating for small τ are rendered in bold. This analysis was first done by Kovchegov and Taliotis [58], who deduced that for *generic* s the dominant terms at small τ are of the form

$$\frac{z^4}{\tau^s} \cdot f\left(w \equiv \frac{z}{\tau}\right) \tag{5.62}$$

In [58], the scaling solution was found, but due to its complex branch cut structure,⁸ Kovchegov and Taliotis argued that the only acceptable solution had $s = 0$, which is a very interesting result.

But if we again look at (5.61), we see that the terms resummed by the scaling variable *vanish* for $s = 0$ and are no longer dominant. Hence there is no place for a scaling variable at small τ and for $s = 0$ one has to reanalyze the Einstein equations in order to describe the full solution at $\tau \sim 0$.

5.11.2 The Existence of a Regular Initial Condition

One can reach the same conclusion, as well as some more detailed information, on the small τ dependence of $\varepsilon(\tau)$ *assuming* that at $\tau = 0$ we have a regular initial condition.⁹ Recall the expression (5.59) and substitute $\tau = 0$. Firstly, we see that

⁸ And an additional physical bound on s , see [58].

⁹ This is an assumption which may, or may not be realistic for ultraenergetic collisions. We prefer to keep the options open and analyze boost invariant flow with regular initial conditions as an interesting nonequilibrium dynamical system for its own sake.

the assumption that the metric coefficients are finite leads to a finite limit of $\varepsilon(\tau)$ as $\tau \rightarrow 0$, consistent with $s = 0$. Secondly, the inverse powers of τ appearing in the higher order terms do not lead to a singularity if and only if $\varepsilon(\tau)$ has an expansion only in *even* powers of τ :

$$\varepsilon(\tau) = \varepsilon_0 + \varepsilon_2\tau^2 + \varepsilon_4\tau^4 + \dots \quad (5.63)$$

A closer analysis reveals that the coefficients ε_{2n} are uniquely determined, through the Einstein's equations, in terms of the coefficients of the initial condition for the metric:

$$a(\tau = 0, z) = a_0z^4 + a_2z^6 + a_4z^8 + \dots \quad (5.64)$$

In order to complete the analysis of the gravitational setup, we have to analyze what are the admissible initial conditions (5.64). Once this is done, one can set up the analysis of the system by evolving the geometry from (5.64) using Einstein's equations and read off $\varepsilon(\tau)$ from the metric. This can be done either numerically solving Einstein's equations [59], or analytically by expressing the coefficients ε_{2n} directly in terms of the coefficients of the initial condition a_{2n} [57].

5.11.3 The Classification of Possible Initial Conditions

As is well known, in General Relativity, the initial conditions cannot be arbitrary but have to satisfy some nonlinear constraint equations. This causes the gravitational initial value problem to be quite nontrivial in general. Fortunately, for the case of the $\tau = 0$ hypersurface in the Fefferman–Graham coordinates, the constraints can be solved exactly.

Let us denote by $E_{\alpha\beta}$, the components of Einstein's equations written in the form

$$E_{\alpha\beta} \equiv R_{\alpha\beta} + 4g_{\alpha\beta} = 0 \quad (5.65)$$

Then the constraints are contained in equations $E_{\tau z}$ and E_{zz} . Denoting $a_0(z) \equiv a(\tau = 0, z)$ etc. we get at once

$$a_0(z) = b_0(z) \quad \dot{a}_0 = \dot{b}_0 = \dot{c}_0 = 0 \quad (5.66)$$

and the only remaining constraint is the single nonlinear equation

$$a_0'' + c_0'' + \frac{1}{2}(a_0')^2 + \frac{1}{2}(c_0')^2 - \frac{1}{z}(a_0' + c_0') = 0 \quad (5.67)$$

Let us note an extremely surprising feature of the above equation. At the linearized level, it has a trivial solution $a_0(z) = -c_0(z)$. So one may expect that for infinitesimal a_0 , the function c_0 would be also very small and only slightly deformed from $-a_0$ by taking into account the effect of the nonlinear terms. It turns out, however, that this is never true, and the nonlinearity always causes a blowup of the

solution for some finite z . To see this introduce $v(z^2) = \frac{1}{4z} a'_0(z)$ and similarly $w(z^2)$ for c_0 . Then the constraint Eq. 5.67 takes the simple form

$$v' + w' + v^2 + w^2 = 0 \quad (5.68)$$

Now it is easy to see, that there does not exist an everywhere bounded ($v = w = 0$ at infinity) solutions of the constraint equations. To this end it is enough to integrate (5.68) to get

$$0 = \int_0^\infty (v' + w') + \int_0^\infty (v^2 + w^2) = \int_0^\infty (v^2 + w^2) \quad (5.69)$$

Therefore, at $\tau = 0$ with boost invariant symmetry, gravity leads to inherently nonlinear dynamics—a linearized regime does not exist at all!

It is not difficult to impose conditions on v and w so that the blowup is not a curvature singularity—at least $\mathfrak{R}$ stays finite there (see [57] for details). Moreover one can solve the constraint (5.68) analytically. Indeed, defining $v_+ = -w - v$, $v_- = w - v$,

$$v_- = \sqrt{2v'_+ - v_+^2} \quad (5.70)$$

solves (5.68) for any v_+ . Let us conclude with an example of a simple particular solution of the initial value constraints:

$$a_0(z) = b_0(z) = 2 \log \cos az^2 \quad c_0(z) = 2 \log \cosh az^2 \quad (5.71)$$

The huge range of initial conditions that can be imposed is in fact quite natural. On the gauge theory side we may also expect to be able to prepare an initial state with the same energy density in a multitude of ways. Let us contrast this with the large τ expansion (5.58) of $\varepsilon(\tau)$, which only depends on a *single* scale.¹⁰ In other words, once the dominant asymptotics of $\varepsilon(\tau)$ is known, all subleading power like terms are uniquely determined.

The physical interpretation of this difference is quite clear. We expect dissipative effects to wash out differences in initial conditions leaving only a single scale (under the present symmetry assumptions) governing the large proper time expansion of $\varepsilon(\tau)$. On the gravitational side, these effects can be understood as nonlinear generalizations of higher quasinormal modes which die off exponentially (see [60] for some analysis along these lines in the boost invariant setting).

¹⁰ In (5.58) this scale has been set to unity, but may be reinstated unambiguously by dimensional analysis.

5.11.4 An Analysis of Some Aspects of the Small Proper Time Behaviour of $\varepsilon(\tau)$

Once we have the allowed initial conditions at $\tau = 0$, we need to solve Einstein's equations with these initial data. Then, as explained in Sect. 5.4, we may read off $\varepsilon(\tau)$ from the solution of Einstein's equations. Since we do not have a scaling variable at our disposal we have to do it exactly. The ideal way to proceed would be to solve Einstein's equations numerically. This study is currently under way [59]. The route followed in [57] was to solve these equations for the metric in a power series in z and τ , obtaining a power series expression for $\varepsilon(\tau)$ up to the order τ^{100} for some initial conditions. A drawback of the above method is that the power series in question has a finite radius of convergence, necessitating the use of Pade approximations as an extrapolation method. Below we will present some analysis of these extrapolated profiles [57]. We have to emphasize, however, that one would require a real numerical solution of Einstein's equation to be sure of all the details.

First let us discuss the transition to hydrodynamics. One possibility of quantifying this is by considering an 'effective exponent' of the power law dependence of $\varepsilon(\tau)$ defined as

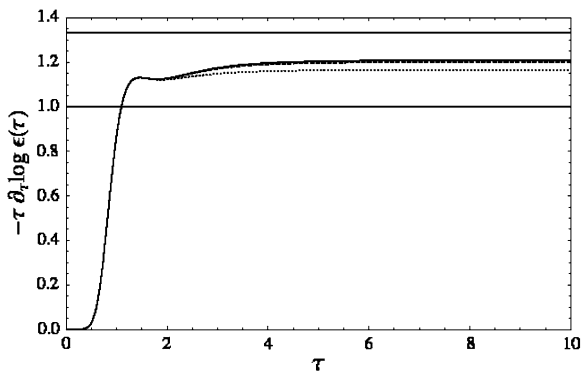
$$-\tau \frac{d}{d\tau} \log \varepsilon(\tau) \quad (5.72)$$

Initially it is zero, and it should rise up to $4/3$ for late time expansion. In order to evaluate it we plot a Pade approximant (of constant large τ asymptotics) of the expression (5.72). The result, for the initial condition (5.71) is shown in Fig. 5.2.

We see that it definitely crosses $s = 1$ of free streaming and moves upwards. However, to be sure that it would reach $4/3$ a numerical solution for $\varepsilon(\tau)$ would be needed.

Now assuming the late time exponent $4/3$, we may perform a Pade approximation of $\varepsilon(\tau)$ with this asymptotics to see the profiles of $\varepsilon(\tau)$ for a set of initial conditions. Example plots are shown in Fig. 5.3.

Fig. 5.2 The effective power (5.72) of $\varepsilon(\tau)$ corresponding to the initial condition (5.71)



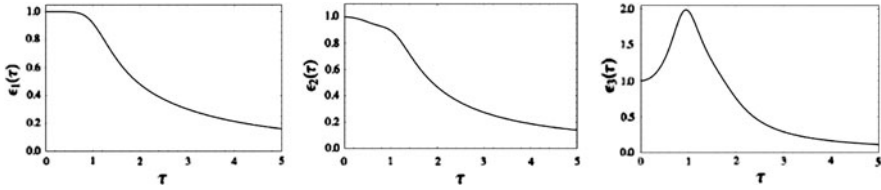


Fig. 5.3 Pade resummed profiles of $\varepsilon(\tau)$ for a set of initial conditions [57]

The main interest of the knowledge of the exact profile of $\varepsilon(\tau)$ for various initial conditions is that then we would be able to study the transition to hydrodynamics and its dependence on various features of the initial conditions. Since the Pade extrapolation introduces some serious systematic uncertainties, we refrain from doing so until we will have at our disposal a direct numerical solution of Einstein equations for these various initial conditions. Apart from the reasons mentioned above, the numerical solution would also allow to analyze the nature of the apparent singularity in the initial data and, more importantly, analyze the geometry for the presence of apparent (dynamical) horizons, relevant for the thermodynamic interpretation. Some of these issues are currently under investigation [59].

Before we close this last part of the lectures let us note a complementary numerical investigation of boost invariant plasma in [61]. The setup of [61] was different from the one presented in these lectures in that the *gauge theory metric* was perturbed in a boost invariant way at some $\tau \sim \tau_0 > 0$ and then set again to flat Minkowski. The metric perturbation produced a change in the geometry which induced a boost invariant flow on the boundary. The main results observed in [61] were a transition to hydrodynamics and a formation of an apparent horizon which moved in from infinity.

Another related work¹¹ was [62], where a perturbation by a boundary scalar source induced a black hole formation in the bulk.

Looking at all these examples, one sees that Einstein's equations, through the AdS/CFT correspondence, have the potential of describing a multitude of far from equilibrium strongly coupled gauge theory phenomena. It is however clear that the majority of problems remain still unsolved.

5.12 Conclusions

In these lectures we have described an approach using the AdS/CFT correspondence as a tool for studying real time dynamics of strongly coupled gauge theory plasma. The basic theoretical tool is the possibility of translating, in a completely explicit and constructive way, between the spacetime energy-momentum tensor

¹¹ This time not in the boost-invariant setting discussed here.

characterizing the gauge theory configuration in question and the five-dimensional metric of the dual geometry. Then one uses the fact, that at strong coupling, the dynamics of the dual geometry is governed by Einstein's equations (with a cosmological constant following from the full ten dimensional supergravity solution). A further dynamical input is the requirement of the absence of naked singularities in the gravity background. This restricts very strongly the allowed spacetime profiles of the gauge theory energy momentum tensor, and consequently the possible gauge theory dynamics.

Using these methods, one may establish the appearance of nonlinear hydrodynamics, as exhibited here in the form of near perfect fluid dynamics in the large proper time asymptotics of boost invariant plasma expansion. Moreover, Einstein equations together with the nonsingularity criterion unambiguously predict viscous first- and higher-order deviations from perfect fluid dynamics with specific values of the transport coefficients appropriate to the case of the $\mathcal{N} = 4$ SYM theory studied here.

Let us note, that we arrived at these results without presupposing anything even about the general form of the gauge theory energy momentum tensor like the presence of something like a flow velocity u^μ etc. This flexibility comes from the fact that Einstein's equations on the string side of the AdS/CFT correspondence require only that the gauge theory energy-momentum tensor is conserved and traceless (throughout these lectures we are dealing exclusively with the conformal case of $\mathcal{N} = 4$ SYM). Therefore one can use the same techniques to address the question of far from equilibrium dynamics where hydrodynamics cannot be used as a starting point of approximation. We exhibited an example of such a study by describing some aspects of the behaviour of the boost invariant plasma expansion in the region of small proper time.

It should be obvious that the majority of questions concerning far from equilibrium dynamics of strongly coupled plasma remain still unanswered even for the case of $\mathcal{N} = 4$ plasma. Once more details are understood, it would be very interesting to address similar problems in more complicated versions of the AdS/CFT correspondence involving gauge theories which might be closer to QCD. Apart from this direct 'application driven' motivation, the study of such time-dependent systems leads to fascinating interrelations with General Relativity. On the one hand, it provides a new setting for investigating such GR concepts as dynamical apparent horizons, black hole formation, providing these GR phenomena with new interpretation. On the other hand, the well developed technology of numerical relativity might be applied to learn more about the properties of far from equilibrium strongly coupled gauge theory systems.

Last but not least, let us note that the gravity backgrounds obtained as dual descriptions of evolving plasma systems can very well by themselves form a scene for the AdS/CFT correspondence enabling one to study various kind of physics questions. In this way one may study how the expanding plasma system influences properties of mesons, Wilson loops, various correlation functions. Of course, due to the time-dependent nature of those backgrounds this may be quite difficult to do

in practice, but the possibility of doing so is certainly very interesting and the results may be rewarding.

Acknowledgement The approach presented in these lectures was introduced with Robi Peschanski, and developed further in collaboration with Micha, Heller, Dongsu Bak, Alex Buchel, Paolo Benincasa, Guillaume Beuf. To all of whom I am grateful for enjoyable collaboration and numerous discussions. I would like to thank the organizers and participants of the *Fifth Aegean Summer School “From Gravity to Thermal Gauge Theory: The AdS/CFT Correspondence”* for a very interesting school. I would also like to thank the *New Frontiers in QCD 2010* program and the Yukawa Institute for Theoretical Physics, Kyoto for hospitality when these lectures were written up. This work was supported in part by Polish science funds as a research project N N202 105136 (2009–2011) and the Marie Curie ToK KraGeoMP (SPB 189/6.PRUE/2007/7).

Appendix

A.1 Topics Not Covered in the Main Text: A Brief Guide to the Literature

In this appendix we would like to give pointers to the literature on other developments related to the approach presented in these lectures.

- Leading α' corrections (i.e. beyond strong coupling in $\mathcal{N} = 4$ SYM theory) to the transport coefficients were computed using the boost invariant flow [63, 64]. This task involved using α' corrected Einstein’s equations.
- Beyond $\mathcal{N} = 4$ SYM. A class of general conformal field theories parametrized by higher curvature terms in the dual gravitational action was considered in [65].
- Extension to hydrodynamics with conserved charge(s) was considered in [66–68]. Electric-magnetic equilibration at large proper times was found in [69]. In addition, dilaton driven hydrodynamics was considered in a general way in [68] and the onset of turbulence was observed [70].
- Lower dimensional examples. The case of a 1+1 dimensional conformal field theory allows for an explicit exact dual gravitational solutions [71, 72]. Other investigations in different number of dimensions were performed in [73].
- Various (exact) solutions for $\mathcal{N} = 4$ gauge theory in curved and possibly time-dependent backgrounds were obtained [74–76]. The exact solutions which were found do not involve viscosity effects. An exact gravitational description of a shearless flow in $\mathcal{N} = 4$ in flat space was obtained in [77].
- Further properties of solutions with boost invariant symmetries were studied in [78–82].
- Physics in the expanding plasma. Fundamental flavours were introduced (through D7 brane embeddings) into the late proper time boost invariant geometry [83], diffusion constant was evaluated [84], drag force was computed [85]. In addition, various physical questions were addressed in the case of the

shock wave solutions described in Sect. 5.4 Deep Inelastic Scattering (DIS) was analyzed [86–89] as well as heavy quark energy loss [90, 91].

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Chapter 6

Fluid Dynamics from Gravity

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Abstract We give a brief review of the recently-formulated fluid/gravity correspondence. Originating from the AdS/CFT correspondence, this constitutes a one-to-one map between configurations of a conformal fluid dynamics in d dimensions and solutions to Einstein's equations in $d + 1$ dimensions. The map is fully constructive; for a given fluid configuration, it is completely algorithmic to write down the spacetime geometry and deduce its causal properties. In particular, the bulk solutions describe a regular generic, non-uniform and dynamical, black hole which at late times settles down to a stationary planar black hole. We briefly indicate the iterative construction of such solutions, extract the key physical properties, and discuss further generalizations and open questions.

6.1 Introduction

In these lectures I will review the recent developments in formulating a relation between fluid dynamics and gravity, which came to be called the fluid/gravity correspondence. This correspondence is rooted in the well-known AdS/CFT (or more generally the gauge/gravity) correspondence, reviewed elsewhere in these proceedings.

As is often the case with important developments, there are several diverse and complementary takes on the underlying motivation for formulating and using such a correspondence. Iconically speaking, we were motivated by the need to:

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- probe nature of spacetime in quantum gravity,
- extract universal dynamics of wide class of gravitational theories,
- explore properties of strongly-coupled gauge theories (with particular view to exploring QCD), and
- elucidate long-standing questions in fluid dynamics.

I will now briefly elaborate on each of these in turn, simultaneously reminding the reader of the key ingredients of the AdS/CFT correspondence to be used later.

6.1.1 Quantum Gravity

One of the most important long-standing goals of the past few decades has been to unravel the underlying quantum theory of gravity. Since the classical theory of gravity, Einstein’s general relativity, identifies gravity with the curvature of spacetime, the key question of quantum gravity is *what is the fundamental nature of spacetime?* Serendipitously, one invaluable tool that came to our aid in tackling this question during the last decade is the celebrated AdS/CFT correspondence, for the simple reason that it allows us, in principle, to recast the long-standing quantum gravitational questions in terms of non-gravitational quantum field theory.

In its simplest formulation, string theory (which in particular contains gravity while being a consistent quantum theory), on asymptotically five-dimensional Anti-de Sitter (AdS) spacetime times a five-sphere, is dual to Super Yang-Mills gauge theory which is a four-dimensional conformal field theory (CFT). Such a correspondence is holographic; the CFT may be thought of as living on the boundary of AdS. We will therefore refer to the AdS side as the “bulk” and the CFT side as the “boundary” theory. According to the AdS/CFT dictionary, different asymptotically AdS spacetimes manifest themselves by different states in the boundary theory. For example, the vacuum state in the CFT corresponds to pure AdS. Metric perturbations from AdS are related to the stress-energy-momentum tensor expectation value in the CFT.

More relevantly, putting a black hole in the bulk has the effect of heating up the boundary theory. Specifically, a large¹ Schwarzschild-AdS black hole corresponds to (approximately) thermal state in the gauge theory. Conceptually, this can be understood as the late-time configuration that a generic state evolves to: in the bulk, gravity, as well as the negative curvature, tends to make a generic large-energy configuration collapse and form a black hole which then quickly settles down to the Schwarzschild-AdS geometry. Correspondingly, in the field theory, a generic large-energy excitation will eventually thermalize.

¹ AdS is a space of constant negative curvature, which introduces a length scale, called the AdS scale R_{AdS} , corresponding to the radius of curvature. The black hole size is then measured in terms of this AdS scale; large black holes have horizon radius $r_+ > R_{\text{AdS}}$.

While appealingly simple, this level of understanding is far too coarse to allow us to extract the more interesting aspects. We need to probe the AdS/CFT dictionary further to uncover what happens in regions where the classical description of the black hole breaks down, such as near the curvature singularity, or in the more general, dynamical, situations. Ultimately, we would like to answer such questions as: Which CFT configurations admit a dual spacetime description? or What types of spacetime singularities are physically allowed? More generally, we wish to probe spacetime dynamics. Although the fluid/gravity correspondence will not enable us to answer these intriguing questions fully, it will take the first step in this direction.

6.1.2 *Universal Implications*

While it is heartening to have a framework wherein we can explore the questions of interest, it is much more gratifying for that framework to exhibit certain degree of universality. The fluid/gravity framework does indeed achieve this. Specifically, the setup which we will consider in the bulk,² namely Einstein's gravity with negative cosmological constant (and no other bulk matter content), is in fact relevant for a much wider class of theories. This is because Einstein's equations (with negative cosmological constant) constitute a consistent truncation of all two-derivative gravitational theories interacting with other fields (with spin < 2) which have AdS as a solution. This means that for every CFT having SUGRA bulk dual description, we have a decoupled sector exhibiting universal dynamics for the stress tensor. In particular, the stress tensor correlators (from which we can extract the transport coefficients) are in this sense universal. More importantly, since uncharged planar black holes solve our theory, they likewise lie in this universal sector, so this universality holds at any temperature.

6.1.3 *QCD*

A complementary long-standing goal of theoretical physics is to solve QCD. As observed elsewhere, AdS/CFT provides a convenient window of opportunity even for this programme, since it allows us to get a handle on certain strongly coupled field theories. Unfortunately, QCD is in many respects markedly different from the super Yang-Mills CFT, so it is only for certain universal features that the AdS/CFT correspondence has any direct relevance. Nevertheless, as remarked elsewhere in these proceedings, many quantities of interest in QCD, such as its

² In what follows, the five-sphere of $\text{AdS}_5 \times S^5$ will play a passive role, so we will ignore it and consider the five-dimensional (asymptotically) AdS bulk spacetime only.

transport coefficients, do fall into this category, so that while not quantitatively accurate, the predictions from fluid/gravity do give a reasonable indication of the qualitative features. This is especially useful since presently we do not have other means, apart from possibly lattice calculations, to calculate in the strong coupling regime. This makes the AdS/CFT, and specifically the fluid/gravity, correspondence relevant to high-energy experiments. Indeed, the predictions which will be described later in these lectures have been used in analysing data from the Relativistic Heavy Ion Collider.

6.1.4 *Fluid Dynamics*

Although many detailed questions about the fundamental degrees of freedom in a given strongly coupled field theory remain well beyond our reach, a more macroscopic effective description is not only more easily accessible, but also is interesting in its own right. The key observation is that any interacting field theory admits an effective description in terms of fluid dynamics. Fortuitously, fluid dynamics is a subject that has been well-studied for more than a century, theoretically as well as observationally/experimentally and numerically. Nevertheless, there are important questions in fluid dynamics which have yet to be understood. While we have a good handle on near-equilibrium physics, dynamics away from thermodynamic equilibrium is more difficult to tackle. For example, one of the famous Clay Millenium Prize Problems concerns the global regularity (existence and smoothness) of the Navier–Stokes equations. Intriguingly, the solutions often include turbulence, which, despite its importance in science and engineering, still remains one of the greatest unsolved problems in physics. Furthermore, there are causality issues in the conventional Navier–Stokes equations which plague numerical simulations. As already suggested by its name, the fluid/gravity correspondence is indeed relevant for addressing such important questions in fluid dynamics.

The structure of these lectures is as follows. We will first review the necessary background for constructing the fluid/gravity map in [Sect. 6.2](#). In particular, we will briefly discuss the two sides of the correspondence, conformal fluid dynamics on the boundary and gravity in the bulk, and then indicate the main idea of constructing the map between them. In [Sect. 6.3](#) we will sketch the actual construction, which involves expanding in boundary derivatives; in particular, we will explicitly correct the zeroth order configuration to first order. The second order solution, calculated in [\[5\]](#), is discussed in [Sect. 6.4](#); in particular, we will present the boundary stress tensor and transport coefficients for our conformal fluid, followed by a discussion of the bulk geometry from which we extract the location and area of the event horizon. Finally, [Sect. 6.5](#) summarizes the salient points and indicates several open questions.

These lectures are based primarily on [\[5\]](#) which first formulated the nonlinear fluid/gravity correspondence and constructed the second order solution, and on [\[6\]](#)

which analysed the causal structure of this solution, focussing on the event horizon area and corresponding entropy current in the boundary fluid. This work builds on earlier derivations of linearized fluid dynamics from linearized gravity [18] (see [22] for an excellent review), and on earlier examples of the duality between non-linear fluid dynamics and gravity [14, 4]. Since its inception, the fluid/gravity correspondence has flourished into an active line of research, with large number of extensions and generalizations. Some of the key developments and reviews of further progress include [7, 8, 19], among many others. Since these lectures are intended as a brief overview serving rather as an invitation into the subject than as a full, self-contained review (excellent samples of which already exist in the literature), I am purposefully keeping the list of references to the bare minimum; please see the above-cited works for a more complete set of background references.

6.2 Background

We begin with a brief review of conformal fluid dynamics, proceed to discuss the dual gravitational solutions, and then motivate the construction of the explicit mapping between them. To keep the discussion and formulas as clean as possible, we will restrict attention to the simplest case of uncharged conformal fluid on four-dimensional Minkowski spacetime $\mathbf{R}^{3,1}$ (though initially we will quote the d -dimensional results). Several generalizations to this setup will be mentioned in Sect. 6.5.

Before proceeding, we make few remarks on notation: the bulk metric will be denoted by g_{MN} with the capital Latin indices taking values over the $d + 1$ bulk dimensions; we will separate the coordinates into the radial coordinate r and the remaining “boundary coordinates” x^μ , where the μ index ranges over the d boundary directions (which includes time). The stress tensor in the boundary theory is denoted $T^{\mu\nu}$, and in writing its conservation ($\nabla_\mu T^{\mu\nu} = 0$), the ∇_μ is the covariant derivative with respect to the boundary metric $\eta_{\mu\nu}$, which in this case simply reduces to the partial derivative ∂_μ .

6.2.1 Conformal Fluid Dynamics

Fluid dynamics is the continuum effective description of any (interacting) microscopic quantum field theory. In order to meaningfully describe the system in terms of the fluid variables, the fluid description assumes that the system achieves local thermodynamic equilibrium. This means that the regime of validity where such a description is valid requires that the scale of variation of the dynamical degrees of freedom, L , be much larger than the microscopic scale ℓ_{mfp} , typically set by the

temperature, T . In this *long-wavelength approximation*, local equilibrium then demands that $LT \equiv \frac{1}{\epsilon} \gg 1$.

A conformal fluid is characterized by a traceless symmetric stress tensor, which in d spacetime dimensions has $\frac{d(d+1)}{2} - 1$ degrees of freedom, along with a collection of charge currents (which for simplicity we have set to zero). In a fluid dynamical characterization of the same system, the number of basic degrees of freedom is drastically reduced. The conformal invariance fixes the equation of state, thereby determining the pressure in terms of the energy density, which can in turn be expressed in terms of the temperature. Hence the basic variables are the local temperature $T(x)$ and velocity $u_\mu(x)$ (unit-normalized so that $u_\mu u^\mu = -1$), which constitute just d degrees of freedom.

The equations of fluid dynamics are then simply the equations of local conservation of the stress tensor (as well as the charge currents in more general situations), supplemented by constitutive relations that express these currents as functions of the fluid dynamical variables. As fluid dynamics is a long wavelength effective theory, such constitutive relations are usually specified in a derivative expansion. At any given order, thermodynamics plus symmetries determine the form of this expansion up to a finite number of undetermined coefficients. In general, the coefficients can be obtained either from measurements or from microscopic computations. However, as we will see, in the present framework these coefficients are fully determined by the gravity side (which in a sense knows about the microscopics of the boundary field theory).

Purely based on the symmetries, we can then write down an expression for the stress tensor of a d -dimensional conformal fluid:

$$T^{\mu\nu} = \alpha T^d (\eta^{\mu\nu} + du^\mu u^\nu) + \pi_{\text{dissipative}}^{\mu\nu}.$$

The first two terms describe the ideal conformal fluid stress tensor, while $\pi_{\text{dissipative}}^{\mu\nu}$ incorporates all the dissipative terms, and α is a constant giving the overall normalization. As variations of $T(x)$ and $u_\mu(x)$ are small, we can expand $\pi^{\mu\nu}$ in a derivative expansion $\partial_\mu \equiv \frac{\partial}{\partial x^\mu}$ in the boundary directions; the leading term will turn out to be proportional to the shear viscosity. The dynamical content of the fluid equations is encoded in the conservation of the stress tensor

$$\partial_\mu T^{\mu\nu} = 0. \quad (6.1)$$

Fluid dynamics viewed in this derivative expansion constructs an effective field theory for the slowly varying modes $T(x)$ and $u_\mu(x)$, analogously to the chiral Lagrangian for pions.

6.2.2 Gravity in the Bulk

We now turn to the gravitational solutions in asymptotically AdS spacetime. Motivated by the AdS/CFT correspondence, we will consider two-derivative

theories of gravity with an AdS_{d+1} “vacuum”, such as the IIB SUGRA on $\text{AdS}_5 \times S^5$. As mentioned above, the solution space has a universal sub-sector, pure gravity with negative cosmological constant, for which the bulk field equations are simply Einstein’s equations,

$$E_{MN} \equiv R_{MN} - \frac{1}{2}Rg_{MN} + \Lambda g_{MN} = 0. \quad (6.2)$$

(Note that without loss of generality we can measure distances in AdS units, which amounts to taking $R_{\text{AdS}} = 1$ and thereby sets $\Lambda = -6$ in five dimensions.) We will focus on this sub-sector in the long-wavelength limit. Apart from the pure AdS_5 solution, there is a 4-parameter family of solutions representing asymptotically- AdS_5 boosted planar black holes. We will use these solutions to construct general dynamical spacetimes characterized by fluid-dynamical configurations.

Roughly speaking, our construction may be thought of as a “collective coordinate method” for black hole horizons. Recall that the isometry group of AdS_5 is $SO(4, 2)$. The Poincare algebra plus dilatations form a distinguished subalgebra of this group (one that preserves the boundary). Out of these, the $SO(3)$ rotations and translations in world-volume $\mathbf{R}^{3,1}$ leave the static planar AdS black hole invariant, but the remaining symmetry generators, dilatations and boosts, act nontrivially on this solution, generating a 4-parameter family of boosted planar black holes, parameterized by the temperature T and the velocity u^μ of the brane. Our construction effectively promotes these parameters to Goldstone fields (or collective coordinate fields) $T(x)$ and $u^\mu(x)$, and determines their dynamics, order by order in the boundary derivative expansion. Note that this is distinct from linearization: we make no assumptions about the amplitudes of these slow variations.

6.2.3 Fluid/Gravity Map

Before proceeding to sketch the construction in more detail, we pause to stress an important point in mapping these long-wavelength gravity solutions to corresponding fluid configurations. A well-known procedure of holographic renormalization (see e.g. [3, 11]) links the boundary stress tensor to the behaviour of the bulk metric near the AdS boundary. Given any asymptotically AdS spacetime, we can read-off the induced stress tensor on the boundary, since the latter is related to the normalizable modes of the gravitational field in AdS. In particular, expanding the bulk metric in the Fefferman–Graham form near the boundary $z = 0$,

$$ds^2 = \frac{dz^2 + (\eta_{\mu\nu} + \alpha z^d T_{\mu\nu})dw^\mu dw^\nu}{z^2},$$

the stress tensor is simply given by $T_{\mu\nu}$. Conversely, given a boundary stress tensor, there is a procedure to holographically reconstruct the bulk metric in a radial expansion around the boundary.

Naively, this might seem puzzling: as mentioned above, a conformally invariant stress tensor in d dimensions has $\frac{d(d+1)}{2} - 1$ degrees of freedom. If any such stress tensor yielded a regular bulk spacetime, we would have a discrepancy between the fluid side which has only d degrees of freedom and whose dynamics is correspondingly specified by only d equations, and the gravity side that would seemingly allow more degrees of freedom. In other words, passing from a generic quantum conformal field theory stress tensor to the stress tensor of its effective description in terms of fluid dynamics constitutes a drastic reduction in the number of degrees of freedom required to specify the spacetime. How is this manifested in the bulk? The answer lies in regularity. As a series expansion around the boundary, the holographic reconstruction cannot guarantee that the metric does not become nakedly singular at some finite radial value in the bulk. In fact, for a generic stress tensor it will. The fluid/gravity construction demonstrates that the regular solutions are given precisely by such stress tensors which are fluid dynamical. Moreover, we claim that the gravity solutions thus constructed are the most general regular long-wavelength³ solutions to Einstein's equations with negative cosmological constant. They typically correspond to deformed and dynamical black holes; i.e. the solutions admit a regular event horizon which shields a curvature singularity.

The heuristic picture of a generic evolution, on the two sides of the fluid/gravity correspondence, is as follows. Suppose we start with some generic high energy initial conditions. On the CFT side, the system quickly settles down to local thermodynamic equilibrium, whose bulk dual is described by a dynamical, non-uniform (planar) black hole. On both sides, such configuration is described by local velocity and temperature fields which exhibit slow variation in the boundary directions. The subsequent evolution is described by equations of fluid dynamics on the boundary, which originate from Einstein's equations governing the bulk evolution. Finally, at late times, the system relaxes to global thermal equilibrium, given by a stationary state parameterized by a constant temperature and velocity. In the bulk, this is one of the well-known stationary solutions describing a planar black hole in AdS mentioned in the previous subsection and given explicitly below.

Our construction utilizes the long-wavelength regime of fluid dynamics: we write Einstein's equations as a perturbative expansion in boundary derivatives (however keeping the exact radial dependence). This allows us to solve the equations order by order in this boundary derivative expansion. It turns out that Einstein's equations at a given order implement the fluid stress tensor conservation equations at lower order. Therefore, order by order, we can use the lower-order fluid dynamical solution to construct the bulk metric, and then read off the corrected fluid dynamical stress tensor. In [5], we constructed the boundary stress tensor $T^{\mu\nu}$ and corresponding bulk metric g_{MN} to second order in boundary

³ Note, however, that there are regular solutions which do not fall into the long-wavelength category, such as small black holes in AdS, which correspondingly are not described by fluid configurations.

derivative expansion. This yields a map between fluid dynamics and gravity, which we now proceed to sketch in more detail.

6.3 Construction of Bulk Metric and Boundary Stress Tensor

To explain our iterative procedure, we will first review the zeroth order configuration, and then discuss the deformation of this solution to first order.

6.3.1 0th Order

We start with the stationary solution to Einstein's equations with negative cosmological constant corresponding to the boosted planar Schwarzschild-AdS black hole:

$$ds^2 = -2u_\mu dx^\mu dr + r^2(\eta_{\mu\nu} + [1 - f(r/\pi T)u_\mu u_\nu] dx^\mu dx^\nu), \quad (6.3)$$

where $f(r) \equiv 1 - \frac{1}{r^4}$. Albeit perhaps less familiar, this form of the solution is very convenient since the metric is manifestly regular on the event horizon at $r = \pi T$ as well as being boundary-covariant. To obtain it from a more familiar form, we can start with the static Schwarzschild-AdS black hole in the planar limit:

$$ds^2 = r^2 \left(-f(r) dt^2 + \sum_i (dx^i)^2 \right) + \frac{dr^2}{r^2 f(r)},$$

change to ingoing Eddington coordinates to avoid the coordinate singularity on the horizon: $v = t + r_*$ where $dr_* = \frac{dr}{r^2 f(r)}$, and finally “covariantize” by boosting: $v \rightarrow u_\mu x^\mu$, $x_i \rightarrow P_{i\mu} x^\mu$, where $P_{\mu\nu}$ is the spatial projector, $P_{\mu\nu} = \eta_{\mu\nu} + u_\mu u_\nu$.

This solution is now parameterized by four parameters: the temperature T and the boosts u^μ (which have three independent components in four-dimensions due to normalization). The metric given in Eq. 6.3 constitutes the 0th order solution in our iterative procedure. The causal structure of this solution is identical to that of the static Schwarzschild-AdS black hole: there is a spacelike curvature singularity at $r = 0$, cloaked by a regular event horizon, and a timelike boundary. The AdS/CFT correspondence maps this bulk solution to an ideal fluid characterized by temperature T and fluid velocity u^μ . In particular, the induced stress tensor on the boundary for $d = 4$ is

$$T^{\mu\nu} = \pi^4 T^4 (\eta^{\mu\nu} + 4u^\mu u^\nu). \quad (6.4)$$

Note that this is an exact solution since the conservation equation (6.1) is automatically satisfied for this constant stress tensor, and correspondingly the metric (6.3) satisfies the bulk Einstein's equations (6.2) for any constant T and u^μ .

6.3.2 1st Order

A general fluid configuration in local, but not global, equilibrium can be described by promoting the four parameters T and u^μ to physical fields dependent on the boundary coordinates x^μ , i.e., to $T(x)$ and $u^\mu(x)$. If these fields vary slowly compared to the microscopic scale ℓ_{mfp} , i.e. if

$$\frac{\partial_\mu \log T}{T} \sim \mathcal{O}(\epsilon), \quad \frac{\partial_\mu u^\mu}{T} \sim \mathcal{O}(\epsilon)$$

for small ϵ , the fluid configuration still satisfies the conditions of local equilibrium. In each local domain of slow variation, which we refer to as tube, the bulk gravitational solution is approximately that of a uniform black brane. Remarkably, the bulk solution can be constructed by patching together these tubular domains! Of course, if we just replace u_μ and T in the metric (6.3) by $T(x)$ and $u^\mu(x)$, the resulting metric (call it $g_{MN}^{(0)}$) will no longer solve Einstein's equations (6.2). Instead, the metric $g_{MN}^{(0)}$ will need to be corrected by higher-order piece ($g_{MN}^{(1)}$, etc.), which we can obtain iteratively as an expansion in ϵ . We will find that the resulting corrected metric can be constructed systematically to any desired order, and is valid well inside the event horizon, thus allowing verification of its regularity. It is worthwhile to stress that the success of such a procedure rests on the fact the our seed metric $g_{MN}^{(0)}$ is manifestly regular on the horizon, since otherwise the expansion would break down near the coordinate singularity at horizon.

To implement the construction algebraically, we express the line element in a boundary derivative expansion of the fields $u_\mu(x)$ and $T(x)$, and use ϵ as a book-keeping parameter (counting the number of x^μ derivatives):

$$g_{MN} = \sum_{k=0}^{\infty} \epsilon^k g_{MN}^{(k)}, \quad T = \sum_{k=0}^{\infty} \epsilon^k T^{(k)}, \quad u_\mu = \sum_{k=0}^{\infty} \epsilon^k u_\mu^{(k)}. \quad (6.5)$$

The term $g_{MN}^{(k)}$ corrects the metric at the k th order, such that Einstein's equations will be satisfied to $\mathcal{O}(\epsilon^k)$ provided the functions $T(x)$ and $u^\mu(x)$ obey a certain set of equations of motion, which turn out to be precisely the stress tensor conservation equations of boundary fluid dynamics at $\mathcal{O}(\epsilon^{k-1})$.

Specifically, we can obtain the equations for $g_{MN}^{(k)}$ by substituting the expansion (6.5) into Einstein's equations (6.2), and extracting the coefficient of $\mathcal{O}(\epsilon^k)$. Schematically, these take the form

$$\mathcal{H}[g^{(0)}(u_\mu^{(0)}, T^{(0)})]g^{(k)}(x^\mu) = s_k \quad (6.6)$$

where $\mathcal{H}$ is a second-order linear differential operator in the variable r alone and s_k are regular source terms which are built out of $g^{(k)}$ with $n < k - 1$. Since $g^{(k)}(x^\mu)$ is already of $\mathcal{O}(\epsilon^k)$, and since every boundary derivative appears with an additional power of ϵ , $\mathcal{H}$ is an ultralocal operator in the field theory directions. Moreover, at a given x^μ , the precise form of this operator $\mathcal{H}$ depends only on the local values of T and u^μ but not on their derivatives at x^μ . Furthermore, the operator $\mathcal{H}$ is independent of k ; we have the same homogeneous operator at every order in perturbation theory. This allows us to find an explicit solution of (6.6) systematically at any order. The source term s_k however gets more complicated with each order, and reflects the nonlinear nature of the theory.

Bit more explicitly, the equations of motion split up into two kinds: Constraint equations, $E_{r\mu} = 0$ which implement stress-tensor conservation (at one lower order), and Dynamical equations $E_{\mu\nu} = 0$ and $E_{rr} = 0$ which allow determination of $g^{(k)}$. We solve the dynamical equations

$$g^{(k)} = \text{particular}(s_k) + \text{homogeneous}(\mathcal{H})$$

subject to regularity in the interior and asymptotically AdS boundary conditions. Using the rotational symmetry group of the seed solution (6.3) it turns out to be possible to make a judicious choice of variables such that the operator $\mathcal{H}$ is converted into a decoupled system of first order differential operators. It is then simple to solve the (6.6) for an arbitrary source s_k by direct integration. For the details of the procedure, as well as discussion of convenient gauge choice, etc., we refer the reader to the original work [5] or the review [19].

Instead, here we simply quote the result for the bulk metric and boundary stress tensor, corrected to first order in ϵ . To first order the bulk metric takes the form

$$ds^2 = -2u_\mu dx^\mu dr + r^2(\eta_{\mu\nu} + [1 - f(r/\pi T)]u_\mu u_\nu) dx^\mu dx^\nu \\ + 2r \left[\frac{r}{\pi T} F(r/\pi T) \sigma_{\mu\nu} + \frac{1}{3} u_\mu u_\nu \partial_\lambda u^\lambda - \frac{1}{2} u^\lambda \partial_\lambda (u_\nu u_\mu) \right] dx^\mu dx^\nu, \quad (6.7)$$

where $T(x)$ and $u_\mu(x)$ are any slowly-varying functions which satisfy the conservation equation (6.1) for the zeroth order ideal fluid stress tensor (6.4),

$$F(r) \equiv \int_r^\infty dx \frac{x^2 + x + 1}{x(x+1)(x^2+1)} = \frac{1}{4} \left[\ln \left(\frac{(1+r)^2(1+r^2)}{r^4} \right) - 2 \arctan(r) + \pi \right] \quad (6.8)$$

and $\sigma^{\mu\nu}$ is the transverse traceless symmetric part of $\partial^\mu u^\nu$ called shear, i.e.

$$\sigma^{\mu\nu} = P^{\mu\alpha} P^{\nu\beta} \partial_{(\alpha} u_{\beta)} - \frac{1}{3} P^{\mu\nu} \partial_\alpha u^\alpha.$$

Note that the first line of (6.7) is simply the zeroth order (boundary-derivative-free) solution (6.3), whereas each of the terms in the second line have exactly one boundary derivative.⁴

The induced fluid stress tensor on the boundary, which can be easily obtained⁵ from the bulk metric (6.7), is given by

$$T^{\mu\nu} = \pi^4 T^4 (4u^\mu u^\nu + \eta^{\mu\nu}) - 2\pi^3 T^3 \sigma^{\mu\nu}. \quad (6.9)$$

Here the first two (derivative-free) terms describe a perfect fluid with pressure (or negative free energy density) $\pi^4 T^4$, and correspondingly (using thermodynamics) entropy density $s = 4\pi^4 T^3$. The shear viscosity η of this fluid may be read off from the coefficient of $\sigma^{\mu\nu}$ and is given by $\pi^3 T^3$. Notice that $\eta/s = 1/(4\pi)$, in agreement with the well-known result of [18].

6.4 Solution to Second Order

In the previous section we have illustrated at first order how our iterative procedure can be implemented (in principle systematically to any order) to construct a generic long-wavelength solution. Such a procedure was carried out in [5], where the bulk metric and boundary stress tensor was calculated explicitly to second order in the boundary derivative expansion. In this section we will discuss the new physics which can be extracted from such a construction.

Note that already at first order, the bulk metric (6.7) was a much lengthier expression than the boundary stress tensor (6.9). This remains true in general; in fact, already at second order the expression for the metric is far too unwieldy to write down here. In the following subsections, we will therefore only write the second order boundary stress tensor explicitly but indicate the bulk metric only schematically.

⁴ Note that (6.7) does not have any $\partial_\mu T$ terms appearing explicitly, since by implementing the zeroth order stress tensor conservation, we have expressed the temperature derivatives in terms of the velocity derivatives.

⁵ We use the standard prescription (cf. [3]),

$$T_v^\mu = -2 \lim_{r \rightarrow \infty} r^4 (K_v^\mu - \delta_v^\mu)$$

where $K_{\mu\nu}$ is the extrinsic curvature tensor on a constant- r surface.

6.4.1 The Four-Dimensional Conformal Fluid from AdS_5

The second order stress tensor can be written as⁶

$$T^{\mu\nu} = (\pi T)^4 (\eta^{\mu\nu} + 4u^\mu u^\nu) - 2(\pi T)^3 \sigma^{\mu\nu} + (\pi T)^2 \left((\ln 2) T_{2a}^{\mu\nu} + 2T_{2b}^{\mu\nu} + (2 - \ln 2) \left[\frac{1}{3} T_{2c}^{\mu\nu} + T_{2d}^{\mu\nu} + T_{2e}^{\mu\nu} \right] \right) \quad (6.10)$$

where the expressions in the second line are given in terms of the fluid velocity and derived quantities,

$$\begin{aligned} T_{2a}^{\mu\nu} &= \epsilon^{\alpha\beta\gamma(\mu} \sigma_{\gamma}^{\nu)} u_\alpha \lambda_\beta, & T_{2b}^{\mu\nu} &= \sigma^{\mu\alpha} \sigma_\alpha^\nu - \frac{1}{3} P^{\mu\nu} \sigma^{\alpha\beta} \sigma_{\alpha\beta}, \\ T_{2c}^{\mu\nu} &= \partial_\alpha u^\alpha \sigma^{\mu\nu}, & T_{2d}^{\mu\nu} &= \mathcal{D} u^\mu \mathcal{D} u^\nu - \frac{1}{3} P^{\mu\nu} \mathcal{D} u^\alpha \mathcal{D} u_\alpha, \\ T_{2e}^{\mu\nu} &= P^{\mu\alpha} P^{\nu\beta} \mathcal{D}(\partial_{(\alpha} u_{\beta)}) - \frac{1}{3} P^{\mu\nu} P^{\alpha\beta} \mathcal{D}(\partial_\alpha u_\beta) \end{aligned} \quad (6.11)$$

with $\mathcal{D} \equiv u^\mu \partial_\mu$, $\lambda_\mu \equiv \epsilon_{\alpha\beta\gamma\mu} u^\alpha \partial^\beta u^\gamma$ where $\epsilon_{\alpha\beta\gamma\mu}$ is the Levi-Civita tensor, and as before $P^{\mu\nu}$ is the spatial projector and $\sigma^{\mu\nu}$ is the shear. Note that all of these expressions are second order in boundary derivatives (denoted by the 2 in the subscript).

In fact, one can package the expressions in (6.11) more compactly by using the decomposition of the 4-velocity gradient $\partial^\nu u^\mu$ into transverse, traceless and trace parts,

$$\partial^\nu u^\mu = -a^\mu u^\nu + \sigma^{\mu\nu} + \omega^{\mu\nu} + \frac{1}{3} \theta P^{\mu\nu},$$

where expansion, acceleration, and vorticity, are respectively defined as:

$$\theta = \partial_\mu u^\mu, \quad a^\mu = \mathcal{D} u^\mu, \quad \omega^{\nu\mu} = P^{\mu\alpha} P^{\nu\beta} \partial_{[\alpha} u_{\beta]}.$$

We can then rewrite the expressions (6.11) as:

$$\begin{aligned} T_{2a}^{\mu\nu} &= -2\omega^{\rho(\mu} \sigma_{\rho}^{\nu)} \\ T_{2b}^{\mu\nu} &= \sigma^{\rho(\mu} \sigma_{\rho}^{\nu)} \\ T_{2c}^{\mu\nu} &= \theta \sigma^{\mu\nu} \\ T_{2d}^{\mu\nu} &= a^{(\mu} a^{\nu)} \\ \frac{1}{3} T_{2c}^{\mu\nu} + T_{2d}^{\mu\nu} + T_{2e}^{\mu\nu} &= \langle \mathcal{D} \sigma^{\mu\nu} \rangle + \frac{1}{3} \theta \sigma^{\mu\nu} \end{aligned} \quad (6.12)$$

⁶ Here we quote the stress tensor as written originally in [5]; subsequently, slightly simpler and more general expressions were found; cf. e.g. [19] for a review. (Also, for simplicity we have absorbed an overall factor of $16\pi G_N^{(5)}$ which would appear in the conventionally defined stress tensor as conjugate to boundary time translations.)

Here we have adopted the notation employed in e.g. [1], using the angular brackets around the indices $A^{(\mu\nu)}$ to denote the symmetric, transverse, traceless part of $A^{\mu\nu}$.

It however turns out to be most useful to write the second order stress tensor in a different basis of operators, $\mathcal{T}_k^{\mu\nu}$, which are manifestly conformally covariant, cf. [1, 16]:

$$T_{(2)}^{\mu\nu} = \tau_\pi \eta \mathcal{T}_1^{\mu\nu} + \kappa \mathcal{T}_2^{\mu\nu} + \lambda_1 \mathcal{T}_3^{\mu\nu} + \lambda_2 \mathcal{T}_4^{\mu\nu} + \lambda_3 \mathcal{T}_5^{\mu\nu}$$

and whose coefficients are the physically meaningful transport coefficients.

At first order the only nontrivial transport coefficient is the shear viscosity,

$$\eta = \frac{N^2}{8\pi} (\pi T)^3$$

whereas at second order the fluid parameters (relaxation timescales, etc.) extracted from the stress tensor (6.10) are

$$\tau_\pi = \frac{2 - \ln 2}{\pi T}, \quad \lambda_1 = \frac{2\eta}{\pi T}, \quad \lambda_2 = \frac{2\eta \ln 2}{\pi T}, \quad \lambda_3 = 0.$$

These coefficients agree with the independent results of [1], who in addition derive the curvature coupling term, $\kappa = \frac{\eta}{\pi T}$. These coefficients correspond to the various relaxation timescales discussed in the literature (see e.g. [17]) in context of high energy collider physics. As indicated earlier, the fact that we are dealing with a particular conformal fluid, namely one that is dual to gravitational dynamics in asymptotically AdS spacetimes, leads to these coefficients being determined as fixed numbers.

6.4.2 The Spacetime Geometry Dual to Fluids

Let us now turn to discuss the bulk geometry obtained at the second order in boundary derivative expansion (whose first order part is given by (6.7)). As mentioned previously, this bulk solution is “tubewise” approximated by a planar black hole. This means that in each tube, defined by a small neighbourhood of given x^μ , but fully extended in the radial direction r , the radial dependence of the metric is approximately that of a boosted planar black hole at some temperature T and horizon velocity u^μ . These parameters vary from one position x^μ to another in a manner consistent with fluid dynamics. Our choice of coordinates is such that each tube extends along an ingoing radial null geodesic. Apart from technical advantages, this is conceptually rather pleasing, since it suggests a mapping between the boundary and the bulk which is natural from causality considerations.

It is worth stressing that although we refer to the metric written in (6.7) and its second order extension presented in [5] as “a solution” in the singular form rather than the plural, these expressions actually correspond to not just a single solution or finite family of solutions, but rather a continuously infinite family of solutions,

specified by the four functions $T(x)$ and $u^\mu(x)$ of four variables. The flip side of the coin is that while very general, such a metric is not fully explicit: in order to be so, we need to use a given solution to fluid dynamics as input.

However, even in the absence of the explicit functional dependence of $T(x)$ and $u^\mu(x)$, it is possible to extract certain salient features of any such geometry. The most important feature of our geometry is the presence of an event horizon. In [6] we have demonstrated explicitly that the event horizon is regular, and determined its location in terms of the functions $T(x)$ and $u^\mu(x)$. Here we only schematically motivate these results. Intriguingly, it turns out that the location of the event horizon $r_H(x^\mu)$ in the bulk is determined locally by the fluid dynamical data at a point x^μ (within the derivative expansion), rather than globally as usual in general relativity.

To motivate this physically, within each tube characterized by a given x^μ , the position of the horizon is approximately at $r_H \approx \pi T$ corresponding to that tube. Since T varies as a function of x^μ , so will the horizon position $r_H(x^\mu)$. In the corrected solution, the surface $r = \pi T(x)$ is not the event horizon (for example, it is not a null surface in general), but if the variation $T(x)$ is slow, the deviation from the true event horizon is likewise small.

One can determine the position of the event horizon within our perturbation scheme using the fact that the solution settles down at late times to a uniformly boosted planar black hole. In particular, if we expand the horizon location as a series in boundary derivatives, tagged as before by ϵ ,

$$r = r_H(x) = \pi T(x) + \sum_{k=1}^{\infty} \epsilon^k r_{(k)}(x),$$

then the coefficient functions $r_{(k)}(x)$ are determined algebraically by demanding that the surface given by $r = r_H(x)$ be null.

A very simple toy model which captures the gist of this argument is given by a time-dependent but spherically symmetric black hole, the Vaidya spacetime:

$$ds^2 = -\left(1 - \frac{2m(v)}{r}\right) dv^2 + 2 dv dr + r^2 d\Omega^2.$$

This metric describes a four-dimensional asymptotically flat black hole accreting null dust, so that the mass $m(v)$ increases with time. Assuming that at late times the black hole settles down to Schwarzschild, $m(v) \rightarrow m_f$ as $v \rightarrow \infty$, and denoting the location of event horizon by $r = r_H(v)$, we can find its position by demanding that it describes the null surface which at late times approaches the correct event horizon $r_H \rightarrow 2m_F$ as $v \rightarrow \infty$. Note that the normal n to a null surface will be simultaneously tangent to that surface and likewise null. For Vaidya, the normal 1-form $n = dr - \dot{r} dv$ (where $\dot{} \equiv \frac{d}{dv}$) is null when

$$r_H(v) = 2m(v) + 2r_H(v)\dot{r}_H(v).$$

Of course, the exact solution to this equation yields the horizon $r_H(v)$ non-locally in terms of $m(v)$, requiring the knowledge of $m(v)$ for all $v < \infty$. However, when

$m(v)$ varies slowly, so that $\dot{m} = \mathcal{O}(\epsilon)$, $m\ddot{m} = \mathcal{O}(\epsilon^2)$, etc., we can determine this location in terms of an ϵ expansion. For the ansatz

$$r_H(v) = 2m(v) + am(v)\dot{m}(v) + bm(v)\dot{m}(v)^2 + cm(v)^2\ddot{m}(v) + \dots$$

this expansion gives $a = 8, b = 64, c = 32, \dots$. This toy model illustrates that in spite of the event horizon being defined globally (as the boundary of the causal past of the future null infinity) and therefore requiring knowledge of the mass $m(v)$ for all time $v < \infty$, for slowly varying $m(v)$ we can nevertheless express $r_H(v_0)$ solely in terms of m its derivatives at $v = v_0$.

Returning to the problem of interest, we can similarly locate the event horizon $r_H(x^\mu)$ in our dynamical non-uniform planar black hole geometry in terms of T, u^μ , and all their derivatives, at the given point x^μ . At first order, the position of the horizon is unchanged, whereas at second order it is corrected by terms which scale with square of the shear and vorticity (see [6, 19] for explicit expressions.)

Once we identify the position of the event horizon in our geometry, it is easy to check that this horizon is regular. In fact, our construction manifestly guarantees regularity: the only curvature singularity of the seed metric is at $r = 0$, and the source terms which appear in correcting the metric order by order do not introduce any additional singularities. The final issue to check is the regime of validity of our expansion, and this can be seen to extend well inside the event horizon.

Therefore we have an explicit one-to-one map relating conformal fluid configurations on $\mathbf{R}^{3,1}$ to asymptotically AdS₅ inhomogeneous black brane solutions having regular event horizons. This is a remarkable statement about gravity, suggesting that fluid dynamical configurations within the AdS/CFT correspondence naturally uphold Cosmic Censorship.

To extract further physics from the position of the event horizon, let us consider the proper area of its spatial slices. By the second law of black hole mechanics, the horizon area cannot decrease with time; or equivalently, the expansion of the horizon generators must be non-negative. A well-known identification with thermodynamics translates this statement to that of the entropy increasing, or more locally, the entropy current having non-negative divergence. Having obtained the event horizon for our geometry explicitly in terms of the metric functions $u^\mu(x)$ and $T(x)$, we can verify these statements, and identify the entropy current naturally induced on the boundary.

To obtain the boundary entropy current J_S^μ from the bulk geometry, we can pull-back the area form A on the event horizon to the boundary. We perform this pull-back along a tube of constant x^μ , i.e. along ingoing radial null geodesics. This yields the expression

$$\begin{aligned} (4\pi\eta)^{-1}J_S^\mu &= [1 + b^2(A_1\sigma_{\alpha\beta}\sigma^{\alpha\beta} + A_2\omega_{\alpha\beta}\omega^{\alpha\beta} + A_3\mathcal{R})]u^\mu \\ &\quad + b^2[B_1\mathcal{D}_\lambda\sigma^{\mu\lambda} + B_2\mathcal{D}_\lambda\omega^{\mu\lambda}] \\ &\quad + C_1b\lambda^\mu + C_2b^2u^\lambda\mathcal{D}_\lambda\lambda^\mu + \dots \end{aligned} \tag{6.13}$$

with $b \equiv \frac{1}{\pi T}$ and

$$\begin{aligned} A_1 &= \frac{1}{4} + \frac{\pi}{16} + \frac{\ln 2}{4}, & A_2 &= -\frac{1}{8}, & A_3 &= \frac{1}{8}, \\ B_1 &= \frac{1}{4}, & B_2 &= \frac{1}{2}, & C_1 &= C_2 = 0 \end{aligned} \quad (6.14)$$

which can be easily verified to have non-negative divergence. At leading order, the divergence is proportional to $\sigma_{\mu\nu}\sigma^{\mu\nu}$ which is manifestly positive for any non-zero shear. In case of zero shear, the relations $B_1 = 2A_3$ and $C_1 + C_2 = 0$ guarantee non-negativity. We can readily see that these relations are indeed satisfied for the entropy current given by our gravity dual construction (6.14).

The expression (6.13) was written suggestively in a most general form yielding a Weyl-covariant entropy current for any set of constant coefficients A_i, B_i, C_i . In general, to the order we work, the seven coefficients get reduced to five independent ones allowing for Weyl-covariant entropy current with non-negative divergence. From the field theory side, this would therefore suggest a 5-parameter ambiguity in constructing a sensible entropy current, purely based on the symmetries and the requirement that it correctly reproduces equilibrium physics. This is at first sight puzzling, since our gravity construction seemingly fixed all these parameters. In fact, this was not quite the case: there is still an ambiguity even on the gravity side, corresponding to the freedom to add total derivative terms without changing the horizon area, and the pull-back being ambiguous to boundary diffeomorphisms. However, at second order this results in a 2-parameter ambiguity for Weyl covariant current with positive divergence, so that some mismatch remains.⁷

6.5 Summary

One of the most intriguing features of the fluid/gravity correspondence is that it provides us with a window into the *generic* behavior of gravity in a nonlinear regime, mapping long-wavelength (but arbitrary amplitude) perturbations of AdS black holes to the more familiar physics of fluid dynamics. Apart from the obvious conceptual advantages, one has a tremendous computational simplification for numerical studies of gravitational solutions since the fluid dynamics lives in one lower dimension.

The bulk spacetime solutions discussed here describe a generic (time-dependent and non-uniform) planar black hole. We have demonstrated that each of these solutions (with regular fluid data $T(x)$ and $u_\mu(x)$) has its singularities hidden from the boundary by a regular event horizon. In this sense, all gravitational solutions dual to regular solutions of fluid dynamics uphold the cosmic censorship

⁷ Although including higher orders may remedy this discrepancy [21].

conjecture. Moreover, our construction seems to imply a new variant of Uniqueness theorem: for a given boundary fluid configuration, there corresponds a unique regular bulk metric.

At the technical level, the key to our construction lies in utilizing the long-wavelength regime of fluid dynamics. This allows us to write the bulk metric, and corresponding boundary stress tensor, to any order in a boundary derivative expansion in a completely procedural manner. Furthermore, once having obtained the bulk geometry, this expansion likewise enables us to determine the radial position of its event horizon, remarkably expressed locally in terms of the fluid dynamical data. We can then easily verify that the area of this event horizon is necessarily non-decreasing, in accordance with the Area Theorem. The corresponding entropy current induced from the bulk solution is then guaranteed to satisfy the Second Law of thermodynamics.

The boundary fluid stress tensor contains new quantities of interest, namely the various transport coefficients which characterise the fluid. At first order we have recovered the previously-known value of viscosity and verified that $\frac{\eta}{s} = \frac{1}{4\pi}$. More importantly, we have predicted the second order fluid parameters $(\tau_\pi, \lambda_1, \lambda_2, \lambda_3)$. This has been of interest in QCD phenomenology, especially in understanding certain characteristic features of the quark-gluon plasma.

Although (in interest of familiarity and brevity of presentation) we have sketched the fluid/gravity correspondence in the simplest setting, many useful generalizations have already been carried out. One of the earliest such generalizations involved extending the correspondence to other dimensions, relating a d -dimensional conformal fluid to asymptotically AdS_{d+1} black hole (see [23] for the interesting case of $d = 3$ and [8, 13] for general d). More ambitiously, one may also consider fluids on curved manifolds (rather than just the Minkowski spacetime $\mathbf{R}^{d-1,1}$), as has been initiated in [7]. In addition, one can include matter in the bulk. This allows for richer dynamics, but typically at the expense of losing universality. Early examples of such extensions include considering the dilaton (which corresponds to forcing of the fluid) in [7], Maxwell $U(1)$ field [2, 12], multiple Maxwell fields and scalars, magnetic and dyonic charges, as well as more exotic models (see e.g. [19] for further references). Moreover, one can even extend the correspondence to con-conformal fluids [10, 15] as well as to non-relativistic fluids [9, 20], which allows us to make closer contact with familiar everyday systems.

Nevertheless, many future directions and puzzles remain, as well the need for further generalizations. For example, of particular current interest is to understand the fluid/gravity correspondence for extremal fluids (and in particular superfluids) which are presently attracting much attention. Also, to mimic many of the familiar aspects of fluid flows, we need to understand how to confine the fluid within walls in the gravity dual. Still more ambitiously, to understand the rich phenomena rooted in quantum processes, we would like to get a better handle on finite- N effects.

More prosaically, even within the original context, several intriguing issues deserve further investigation. We have established one-to-one map between long-wavelength gravitational solutions and solutions of fluid dynamics. Naively, one

might think of this as merely a low-energy effective description of the AdS/CFT correspondence. However it is not true that any gravitational solution in AdS admits a fluid description. In particular, it would be useful to understand the role of non-long-wavelength bulk classical or semiclassical solutions, such as small AdS black holes. Likewise, there is room for more detailed bulk analysis, such as probing the allowed horizon topology and dynamics in more general situations, the nature of curvature singularity, or Cosmic Censorship.

Another intriguing aspect is the striking difference between the phenomenology of turbulent flows in $3 + 1$ and $2 + 1$ dimensions, as pointed out in [23]. In the $3 + 1$ dimensional turbulent energy cascade, large scale eddies give rise to smaller scale eddies, eventually transferring energy down to scales where viscosity becomes important and energy is dissipated. In contrast, the $2 + 1$ dimensional turbulent flows are characterized by an “inverse cascade,” in which smaller scale eddies merge into large scale eddies, creating large long-lived vortical structures. If these qualitative differences extend to relativistic fluids, they would suggest a profound difference in gravitational dynamics between four and five dimensional gravity. In particular, we might predict that black holes in AdS_4 would take much longer to equilibrate owing to the fact that the fluctuations on the horizon could coalesce in macroscopic structures. From the gravitational standpoint, this would certainly seem very surprising and counter-intuitive.

More generally, our correspondence offers new insight into the black hole membrane paradigm. The conventional membrane paradigm provides a simple picture of black hole dynamics in terms of classical physics of fluid living on a “membrane” (or stretched horizon) just outside the event horizon. Taking a more general view of trying to encode the black hole dynamics by fluid dynamics localized on a membrane in the spacetime, the immediate natural question is: where should such a membrane live? Perhaps the most obvious candidate is the event horizon; but this is problematic due to its null nature, and more importantly, because it is globally defined so we cannot fix its position without knowing the full future evolution of the spacetime. Alternately, several (quasi)local notions of a black hole have been proposed, such as the so-called dynamical horizon, which however are spacelike surfaces inside the event horizon, and therefore do not admit the standard notion of evolution. A more popular suggestion is the stretched horizon, which is the formulation given by the membrane paradigm. However, there likewise remain ambiguities in localizing stretched horizon. Within the fluid/gravity correspondence, the full spacetime dynamics is mapped to the dynamics of a conformal fluid, which albeit reminiscent of the membrane paradigm, has one important twist: the membrane lives on the boundary of the spacetime (which is unambiguously defined and admits a fluid description with well-defined dynamics), and gives a perfect mirror of the bulk physics. This “membrane at the end of the universe” picture is a natural consequence of the holographic nature of the fluid/gravity correspondence.

Finally, we have mentioned in the Introduction that one of the outstanding problems in fluid dynamics is understanding turbulence. We might therefore ask how relevant is the fluid/gravity correspondence (appropriately generalized) to

tackling such a problem. We argue that the correspondence is indeed relevant and worth vigorously pursuing further (with a handful of proposals already appearing in the literature). The important point is that the regime where our derivative expansion is valid corresponds to large Reynolds number (where the viscous terms are small relative to the leading order terms), so the phenomenology of turbulence should be directly relevant to the study of near-equilibrium AdS black hole dynamics. At least for the widely studied case of non-relativistic fluids described by the Navier–Stokes equations, turbulence has many striking qualitative features, including the tendency to dynamically break symmetries as Reynolds number is increased, the sharp onset of turbulence at critical Reynolds numbers, and an “energy cascade” in fully developed turbulence in which energy is transferred in a predictable way between eddies at different scales. Less is known about turbulence for the microscopically relativistic fluids relevant in our context, but it would be fascinating to understand the gravitational interpretation for those features that do generalize to the relativistic case.

Acknowledgements It is a pleasure to thank my collaborators, Sayantani Bhattacharyya, R. Loganayagam, Gautam Mandal, Shiraz Minwalla, Takeshi Morita, Harvey Reall, Mukund Rangamani, and Mark Van Raamsdonk for marvelous collaborations on various aspects of fluid dynamics. I would also like to thank the participants of the Fifth Aegean summer school for their enthusiasm and questions, and the organizers, especially Eleftherios Papantonopoulos, for putting together an excellent summer school. This work was supported in part by STFC Rolling grant and by the National Science Foundation under the Grant No. NSF PHY05-51164.

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Chapter 7

The Gauge-Gravity Duality and Heavy Ion Collisions

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Abstract This chapter provides a review of two particular applications of the gauge-gravity duality to heavy ion collisions. The first involves a study of the wake of a quark as it travels through the quark gluon plasma and its possible connection to measurements of jet correlations carried out at the relativistic heavy ion collider at Brookhaven. The second section provides, via the gauge/gravity duality, a lower bound on the entropy produced in a collision of two energetic distributions. This is then compared to particle multiplicity in gold–gold collisions.

In its simplest form, the gauge-gravity duality [21, 47, 61] relates the planar limit of $\mathcal{N} = 4$ super-Yang–Mills (SYM) at large 't Hooft coupling to classical gravity in an asymptotically anti-de-Sitter (AdS) geometry. The particle content of $\mathcal{N} = 4$ super-Yang–Mills is completely determined by super-symmetry and includes gluons, four Majorana fermions in the adjoint representation of the gauge group and six real scalars also in the adjoint. While some deformations of the $\mathcal{N} = 4$ SYM theory and their dual are well understood, the string theory dual of quantum chromodynamics (QCD) is currently unavailable. Nevertheless, there have been numerous attempts to extract information about the strongly coupled deconfined phase of QCD using AdS/CFT.

A critic may argue that any attempt to compare QCD with $\mathcal{N} = 4$ SYM or its variants is destined to fail since there is no control parameter for such a comparison. However, even though there is no direct means of comparing the two, one may be able to address open problems in QCD using the gauge-gravity duality. This can be achieved either by uncovering universal properties of strongly coupled gauge theories—ones that are shared by all gauge theories with a holographic dual,

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or by providing a precise answer to a question for which even a qualitative solution is lacking.

In this chapter, two particular applications of AdS/CFT to heavy ion collisions will be reviewed. The first involves a study of the wake of a quark as it travels through the quark gluon plasma and its possible connection to measurements of jet correlations carried out at the relativistic heavy ion collider at Brookhaven. The second section provides, via the gauge/gravity duality, a lower bound on the entropy produced in a collision of two energetic distributions. This is then compared to particle multiplicity in gold–gold collisions.

7.1 The Wake of a Quark

7.1.1 Jet Correlations at RHIC

At the relativistic heavy ion collider at brookhaven (RHIC) two gold ions collide with a total center of mass energy of about 39 TeV. Each gold ion has 197 nucleons. So, in the center of mass frame each nucleon has an energy of roughly

$$E_n \sim 39 \text{ TeV} / (2 \times 197) \sim 100 \text{ GeV}. \quad (7.1)$$

Given that the mass of a nucleon is roughly 1 GeV, the Lorentz factor for the ions is roughly a hundred,

$$\gamma \sim E_n / M_n \sim 100, \quad (7.2)$$

which implies that the ions will be Lorentz contracted. For this reason the moving ions are usually depicted as flat “pancakes”. See Fig. 7.1. Shortly after the collision thousands of hadrons reach the detector and it is an experimental challenge to extract information about the intermediate stages of the collision from the final particle distribution at the detector. Presumably, at the late stages of the collision a

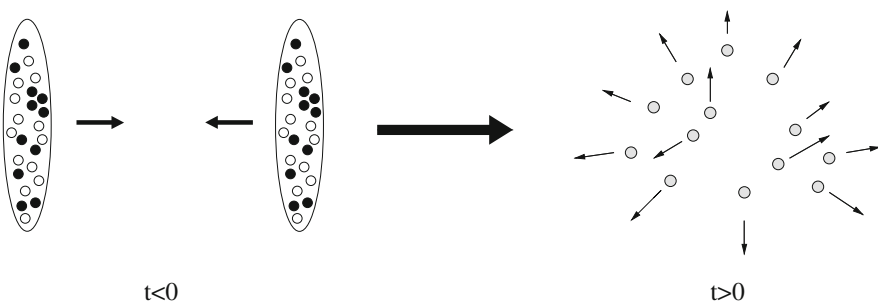


Fig. 7.1 A head on collision of two gold ions. Before the collision there are roughly 400 nucleons, each one has a center of mass energy of approximately 100 GeV. This is why the heavy ions are usually depicted as flattened ‘pancakes’. After the collision thousands of hadrons reach the detector. Presumably a quark-gluon plasma is formed in the intermediate stages of the collision

strongly coupled quark-gluon plasma (QGP) is formed—a configuration in which the quarks and gluons are strongly coupled but deconfined. Some indication for the existence of a QGP is the Boltzman distribution of hadrons reaching the detector at a temperature which is roughly the deconfinement temperature [37].

One method to quantify the resulting distribution of hadrons is a measurement of the degree of correlation between the emerging hadrons. In proton–proton collisions it is expected that correlated hadrons are a result of a pair being created in the collision region: from momentum conservation, the jets reaching the detector must come out back to back, as shown in Fig. 7.2. In gold–gold collisions, at mid-rapidity, and when considering trigger jets with transverse momentum in the range $2.5 \text{ GeV} < p_T < 4 \text{ GeV}$ and away-side jets of transverse momentum $1 \text{ GeV} < p_T < 2.5 \text{ GeV}$, instead of back-to-back jets one finds a local minimum in the correlation function at angles of π , and a maximum at an angle of, roughly $\pi - 1$ [2–4, 32, 36, 58, 59]. As depicted in Fig. 7.3, this implies that instead of back-to-back jets, the jets are “split”.

A possible explanation of this phenomenon involves the hydrodynamic excitations of the parton pair as it moves through the quark-gluon plasma [14, 15]. Imagine a pair produced very close to the surface of the plasma such that one of the quarks/gluons immediately leaves the QGP while its partner has to plow through the plasma. As the partner particle traverses the plasma it will loose energy and eventually slow to a halt. But, if its initial velocity is higher than the speed of sound of the plasma it will create a shock wave. Once hadronization occurs the energy contained in the shock wave will reach the detector as particles and the observed signal will be that of a split jet. While such a scenario is consistent with observations, its drawback is that point like excitations such as quarks also generate a laminar wake far behind them. In the setup we are considering, such a wake, if it exists, will upon hadronization generate a broad-shoulder like signal which will be observed, at the detector, at an angle of π instead of the observed minimum. This is depicted in Fig. 7.4.

While it is true that a point-like source will generate a laminar wake behind it, it is possible that the collective excitation of the away-side quark, which is what sources the hydrodynamic modes, is not point-like. It is possible that very close to the quark, the dynamics of the plasma is not governed by hydrodynamics. On the contrary,

Fig. 7.2 A schematic diagram of the detector (shaded) and two correlated hadrons reaching it at an angle of π relative to one another. One should keep in mind that this is a highly oversimplified picture of the detector and that in practice the experimental situation is much more complex

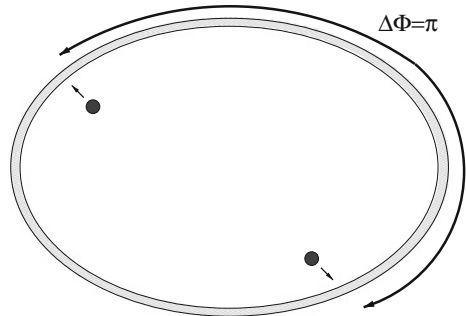


Fig. 7.3 A schematic diagram of the detector (shaded) and two correlated hadrons reaching it at an angle of $\pi - 1$ relative to one another. One should keep in mind that this is a highly oversimplified picture of the detector and that in practice the experimental situation is much more complex

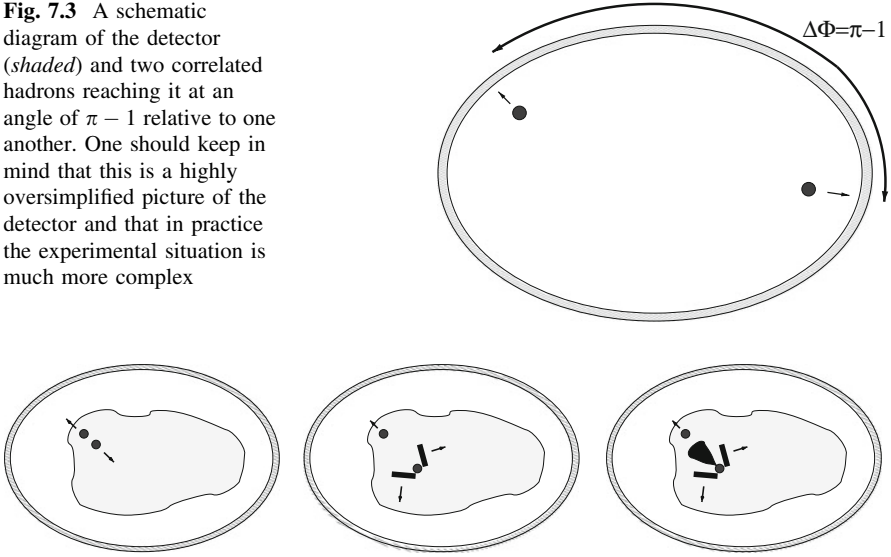


Fig. 7.4 A simplified picture of a proposed mechanism for the generation of the jet-splitting effect. The detector is depicted by the gray ellipsoid, and the quark gluon plasma is the light gray region in the center. A quark pair is created very close to the surface of the plasma (far left). One quark exits the QGP and appears as a trigger jet at the detector. The other quark traverses the plasma generating a shock wave (middle), or a shock wave and wake (far right). In the first case, the signal arriving at the detector will have a local minimum at an angle of π relative to the trigger jet whereas in the second case the signal will have a broad shoulder-like structure

hydrodynamics describes the long wavelength excitations of the QGP far away from the moving source. What sources the hydrodynamic modes is not the moving quark (or parton) but rather the quark plus near-field excitation. As was shown in [14, 15], it is possible that such an effective source for hydrodynamics will not behave like a point-like particle and therefore will not generate a wake behind it.

While the phenomenological model in [14, 15] seems to explain some of the data, it is unsatisfying that one needs to assume that the effective excitation of a quark moving in a strongly coupled QGP is tuned precisely so that a wake will not be generated behind the quark. Without some handle over the interaction between the quark and the QGP it is impossible to know whether or not such a model is viable. This is where the AdS/CFT correspondence comes in. Some attempts at treating this problem by more conventional methods can be found in, for example [10, 35, 46–49, 55, 56].

7.1.2 A Holographic Computation

As explained in the previous section, it is difficult to compute properties of the wake of a quark moving in a QCD plasma at temperatures above the deconfinement

temperature where QCD is presumably strongly coupled. However, it is possible to carry out an exact computation of the linear response of the QGP to a heavy quark in the $\mathcal{N} = \text{SYM}$ theory. As emphasized earlier, there is no control parameter which allows us to go from $\mathcal{N} = 4$ SYM to QCD. Nevertheless, we will see that the results of the $\mathcal{N} = 4$ SYM theory computation will provide important information regarding the possible validity of the scenario described above where the quark presumably does not create a wake behind it.

The planar limit of strongly coupled $\mathcal{N} = 4$ SYM is equivalent to the supergravity limit of type *IIB* superstring theory in an asymptotically $\text{AdS}_5 \times S^5$ geometry. The string tension α' is related to the 't Hooft coupling, λ , in the gauge theory, and the number of colors in the gauge theory, N , is related to the gravitational constant in the bulk G_N ,

$$G_N \sim 1/N^2 \quad \alpha' \sim \sqrt{\lambda}. \quad (7.3)$$

The duality between $\mathcal{N} = 4$ SYM and type *IIB* string theory can be succinctly stated as an equality between the partition function of string theory and the generating function of the gauge theory. In the supergravity limit this implies that the on-shell supergravity action of *IIB* supergravity gives the generating function of the planar limit of $\mathcal{N} = 4$ SYM. Explaining the duality in detail will take us far afield, and is beyond the scope of this work. There is an immense body of literature which explains the internal working of the duality to which the interested reader is referred to. Some canonical works are [5, 21, 47, 61].

Since the duality provides the generating function for a strongly coupled gauge theory, we can use it to compute a particular one-point function: that of the stress tensor $\langle T_{\mu\nu} \rangle$. The AdS/CFT prescription for carrying out this computation is as follows [12, 13, 21, 41, 61]. We work in a gauge where the line element takes the form:

$$ds^2 = \frac{L^2}{z^2} g_{\mu\nu} dx^\mu dx^\nu + \frac{L^2}{z^2} dz^2 \quad (7.4)$$

where we have restricted our attention to the five non-compact dimensions and have omitted an S^5 part of the line element. The S^5 will play a minor role in the current analysis. The energy momentum tensor of the boundary theory can be read off a near boundary ($z \rightarrow 0$) expansion of the metric

$$g_{\mu\nu} = \eta_{\mu\nu} + \dots + \frac{3}{16\pi G_N} \langle T_{\mu\nu} \rangle z^4 + \mathcal{O}(z^5) \quad (7.5)$$

where η is the Minkowski metric and by $\dots$ we mean intermediate terms in the series expansion.

Let us see how the prescription (7.5) works in practice. Consider first AdS_5 space which, according to the AdS/CFT dictionary is dual to the vacuum of the SYM gauge theory. The line element for a Poincaré patch of AdS_5 can be written in the form

$$ds^2 = \frac{L^2}{z^2} \left(-dt^2 + \sum_i (dx^i)^2 \right) + \frac{L^2}{z^2} dz^2. \quad (7.6)$$

where z runs from 0 to ∞ and the other coordinates run from $-\infty$ to ∞ . Note that apart from the overall warp factor $\frac{L^2}{z^2}$ the line element of AdS_5 space is identical to the Minkowski space line element; if we look at a surface of constant $z = z_0$ then the induced metric on this surface is precisely that of Minkowski space stretched by an overall warp factor $\frac{L^2}{z_0^2}$. See Fig. 7.5. Going back to the prescription in (7.5) we obtain immediately

$$\langle T_{\mu\nu} \rangle = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad (7.7)$$

as expected for the vacuum state of $\mathcal{N} = 4$ SYM.

Moving on to a more complicated example, we consider the state dual to a Schwarzschild black hole in AdS which is, presumably, a thermal state with temperature T [61]. Before carrying out such a computation let us see what sort of stress-tensor we expect for such a state. From a field theory point of view, we expect that the stress energy tensor for a large N $SU(N)$ gauge theory be proportional to N^2 the number of degrees of freedom, and in a conformal theory, in the absence of any other scales, it should also be proportional to T^4 , from dimensional arguments. Finally, by studying the variation of the action by an infinitesimal dilatation one finds that the stress energy tensor is traceless. Hence, in a conformal theory one expects that

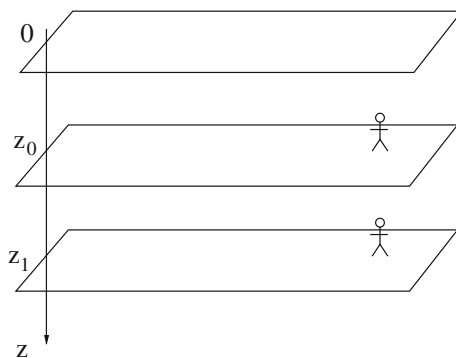
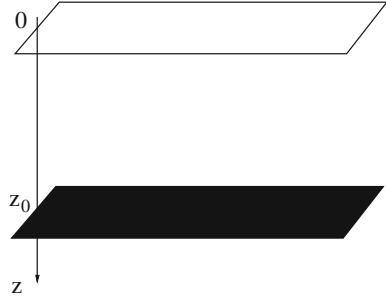


Fig. 7.5 A cartoon of AdS space. Using the parameterization in (7.6), an observer confined to $z = z_0$ would not be able to distinguish the surface she lives on from Minkowski space. However, when moving in the z direction the observer will notice that distances transverse to the z direction are warped. The boundary of AdS is located at $z = 0$

Fig. 7.6 A cartoon of an AdS black hole. The patch covered by the coordinate system in (7.9) covers the region of space from the boundary to the event horizon which is located at $z = z_0$



$$\langle T_{\mu\nu} \rangle = CN^2 T^4 \begin{pmatrix} 3 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad (7.8)$$

where C is a number which depends on the details of the theory.

A thermal state in the gauge theory is dual to a black hole [61]. The line element of an AdS₅ black hole is given by

$$ds^2 = \frac{L^2}{z^2} \left(-f(z) dt^2 + \sum_{i=1}^3 (dx^i)^2 + \frac{dz^2}{f(z)} \right) \quad (7.9)$$

where

$$f(z) = 1 - \left(\frac{z}{z_0} \right)^4 \quad (7.10)$$

The Hawking temperature of this black hole is $T = 1/(4\pi z_0)$ and is equal to the temperature of the thermal state of the gauge theory. If $f(z) = 1$, we obtain the AdS geometry given in (7.6). When $f(z)$ is as given in (7.10), the patch of spacetime we are interested in ends at $z = z_0$, where the event horizon of the black hole is located. See Fig. 7.6.

To use (7.5) we need to bring the metric in (7.9) to the form (7.4).¹ This can be achieved by using a coordinate ρ such that

$$\frac{L^2}{z^2} \frac{dz^2}{f} = \frac{L^2}{\rho^2} d\rho^2 \quad (7.11)$$

in which case the line element takes the form

$$ds^2 = \frac{L^2}{\rho^2} \left(\frac{\rho^4 - 4}{4(\rho^4 + 4)z_0^2} \right) dt^2 + \left(1 + \frac{\rho^4}{4z_0^2} \right) \sum_i (dx^i)^2 + \frac{L^2 d\rho^2}{\rho^2} \quad (7.12)$$

¹ Alternately, we can work in a covariant formalism such as the one discussed in [41].

Taking the appropriate $\rho \rightarrow 0$ limit of the metric components, we find, in the notation of (7.4), that

$$g_{tt} = -1 + \frac{3\rho^4}{4z_0^2} + \mathcal{O}(\rho^5) \quad g_{x^i x^i} = 1 + \frac{\rho^4}{4z_0^2}. \quad (7.13)$$

Thus, the energy momentum tensor is given by

$$\langle T_{\mu\nu} \rangle = \frac{\pi^2}{8} N^2 T^4 \begin{pmatrix} 3 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad (7.14)$$

which is precisely the form given in (7.14) with $C = \pi^2/8$.

Recall that our goal in this section is to compute the response of a thermal medium to a moving quark and to check whether the laminar wake is strong or weak. We know what the dual of a thermal state is—it is given by the black hole geometry in (7.9). What we need to do next is place a quark in the thermal medium. According to [21, 31, 41], a heavy (infinitely massive) quark is dual to the endpoint of an open string located at the asymptotically AdS boundary. A stationary quark, for example, would be dual to a string whose endpoint is standing still at the boundary of AdS. By symmetry, the string must hang straight down into the bulk of AdS. To formally derive the profile of the string we need to solve the string equation of motion, with appropriate boundary conditions.

The dynamics of the string are governed by the Nambu–Goto action:

$$S_{\text{NG}} = -\frac{1}{2\pi\alpha'} \int \sqrt{-|G_{\mu\nu} \partial_\alpha X^\mu \partial^\alpha X^\nu|} d\tau d\sigma \quad (7.15)$$

where G is the spacetime metric and X is the string embedding function. The boundary conditions we impose on the string is that its endpoint on the boundary of AdS is stationary. Just from symmetry, it should be clear that the only solution to the equations of motion is one where the string hangs straight down into AdS (see Fig. 7.7). Thus, if we use a gauge where $\tau = X^0$ and $\sigma = X^5$, we find that the solution to the string equations of motion must be:

$$X^\mu = (\tau \quad 0 \quad 0 \quad 0 \quad \sigma). \quad (7.16)$$

Of more relevance to the problem at hand is a moving quark. i.e., a string whose endpoint is not stationary. The simplest case to consider is a configuration where the string endpoint is moving at constant velocity. In this case, if we insert the ansatz

$$X^\mu = (t \quad vt + \xi(z) \quad 0 \quad 0 \quad z) \quad (7.17)$$

into the Nambu–Goto action (7.15), we find

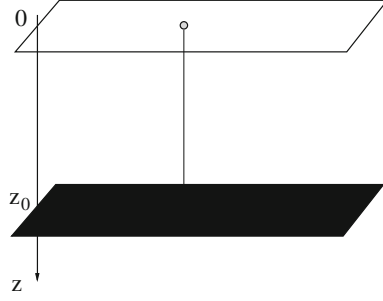


Fig. 7.7 A sketch of an open string hanging into AdS space. The string endpoint located at the AdS boundary is stationary and its other end stretches into the black hole. Since the boundary conditions and the equations of motion are rotationally invariant then the solution should have the same symmetry. Thus, we conclude that the string hangs from its endpoint straight down into AdS space

$$S_{\text{NG}} = -\frac{1}{2\pi\alpha'} \int \frac{L^2}{z^2} \sqrt{f(z)(\xi'(z))^2 + 1 - \frac{v^2}{f(z)}} dz dt \quad (7.18)$$

$$= -\frac{1}{2\pi\alpha'} \int \mathcal{L}_{\text{NG}} dz dt. \quad (7.19)$$

Since the action (7.15) does not depend explicitly on ξ but only on ξ' , then there is a conserved quantity Π_ξ given by

$$\Pi_\xi \equiv \frac{\partial \mathcal{L}_{\text{NG}}}{\partial \xi'} = \frac{L^2 f(z) \xi'(z)}{z^2 \sqrt{f(z)(\xi')^2 + 1 - v^2/f(z)}}. \quad (7.20)$$

Inverting (7.20) we obtain the following equation of motion for ξ' ,

$$(\xi')^2 = \frac{\Pi_\xi^2 (v^2 - f(z))}{f(z)^2 \left(\Pi_\xi^2 - \frac{L^4}{z^4} f(z) \right)} \quad (7.21)$$

Unless the solution is trivial, $\Pi_\xi = 0$, the only way in which $(\xi')^2$ can be real is that the numerator and denominator on the right hand side of (7.21) flip sign simultaneously. Thus,

$$\Pi_\xi = \frac{L^4}{z_*^4} v^2 \quad (7.22)$$

where z_* is defined through

$$f(z_*) = v^2. \quad (7.23)$$

With this value of Π_ξ the solution to (7.21) is

$$\xi(z) = \frac{v z_0}{4} \left(\ln \frac{1 - \frac{z}{z_0}}{1 + \frac{z}{z_0}} + 2 \arctan \frac{z}{z_0} \right) \quad (7.24)$$

Geometrically what we find is that the string is trailing behind its endpoint. See Fig. 7.8.

At this point we have all the ingredients to complete our computation: we know how to compute the stress tensor of the boundary theory QGP and we know how to represent a moving quark in this background. What we are looking for is the response of the QGP to the moving quark. From the previous discussion this maps into the response of the metric to the moving string: like a point particle, the string warps space-time as it moves through it. In other words, as the string moves through AdS-space it will source the metric. The metric fluctuations near the boundary map into the response of the medium to the moving quark. See Fig. 7.9.

Fig. 7.8 A sketch of an open string hanging into AdS space. The string endpoint located at the AdS boundary and is moving at constant velocity v to the left. The causal solution to the equations of motion is that the rest of the string trails behind its endpoint

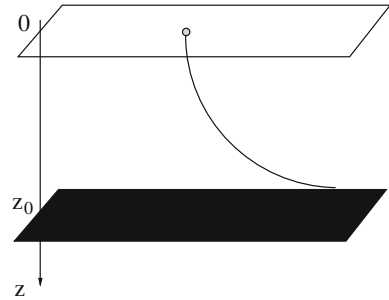
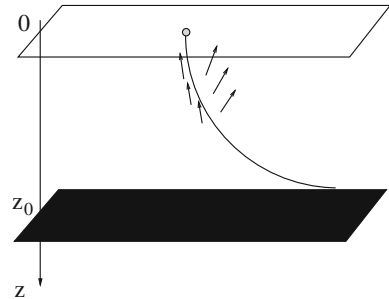


Fig. 7.9 A sketch of an open string sourcing the metric as it moves through AdS space. The near boundary value of the response of the metric to the moving string translates into the response of the medium to the moving quark



To solve the coupled, string + gravity, problem, consider the total action

$$\begin{aligned}
 S &= \frac{1}{16\pi G_N} \int \sqrt{-G} \left(R + \frac{12}{L^2} \right) d^5x - \frac{1}{2\pi\alpha'} \int \sqrt{-G\partial X\partial\bar{X}} d\tau d\sigma \\
 &= \frac{1}{16\pi G_N} \int \left[\sqrt{-G} \left(R + \frac{12}{L^2} \right) \right. \\
 &\quad \left. - \frac{16\pi G_N}{2\pi\alpha'} \int \sqrt{-G\partial X\partial\bar{X}} \delta^{(5)}(x^\mu - X^\mu) d\tau d\sigma \right] d^5x.
 \end{aligned} \tag{7.25}$$

The Einstein equations are

$$R^{\mu\nu} - \frac{1}{2} G^{\mu\nu} R - \frac{6}{L^2} G_{\mu\nu} = \mathcal{T}^{\mu\nu} \tag{7.26}$$

where

$$\mathcal{T}^{\mu\nu} = -\frac{8\pi G_N}{2\pi\alpha'} \int \frac{\sqrt{-|G\partial X\partial\bar{X}|}}{\sqrt{-G}} \delta^{(5)}(x^\mu - X^\mu) \partial_\alpha X^\mu \partial_\alpha X^\nu d\tau d\sigma. \tag{7.27}$$

We will not write out explicitly the equations of motion for the string. The AdS/CFT dictionary tells us that

$$G_N = \frac{\pi L^3}{2N^2} \quad \alpha' = \frac{L^2}{\sqrt{\lambda}}, \tag{7.28}$$

so

$$\frac{G_N}{\alpha'} \sim \frac{\sqrt{\lambda}}{N^2}. \tag{7.29}$$

In what follows we will work in the limit where

$$\frac{\sqrt{\lambda}}{N^2} \ll 1. \tag{7.30}$$

The limit (7.30) is useful since it implies that the energy momentum tensor of the string $\mathcal{T}^{\mu\nu}$ decouples from gravity and we can treat it as a linear perturbation. From the boundary theory point of view we will be looking at the linear response of the medium to the moving quark.

To proceed consider the expansion

$$G_{\mu\nu} = G_{\mu\nu}^{(0)} + \frac{\sqrt{\lambda}}{N^2} G_{\mu\nu}^{(1)} + \mathcal{O}\left(\frac{\lambda}{N^4}\right). \tag{7.31}$$

To leading order, the Einstein equation reduces to

$$R^{(0)\mu\nu} - \frac{1}{2} G^{(0)\mu\nu} R^{(0)} - \frac{6}{L^2} G^{(0)\mu\nu} = 0 \tag{7.32}$$

whose solution is the AdS black hole we have encountered previously,

$$G_{\mu\nu}^{(0)} dx^\mu dx^\nu = \frac{L^2}{z^2} \left(-f dt^2 + \sum_{i=1}^3 (dx^i)^2 + \frac{dz^2}{f} \right). \quad (7.33)$$

Since at this order the metric does not back-react on the string, the solution to the string equations of motion is the same as before,

$$X^{(0)\mu} = (t, vt + \zeta(z), 0, 0, z) \quad (7.34)$$

where we have expanded the embedding function X^μ in the same way we have expanded the metric, c.f., Eq. 7.33. The function $\zeta(z)$ is given in (7.24).

At the next order in $\sqrt{\lambda}/N^2$ the equation of motion for the metric takes the form

$$\begin{aligned} & \frac{\sqrt{\lambda}}{N^2} \left(R^{(1)\mu\nu} - \frac{1}{2} G^{(1)\mu\nu} R^{(0)} - \frac{1}{2} G^{(0)\mu\nu} R^{(1)} - \frac{6}{L^2} G^{(1)\mu\nu} \right) \\ &= -\frac{4\pi G_N}{\pi\alpha'} \int \frac{\sqrt{-|G^{(0)} \partial X^{(0)} \partial X^{(0)}|}}{\sqrt{-|G^{(0)}|}} \delta^{(5)}(X^\mu - x^\mu) \partial_\alpha X^{(0)\mu} \partial^\alpha X^{(0)\nu} d\tau d\sigma. \end{aligned} \quad (7.35)$$

Written in components, the resulting equation is a rather complicated second order linear differential equation. There are many good references that explain how to decouple the various equations [16, 20, 23, 24]. One straightforward way to simplify them is to

- Work in Fourier space,

$$\widehat{G}_{\mu\nu}^{(1)} = \int G_{\mu\nu}^{(1)} e^{-i(k_1(x^1 - vt) + k_2 x^2 + k_3 x^3)}, \quad (7.36)$$

so that the partial differential equation reduces to an ordinary differential equation.

- Fix a gauge. A useful gauge is $G_{z\mu}^{(1)} = 0$ which one might call the axial gauge.
- Use the rotational symmetry in the x^2, x^3 plane to set $G_{\mu 3}^{(1)} = 0$ ($\mu \neq 3$).

Even after implementing these points (or using the fancier methods described in [24]) the equations of motion are still analytically intractable. (Though solutions can be obtained for very small or very large momentum see [20, 23, 62, 63].) To proceed, one must resort to numerics.

In most of what remains of this section we will present the final result of the numerical analysis. Equation 7.35 was solved numerically and the near boundary asymptotics of the metric (7.5),

$$\langle T_{mn} \rangle = \frac{L^3}{4\pi G_N} g_{mn}^{(4)}, \quad (7.37)$$

where used in order to translate the numerical solution into a boundary theory stress tensor of the form

$$\langle T_{mn} \rangle = \langle T_{mn}^{(0)} \rangle + \frac{\sqrt{\lambda}}{N^2} \langle T_{mn}^{(1)} \rangle + \dots \quad (7.38)$$

In Fig. 7.10 we have plotted the numerical expression for the energy density $\langle T_{00}^{(1)} \rangle$ for $v = 3/4 > 1/\sqrt{3}$. In this contour plot dark regions signify a positive energy density relative to the background value $\langle T_{00}^{(0)} \rangle$ which has been subtracted. Light regions correspond to negative energy. The location of the Mach cone is signified by the dashed line. Clearly, the large distance asymptotics match onto the expected hydrodynamic behavior, a property which can be confirmed analytically by carrying out a large momentum expansion. Since we have good control over the asymptotic behavior of the solution, we can zoom in on the region near the quark. At scales which are roughly three orders of magnitude smaller than the ones in Fig. 7.10 we find that a new structure emerges. This can structure is depicted in Fig. 7.11. The dashed line in this figure signifies the location of the Mach cone. The fact that the features of this plot are not associated with the Mach cone provides evidence that the physical mechanism responsible for the energy distribution close to the quark does not have a hydrodynamical origin. Further evidence for non hydrodynamical behavior is given by the extra lobes, appearing at velocities above the speed of sound. In fact, there is an apparent transition from a configuration where there is a region of energy depletion behind the quark to a configuration where there is a region of energy depletion in front of the quark. The physical mechanism responsible for this behavior is unclear at the moment. But, as we will see shortly, it might play an important role in the jet-splitting effect.

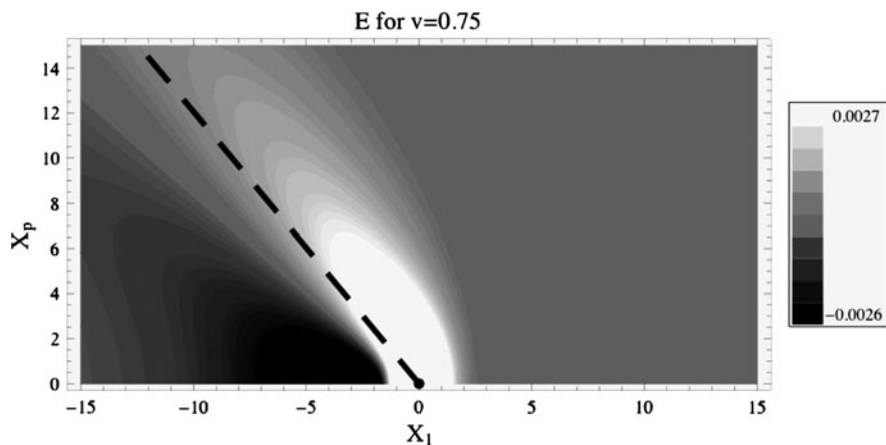


Fig. 7.10 (Reproduced from [24].) A contour plot of the energy density $\langle T^{00} \rangle$ surrounding a massive quark moving in the $\mathcal{N} = 4$ SYM plasma. The quark is located at the origin. *Dark regions* correspond to a higher energy density the surrounding plasma and *light regions* to a lower energy density. The *dashed line* signifies the location of the Mach cone

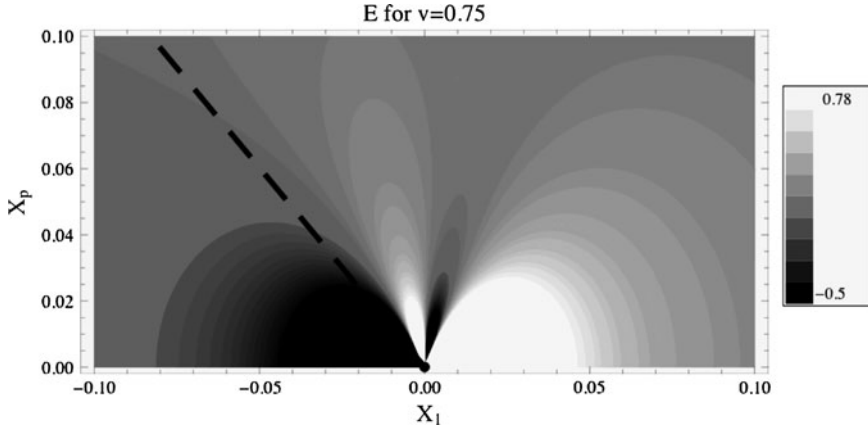


Fig. 7.11 (Reproduced from [24].) Energy density for a massive quark moving through an $\mathcal{N} = 4$ SYM plasma. The quark is located at the origin. The extra lobes which can be observed in the figure and the mismatch between the location of the Mach cone and the regions of over-energy indicate that the physical mechanism responsible for the near field is not hydrodynamical in origin

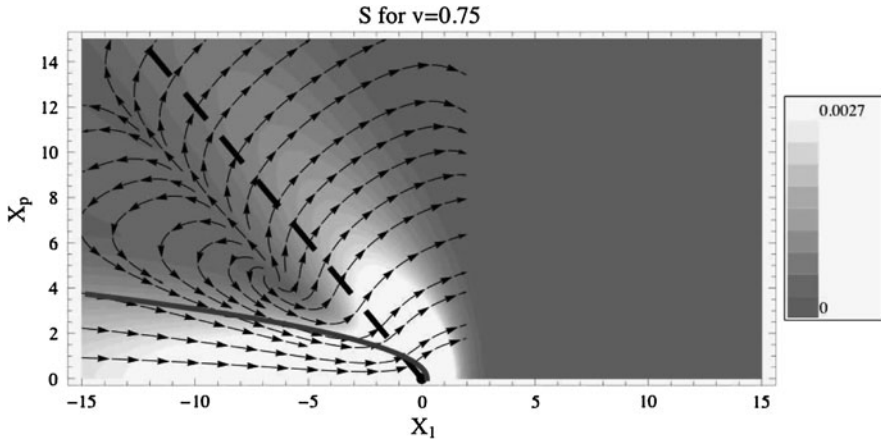


Fig. 7.12 (Reproduced from [24].) The Poynting vector $S^i = \langle T^{0i} \rangle$ for a massive quark moving through an $\mathcal{N} = 4$ SYM plasma. The quark is located at the origin. The magnitude of the Poynting vector is color coded from light to dark. The *arrows* signify its direction. The *dashed* and *thick lines* specify the location of the Mach cone and laminar wake—according to a hydrodynamic analysis

We had set out to find the strength of the wake behind the moving quark. This wake manifests itself as a flux of energy far behind it. It can be observed by computing the Poynting vector $\langle T^{0i} \rangle$. From Fig. 7.12, we see that there is a strong wake behind the quark. A closer study shows that the far-field behavior of the

wake is precisely the one expected of a point particle [25, 27, 29, 30]. Thus, a heavy quark in $\mathcal{N} = 4$ SYM generates a strong wake behind it—so strong that once the QGP hadronizes, instead of a minimum at $\Delta\Phi = \pi$ a broad-shoulder like structure should appear. At this point it should be mentioned that this sort of behavior is not unique to the $\mathcal{N} = 4$ theory and that a strong wake will appear for any theory with a holographic dual [29, 30].

In [11, 51], the AdS/CFT analysis described so far was taken one step further. It is possible to estimate the final distribution of hadrons reaching the detector from the energy distribution of the QGP using a mathematical prescription which goes under the name of ‘Cooper–Frye hadronization’ [17]. Of course, the $\mathcal{N} = 4$ SYM theory does not have a confined phase, but nevertheless, it is interesting to ask what the hadron distribution would look like had the $\mathcal{N} = 4$ SYM plasma hadronized. Following the discussion in this section, one would expect that the appearance of a wake in these theories will result in a high degree of correlation between events reaching the detector at an angle of π (and with appropriate transverse momenta) and the trigger jet. It turns out that this is not the case. Surprisingly, even though there’s a strong wake, jet splitting does seem to occur. As discussed in [50], this effect comes about in an unexpected way: it is not the far field behavior that is responsible for the effect but rather the near field which was observed in [23, 62]. The relation between this observation and the actual data at RHIC is unclear. One might treat it as an indication that the correct physics responsible for the jet-splitting effect lies in the near field energy distribution around the quark and not in the far-field hydrodynamic behavior. Certainly, further study is warranted.

7.2 Entropy Production

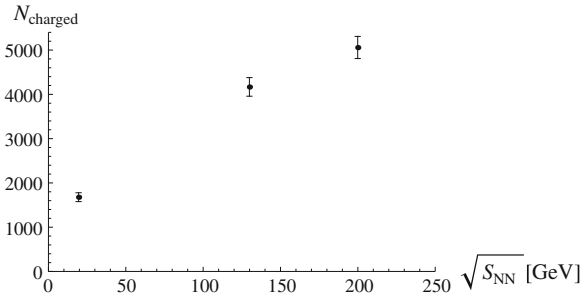
Information about the entropy produced during the collision process can be obtained by measuring the total particle multiplicity. Recall that before the collision we have two pancake shaped gold ions, and after the collision thousands of particles are produced (see Fig. 7.1). Certainly, a large amount of entropy is created in this process. One method of estimating the amount of entropy produced is to compute the entropy per hadron for a gas of hadrons at the deconfinement temperature, which according to [26, 52] is

$$s/N \sim 7.5, \quad (7.39)$$

and multiply this ratio by the total amount of charged hadrons reaching the detector. Figure 7.13 shows a plot of the total number of charged particles produced in the collision process as a function of the center of mass energy.

In this section we would like to construct a process dual to a collision and then use the AdS/CFT correspondence to estimate the amount of entropy produced. As is well known, the $\mathcal{N} = 4$ SYM theory is deconfined, so it would be difficult to

Fig. 7.13 Total number of charged particles reaching the detector as a function of center of mass energy. The total multiplicity is proportional to the amount of entropy produced during the collision process. Data taken from [7]



generate a collision of ions. But, one can consider a collision of two energetic distributions. If these distributions thermalize and generate a thermal state then entropy will be produced.

There are many setups one can use to generate a boundary theory configuration where a thermal state is produced. In this section we use the simplest setup we could think of: the collision of two light-like particles in the bulk of AdS (see Fig. 7.14). This is the simplest case to consider since, as we will see shortly, one has some control of the back-reacted geometry.

Consider the action of gravity coupled to a light-like particle:

$$S = \int \frac{1}{16\pi G_N} \int \sqrt{-|G|} \left(R + \frac{12}{L^2} \right) d^5x + \int \frac{1}{2e} G_{\mu\nu} \partial_\eta X^\mu X^\nu d\eta \quad (7.40)$$

where the term on the right hand side is the action for a point particle: e is the einbein on the particle worldline and η parameterizes it. To start, we wish to describe a single light like particle moving in the bulk of AdS at some constant depth $z = z_*$ (see Fig. 7.15).

The equation of motion for a light-like particle in AdS admits a solution in which the particle moves along a line of constant $z = z_*$. The equations of motion for the metric are

$$R_{\mu\nu} - \frac{1}{2} G_{\mu\nu} R - \frac{6}{L^2} G_{\mu\nu} = 8\pi G_N T_{\mu\nu}. \quad (7.41)$$

Fig. 7.14 A head-on collision of two light-like particles moving along the light-like geodesic $z = z_*$ in an AdS background

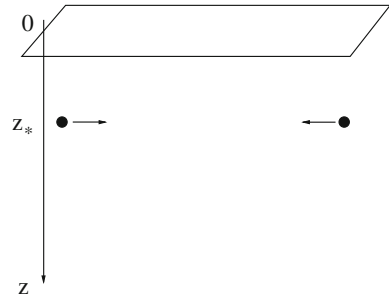
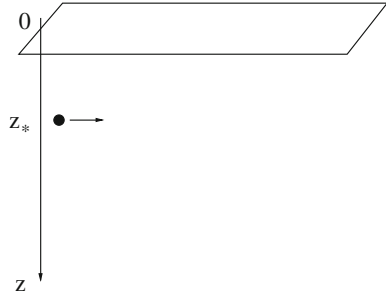


Fig. 7.15 A single lightlike particle moving along the lightlike geodesic $z = z_*$ in an AdS background



Working in light-like coordinates:

$$u = t - x^3, \quad v = t + x^3 \quad (7.42)$$

we find that

$$\mathcal{T}_{uu} = \frac{Ez^3}{L^3} \delta(u) \delta(z - z_*) \delta(x^1) \delta(x^2) \quad (7.43)$$

and all other components of the energy momentum tensor vanish. Plugging the ansatz

$$ds^2 = \frac{L^2}{z^2} \left(-dudv + \sum_{i=1}^2 (dx^i)^2 + dz^2 + \frac{z}{L} \Phi(x^1, x^2, z) \delta(u) du^2 \right) \quad (7.44)$$

into the equations of motion (7.41), one finds a single non trivial equation:

$$\left(\frac{z^3}{L^3} \partial_z \frac{1}{z} \partial_z + \frac{z^2}{L^2} (\partial_1^2 + \partial_2^2) - \frac{3}{L^2} \right) \Phi = - \frac{16\pi G_N E z^4}{L^4} \delta(z - z_*) \delta(x^1) \delta(x^2). \quad (7.45)$$

The solution is

$$\Phi(x^1, x^2, z) = \frac{2G_N}{L} \left(\frac{1 + 8q(1 + q) - 4\sqrt{q}1 + q(1 + 2q)}{\sqrt{q(1 + q)}} \right) \quad (7.46)$$

where

$$q = \frac{(x^1)^2 + (x^2)^2 + (z - z_*)^2}{4z_*z}. \quad (7.47)$$

Equation 7.46 gives the response of the metric to a moving light like particle. Moreover, it is a full solution to the fully back-reacted gravity plus lightlike particle system. One way to see this is to take a black hole solution and boost it to the speed of light [26]. The first study of such configurations in flat space-time can be found in [6]. Shock-waves in the context of AdS/CFT were discussed in [19, 33, 34, 54, 57].

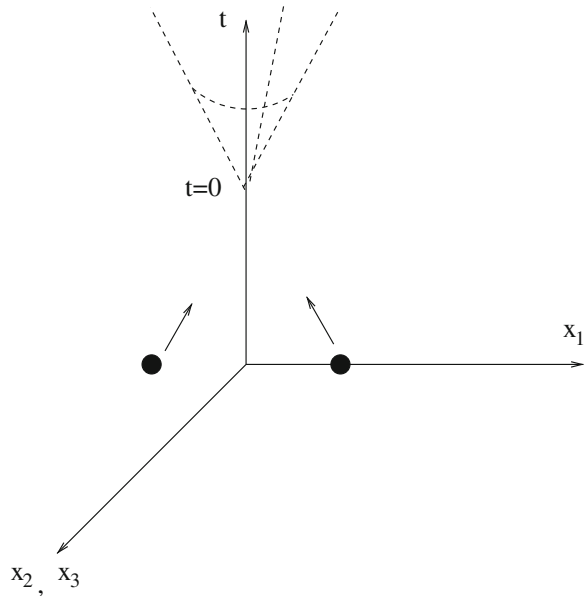
With the solution for a single point-like particle at hand we can easily construct a solution involving two lightlike particles, as depicted in Fig. 7.14—at least for spacetime points outside the future light cone of the collision. See Fig. 7.16. Outside this region causality tells us that we can add the solutions for the two lightlike particles. In other words the line element

$$ds^2 = \frac{L^2}{z^2} \left(-dudv + \sum_{i=1}^2 (dx^i)^2 + dz^2 + \frac{z}{L} \Phi \delta(v) dv^2 + \frac{z}{L} \Phi \delta(u) du^2 \right) \quad (7.48)$$

with Φ given in (7.46) is a solution to the Einstein equations in the presence of two lightlike particles in the region outside the future light-cone of the collision point.

We would eventually like to see a black hole form, and to compute its area which is dual to the entropy produced in the collision. To that end we need control over the metric in the future light cone of the collision. This is because the definition of an event horizon is non local and requires information on geodesics reaching future null infinity. Heuristically, the event horizon can be defined as the boundary of all the causal curves reaching future null infinity. Unfortunately, it is a difficult problem to solve for the metric in the future light cone of the collision point. So, instead of carrying out a direct computation, we follow a different strategy. There is another notion of a horizon called an apparent horizon or a marginally trapped surface. A marginally trapped surface (in a five dimensional spacetime) can be heuristically defined as a three dimensional surface for which the outward pointing null vector propagates neither inward nor outward and the other null vector propagates inward. A more formal definition of a marginally

Fig. 7.16 A spacetime diagram of a constant z slice of AdS space. Causality dictates that the two lightlike particles affect each other only on or inside the future light cone of the collision point



trapped surface is the following: let ℓ_μ and n_μ be the null normal vectors to the surface. Then, a marginally trapped surface satisfies:

$$\Theta = h^{\mu\nu} D_\nu \ell_\mu = 0. \quad (7.49)$$

Θ is called the expansion, and $h^{\mu\nu}$ is the induced metric on the surface. The virtue of a trapped surface is that its definition is local, so one can compute it even without knowing the full geometry. Barring some issues regarding cosmic censorship, one can argue that a trapped surface will also always be on or inside an event horizon. The latter property turns out to be very useful; the area increase theorems of general relativity tell us that the area of an event horizon can only increase. Thus, we can use the following strategy to obtain a lower bound on the area (and thus the entropy) of the black hole which forms in a collision of two lightlike particles: We find a marginally trapped surface and compute its area. Since the marginally trapped surface is located inside or on the event horizon of the black hole, the area of the black hole should be bigger or equal to the area of the trapped surface. Following the area increase theorem, this area provides a lower bound on the final area of the black hole and so on the entropy of the configuration.

Thus, our main challenge reduces to finding a marginally trapped surface. In what follows we use an ansatz first suggested by Penrose [53], and elaborated on in [18, 64]. The ansatz is that the trapped surface is composed of two parts. Part I is the surface $u = 0$ and $v = -\psi(q)$, where q is as given in (7.47) and ψ is a function which will be determined shortly. Part II of the surface is given by $v = 0$ and $u = -\psi(q)$. We will find ψ by requiring that the expansion, defined in (7.49), vanishes on the surface. A null normal to part I of the surface is given by

$$\ell_\mu^{(I)} dx^\mu = A du + B(dv + d\psi). \quad (7.50)$$

Requiring that this vector is lightlike and outward pointing implies that²

$$A = -\frac{1}{4}(\partial\psi)^2 \quad B = -\frac{z^2}{L^2}. \quad (7.51)$$

The inward pointing null vector is given by

$$n_\mu^{(I)} dx^\mu = -\frac{1}{4}(\partial\psi)^2 du + \frac{z^2}{L^2}(dv + d\psi). \quad (7.52)$$

From symmetry the normals in part II of the surface will take the form

$$n_\mu^{(II)} dx^\mu = -\frac{1}{4}(\partial\psi)^2 dv + \frac{z^2}{L^2}(du + d\psi). \quad (7.53)$$

² Note that the metric is singular at $u = 0$ and $v < 0$. In order for the metric to be finite we used the coordinate transformation $v \rightarrow v + \frac{z}{L}\phi(q)\Theta(u)$ with Θ the Heaviside step function which is unity when its argument is positive.

$$\ell_\mu^{(II)} dx^\mu = -\frac{1}{4}(\partial\psi)^2 dv - \frac{z^2}{L^2}(du + d\psi). \quad (7.54)$$

Once we have the null normals, to compute the expansion we need the induced metric $h_{\mu\nu}$ which, by definition, is orthogonal to both normals. Using an ansatz

$$h_\mu^v = \delta_\mu^v + A\ell_\mu\ell^v + B(\ell_\mu n^v + n_\mu\ell^v) + Cn_\mu n^v \quad (7.55)$$

for the induced metric, the unknown functions A , B and C can be determined through the condition that

$$h_\mu^v \ell_v = 0 \quad \text{and} \quad h_\mu^v \ell_v = 0. \quad (7.56)$$

With ℓ and h at hand we can compute the expansion and find a function ψ for which the expansion vanishes. Defining $\Psi = \frac{L}{z}\psi$ the vanishing of the expansion, $\Theta = 0$, is equivalent to

$$\left(q(1+q)\partial_q^2 + \frac{3}{2}(1+2q)\partial_q - 3 \right) (\Psi - \Phi) = 0 \quad (7.57)$$

where Φ is given by (7.46) and q is as in (7.47).³ The boundary conditions implied by continuity of the normals are

$$\Psi|_C = 0 \quad \text{and} \quad (\partial\Psi)^2|_C = 4. \quad (7.58)$$

where C is the mutual boundary of regions I and II. Note (7.57) and (7.58) constitute a rather unusual boundary value problem since we are imposing both Dirichlet and Neumann boundary conditions. The equivalent electrostatic problem would be to solve the Laplace equation on a surface C where the potential vanishes and the electric field has a specified non zero value. The problem isn't over-determined since to solve such a problem we need to find both the surface C and the electric potential. Going back to the problem at hand, we note that the homogenous equation (7.57) admits two solutions only one of which is finite at the origin:

$$\Psi - \Phi = (1+2q)K \quad (7.59)$$

where K is an undetermined integration constant. From symmetry, the boundary C should be a surface of constant q , $q = q_C$. Plugging the general solution (7.59) into the boundary conditions (7.58) we find that the latter turn into algebraic constraints,

³ It is no coincidence that the resulting equation depends only on q . The variable q can be thought of as a radial variable of the AdS hyperboloid which one finds when embedding AdS₅ in $\mathbf{R}^{4,2}$.

$$\frac{EG_N}{L} = 2q_c(1 + q_c)(1 + 2q_c) \quad K = -\frac{\Phi(q_c)}{1 + 2q_c}. \quad (7.60)$$

Solving (7.60) we can find both Ψ and the surface C .

Recall that our goal was to find a marginally trapped surface, compute its area, and use that as a lower bound on the entropy of the black hole produced. The area of the trapped surface is given by

$$A = \int \sqrt{h_S} d^3x = \cdots = 4\pi L^3 \int_0^{x_c} \frac{x^2 dx}{\sqrt{1 + x^2}} \quad (7.61)$$

where $x_c = 2\sqrt{q_c(1 + q_c)}$. This integral can be computed explicitly, but we will only need its asymptotic form:

$$\frac{2A}{4G_N} = \pi \left(\frac{L^3}{G_N} \right) (2Ez_*)^{2/3} \left(1 + \mathcal{O}\left((Ez_*)^{-1}\right) \right). \quad (7.62)$$

Using the Bekenstein–Hawking law for the black hole entropy we find that a lower bound on the entropy of the resulting black hole is given by (7.62),

$$S_b = \frac{A}{2G_N} \quad (7.63)$$

We now have an exact solution to a well defined problem in classical gravity. The next step in our analysis is to map this into a boundary quantities. First, consider the early stages of the collision. Recall from (7.5) that the boundary theory stress tensor is related to the metric through

$$\langle T_{mn} \rangle = \frac{L^3}{4\pi G_N} g_{mn}^{(4)}, \quad (7.64)$$

Expanding the metric in (7.44) in a near boundary Taylor series and extracting the appropriate coefficient in the series expansion we find that

$$\langle T_{uu} \rangle = \frac{2Ez_*^4}{\pi(x_\perp^2 + z_*^2)^3} \delta(u) \quad \langle T_{vv} \rangle = \frac{2Ez_*^4}{\pi(x_\perp^2 + z_*^2)^3} \delta(v). \quad (7.65)$$

Thus, from the boundary point of view we are looking at the collision of two energetic objects moving at the speed of light and with an energy distribution which is spread in the transverse directions. The total energy of such a configuration is given by E and the energy averaged RMS size of the distribution is given by $z_*^2 = \int \langle T_{uu} \rangle x_\perp^2 d^3x$. It is natural to identify the energy of the lightlike particle E with the beam energy,

$$E = 19.7 \text{ TeV} \quad (7.66)$$

and to identify z_* with its QCD counterpart which can be estimated using a standard exponential-like distribution of the nucleus, see, for example [1, 40]. We find

$$z_* = 4.3 \text{ fm} \quad (7.67)$$

To fully convert the entropy bound (7.63) to a boundary quantity we need to identify G_N with N^2 . For the $\mathcal{N} = 4$ SYM theory one has

$$G_N = \frac{\pi L^3}{2N^2}. \quad (7.68)$$

A more general version of (7.68) is

$$G_N = \frac{V_5}{2\pi N^2} \quad (7.69)$$

FIX! where V_5 is the volume of a five dimensional compact space. Assuming V_5 is tunable, we have some freedom in choosing G_5/L^3 .⁴ Recall from (7.5) that

$$\langle T_{00} \rangle = \frac{3L^3}{16\pi G_N z_0^4}. \quad (7.70)$$

This can be compared with the lattice prediction,

$$\langle T_{00} \rangle_{\text{lattice}} \sim 11T^4 \quad (7.71)$$

for QCD. Comparing (7.70) with (7.71) we obtain

$$\frac{L^3}{G_5} \sim 2 \quad (7.72)$$

(which is unfortunately small.)

Using (7.66), (7.67) and (7.72) in (7.62), we find

$$S_b = 35,000 \left(\frac{\sqrt{s_{NN}}}{200 \text{ GeV}} \right)^{2/3}. \quad (7.73)$$

Figure 7.17 shows a comparison of the data obtained at RHIC with the lower bound on the entropy [converted to total multiplicity using (7.39)]. While the bound is in surprisingly good agreement with the data, there is some cause for worry. More traditional models relating the total entropy produced to the beam energy predict a scaling behavior of the form $S_b \sim E^{1/2}$ instead of the $E^{2/3}$ scaling given in (7.73). While the current data agrees with both the $2/3$ scaling and the $1/2$ scaling, at energies as high as the LHC energies, the prediction of (7.73) is

⁴ Until now we have ignored the compact five dimensional manifold attached to every point in AdS_5 space.

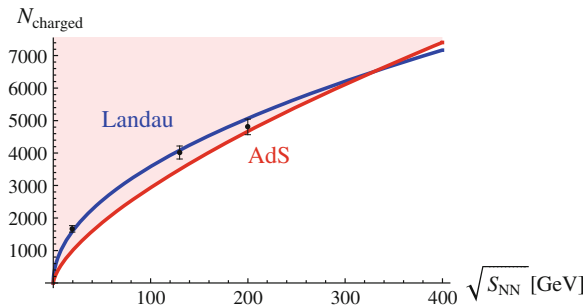


Fig. 7.17 (Reproduced from [26].) Various models which predict total multiplicity as a function of center of mass energy. The bound on the *shaded region* marks the allowed region according to the AdS/CFT model described in the main text. The *thick blue line* follows from Landau's model [41] which predicts an $S_b \sim E^{1/2}$ behavior. The data points were taken from [7]

150% higher than the traditional predictions. It is tempting to speculate that this is not a problem but a prediction of AdS/CFT. However, a closer look at the approximation being made shows us that, in the particular problem we are dealing with, the relation between the gauge-gravity duality and QCD is, perhaps, being pushed too hard. There are two important features of QCD that the AdS/CFT duality does not capture. One is asymptotic freedom and the other is confinement. Both these features of QCD may play an important role in the early stages of the collision.

So how can we estimate what effect these features have on the entropy bound? As a general rule of thumb the infrared physics is captured by the geometry far from the boundary (large values of z) while the ultraviolet physics are captured by the near boundary geometry. Thus, as a very (very) coarse approximation of UV and IR effects we can simply chop off the parts of the trapped surface below a certain IR cutoff and above a UV cutoff. As depicted in Fig. 7.18, the result is that

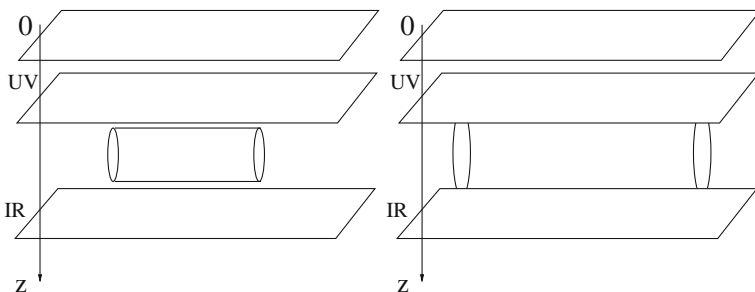


Fig. 7.18 The effects of the UV and IR cutoffs on the area of the trapped surface. The surface is depicted as a cylinder. Once the energy of the collision is large enough, the trapped surface enters the UV and IR regions where its area gets modified

Fig. 7.19 A caricature of a collision of two gold ions with a non trivial impact parameter b

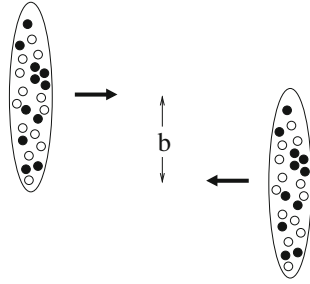
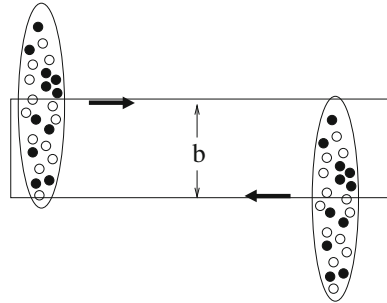


Fig. 7.20 A cartoon of an off center collision of two heavy ions. The nucleons outside the interaction region (*boxed*) are “spectators” and do not participate in the collision process



the area of the trapped surface will decrease as it enters the UV and IR region.⁵ With $\Lambda_{\text{UV}} = 2 \text{ GeV}$ and $\Lambda_{\text{IR}} = 0.2 \text{ GeV}$, we find that at RHIC energies $S_b \sim E^{2/3}$ as before, but at LHC energies the power law dependence of S_b on E decreases appreciably until it reaches $S_b \sim E^{1/3}$ [28].

Until now we have focused on central collisions. It is also possible to consider off center ones, i.e., ones where the impact parameter is non zero (see Fig. 7.19). Ideally one would like to measure the total particle multiplicity as a function of the impact parameter. In practice the impact parameter is difficult to measure. What is measured instead is the number of nucleons participating in the collision N_{part} . Due to confinement, only those nucleons that are in the interaction region will participate in the collision. This is depicted in Fig. 7.20. The other nucleons are “spectators” and are detected by the paddle detectors located along the beam axis. In practice, the data is usually presented in a plot such as the one in 7.21 where the dependance of the ratio of the total multiplicity to the number of participating nucleons is presented as a function of the number of participating nucleons.

From the point of view of the gauge-gravity duality, generating an off center collision of light-like particles is straightforward: One considers colliding light-like particles with a non trivial impact parameter. From the bulk point of view the impact parameter can be both in the radial AdS direction z , and in the

⁵ Actually, a closer analysis shows that the UV region is more important than the IR region. This is because the AdS warp factor L/z diverges near the boundary.

transverse x^2 , or x^3 directions. The former would correspond to a head-on collision of different sized energy distributions and the latter to an off center collision of identical nuclei. It is interesting that in a conformal theory these two setups are related by a conformal transformation. In what follows we will focus on an impact parameter in the x^2, x^3 plane.

When dealing with an off center collision, one complication is that the initial configuration, and the shape of the trapped surface have less symmetry. One way around this is to solve the problem numerically [42]. Another is to work perturbatively in the impact parameter [28]. In the latter case, one finds that

$$S_b = 35,000 \left(\frac{\sqrt{s_{NN}}}{200 \text{ GeV}} \right)^{2/3} \frac{\sinh^{-1} \beta}{\beta \sqrt{1 + \beta^2}} \quad (7.74)$$

where β is related to the impact parameter b in the boundary theory through

$$\beta = 0.12b/\text{fm}. \quad (7.75)$$

To compare equation (7.74) with the data, we need to somehow convert the value of the impact parameter to the number of participating nucleons. This can be done using a standard optical-Glauber method. We once again refer the reader to the literature [28, 38, 39, 45] for details.

In Fig. 7.21 we have plotted the lower bound on the entropy given in (7.74) together with the data. As can be clearly seen, once the impact parameter becomes large the dependance of N on N_{part} deviates from a linear one. Even if we include a UV and IR cutoff the situation does not improve by much. One possible reason for the deviation from linearity is that in the strongly interacting conformal theory there are no spectator particles: as opposed to a collision of two ions, after the collision, spatially separated distributions will still interact. To somehow take the “spectators effect” into account we can identify the photon energy E not with

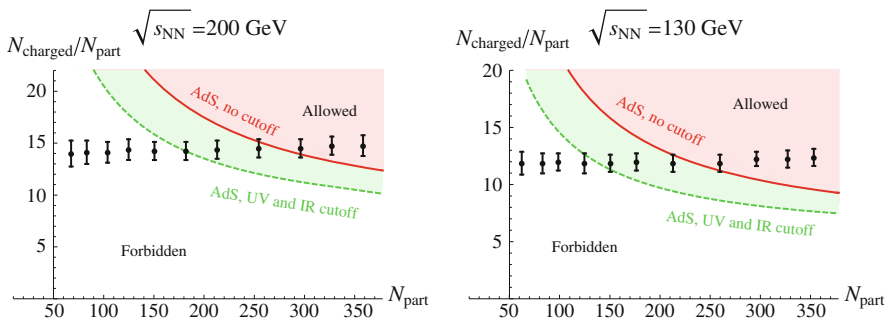


Fig. 7.21 (Reproduced from [28].) Ratio of the total multiplicity N_{charged} to the number of participating nucleons N_{part} as a function of N_{part} . The inner red region marks the allowed region according to the AdS/CFT model without imposing a UV and IR cutoff. The outer green region marks the region which is allowed once a cutoff is introduced. The data was taken from [8]

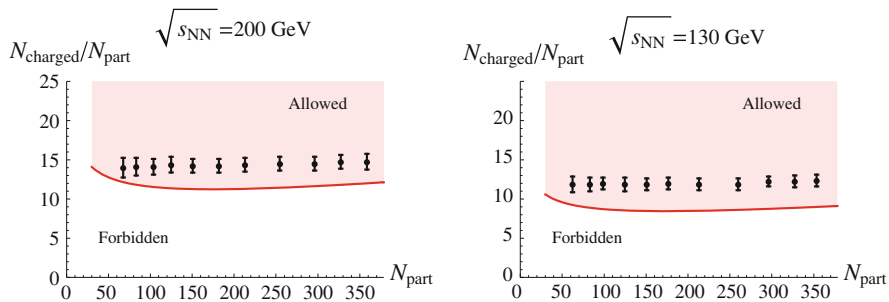


Fig. 7.22 (Reproduced from [28].) Ratio of the total multiplicity N_{charged} to the number of participating nucleons N_{part} as a function of N_{part} . The red region marks the allowed region according to the AdS/CFT model once one makes the replacement (7.76) which takes into account the non-participating nucleons, or spectators. See the main text for details. The data was taken from [8]

the beam energy but with the energy of the nucleons participating in the collision. Thus in (7.74) we should make the substitution

$$\sqrt{s_{NN}} \rightarrow \sqrt{s_{NN}} N_{\text{part}} / 394. \quad (7.76)$$

With this substitution the fit to the data seems much better. See Fig. 7.22. However, one can not escape the feeling that the manipulation that lead to this sort of fit is somewhat contrived and that one can do better. It would be interesting to carry out this analysis in a background which does capture effects such as confinement and asymptotic freedom in a controlled way.

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Chapter 8

AdS/CFT on the Brane

Jiro Soda

Abstract It is widely recognized that the AdS/CFT correspondence is a useful tool to study strongly coupled field theories. On the other hand, Randall-Sundrum (RS) braneworld models have been actively discussed as a novel cosmological framework. Interestingly, the geometrical set up of braneworlds is quite similar to that in the AdS/CFT correspondence. Hence, it is legitimate to seek a precise relation between these two different frameworks. In this lecture, I will explain how the AdS/CFT correspondence is related to the RS braneworld models. There are two different versions of RS braneworlds, namely, the single-brane model and the two-brane model. In the case of the single-brane model, we reveal the relation between the geometrical and the AdS/CFT correspondence approach using the gradient expansion method. It turns out that the high energy and the Weyl term corrections found in the geometrical approach correspond to the CFT matter correction found in the AdS/CFT correspondence approach. In the case of two-brane system, we also show that the AdS/CFT correspondence play an important role in the sense that the low energy effective field theory can be described by the conformally coupled scalar-tensor theory where the radion plays the role of the scalar field. We also discuss dilatonic braneworld models from the point of view of the AdS/CFT correspondence.

8.1 Introduction

It is believed that string theory is a candidate of the unified theory of everything. Remarkably, string theory can be consistently formulated only in 10 dimensions [1].

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This fact requires a mechanism to fill the gap between our real world and the higher dimensions. Conventionally, the extra dimensions are considered to be compactified to a small compact space of the order of the Planck scale. However, recent developments of superstring theory invented a new idea, the so-called braneworld where matter resides on the hypersurface in higher dimensional spacetime [2–6] (see also earlier independent works [7, 8]). This hypersurface is called (mem)brane. This idea originates from D-brane solutions in string theory. Interestingly, the D-brane solution also gives rise to the AdS/CFT correspondence which claims that classical gravity in an anti-de Sitter(AdS) spacetime is equivalent to a strongly coupled conformally invariant field theory (CFT). Since the origin is the same, braneworlds and the AdS/CFT correspondence may be related to each other. In particular, Randall and Sundrum (RS) braneworld models [9, 10] have a similar geometrical setup to that in the AdS/CFT correspondence. Hence, in this lecture, I will try to reveal relations between the AdS/CFT correspondence and RS braneworld models [11].

The method we will use is the gradient expansion method. Physically, it is a low energy expansion method. Historically, the method has been used in the cosmological context [12–15]. In particular, it is known to be useful for analyzing the evolution of cosmological perturbations during inflation. Since the AdS spacetime can be regarded as the inflating universe in the spatial direction, the gradient expansion method is also expected to be useful in the AdS spacetime. First, by utilizing the gradient expansion method, we approximately solve Einstein equations in the bulk. Then, the junction conditions at the brane give the effective equations of motion on the brane. Thus, we can understand the low energy physics in the braneworld. The similar but slightly different method is also used in the AdS/CFT correspondence. We will identify a concrete relation between the geometrical and the AdS/CFT correspondence approach by detailed comparison. The difference shows up when we consider two brane systems. Indeed, we do not have the conventional AdS/CFT correspondence for the two-brane system. Instead, we have a conformally coupled radion on the brane which reflects the conformal symmetry of the theory. This observation is useful for understanding why brane inflation suffers from the eta problem. It is apparent that the gradient expansion method can be applicable to various braneworld models. Moreover, as we will see, the gradient expansion method provides a unified view of braneworlds and a useful tool to make cosmological predictions.

The organization of this lecture is as follows: In [Sect. 8.2](#), we introduce RS models and derive the effective Friedman equation on the brane. Here, two important corrections, i.e., the dark radiation and the high energy corrections, are identified. This sets our starting point. In [Sect. 8.3](#), we review two different views from the brane, namely, the AdS/CFT correspondence and the geometrical holography. These two frameworks give us a complementary picture of the braneworlds. In [Sect. 8.4](#), we present key questions to make our concerns manifest. In [Sect. 8.5](#), we review the gradient expansion method. In [Sects. 8.6](#) and [8.7](#), we apply the gradient expansion method to the single-brane model and to the two-brane model, respectively. We obtain the effective theory for both cases. In [Sect. 8.8](#), we give answers to the key questions. This completes explanation of

relations between the AdS/CFT correspondence and RS braneworld models. In Sect. 8.9, we extend our analysis to models with a bulk scalar field. Since the presence of bulk fields would break the conformal invariance, it is interesting to consider dilatonic braneworlds in conjunction with the AdS/CFT correspondence. The final section is devoted to the conclusion.

8.2 Braneworlds in AdS Spacetime

In this section, we will introduce RS braneworld models. We will derive the effective Friedmann equation and identify the effects of extra-dimensions. In this lecture, we will concentrate on cosmology although we can apply the results to black hole physics.

8.2.1 RS Models

Randall and Sundrum proposed a simple model where a four-dimensional brane with the tension σ is embedded in the five-dimensional asymptotically anti-de Sitter (AdS) bulk with a curvature scale ℓ . This single-brane model is described by the action [10]

$$S = \frac{1}{2\kappa^2} \int d^5x \sqrt{-g} \left(\mathcal{R} + \frac{12}{\ell^2} \right) - \sigma \int d^4x \sqrt{-h} + \int d^4x \sqrt{-h} \mathcal{L}_{\text{matter}}, \quad (8.1)$$

where $\mathcal{R}$ and κ^2 are the scalar curvature and gravitational constant in five-dimensions, respectively. We impose Z_2 symmetry on this spacetime, with the brane at the fixed point. The matter $\mathcal{L}_{\text{matter}}$ is confined to the brane. Throughout this lecture, $h_{\mu\nu}$ represents the induced metric on the brane. Remarkably, the internal dimension is non-compact in this model. Hence, we do not have to care about the stability problem. The basic equations consist of the equations of motion in the bulk and junction conditions at the brane position due to the presence of the brane. Alternatively, the basic equations can be regarded as the 5-dimensional Einstein equations with singular sources. Let us recall the 4-dimensional components of 5-dimensional Einstein tensor can be expressed by

$$G_{\mu\nu}^{(5)} = G_{\mu\nu}^{(4)} + \mathcal{L}_{n^\mu} [K_{\mu\nu} - g_{\mu\nu} K] + \cdots, \quad (8.2)$$

where $\mathcal{L}_{n^\mu}$ denotes the Lie derivative along the unit normal vector to the brane, n^μ . Here, we defined the extrinsic curvature by

$$K_{\mu\nu} = -\left(\delta_{\mu}^{\rho} - n_{\mu}n^{\rho}\right)\nabla_{\rho}n_{\nu}. \quad (8.3)$$

By integrating Einstein equations along the normal to the brane, we obtain the jump of the extrinsic curvature $(K_{\mu\nu}^{+} - g_{\mu\nu}K^{+}) - (K_{\mu\nu}^{-} - g_{\mu\nu}K^{-})$ from the left hand side and the total energy momentum tensor on the brane from the right hand side due to the delta function sources. Thus, taking into account the Z_2 symmetry $K_{\mu\nu} \equiv K_{\mu\nu}^{+} = -K_{\mu\nu}^{-}$, we obtain the junction conditions

$$\left[K_{\mu\nu}^{+} - \delta_{\mu\nu}K^{+}\right]\Big|_{\text{at the brane}} = \frac{\kappa^2}{2}(-\sigma\delta_{\mu\nu}^{+} + T_{\mu\nu}^{+}) \quad (8.4)$$

Here, $T_{\mu\nu}$ represents the energy-momentum tensor of the matter.

Originally, they proposed the two-brane model as a possible solution of the hierarchy problem [9]. The action reads

$$S = \frac{1}{2\kappa^2} \int d^5x \sqrt{-g} \left(\mathcal{R} + \frac{12}{\ell^2} \right) - \sum_{i=\oplus, \ominus} \sigma_i \int d^4x \sqrt{-h_i} + \sum_{i=\oplus, \ominus} \int d^4x \sqrt{-h} \mathcal{L}_{\text{matter}}^i, \quad (8.5)$$

where $\oplus$ and $\ominus$ represent the positive and the negative tension branes, respectively. In principle, one can consider multiple-branes although they are not discussed in this lecture.

8.2.2 Cosmology

The homogeneous cosmology of the single-brane model is easy to analyze [16]. Because of the Birkoff theorem due to the symmetry on the brane, it is sufficient to consider AdS black hole spacetime:

$$ds^2 = -h(r)dt^2 + \frac{dr^2}{h(r)} + r^2[d\chi^2 + f_k^2(\chi)(d\theta^2 + \sin^2\theta d\phi^2)], \quad (8.6)$$

where

$$f_k = \begin{cases} \sin \chi & \text{for } k = 1 \\ \chi & \text{for } k = 0 \\ \sinh \chi & \text{for } k = -1 \end{cases} \quad (8.7)$$

and

$$h(r) = k - \frac{M}{r^2} + \frac{r^2}{\ell^2}. \quad (8.8)$$

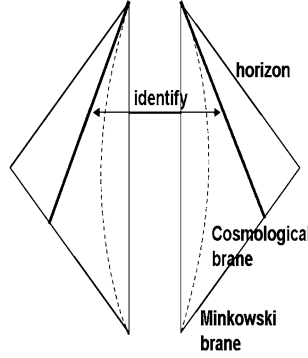


Fig. 8.1 The Minkowski brane represented by the dotted line is a static brane in the Poincaré coordinate system of the AdS spacetime. While, the cosmological brane represented by the thick line is moving in the bulk. The motion of the brane induces the expansion of the brane universe. The Cauchy horizon of AdS spacetime corresponds to the big-bang singularity. In the case of AdS-Schwarzschild spacetime, the horizon should be the past horizon of the black hole. The big bang is located beyond the horizon

Note that M is the mass of the black hole, k is the curvature of the horizon and ℓ is the AdS curvature radius. Suppose the brane is moving in this spacetime with a trajectory $t = t(\tau)$, $r = a(\tau)$, where τ is a proper time of the brane (see Fig. 8.1).

The induced metric on the brane becomes

$$ds_{ind}^2 = -d\tau^2 + a^2(\tau)[d\chi^2 + f_k^2(\chi)(d\theta^2 + \sin^2\theta d\phi^2)]. \quad (8.9)$$

This is nothing but the Friedman-Robertson-Walker spacetime where a is the scale factor. The motion of the brane cannot be arbitrary. It is constrained by the junction condition:

$$K^\chi_\chi = -\frac{\kappa^2\sigma}{6} - \frac{\kappa^2}{2}\left[T^\chi_\chi - \frac{1}{3}T\right], \quad (8.10)$$

where K^χ_χ , T^χ_χ , T are a $\chi\chi$ component of the extrinsic curvature, a $\chi\chi$ component and the trace part of the energy momentum tensor of matter on the brane, respectively. From the normalization condition $n^\mu n_\mu = 1$ of the unit normal vector $n^\mu = (-\dot{a}, \dot{t})$, we obtain

$$\dot{t} = \sqrt{\frac{1}{h(a)} + \frac{\dot{a}^2}{h^2(a)}}. \quad (8.11)$$

Here, the dot is a derivative with respect to the proper time τ . Now, one can calculate $K_{\chi\chi}$ as

$$K_{\chi\chi} = -\nabla_\chi n_\chi = -\partial_\chi n_\chi + \Gamma^r_{\chi\chi} = -\frac{1}{2}n^r \frac{\partial}{\partial\chi} g_{\chi\chi}. \quad (8.12)$$

Hence, from Eqs. 8.10, 8.11 and 8.12, we have an equation

$$\frac{1}{a} \sqrt{h(a) + \dot{a}^2} = \frac{\kappa^2}{6} (\sigma + \rho). \quad (8.13)$$

Thus, we get

$$\frac{k}{a^2} - \frac{M}{a^4} + \ell^2 + \frac{\dot{a}^2}{a^2} = \frac{\kappa^4}{36} (\sigma + \rho)^2, \quad (8.14)$$

or

$$H^2 = \frac{\kappa^4 \sigma^2}{36} - \frac{1}{\ell^2} + \frac{\kappa^4 \sigma}{18} \rho - \frac{k}{a^2} + \frac{M}{a^4} + \frac{\kappa^4}{36} \rho^2. \quad (8.15)$$

By setting $\kappa^2 \sigma \ell = 6$, we finally derived the effective Friedmann equation as [17–21]

$$H^2 = -\frac{k}{a^2} + \frac{\kappa^2}{3\ell} \rho + \frac{M}{a^4} + \frac{\kappa^4}{36} \rho^2, \quad (8.16)$$

where $H = \dot{a}/a$ is the Hubble parameter. The Newton's constant can be identified as $8\pi G_N = \kappa^2/\ell$. The curvature of the horizon k corresponds spatial curvature of the universe. The black hole mass M is referred to as the dark radiation [22] which is not real radiation fluid but a reflection of the bulk geometry. This effect exists even in the low energy regime. The last term represents the high energy effect of the braneworld [23].

As to the two-brane model, the same effective Friedmann equation (8.16) can be expected on each brane because the above Eq. 8.16 has been deduced without referring to the bulk equations of motion.

Given this cosmological background, it is natural to investigate cosmological perturbation in the braneworld [24]. In the case of the single-brane model, it is shown that the gravity in Minkowski brane is localized on the brane in spite of the noncompact extra dimension. Consequently, it turned out that the conventional linearized Einstein equation approximately holds at scales large compared with the curvature scale ℓ . It should be stressed that this result can be attained by imposing the outgoing boundary conditions. It turns out that this is also true in the cosmological background [25].

In the case of the two-brane model, Garriga and Tanaka analyzed linearized gravity and have shown that the gravity on the brane behaves as the Brans-Dicke theory at low energy [26]. Thus, the conventional linearized Einstein equations do not hold even on scales large compared with the curvature scale ℓ in the bulk. Charmousis et al. have clearly identified the Brans-Dicke field as the radion mode [27].

In the end, we would like to know how nonlinear gravity in the braneworld is deviated from the conventional Einstein gravity. A partial answer will be given in the following sections.

8.3 View from the Brane

In the previous section, we have considered an isotropic and homogeneous universe and seen that the effective Friedmann equation on the brane can be regarded as the conventional Friedmann equation with two kind of corrections, i.e., the dark radiation and high-energy corrections. Here, we review two different approaches to extend the above result to more general cases.

8.3.1 AdS/CFT Correspondence

Let us start with the AdS/CFT correspondence [28–31]. After solving the equations of motion in the bulk with the boundary value fixed and substituting the solution g_{cl} into the 5-dimensional Einstein-Hilbert action $S_{5\text{d}}$, we obtain the effective action for the boundary field $h = g_{\text{cl}}|_{\text{boundary}}$. The statement of the AdS/CFT correspondence is that the resultant effective action can be equated with the partition functional of some conformally invariant field theory (CFT), namely

$$\exp[iS_{5\text{d}}[g_{\text{cl}}]] \approx \langle \exp \left[i \int h \mathcal{O} \right] \rangle_{\text{CFT}}, \quad (8.17)$$

where $\mathcal{O}$ is the field in CFT. In the right hand side, h should be interpreted as a source field. This action must be defined at the AdS infinity where the conformal symmetry exists as the asymptotic symmetry. Hence, there exist infrared divergences which must be subtracted by counter terms. Thus, the correct formula becomes

$$\exp[iS_{5\text{d}}[g_{\text{cl}}] + iS_{\text{ct}}] = \langle \exp \left[i \int h \mathcal{O} \right] \rangle_{\text{CFT}}, \quad (8.18)$$

where we added the counter terms

$$S_{\text{ct}} = S_{\text{brane}} - S_{4\text{d}} - [R^2 \text{terms}], \quad (8.19)$$

where S_{brane} and $S_{4\text{d}}$ are the brane action and the 4-dimensional Einstein-Hilbert action, respectively. Here, the higher curvature terms $[R^2 \text{terms}]$ should be understood symbolically.

In the case of the braneworld, the brane acts as the cutoff. Therefore, there is no divergences in the above expressions. In other words, no counter term is necessary. We can regard the above relation as the definition of the “cut off” CFT. Thus, we can freely rearrange the terms as follows

$$S_{5\text{d}} + S_{\text{brane}} = S_{4\text{d}} + S_{\text{CFT}} + [R^2 \text{terms}], \quad (8.20)$$

where we have defined

$$\exp iS_{\text{CFT}} \equiv \langle \exp \left[i \int h \mathcal{O} \right] \rangle_{\text{CFT}}. \quad (8.21)$$

This tells us that the brane models can be described as the conventional Einstein theory with the cutoff CFT and higher order curvature terms [32–35]. In terms of the equations of motion, the AdS/CFT correspondence reads

$$G_{\mu\nu}^{(4)} = \frac{\kappa^2}{\ell} \left(T_{\mu\nu} + T_{\mu\nu}^{\text{CFT}} \right) + [R^2 \text{terms}], \quad (8.22)$$

where the R^2 terms represent the higher order curvature terms and $T_{\mu\nu}^{\text{CFT}}$ denotes the energy-momentum tensor of the cutoff version of conformal field theory. When we apply this result to cosmology, we see CFT corresponds to the dark radiation in the braneworld and the higher curvature terms can be reduced to the high-energy corrections.

8.3.2 Geometrical Holography

Here, let us review the geometrical approach [36]. In the Gaussian normal coordinate system:

$$ds^2 = dy^2 + g_{\mu\nu}(y, x^\mu) dx^\mu dx^\nu, \quad (8.23)$$

we can write the 5-dimensional Einstein tensor $G_{\mu\nu}^{(5)}$ in terms of the 4-dimensional Einstein tensor $G_{\mu\nu}^{(4)}$ and the extrinsic curvature as

$$\begin{aligned} G_{\mu\nu}^{(5)} &= G_{\mu\nu}^{(4)} + K_{\mu\nu,y} - g_{\mu\nu} K_{,y} - K K_{\mu\nu} + 2K_{\mu\lambda} K^\lambda_{\nu} \\ &\quad + \frac{1}{2} g_{\mu\nu} (K^2 + K^\alpha_\beta K^\beta_\alpha) = \frac{6}{\ell^2} g_{\mu\nu}, \end{aligned} \quad (8.24)$$

where we have introduced the extrinsic curvature

$$K_{\mu\nu} = -\frac{1}{2} g_{\mu\nu,y}, \quad (8.25)$$

and the last equality comes from the 5-dimensional Einstein equations. On the other hand, the Weyl tensor in the bulk can be expressed as

$$C_{y\mu y\nu} = K_{\mu\nu,y} - g_{\mu\nu} K_{,y} + K_\mu{}^\lambda K_{\lambda\nu} + g_{\mu\nu} K^{\alpha\beta} K_{\alpha\beta} - \frac{3}{\ell^2} g_{\mu\nu}. \quad (8.26)$$

Now, one can eliminate $K_{\mu\nu,y} - g_{\mu\nu} K_{,y}$ from (8.24) using (8.26) and obtain

$$\begin{aligned}
G_{\mu\nu}^{(4)} = & -C_{y\mu y\nu} + KK_{\mu\nu} - K_{\mu\lambda}K^{\lambda}_{\nu} \\
& - \frac{1}{2}g_{\mu\nu}(K^2 - K^{\alpha}_{\beta}K^{\beta}_{\alpha}) + \frac{3}{\ell^2}g_{\mu\nu}.
\end{aligned} \tag{8.27}$$

Taking into account the Z_2 symmetry, we also obtain the junction conditions

$$[K^{\mu}_{\nu} - \delta^{\mu}_{\nu}K] \Big|_{y=0} = \frac{\kappa^2}{2}(-\sigma\delta^{\mu}_{\nu} + T^{\mu}_{\nu}). \tag{8.28}$$

Here, $T_{\mu\nu}$ represents the energy-momentum tensor of the matter. Evaluating Eq. 8.27 at the brane and substituting the junction condition into it, we have the “effective” equations of motion

$$G_{\mu\nu}^{(4)} = \frac{\kappa^2}{\ell}T_{\mu\nu} + \kappa^4\pi_{\mu\nu} - E_{\mu\nu} \tag{8.29}$$

where we have defined the quadratic of the energy momentum tensor

$$\pi_{\mu\nu} = -\frac{1}{4}T_{\mu}^{\lambda}T_{\lambda\nu} + \frac{1}{12}TT_{\mu\nu} + \frac{1}{8}g_{\mu\nu}\left(T^{\alpha\beta}T_{\alpha\beta} - \frac{1}{3}T^2\right) \tag{8.30}$$

and the projection of Weyl tensor $C_{y\mu y\nu}$ onto the brane

$$E_{\mu\nu} = C_{y\mu y\nu}|_{y=0}.$$

Here, we assumed the relation

$$\kappa^2\sigma = \frac{6}{\ell} \tag{8.31}$$

so that the effective cosmological constant vanishes.

Because of the traceless property of $E_{\mu\nu}$, when we consider an isotropic and homogeneous universe, it is easy to show that this gives the dark radiation component $\propto 1/a^4$. The existence of the high-energy corrections $\propto \rho^2$ is apparent in this approach.

The geometrical approach is useful to classify possible corrections to the conventional Einstein equations. One defect of this approach is the fact that the projected Weyl tensor $E_{\mu\nu}$ can not be determined without solving the equations in the bulk.

8.4 Does AdS/CFT Play Any Role in Braneworld?

To make our concerns explicit, we give a sequence of questions. We treat the single-brane model and two-brane model, separately.

8.4.1 Single-Brane Model

Is the Einstein theory recovered even in the non-linear regime?

In the case of the linear theory, it is known that the conventional Einstein theory is recovered at low energy. On the other hand, the cosmological consideration suggests the deviation from the conventional Friedmann equation even in the low energy regime. This is due to the dark radiation term. Therefore, we need to clarify when the conventional Einstein theory can be recovered on the brane.

How does the AdS/CFT come into the braneworld?

It was argued that the cutoff CFT comes into the braneworld. However, no one knows what is the cutoff CFT. It is a vague concept at least from the point of view of the classical gravity. Moreover, it should be noted that the AdS/CFT correspondence is a specific conjecture. Indeed, originally, Maldacena conjectured that the supergravity on $AdS_5 \times S^5$ is dual to the four-dimensional $\mathcal{N} = 4$ super Yang-Mills theory [28, 29]. Nevertheless, the AdS/CFT correspondence seems to be related to the brane world model as has been demonstrated by several people [33–42]. Hence, it is important to reveal the role of the AdS/CFT correspondence starting from the 5-dimensional general relativity.

How are the AdS/CFT and geometrical approach related?

The geometrical approach gives the effective equations of motion (8.29)

$$G_{\mu\nu}^{(4)} = \frac{\kappa^2}{\ell} T_{\mu\nu} + \kappa^4 \pi_{\mu\nu} - E_{\mu\nu}.$$

On the other hand, the AdS/CFT correspondence yields the other effective equations of motion (8.22)

$$G_{\mu\nu}^{(4)} = \frac{\kappa^2}{\ell} \left(T_{\mu\nu} + T_{\mu\nu}^{\text{CFT}} \right) + [R^2 \text{terms}].$$

An apparent difference is remarkable.

It is an interesting issue to clarify how these two descriptions are related. Shiromizu and Ida tried to understand the AdS/CFT correspondence from the geometrical point of view [43]. They argued that π_μ^μ corresponds to the trace anomaly of the cutoff CFT on the brane. However, this result is rather paradoxical because there exists no trace anomaly in an odd dimensional brane although π_μ^μ exists even in that case. Thus, the more precise relation between the geometrical and the AdS/CFT approaches should be given.

Moreover, since both the geometrical and AdS/CFT approaches seem to have their own merit, it would be beneficial to understand the mutual relationship.

8.4.2 Two-Brane Model

How is the geometrical approach consistent with the Brans-Dicke picture?

Irrespective of the existence of other branes, the geometric approach gives the same effective equations (8.29). The effect of the bulk geometry comes into the brane world only through $E_{\mu\nu}$. In this picture, the two-brane system can be regarded as the Einstein theory with some corrections due to the Weyl tensor in the bulk.

On the other hand, the linearized gravity on the brane behaves as the Brans-Dicke theory on scales large compared with the curvature scale ℓ in the bulk [26]. Therefore, the conventional linearized Einstein equations do not hold at low energy.

In the geometrical approach, no radion appears. While, from the linear analysis, it turns out that the system can be described by the Brans-Dicke theory where the extra scalar field is nothing but the radion. How can we reconcile these seemingly incompatible pictures?

What replaces the AdS/CFT correspondence in the two-brane model?

In the single-brane model, there are continuum Kaluza-Klein (KK)-spectrum around the zero mode. They induce the CFT matter in the 4-dimensional effective action. In the two-brane system, the spectrum become discrete and then a mass gap exists. Hence, we can not expect CFT matter on the brane, although KK-modes exist and affect the physics on the brane. So, it is interesting to know if the AdS/CFT correspondence play a role in the two-brane system.

8.5 Gradient Expansion Method

Our aim in this lecture is to show the gradient expansion method gives the answers to all of the questions presented in the previous section. Here, we review the formalism developed by us [44–47].

We use the Gaussian normal coordinate system (8.23) to describe the geometry of the brane world. Note that the brane is located at $y = 0$ in this coordinate system. Decomposing the extrinsic curvature into the traceless part and the trace part

$$K_{\mu\nu} = \Sigma_{\mu\nu} + \frac{1}{4}h_{\mu\nu}K, \quad K = -\frac{\partial}{\partial y} \log \sqrt{-g}, \quad (8.32)$$

we obtain the basic equations which hold in the bulk;

$$\Sigma^\mu_{\nu,y} - K\Sigma^\mu_{\nu} = -\left[R^{(4)}_{\nu}{}^\mu - \frac{1}{4}\delta_\nu^\mu R^{(4)}\right], \quad (8.33)$$

$$\frac{3}{4}K^2 - \Sigma^\alpha_\beta \Sigma^\beta_\alpha = \left[R^{(4)}\right] + \frac{12}{\ell^2}, \quad (8.34)$$

$$K_{,y} - \frac{1}{4}K^2 - \Sigma^{\alpha\beta}\Sigma_{\alpha\beta} = -\frac{4}{\ell^2}, \quad (8.35)$$

$$\nabla_\lambda \Sigma_\mu{}^\lambda - \frac{3}{4} \nabla_\mu K = 0, \quad (8.36)$$

where $R^\mu{}_\nu$ is the curvature on the brane and ∇_μ denotes the covariant derivative with respect to the metric $g_{\mu\nu}$. One also have the junction condition

$$\left[K^\mu{}_\nu - \delta^\mu_\nu K \right] \Big|_{y=0} = \frac{\kappa^2}{2} (-\sigma \delta^\mu_\nu + T^\mu{}_\nu). \quad (8.37)$$

Recall that we are considering the Z_2 symmetric spacetime.

The problem now is separated into two parts. First, we will solve the bulk equations of motion with the Dirichlet boundary condition at the brane,

$$g_{\mu\nu}(y=0, x^\mu) = h_{\mu\nu}(x^\mu). \quad (8.38)$$

After that, the junction condition will be imposed at the brane. As it is the condition for the induced metric $h_{\mu\nu}$, it is naturally interpreted as the effective equations of motion for gravity on the brane.

Along the normal coordinate y , the metric varies with a characteristic length scale ℓ ; $g_{\mu\nu,y} \sim g_{\mu\nu}/\ell$. Denote the characteristic length scale of the curvature fluctuation on the brane as L ; then we have $R \sim g_{\mu\nu}/L^2$. For the reader's reference, let us take $\ell = 1$ mm, for example. Then, the relation (8.31) give the scale, $\kappa^2 = (10^8 \text{ GeV})^{-3}$ and $\sigma = 1 \text{ TeV}^4$. In this lecture, we will consider the low energy regime in the sense that the energy density of matter, ρ , on the brane is smaller than the brane tension, ie., $\rho/\sigma \ll 1$. In this regime, a simple dimensional analysis

$$\frac{\rho}{\sigma} \sim \frac{\ell \kappa^2 \rho}{\kappa^2 \sigma} \sim \left(\frac{\ell}{L} \right)^2 \ll 1 \quad (8.39)$$

implies that the curvature on the brane can be neglected compared with the extrinsic curvature at low energy. Here, we have used the relation (8.31) and Einstein's equation on the brane, $R \sim g_{\mu\nu}/L^2 \sim \kappa^2 \rho/\ell$. Thus, the anti-Newtonian or gradient expansion method used in the cosmological context is applicable to our problem [12].

At zeroth order, we can neglect the curvature term. Then we have

$$\Sigma^\mu{}_{\nu,y} - K \Sigma^\mu{}_\nu = 0, \quad (8.40)$$

$$\frac{3}{4} K^2 - \Sigma^\alpha{}_\beta \Sigma^\beta{}_\alpha = \frac{12}{\ell^2}, \quad (8.41)$$

$$K_{,y} - \frac{1}{4} K^2 - \Sigma^{\alpha\beta} \Sigma_{\alpha\beta} = -\frac{4}{\ell^2}, \quad (8.42)$$

$$\nabla_\lambda \Sigma^\lambda{}_\mu - \frac{3}{4} \nabla_\mu K = 0. \quad (8.43)$$

Equation 8.40 can be readily integrated into

$$\Sigma^\mu{}_\nu = \frac{C^\mu{}_\nu(x^\mu)}{\sqrt{-g}}, \quad C^\mu{}_\mu = 0, \quad (8.44)$$

where $C^\mu{}_\nu$ is the constant of integration. Equation 8.43 also requires $C^\mu{}_{\nu|\mu} = 0$. If it could exist, it would represent a radiation like fluid on the brane and hence a strongly anisotropic universe. In fact, as we see soon, this term must vanish in order to satisfy the junction condition. Therefore, we simply put $C^\mu{}_\nu = 0$, hereafter. Now, it is easy to solve the remaining equations. The result is

$$K^{(0)} = \frac{4}{\ell}. \quad (8.45)$$

Using the definition of the extrinsic curvature

$$K_{\mu\nu}^{(0)} = -\frac{1}{2} \frac{\partial}{\partial y} g_{\mu\nu}^{(0)}, \quad (8.46)$$

we get the zeroth order metric as

$$ds^2 = dy^2 + b^2(y) h_{\mu\nu}(x^\mu) dx^\mu dx^\nu, \quad b(y) = e^{-2\frac{y}{\ell}}, \quad (8.47)$$

where the tensor $h_{\mu\nu}$ is the induced metric on the brane, which conforms to the boundary condition (8.38).

From the zeroth order solution, we obtain

$$\left[K^\mu{}_\nu^{(0)} - \delta^\mu_\nu K^{(0)} \right] \Big|_{y=0} = -\frac{3}{\ell} \delta^\mu_\nu = -\frac{\kappa^2}{2} \sigma \delta^\mu_\nu \quad (8.48)$$

Then we get the well known relation

$$\kappa^2 \sigma = 6/\ell. \quad (8.49)$$

Here, we will assume that this relation holds exactly. It is apparent that $C^\mu{}_\nu$ is not allowed to exist.

The iteration scheme consists in writing the metric $g_{\mu\nu}$ as a sum of local tensors built out of the induced metric on the brane, the number of gradients increasing with the order. Hence, we will seek the metric as a perturbative series

$$g_{\mu\nu}(y, x^\mu) = b^2(y) \left[h_{\mu\nu}(x^\mu) + g_{\mu\nu}^{(1)}(y, x^\mu) + g_{\mu\nu}^{(2)}(y, x^\mu) + \dots \right], \quad (8.50)$$

where $b^2(y)$ is extracted and we put the Dirichlet boundary condition

$$g_{\mu\nu}^{(i)}(y=0, x^\mu) = 0, \quad (8.51)$$

so that $g_{\mu\nu}(y=0, x) = h_{\mu\nu}(x)$ holds at the brane. Other quantities can be also expanded as

$$\begin{aligned} K^\mu{}_\nu &= \frac{1}{\ell} \delta^\mu_\nu + K^{(1)\mu}{}_\nu + K^{(2)\mu}{}_\nu + \cdots \\ \Sigma^\mu{}_\nu &= +\Sigma^{(1)\mu}{}_\nu + \Sigma^{(2)\mu}{}_\nu + \cdots \end{aligned} \quad (8.52)$$

In our scheme, in contrast to the AdS/CFT correspondence where the Dirichlet boundary condition is imposed at infinity, we impose it at the finite point $y=0$, the location of the brane. Furthermore, we carefully consider the constants of integration, i.e., homogeneous solutions. These homogeneous solutions are ignored in the calculation of AdS/CFT correspondence. However, they play an important role in the braneworld. Note that the scheme can be applicable to other systems [48–51].

8.6 Single Brane Model (RS2)

Now, we will apply the gradient expansion method to the single-brane models and obtain the effective equations on the brane.

8.6.1 Einstein Gravity at Lowest Order

The next order solutions are obtained by taking into account the terms neglected at zeroth order. At first order, Eqs. 8.33–8.36 become

$$\Sigma^{(1)\mu}{}_{\nu,y} - \frac{4}{\ell} \Sigma^{(1)\mu}{}_\nu = - \left[R^\mu{}_\nu - \frac{1}{4} \delta^\mu_\nu R \right]^{(1)}, \quad (8.53)$$

$$\frac{6}{\ell} K^{(1)} = \left[R \right]^{(1)}, \quad (8.54)$$

$$K^{(1)}_{,y} - \frac{2}{\ell} K^{(1)} = 0, \quad (8.55)$$

$$\Sigma^{(1)}{}^\lambda{}_{|\lambda} - \frac{3}{4} K^{(1)}_{|\mu} = 0. \quad (8.56)$$

where the superscript (1) represents the order of the derivative expansion and a stroke | denotes the covariant derivative with respect to the metric $h_{\mu\nu}$. Here, $[R^\mu{}_\nu]^{(1)}$ means that the curvature is approximated by taking the Ricci tensor of $b^2 h_{\mu\nu}$ in place of $R^\mu{}_\nu$. It is also convenient to write it in terms of the Ricci tensor of $h_{\mu\nu}$, denoted $R^\mu{}_\nu(h)$.

Substituting the zeroth order metric into $R^{(4)}$, we obtain

$$K^{(1)} = \frac{\ell}{6b^2} R(h). \quad (8.57)$$

Hereafter, we omit the argument of the curvature for simplicity. Simple integration of Eq. 8.53 also gives the traceless part of the extrinsic curvature as

$$\Sigma^\mu{}_\nu^{(1)} = \frac{\ell}{2b^2} (R^\mu{}_\nu - \frac{1}{4} \delta^\mu_\nu R) + \frac{\chi^\mu{}_\nu(x)}{b^4}, \quad (8.58)$$

where the homogeneous solution satisfies the constraints

$$\chi^\mu{}_\mu = 0, \quad \chi^\mu{}_{\nu|\mu} = 0. \quad (8.59)$$

As we see later, this term corresponds to dark radiation at this order. The metric can be obtained as

$$g^{(1)}_{\mu\nu} = -\frac{\ell^2}{2} \left(\frac{1}{b^2} - 1 \right) \left(R_{\mu\nu} - \frac{1}{6} h_{\mu\nu} R \right) - \frac{\ell}{2} \left(\frac{1}{b^4} - 1 \right) \chi_{\mu\nu}, \quad (8.60)$$

where we have imposed the boundary condition, $g^{(1)}_{\mu\nu}(y=0, x^\mu) = 0$.

Let us focus on the role of $\chi^\mu{}_\nu$ in this part. At this order, the junction condition can be written as

$$\begin{aligned} \left[K^\mu{}_\nu^{(1)} - \delta^\mu_\nu K^{(1)} \right] \Big|_{y=0} &= \frac{\ell}{2} \left(R^\mu{}_\nu - \frac{1}{2} \delta^\mu_\nu R \right) + \chi^\mu{}_\nu \\ &= \frac{\kappa^2}{2} T^\mu{}_\nu. \end{aligned} \quad (8.61)$$

Using the solutions Eqs. 8.57, 8.58 and the formula

$$E^\mu{}_\nu = K^\mu{}_{\nu,y} - \delta^\mu_\nu K_{,y} - K^\mu{}_\lambda K^{\lambda}{}_\nu + \delta^\mu_\nu K^\alpha{}_\beta K^\beta{}_\alpha - \frac{3}{\ell^2} \delta^\mu_\nu, \quad (8.62)$$

we calculate the projective Weyl tensor as

$$E^\mu{}_\nu^{(1)} = \frac{2}{\ell} \chi^\mu{}_\nu. \quad (8.63)$$

Then we obtain the effective Einstein equation

$$R^\mu{}_\nu - \frac{1}{2} \delta^\mu_\nu R = \frac{\kappa^2}{\ell} T^\mu{}_\nu - E^\mu{}_\nu^{(1)}. \quad (8.64)$$

At this order, we do not have the conventional Einstein equations. Recall that the dark radiation exists even in the low energy regime. Indeed, the low energy effective Friedmann equation becomes

$$H^2 = \frac{\kappa^2}{3\ell} \rho + \frac{\mathcal{C}}{a^4}. \quad (8.65)$$

This equation can be obtained from Eq. 8.64 by imposing the maximal symmetry on the spatial part of the brane world and the conditions (8.59). Hence, we observe that $\chi^\mu{}_\nu$ is the generalization of the dark radiation found in the cosmological context.

The nonlocal tensor $\chi_{\mu\nu}$ must be determined by the boundary conditions in the bulk. The natural choice is asymptotically AdS boundary condition. For this boundary condition, we have $\chi_{\mu\nu} = 0$. It is this boundary condition that leads to the conventional Einstein theory in linearized gravity. Assuming this, we have

$$R^\mu{}_\nu - \frac{1}{2} \delta^\mu_\nu R = \frac{\kappa^2}{\ell} T^\mu{}_\nu. \quad (8.66)$$

Thus, the conventional Einstein theory is recovered at the leading order!

8.6.2 AdS/CFT Emerges

In this subsection, we do not include the $\chi_{\mu\nu}$ field because we have adopted the AdS boundary condition. Of course, we have calculated the second order solutions with the contribution of the $\chi_{\mu\nu}$ field. It merely adds extra terms such as $\chi^\mu{}_\nu \chi^\nu{}_\mu$, etc.

At second order, the basic equations can be easily deduced. Substituting the solution up to first order into the Ricci tensor and picking up the second order quantities, we obtain the solutions at second order. The trace part is deduced algebraically as

$$^{(2)}K = \frac{\ell^3}{8b^4} \left(R^\alpha{}_\beta R^\beta{}_\alpha - \frac{2}{9} R^2 \right) - \frac{\ell^3}{12b^2} \left(R^\alpha{}_\beta R^\beta{}_\alpha - \frac{1}{6} R^2 \right). \quad (8.67)$$

By integrating the equation for the traceless part, we have

$$^{(2)}\Sigma^\mu{}_\nu = -\frac{\ell^2}{2} \left(\frac{y}{b^4} + \frac{\ell}{2b^2} \right) \mathcal{S}^\mu{}_\nu - \frac{\ell^3}{24b^2} \left(R R^\mu{}_\nu - \frac{1}{4} \delta^\mu_\nu R^2 \right) + \frac{\ell^3}{b^4} t^\mu{}_\nu, \quad (8.68)$$

where $\mathcal{S}^\mu{}_\nu$ is defined by

$$\delta \int d^4x \sqrt{-h} \frac{1}{2} \left[R^{\alpha\beta} R_{\alpha\beta} - \frac{1}{3} R^2 \right] = \int d^4x \sqrt{-h} \mathcal{S}_{\mu\nu} \delta g^{\mu\nu}. \quad (8.69)$$

The tensor S^μ_v is transverse and traceless,

$$S^\mu_{v|\mu} = 0, \quad S^\mu_\mu = 0. \quad (8.70)$$

The homogeneous solution t^μ_v must be traceless. Moreover, it must satisfy the momentum constraint. To be more precise, we must solve the constraint equation

$$t^\mu_{v|\mu} - \frac{1}{16} R^\alpha_\beta R^\beta_{\alpha|v} + \frac{1}{48} R R_{|v} - \frac{1}{24} R_{|\lambda} R^\lambda_v = 0. \quad (8.71)$$

As one can see immediately, there are ambiguities in integrating this equation. Indeed, there are two types of covariant local tensor whose divergences vanish:

$$\delta \int d^4x \sqrt{-h} \frac{1}{2} R^{\alpha\beta} R_{\alpha\beta} = \int d^4x \sqrt{-h} \mathcal{H}_{\mu\nu} \delta g^{\mu\nu}, \quad (8.72)$$

$$\delta \int d^4x \sqrt{-h} \frac{1}{2} R^2 = \int d^4x \sqrt{-h} \mathcal{K}_{\mu\nu} \delta g^{\mu\nu}. \quad (8.73)$$

Notice that $S^\mu_v = \mathcal{H}^\mu_v - \mathcal{K}^\mu_v/3$. Hence, only S^μ_v and $\mathcal{K}^\mu_v$ are independent. Thanks to the Gauss-Bonnet topological invariant, we do not need to consider the Riemann squared term. In addition to these local tensors, we have to take into account the nonlocal tensor τ^μ_v with the property $\tau^\mu_{v|\mu} = 0$. Thus, we get

$$\begin{aligned} t^\mu_v &= \frac{1}{32} \delta^\mu_v \left(R^\alpha_\beta R^\beta_\alpha - \frac{1}{3} R^2 \right) + \frac{1}{24} \left(R R^\mu_v - \frac{1}{4} \delta^\mu_v R^2 \right) \\ &\quad + \tau^\mu_v + \left(\alpha + \frac{1}{4} \right) S^\mu_v + \frac{\beta}{3} \mathcal{K}^\mu_v, \end{aligned} \quad (8.74)$$

where the constants α and β are free parameters representing the degree of initial conditions in the bulk. Hence, they represent the freedom of gravitational waves in the bulk. The condition $t^\mu_\mu = 0$ leads to

$$\tau^\mu_\mu = -\frac{1}{8} \left(R^\alpha_\beta R^\beta_\alpha - \frac{1}{3} R^2 \right) - \beta \square R. \quad (8.75)$$

This expression is reminiscent of the trace anomaly of the CFT. It is possible to use the result of CFT at this point. For example, we can choose the $\mathcal{N} = 4$ super Yang-Mills theory as the conformal matter. In that case, we simply put $\beta = 0$. This is the way the AdS/CFT correspondence comes into the brane world scenario.

Up to the second order, the junction condition gives

$$R^\mu_v - \frac{1}{2} \delta^\mu_v R + 2\ell^2 \left[\tau^\mu_v + \alpha S^\mu_v + \frac{\beta}{3} \mathcal{K}^\mu_v \right] = \frac{\kappa^2}{\ell} T^\mu_v. \quad (8.76)$$

If we define

$$T_{\mu\nu}^{\text{CFT}} = -2 \frac{\ell^3}{\kappa^2} \tau_{\mu\nu}, \quad (8.77)$$

we can write

$$G_{\mu\nu}^{(4)} = \frac{\kappa^2}{\ell} \left(T_{\mu\nu} + T_{\mu\nu}^{\text{CFT}} \right) - 2\ell^2 \alpha \mathcal{S}_{\mu\nu} - \frac{2\ell^2}{3} \beta \mathcal{K}_{\mu\nu}. \quad (8.78)$$

Let us try to arrange the terms so as to reveal the geometrical meaning of the above equation. We can calculate the projective Weyl tensor as

$$E^\mu{}_\nu^{(2)} = \ell^2 \left[P^\mu{}_\nu + 2\tau^\mu{}_\nu + 2\alpha \mathcal{S}^\mu{}_\nu + \frac{2}{3} \beta \mathcal{K}^\mu{}_\nu \right], \quad (8.79)$$

where

$$P^\mu{}_\nu = -\frac{1}{4} R^\mu{}_\lambda R^\lambda{}_\nu + \frac{1}{6} R R^\mu{}_\nu + \frac{1}{8} \delta^\mu_\nu R^\alpha{}_\beta R^\beta{}_\alpha - \frac{1}{16} \delta^\mu_\nu R^2. \quad (8.80)$$

Substituting this expression into Eq. 8.76 yields our main result

$$G_{\mu\nu}^{(4)} = \frac{\kappa^2}{\ell} T_{\mu\nu} + \ell^2 P_{\mu\nu} - E_{\mu\nu}^{(2)}. \quad (8.81)$$

Notice that $E^\mu{}_\nu$ contains the nonlocal part and the free parameters α and β . On the other hand, $P^\mu{}_\nu$ is determined locally. One can see the relationship in a more transparent way. Within the accuracy we are considering, we can get $P^\mu{}_\nu = \pi^\mu{}_\nu$ using the lowest order equation $R^\mu{}_\nu = \kappa^2 / \ell (T^\mu{}_\nu - 1/2 \delta^\mu_\nu T)$. Hence, we can rewrite Eq. 8.81 as

$$G_{\mu\nu}^{(4)} = \frac{\kappa^2}{\ell} T_{\mu\nu} + \kappa^4 \pi_{\mu\nu} - E_{\mu\nu}^{(2)}. \quad (8.82)$$

Now, the similarity between Eqs. 8.29 and 8.82 is apparent. Thus we get an explicit relation between the geometrical approach and the AdS/CFT approach. However, we note that our Eq. 8.82 is a closed system of equations provided that the specific conformal field theory is chosen.

Now we can read off the effective action as

$$S_{\text{eff}} = \frac{\ell}{2\kappa^2} \int d^4x \sqrt{-h} R + S_{\text{matter}} + S_{\text{CFT}} \\ + \frac{\alpha \ell^2}{2\kappa^2} \int d^4x \sqrt{-h} \left[R^{\mu\nu} R_{\mu\nu} - \frac{1}{3} R^2 \right] + \frac{\beta \ell^2}{6\kappa^2} \int d^4x \sqrt{-h} R^2, \quad (8.83)$$

where we have used the relations Eqs. 8.69, 8.72 and 8.73 and we denoted the nonlocal effective action constructed from $\tau^\mu{}_\nu$ as S_{CFT} .

8.7 Two-Brane Model (RS1)

In this section, we will apply the gradient expansion method to the two-brane models and reveal a role of the AdS/CFT correspondence.

8.7.1 Scalar-Tensor Theory Emerges

We consider the two-brane system depicted in Fig. 8.2. Without matter on the branes, we have the relation $g_{\mu\nu}^{\ominus\text{-brane}} = e^{-2d/\ell} g^{\oplus\text{-brane}} \equiv \Omega^2 g^{\oplus\text{-brane}}$ where d is the distance between the two branes. Although Ω is constant for vacuum branes, it becomes the function of the 4-dimensional coordinates if we put the matter on the brane.

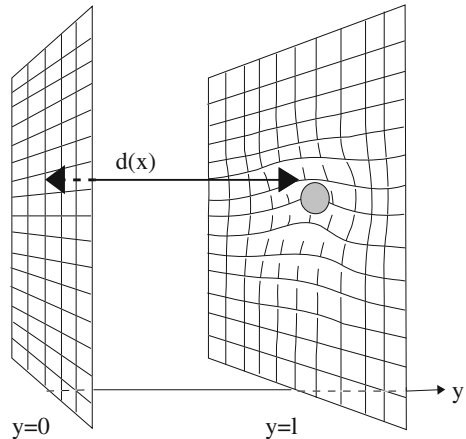
Adding the energy momentum tensor to each of the two branes, and allowing deviations from the pure AdS₅ bulk, the effective (non-local) Einstein equations on the branes at low energies take the form [45],

$$G^\mu{}_\nu(h) = \frac{\kappa^2}{\ell} T^\mu{}_\nu - \frac{2}{\ell} \chi^\mu{}_\nu, \quad (8.84)$$

$$G^\mu{}_\nu(f) = -\frac{\kappa^2}{\ell} \tilde{T}^\mu{}_\nu - \frac{2}{\ell} \frac{\chi^\mu{}_\nu}{\Omega^4}. \quad (8.85)$$

where $h_{\mu\nu} = g_{\mu\nu}^{\oplus\text{-brane}}$, $f_{\mu\nu} = g_{\mu\nu}^{\ominus\text{-brane}} = \Omega^2 h_{\mu\nu}$ and the terms proportional to $\chi_{\mu\nu}$ are 5-dimensional Weyl tensor contributions which describe the non-local 5-dimensional effect. Although Eqs. 8.84 and 8.85 are non-local individually, with undetermined $\chi_{\mu\nu}$, one can combine both equations to reduce them to

Fig. 8.2 In the two brane system, the radion is defined as a distance between two branes. It could depend on the position because the brane could have bending. Hence, for observers on the brane, it appears as a scalar field



local equations for each brane. Since $\chi_{\mu\nu}$ appears only algebraically, one can easily eliminate $\chi_{\mu\nu}$ from Eqs. 8.84 and 8.85. Defining a new field $\Psi = 1 - \Omega^2$, we find

$$G^\mu{}_\nu(h) = \frac{\kappa^2}{\ell\Psi} \overset{\oplus}{T}{}^\mu{}_\nu + \frac{\kappa^2(1-\Psi)^2}{\ell\Psi} \overset{\ominus}{T}{}^\mu{}_\nu + \frac{1}{\Psi} \left(\Psi^{|\mu}{}_{|\nu} - \delta^\mu_\nu \Psi^{|\alpha}{}_{|\alpha} \right) + \frac{3}{2\Psi(1-\Psi)} \left(\Psi^{|\mu}{}_{|\nu} - \frac{1}{2} \delta^\mu_\nu \Psi^{|\alpha}{}_{|\alpha} \right), \quad (8.86)$$

$$\square\Psi = \frac{\kappa^2}{3\ell} (1-\Psi) \left\{ \overset{\oplus}{T} + (1-\Psi) \overset{\ominus}{T} \right\} - \frac{1}{2(1-\Psi)} \Psi^{|\mu}{}_{|\mu}, \quad (8.87)$$

where $|$ denotes the covariant derivative with respect to the metric $h_{\mu\nu}$. Since Ω (or equivalently Ψ) contains the information of the distance between the two branes, we call Ω (or Ψ) the radion.

We can also determine $\chi^\mu{}_\nu$ by eliminating $G^\mu{}_\nu$ from Eqs. 8.84 and 8.85. Then,

$$\chi^\mu{}_\nu = -\frac{\kappa^2(1-\Psi)}{2\Psi} \left(\overset{\oplus}{T}{}^\mu{}_\nu + (1-\Psi) \overset{\ominus}{T}{}^\mu{}_\nu \right) - \frac{\ell}{2\Psi} \left[\left(\Psi^{|\mu}{}_{|\nu} - \delta^\mu_\nu \Psi^{|\alpha}{}_{|\alpha} \right) + \frac{3}{2(1-\Psi)} \left(\Psi^{|\mu}{}_{|\nu} - \frac{1}{2} \delta^\mu_\nu \Psi^{|\alpha}{}_{|\alpha} \right) \right]. \quad (8.88)$$

Note that the index of $\overset{\ominus}{T}{}^\mu{}_\nu$ is to be raised or lowered by the induced metric on the $\ominus$ -brane, $f_{\mu\nu}$.

The effective action for the $\oplus$ -brane which gives Eqs. 8.86 and 8.87 is

$$S_\oplus = \frac{\ell}{2\kappa^2} \int d^4x \sqrt{-h} \left[\Psi R - \frac{3}{2(1-\Psi)} \Psi^{|\alpha}{}_{|\alpha} \right] + \int d^4x \sqrt{-h} \mathcal{L}^\oplus + \int d^4x \sqrt{-h} (1-\Psi)^2 \mathcal{L}^\ominus. \quad (8.89)$$

The above action can be used to make cosmological predictions [52]. It should be stressed that the radion has the conformal coupling. In fact, using the transformation $\Psi = 1 - \psi^2$, we obtain

$$S_\oplus = \frac{6\ell}{\kappa^2} \int d^4x \sqrt{-h} \left[\frac{1}{12} R - \frac{1}{12} \psi^2 R - \frac{1}{2} \psi^{|\alpha}{}_{|\alpha} \right] + \dots \quad (8.90)$$

This is nothing but Einstein theory with a conformally coupled scalar field ψ .

8.7.2 AdS/CFT in Two-Brane System?

In the two-brane case, it is difficult to proceed to the next order calculations. Hence, we need to invent a new method [53]. For this purpose, we shall start with the effective Einstein equation obtained by Shromizu et al. [36]

$$G_{\mu\nu} = T_{\mu\nu} + \pi_{\mu\nu} - E_{\mu\nu} \quad (8.91)$$

where $\pi_{\mu\nu}$ is the quadratic of energy momentum tensor $T_{\mu\nu}$ and $E_{\mu\nu}$ represents the effect of the bulk geometry. Here we have set $8\pi G = 1$. This geometrical projection approach can not give a concrete prediction, because we do not know $E_{\mu\nu}$ without solving the equations of motion in the bulk. Fortunately, in the case of the homogeneous cosmology, the property $E^\mu{}_\mu = 0$ determines the dynamics as

$$H^2 = \frac{1}{3}\rho + \rho^2 + \frac{\mathcal{C}}{a^4}. \quad (8.92)$$

This reflects the interplay between the bulk and the brane dynamics on the brane.

What we want to seek is an effective theory which contains the information of the bulk as finite number of constant parameters like $\mathcal{C}$ in the homogeneous universe. When we succeed in obtaining it, the cosmological perturbation theory can be constructed in a usual way. Although the concrete prediction can not be made, qualitative understanding of the evolution of the cosmological fluctuations can be obtained. This must be useful to make observational predictions.

In the two-brane system, the mass spectrum is known from the linear analysis [26]. At low energy, the propagator for the KK mode with the mass m can be expanded as

$$\frac{-1}{\square - m^2} = \frac{1}{m^2} \left[1 + \frac{\square}{m^2} + \frac{\square^2}{m^4} + \cdots \right]. \quad (8.93)$$

However, massless modes can not be expanded in this way, hence we must take into account all of the massless modes to construct braneworld effective action. It seems legitimate to assume this consideration is valid even in the non-linear regime. Thus, at low energy, the action can be expanded by the local terms with increasing orders of derivatives of the metric $g_{\mu\nu}$ and the radion Ψ [45].

Let us illustrate our method using the following action truncated at the second order derivatives:

$$S_{\text{eff}} = \frac{1}{2} \int d^4x \sqrt{-g} \left[\Psi R - 2\Lambda(\Psi) - \frac{\omega(\Psi)}{\Psi} \nabla^\mu \Psi \nabla_\mu \Psi \right], \quad (8.94)$$

which is nothing but the scalar-tensor theory with coupling function $\omega(\Psi)$ and the potential function $\Lambda(\Psi)$. Note that this is the most general local action which contains up to the second order derivatives and has the general coordinate invariance. It should be stressed that the scalar-tensor theory is, in general, not related to the braneworld. However, we know a special type of scalar-tensor theory corresponds to the low energy braneworld [45, 54–58]. Here, we will present a simple derivation of this known fact.

For the vacuum brane, we can put $T_{\mu\nu} + \pi_{\mu\nu} = -\lambda g_{\mu\nu}$. Hence, the geometrical effective equation reduces to

$$G_{\mu\nu} = -E_{\mu\nu} - \lambda g_{\mu\nu}. \quad (8.95)$$

First, we must find $E_{\mu\nu}$. The above action (8.96) gives the equations of motion for the metric as

$$G_{\mu\nu} = -\frac{\Lambda}{\Psi} g_{\mu\nu} + \frac{1}{\Psi} (\nabla_\mu \nabla_\nu \Psi - g_{\mu\nu} \square \Psi) + \frac{\omega}{\Psi^2} \left(\nabla_\mu \Psi \nabla_\nu \Psi - \frac{1}{2} g_{\mu\nu} \nabla^\alpha \Psi \nabla_\alpha \Psi \right). \quad (8.96)$$

The right hand side of this Eq. 8.96 should be identified with $-E_{\mu\nu} - \lambda g_{\mu\nu}$. Hence, the condition $E^\mu{}_\mu = 0$ becomes

$$\square \Psi = -\frac{\omega}{3\Psi} \nabla^\mu \Psi \nabla_\mu \Psi - \frac{4}{3} (\Lambda - \lambda \Psi). \quad (8.97)$$

This is the equation for the radion Ψ . However, we also have the equation for Ψ from the action (8.94) as

$$\square \Psi = \left(\frac{1}{2\Psi} - \frac{\omega'}{2\omega} \right) \nabla^\alpha \Psi \nabla_\alpha \Psi - \frac{\Psi}{2\omega} R + \frac{\Psi}{\omega} \Lambda', \quad (8.98)$$

where the prime denotes the derivative with respect to Ψ . In order for these two Eqs. 8.97 and 8.98 to be compatible, Λ and ω must satisfy

$$-\frac{\omega}{3\Psi} = \frac{1}{2\Psi} - \frac{\omega'}{2\omega}, \quad (8.99)$$

$$\frac{4}{3} (\Lambda - \lambda \Psi) = \frac{\Psi}{\omega} (2\lambda - \Lambda'), \quad (8.100)$$

where we used $R = 4\lambda$ which comes from the trace part of Eq. 8.95. Equations 8.99 and 8.100 can be integrated as

$$\Lambda(\Psi) = \lambda + \lambda\gamma(1 - \Psi)^2, \quad \omega(\Psi) = \frac{3}{2} \frac{\Psi}{1 - \Psi}, \quad (8.101)$$

where the constant of integration γ represents the ratio of the cosmological constant on the negative tension brane to that on the positive tension brane. Here, one of constants of integration is absorbed by rescaling of Ψ . In doing so, we have assumed the constant of integration is positive. We can also describe the negative tension brane if we take the negative signature.

Thus, we get the effective action

$$S_{\text{eff}} = \int d^4x \sqrt{-g} \left[\frac{1}{2} \Psi R - \frac{3}{4(1 - \Psi)} \nabla^\mu \Psi \nabla_\mu \Psi - \lambda - \lambda\gamma(1 - \Psi)^2 \right]. \quad (8.102)$$

Surprisingly, this completely agrees with the previous result (8.89). Our simple symmetry principle $E^\mu{}_\mu = 0$ has determined the action completely.

As we have shown in [59], if $\gamma < -1$ there exists a static deSitter two-brane solution which turns out to be unstable. In particular, two inflating branes can

collide at $\Psi = 0$. This process is completely smooth for the observer on the brane. This fact led us to the born-again scenario [59, 60]. The similar process occurs also in the ekpyrotic (cyclic) model [61, 62] where the moduli approximation is used. It can be shown that the moduli approximation is nothing but the lowest order truncation of the low energy gradient expansion method developed by us [63–66]. Hence, it is of great interest to see the leading order corrections due to KK modes to this process.

Let us now apply the conformal symmetry method explained above to the higher order case. First, we need to write down the most generic action containing up to fourth order derivatives. Then, we impose the symmetry to determine unknown functionals. The action reads

$$\begin{aligned}
 S_{\text{eff}} = & \frac{1}{2} \int d^4x \sqrt{-g} \left[\Psi R - 2\Lambda(\Psi) - \frac{\omega(\Psi)}{\Psi} \nabla^\mu \Psi \nabla_\mu \Psi \right] \\
 & + \int d^4x \sqrt{-g} \left[A(\Psi) (\nabla^\mu \Psi \nabla_\mu \Psi)^2 + B(\Psi) (\Box \Psi)^2 \right. \\
 & + C(\Psi) \nabla^\mu \Psi \nabla_\mu \Psi \Box \Psi + D(\Psi) R \Box \Psi \\
 & + E(\Psi) R \nabla^\mu \Psi \nabla_\mu \Psi + F(\Psi) R^{\mu\nu} \nabla_\mu \Psi \nabla_\nu \Psi \\
 & + G(\Psi) R^2 + H(\Psi) R^{\mu\nu} R_{\mu\nu} \\
 & \left. + I(\Psi) R^{\mu\nu\lambda\rho} R_{\mu\nu\lambda\rho} + \dots \right], \tag{8.103}
 \end{aligned}$$

where $A, B, \dots$ denote arbitrary functionals of the radion.

Now we impose the conformal symmetry on the fourth order derivative terms in the action (8.103) as we did in the previous example. Starting from the action (8.103), one can read off the equation for the metric from which $E_{\mu\nu}$ can be identified. The compatibility between the equations of motion for Ψ and the equation $E^\mu{}_\mu = 0$ constrains the coefficient functionals in the action (8.103). Surprisingly, every coefficient functionals are determined up to constants.

Thus, we find the 4-dimensional effective action with KK corrections as

$$\begin{aligned}
 S_{\text{eff}} = & \int d^4x \sqrt{-g} \left[\frac{1}{2} \Psi R - \frac{3}{4(1-\Psi)} \nabla^\mu \Psi \nabla_\mu \Psi - \lambda - \lambda\gamma(1-\Psi)^2 \right] \\
 & + \ell^2 \int d^4x \sqrt{-g} \left[\frac{1}{4(1-\Psi)^4} (\nabla^\mu \Psi \nabla_\mu \Psi)^2 \right. \\
 & + \frac{1}{(1-\Psi)^2} (\Box \Psi)^2 + \frac{1}{(1-\Psi)^3} \nabla^\mu \Psi \nabla_\mu \Psi \Box \Psi \\
 & + \frac{2}{3(1-\Psi)} R \Box \Psi + \frac{1}{3(1-\Psi)^2} R \nabla^\mu \Psi \nabla_\mu \Psi \\
 & \left. + jR^2 + kR^{\mu\nu} R_{\mu\nu} \right], \tag{8.104}
 \end{aligned}$$

where constants j and k can be interpreted as the variety of the effects of the bulk gravitational waves. These constants have the same origin as the previous parameters α and β . It should be noted that this action becomes non-local after integrating out the radion field. This fits the fact that KK effects are non-local usually. In principle, we can continue this calculation to any order of derivatives.

8.8 The Answers

We have developed the low energy gradient expansion scheme to give insights into the physics of the braneworld such as the black hole physics and the cosmology. In particular, we have concentrated on the specific questions in this lecture. Here, we summarize our answers obtained by the gradient expansion method. Our understanding of RS braneworlds would be also useful for other brane models.

8.8.1 Single-Brane Model

Is the Einstein theory recovered even in the non-linear regime?

We have obtained the effective theory at the lowest order as

$$G^{(4)\mu}_{\nu} = \frac{\kappa^2}{\ell} T^{\mu}_{\nu} - \frac{2}{\ell} \chi^{\mu}_{\nu}. \quad (8.105)$$

Here we have the correction $\chi_{\mu\nu}$ which can be interpreted as the dark radiation in the cosmological situation.

On the other hand, in the linearized gravity, the conventional Einstein theory is recovered at low energy. This is because the out-going boundary condition is imposed. In other words, the asymptotic AdS boundary condition is imposed. In the nonlinear case, this corresponds to the requirement that the dark radiation term $\chi_{\mu\nu}$ must be zero. For this boundary condition, the conventional Einstein theory is recovered. Hence, the standard Friedmann equation holds.

In this sense, the answer is yes.

How does the AdS/CFT come into the braneworld?

The CFT emerges as the constant of integration which satisfies the trace anomaly relation

$$\tau^{\mu}_{\mu} = -\frac{1}{8} \left(R^{\alpha}_{\beta} R^{\beta}_{\alpha} - \frac{1}{3} R^2 \right) - \beta \square R. \quad (8.106)$$

This constant can not be determined a priori. Here, the AdS/CFT correspondence could come into the braneworld. Namely, if we identify some CFT with $\tau_{\mu\nu}$, then we can determine the boundary condition.

How are the AdS/CFT and geometrical approach related?

The key quantity in the geometric approach is obtained as

$$E_{\nu}^{(2)\mu} = \ell^2 \left[P_{\nu}^{\mu} + 2\tau_{\nu}^{\mu} + 2\alpha S_{\nu}^{\mu} + \frac{2}{3}\beta\mathcal{K}_{\nu}^{\mu} \right]. \quad (8.107)$$

The above expression contains $\tau_{\mu\nu}$ which can be interpreted as the CFT matter. Hence, once we know $E_{\mu\nu}$, no enigma remains. In particular, $P_{\mu\nu} \approx \pi_{\mu\nu}$ is independent of the $\tau_{\mu\nu}$. In odd dimensions, there exists no trace anomaly, but $P_{\mu\nu}$ exists. In 4-dimensions, π_{μ}^{μ} accidentally coincides with the trace anomaly in CFT.

It is interesting to note that the high energy and the Weyl term corrections found in the geometrical approach merge into the CFT matter correction found in the AdS/CFT approach.

8.8.2 Two-Brane Model

How is the geometrical approach consistent with the Brans-Dicke picture?

In the geometrical approach, no radion seems to appear. On the other hand, the linear theory predicts the radion as the crucial quantity. The resolution can be attained by obtaining $E_{\mu\nu}$ ($\chi_{\mu\nu}$ in our notation). The resultant expression

$$\begin{aligned} \chi_{\nu}^{\mu} = & -\frac{\kappa^2(1-\Psi)}{2\Psi} \left(\overset{\oplus}{T}_{\nu}^{\mu} + (1-\Psi)\overset{\ominus}{T}_{\nu}^{\mu} \right) \\ & -\frac{\ell}{2\Psi} \left[\left(\Psi^{|\mu}_{|\nu} - \delta_{\nu}^{\mu}\Psi^{|\alpha}_{|\alpha} \right) + \frac{3}{2(1-\Psi)} \left(\Psi^{|\mu}\Psi_{|\nu} - \frac{1}{2}\delta_{\nu}^{\mu}\Psi^{|\alpha}\Psi_{|\alpha} \right) \right] \end{aligned}$$

contains the radion in an intriguing way. The dark radiation consists of the radion and the matter.

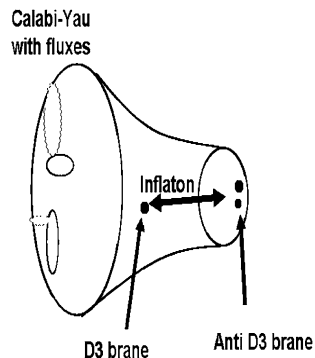
We have shown that the radion transforms the Einstein theory with Weyl correction into the conformally coupled scalar-tensor theory where the radion plays the role of the scalar field. Thus, it turned out that the radion is hidden by the projected Weyl tensor $E_{\mu\nu}$ in the geometrical approach.

What replaces the AdS/CFT correspondence in the two-brane model?

In the case of the single-brane model, the out-going boundary condition at the Cauchy horizon is assumed. This conforms to AdS/CFT correspondence. Indeed, the continuum KK-spectrum are projected on the brane as CFT matter.

On the other hand, the boundary condition in the two-brane system allows only the discrete KK-spectrum. Hence, we can not expect CFT matter on the brane. Instead, the radion controls the bulk/brane correspondence in two-brane model. In fact, the higher derivative terms of the radion mimics the effect of the bulk geometry (KK-effect) as we have shown explicitly. Hence, the conventional AdS/CFT correspondence does not exist. Instead, there exists the

Fig. 8.3 The warped throat is attached to Calabi-Yau manifold. Anti-D3-branes are stacked at the tip of the warped throat. A D3-brane can move in the throat and the radion, i.e. the distance between the D3-brane and anti-D3-branes, plays a role of the inflaton



AdS/CFT correspondence realized by the conformally coupled radion. The conformal coupling can be regarded as a reflection of the symmetry of the bulk geometry.

Here, I would like to mention D-brane inflation models proposed in [67]. There, the inflaton is identified as the radion which is the distance between a D3-brane and anti-D3-branes as is shown in Fig. 8.3. Naively, it seems possible to realize a slow roll inflation due to the warped geometry. However, our result of two-brane system suggests the existence of the conformal coupling of the radion, which ruins the slow roll inflation of the model. In fact, the curvature coupling gives a large mass of the inflaton which causes the notorious eta problem. Hence, a fine tuning is unavoidable.

8.9 AdS/CFT in Dilatonic Braneworld

In the previous sections, we have considered RS braneworlds. If we take into account bulk fields, the pure AdS bulk would not be expected. Hence, it is interesting to see the role of AdS/CFT correspondence in those cases. In this section, we will consider a bulk scalar field and call this kind of models dilatonic braneworlds [68].

In addition to the above theoretical interest, there is an phenomenological interest in dilatonic braneworlds. Let us see how the inflationary universe can be realized in the braneworld. The formula for the effective cosmological constant in the braneworld reads

$$\Lambda_{\text{eff}} = \frac{\kappa^4 \sigma^2}{12} - \frac{3}{\ell^2}, \quad (8.108)$$

where σ and ℓ are the tension of the brane and the curvature scale in the bulk which is determined by the bulk vacuum energy, respectively. For $\kappa^2 \sigma = 6/\ell$, we have Minkowski spacetime. In order to obtain the inflationary universe, we need the

positive effective cosmological constant. In the braneworld model, there are two possibilities. One is to increase the brane tension and the other is to increase ℓ . The brane tension can be controlled by the scalar field on the brane. The bulk curvature scale ℓ can be controlled by the bulk scalar field. The former case is a natural extension of the 4-dimensional inflationary scenario. The latter possibility is a novel one peculiar to the brane model. Recall that, in the superstring theory, scalar fields are ubiquitous. Indeed, the dilaton and moduli exists in the bulk generically, because they arise as the modes associated with the closed string. Moreover, when the supersymmetry is spontaneously broken, they may have the non-trivial potential. Hence, it is natural to consider the inflationary scenario driven by these fields [69, 70]. Therefore, dilatonic braneworlds are phenomenologically interesting. In this section, we would like to discuss this dilatonic braneworld from the point of view of the AdS/CFT correspondence.

8.9.1 Dilatonic Braneworld

We consider a S_1/Z_2 orbifold spacetime with the two branes as the fixed points. In this first Randall-Sundrum (RS1) model, the two flat 3-branes are embedded in AdS_5 and the brane tensions given by $\sigma^\oplus = 6/(\kappa^2\ell)$ and $\sigma^\ominus = -6/(\kappa^2\ell)$. Our system is described by the action

$$S = \frac{1}{2\kappa^2} \int d^5x \sqrt{-g} \mathcal{R} - \int d^5x \sqrt{-g} \left[\frac{1}{2} g^{AB} \partial_A \varphi \partial_B \varphi + U(\varphi) \right] \\ - \sum_{i=\oplus, \ominus} \sigma^i \int d^4x \sqrt{-g^{i-\text{brane}}} + \sum_{i=\oplus, \ominus} \int d^4x \sqrt{-g^{i-\text{brane}}} \mathcal{L}_{\text{matter}}^i, \quad (8.109)$$

where $g_{\mu\nu}^{i-\text{brane}}$ and σ^i are the induced metric and the brane tension on the i -brane, respectively. We assume the potential $U(\varphi)$ for the bulk scalar field takes the form $U(\varphi) = -\frac{6}{\kappa^2\ell^2} + V(\varphi)$, where the first term is regarded as a 5-dimensional cosmological constant and the second term is an arbitrary potential function. The brane tension σ is tuned so that the effective cosmological constant on the brane vanishes. The above setup realizes a flat braneworld after inflation ends and the field φ reaches the minimum of its potential.

Inflation in the braneworld can be driven by a scalar field either on the brane or in the bulk. We derive the effective equations of motion which are useful for both models. Here, we begin with the single-brane system. Since we know the effective 4-dimensional equations hold irrespective of the existence of other branes [47], the analysis of the single-brane system is sufficient to derive the effective action for the two-brane system as we see in the next subsection.

We again adopt the Gaussian normal coordinate system to describe the geometry of the brane model; $ds^2 = dy^2 + g_{\mu\nu}(y, x^\mu) dx^\mu dx^\nu$, where the brane is assumed to be located at $y = 0$. Let us decompose the extrinsic curvature into the

traceless part $\Sigma_{\mu\nu}$ and the trace part K as $K_{\mu\nu} = -\frac{1}{2}g_{\mu\nu,y} = \Sigma_{\mu\nu} + \frac{1}{4}g_{\mu\nu}K$. Then, we can obtain the basic equations off the brane using these variables. First, the Hamiltonian constraint equation leads to

$$\frac{3}{4}K^2 - \Sigma^\alpha_\beta \Sigma^\beta_\alpha = \overset{(4)}{R} - \kappa^2 \nabla^\alpha \varphi \nabla_\alpha \varphi + \kappa^2 (\partial_y \varphi)^2 - 2\kappa^2 U(\varphi), \quad (8.110)$$

where $\overset{(4)}{R}$ is the curvature on the brane and ∇_μ denotes the covariant derivative with respect to the metric $g_{\mu\nu}$. Momentum constraint equation becomes

$$\nabla_\lambda \Sigma^\lambda_\mu - \frac{3}{4} \nabla_\mu K = -\kappa^2 \partial_y \varphi \partial_\mu \varphi. \quad (8.111)$$

Evolution equation in the direction of y is given by

$$\Sigma^\mu_{\nu,y} - K \Sigma^\mu_\nu = - \left[\overset{(4)}{R}_{\mu\nu} - \kappa^2 \nabla^\mu \varphi \nabla_\nu \varphi \right]_{\text{traceless}}. \quad (8.112)$$

Finally, the equation of motion for the scalar field reads

$$\partial_y^2 \varphi - K \partial_y \varphi + \nabla^\alpha \nabla_\alpha \varphi - U'(\varphi) = 0, \quad (8.113)$$

where the prime denotes derivative with respect to the scalar field φ .

As we have the singular source at the brane position, we must consider the junction conditions. Assuming a Z_2 symmetry of spacetime, we obtain the junction conditions for the metric and the scalar field

$$\left[\Sigma^\mu_\nu - \frac{3}{4} \delta^\mu_\nu K \right] \Big|_{y=0} = -\frac{\kappa^2}{2} \sigma \delta^\mu_\nu + \frac{\kappa^2}{2} T^\mu_\nu, \quad (8.114)$$

$$[\partial_y \varphi] \Big|_{y=0} = 0, \quad (8.115)$$

where T^μ_ν is the energy-momentum tensor for the matter fields on the brane.

8.9.2 AdS/Radion Correspondence

We assume the inflation occurs at low energy in the sense that the additional energy due to the bulk scalar field is small, $\kappa^2 \ell^2 V(\varphi) \ll 1$, and the curvature on the brane R is also small, $R\ell^2 \ll 1$. It should be stressed that the low energy does not necessarily implies weak gravity on the brane. Under these circumstances, we can use a gradient expansion scheme to solve the bulk equations of motion.

At zeroth order, we ignore matters on the brane. Then, from the junction condition (8.114), we have

$$\left[\Sigma^\mu_v - \frac{3}{4} \delta^\mu_v K^{(0)} \right] \Big|_{y=0} = -\frac{\kappa^2}{2} \sigma \delta^\mu_v. \quad (8.116)$$

As the right hand side of (8.116) contains no traceless part, we get $\Sigma^\mu_v = 0$. We also take the potential for the bulk scalar field $U(\varphi)$ to be $-6/(\kappa^2 \ell^2)$. We discard the terms with 4-dimensional derivatives since one can neglect the long wavelength variation in the direction of x^μ at low energies. Thus, the equations to be solved are given by

$$\frac{3}{4} K^{(0)2} = \kappa^2 (\partial_y \varphi)^2 + \frac{12}{\ell^2}, \quad (8.117)$$

$$\partial_y^2 \varphi - K \partial_y \varphi = 0. \quad (8.118)$$

The junction condition (8.115) at this order $\left[\partial_y \varphi \right]_{y=0} = 0$ tells us that the solution of Eq. 8.118 must be $\varphi = \eta(x^\mu)$, where $\eta(x^\mu)$ is an arbitrary constant of integration. Now, the solution of Eq. 8.118 yields $K = 4/\ell$. Other Eqs. 8.111 and 8.112 are trivially satisfied at zeroth order. Using the definition $K_{\mu\nu}^{(0)} = -g_{\mu\nu,y}^{(0)}/2$, we have the lowest order metric

$$g_{\mu\nu}^{(0)}(y, x^\mu) = b^2(y) h_{\mu\nu}(x^\mu), \quad b(y) \equiv e^{-y/\ell}, \quad (8.119)$$

where the induced metric on the brane, $h_{\mu\nu} \equiv g_{\mu\nu}(y=0, x^\mu)$, arises as a constant of integration. The junction condition for the induced metric (8.116) merely implies well known relation $\kappa^2 \sigma = 6/\ell$ and that for the scalar field (8.115) is trivially satisfied. At this leading order analysis, we can not determine the constants of integration $h_{\mu\nu}(x^\mu)$ and $\eta(x^\mu)$ which are constant as far as the short length scale ℓ variations are concerned, but are allowed to vary over the long wavelength scale. These constants should be constrained by the next order analysis.

Now, we take into account the effect of both the bulk scalar field and the matter on the brane perturbatively. Our iteration scheme is to write the metric $g_{\mu\nu}$ and the scalar field φ as a sum of local tensors built out of the induced metric and the induced scalar field on the brane, in the order of expansion parameters, that is, $O((R\ell^2)^n)$ and $O(\kappa^2 \ell^2 V(\varphi))^n$, $n = 0, 1, 2, \dots$ [47]. Then, we expand the metric and the scalar field as

$$\begin{aligned} g_{\mu\nu}(y, x^\mu) &= b^2(y) \left[h_{\mu\nu}(x^\mu) + g_{\mu\nu}^{(1)}(y, x^\mu) + g_{\mu\nu}^{(2)}(y, x^\mu) + \dots \right], \\ \varphi(y, x^\mu) &= \eta(x^\mu) + \varphi^{(1)}(y, x^\mu) + \varphi^{(2)}(y, x^\mu) + \dots \end{aligned} \quad (8.120)$$

Here, we put the boundary conditions ${}^{(i)}g_{\mu\nu}(y=0, x^\mu) = 0$, ${}^{(i)}\varphi(y=0, x^\mu) = 0$, $i = 1, 2, 3, \dots$ so that we can interpret $h_{\mu\nu}$ and η as induced quantities. Extrinsic curvatures can be also expanded as

$$K = \frac{4}{\ell} + K^{(1)} + K^{(2)} + \dots, \quad \Sigma^\mu_\nu = \Sigma^\mu_\nu^{(1)} + \Sigma^\mu_\nu^{(2)} + \dots. \quad (8.121)$$

Using the formula such as $R({}^{(4)}g_{\mu\nu}) = R(h_{\mu\nu})/b^2$, we obtain the solution

$$K^{(1)} = \frac{\ell}{6b^2} \left(R(h) - \kappa^2 \eta^{|z} \eta_{|z} \right) - \frac{\ell}{3} \kappa^2 V(\eta), \quad (8.122)$$

where $R(h)$ is the scalar curvature of $h_{\mu\nu}$ and $|$ denotes the covariant derivative with respect to $h_{\mu\nu}$. Substituting the results at zeroth order solutions into Eq. 8.112, we obtain

$$\Sigma^\mu_\nu^{(1)} = \frac{\ell}{2b^2} \left[R^\mu_\nu(h) - \kappa^2 \eta^{| \mu} \eta_{| \nu} \right]_{\text{traceless}} + \frac{\chi^\mu_\nu}{b^4}, \quad (8.123)$$

where $R^\mu_\nu(h)$ denotes the Ricci tensor of $h_{\mu\nu}$ and χ^μ_ν is a constant of integration which satisfies the constraint $\chi^\mu_\mu = 0$. Hereafter, we omit the argument of the curvature for simplicity. Integrating the scalar field equation (8.113) at first order, we have

$$\partial_y {}^{(1)}\varphi = \frac{\ell}{2b^2} \square \eta - \frac{\ell}{4} V'(\eta) + \frac{C}{b^4}, \quad (8.124)$$

where C is also a constant of integration. At first order in this iteration scheme, we get two kinds of constants of integration, χ^μ_ν and C .

Given the matter fields $T_{\mu\nu}$ on the brane, the junction condition (8.114) becomes

$$\left[\Sigma^\mu_\nu^{(1)} - \frac{3}{4} \delta^\mu_\nu K^{(1)} \right] \Big|_{y=0} = \frac{\kappa^2}{2} T^\mu_\nu. \quad (8.125)$$

At this order, the junction condition (8.115) yields

$$\left[\partial_y {}^{(1)}\varphi \right] \Big|_{y=0} = 0. \quad (8.126)$$

These junction conditions give the effective equations of motion on the brane.

Now, we are in a position to discuss the effective equations of motion for the dilatonic two-brane models. The point is the fact that the equations of motion on each brane take the same form if we use the induced metric on each brane [47]. The effective Einstein equations on each positive ($\oplus$) and negative ($\ominus$) tension brane at low-energies yield

$$G^\mu_v(h) = \kappa^2 \left(\eta^{|\mu} \eta_{|v} - \frac{1}{2} \delta_v^\mu \eta^{|\alpha} \eta_{|\alpha} - \frac{1}{2} \delta_v^\mu V \right) - \frac{2}{\ell} \chi^\mu_v + \frac{\kappa^2}{\ell} \overset{\oplus}{T}^\mu_v, \quad (8.127)$$

$$G^\mu_v(f) = \kappa^2 \left(\eta^{|\mu} \eta_{|v} - \frac{1}{2} \delta_v^\mu \eta^{|\alpha} \eta_{|\alpha} - \frac{1}{2} \delta_v^\mu V \right) - \frac{2}{\ell} \chi^\mu_v - \frac{\kappa^2}{\ell} \overset{\ominus}{T}^\mu_v. \quad (8.128)$$

where $f_{\mu\nu}$ is the induced metric on the negative tension brane and ; denotes the covariant derivative with respect to $f_{\mu\nu}$. When we set the position of the positive tension brane at $y = 0$, that of the negative tension brane $\bar{y}$ in general depends on x^μ , i.e. $\bar{y} = \bar{y}(x^\mu)$. Hence, the warp factor at the negative tension brane $\Omega(x^\mu) \equiv b(\bar{y}(x))$ also depends on x^μ . Because the metric always comes into equations with derivatives, the zeroth order relation is enough in this first order discussion. Hence, the metric on the positive tension brane is related to the metric on the negative tension brane as $f_{\mu\nu} = \Omega^2 h_{\mu\nu}$.

Although Eqs. 8.127 and 8.128 are non-local individually, with undetermined χ^μ_v , one can combine both equations to reduce them to local equations for each brane. We can therefore easily eliminate χ^μ_v from Eqs. 8.127 and 8.128, since χ^μ_v appears only algebraically. Eliminating χ^μ_v from both Eqs. 8.127 and 8.128, we obtain

$$\begin{aligned} G^\mu_v &= \frac{\kappa^2}{\ell \Psi} \overset{\oplus}{T}^\mu_v + \frac{\kappa^2 (1 - \Psi)^2}{\ell \Psi} \overset{\ominus}{T}^\mu_v \\ &+ \frac{1}{\Psi} \left[\Psi^{|\mu} \Psi_{|v} - \delta_v^\mu \Psi^{|\alpha} \Psi_{|\alpha} + \frac{3}{2} \frac{1}{1 - \Psi} \left(\Psi^{|\mu} \Psi_{|v} - \frac{1}{2} \delta_v^\mu \Psi^{|\alpha} \Psi_{|\alpha} \right) \right] \\ &+ \kappa^2 \left(\eta^{|\mu} \eta_{|v} - \frac{1}{2} \delta_v^\mu \eta^{|\alpha} \eta_{|\alpha} - \delta_v^\mu V_{\text{eff}} \right) s, \quad V_{\text{eff}} = \frac{2 - \Psi}{2} V, \end{aligned} \quad (8.129)$$

where we defined a new field $\Psi = 1 - \Omega^2$ which we refer to by the name “radion”. The bulk scalar field induces the energy-momentum tensor of the conventional 4-dimensional scalar field with the effective potential which depends on the radion.

We can also determine the dark radiation χ^μ_v by eliminating $G^\mu_v(h)$ from Eqs. 8.127 and 8.128,

$$\begin{aligned} \frac{2}{\ell} \chi^\mu_v &= -\frac{1}{\Psi} \left[\Psi^{|\mu} \Psi_{|v} - \delta_v^\mu \Psi^{|\alpha} \Psi_{|\alpha} + \frac{3}{2} \frac{1}{1 - \Psi} \left(\Psi^{|\mu} \Psi_{|v} - \frac{1}{2} \delta_v^\mu \Psi^{|\alpha} \Psi_{|\alpha} \right) \right] \\ &+ \frac{\kappa^2}{2} (1 - \Psi) \delta_v^\mu V - \frac{\kappa^2}{\ell} \frac{1 - \Psi}{\Psi} \left[\overset{\oplus}{T}^\mu_v + (1 - \Psi) \overset{\ominus}{T}^\mu_v \right]. \end{aligned} \quad (8.130)$$

Due to the property $\chi^\mu_\mu = 0$, we have

$$\square \Psi = \frac{\kappa^2}{3\ell} (1 - \Psi) \left[\overset{\oplus}{T} + (1 - \Psi) \overset{\ominus}{T} \right] - \frac{1}{2(1 - \Psi)} \Psi^{|\alpha} \Psi_{|\alpha} - \frac{2\kappa^2}{3} \Psi (1 - \Psi) V. \quad (8.131)$$

Note that Eqs. 8.129 and 8.131 are derived from a scalar-tensor type theory coupled to the additional scalar field.

Similarly, the equations for the scalar field on branes become

$$\square_h \eta - \frac{V'}{2} + \frac{2}{\ell} C = 0, \quad (8.132)$$

$$\square_f \eta - \frac{V'}{2} + \frac{2}{\ell} \frac{C}{\Omega^4} = 0, \quad (8.133)$$

where the subscripts refer to the induced metric on each brane. Notice that the scalar field takes the same value for both branes at this order. Eliminating the dark source C from these Eqs. 8.132 and 8.133, we find the equation for the scalar field takes the form

$$\square_h \eta - V'_{\text{eff}} = -\frac{\Psi^{|\mu}}{\Psi} \eta_{|\mu}. \quad (8.134)$$

Notice that the radion acts as a source for η . And we can also get the dark source as

$$\frac{2}{\ell} C = -\frac{V'}{2} (1 - \Psi) + \frac{\Psi^{|\mu}}{\Psi} \eta_{|\mu}. \quad (8.135)$$

Now the effective action for the positive tension brane which gives Eqs. 8.129, 8.131 and 8.134 can be read off as

$$\begin{aligned} S = & \frac{\ell}{2\kappa^2} \int d^4x \sqrt{-h} \left[\Psi R - \frac{3}{2(1-\Psi)} \Psi^{|\alpha} \Psi_{|\alpha} - \kappa^2 \Psi \left(\eta^{|\alpha} \eta_{|\alpha} + 2V_{\text{eff}}(\eta, \Psi) \right) \right] \\ & + \int d^4x \sqrt{-h} \mathcal{L}^\oplus + \int d^4x \sqrt{-h} (1 - \Psi)^2 \mathcal{L}^\ominus, \end{aligned} \quad (8.136)$$

where the last two terms represent actions for the matter on each brane. Thus, we found the radion field couples with the induced metric and the induced scalar field on the brane non-trivially. Surprisingly, at this order, the nonlocality of $\chi_{\mu\nu}$ and C are eliminated by the radion. We see the radion has a conformal coupling. However, in the present case, the radion couples to the dilaton field which breaks a conformal invariance. Hence, this gives non-conformal holography.

As this is a closed system, we can analyze a primordial spectrum to predict the cosmic background fluctuation spectrum [71]. Interestingly, χ_ν^μ and C vanishes in the single brane limit, $\Psi \rightarrow 1$, as can be seen from (8.130) and (8.135). The dynamics is simply governed by Einstein theory with the single scalar field. Therefore, we can conclude that the bulk inflaton can drive inflation when the slow role conditions are satisfied.

8.9.3 AdS/CFT and KK Corrections: Single-Brane Cases

It would be important to take into account the KK effects as corrections to the leading order result. It can be accomplished in the single-brane models. Using our approach, in the single brane limit, we can deduce the effective action with KK corrections as [47, 72] (see also [73, 74])

$$S = \frac{\ell}{2\kappa^2} \int d^4x \sqrt{-h} \left[\left(1 + \frac{\ell^2}{12} \kappa^2 V \right) R - \kappa^2 \left(1 + \frac{\ell^2}{12} \kappa^2 V - \frac{\ell^2}{4} V'' \right) \eta^{|\alpha} \eta_{|\alpha} \right. \\ \left. - 2\kappa^2 V_{\text{eff}} - \frac{\ell^2}{4} \left(R^{\alpha\beta} R_{\alpha\beta} - \frac{1}{3} R^2 \right) \right] + \int d^4x \sqrt{-h} \mathcal{L}_{\text{matter}} + S_{\text{CFT}}, \quad (8.137)$$

where the last term comes from the energy-momentum tensor of CFT matter $\tau_{\mu\nu}$ and the effective potential at this order is defined by

$$V_{\text{eff}} = \frac{1}{2} V + \frac{\ell^2 \kappa^2}{48} V^2 - \frac{\ell^2}{64} V'^2. \quad (8.138)$$

It is interesting to note that the effective potential contains the terms which looks like F-terms in supersymmetric models.

Thus, even in the dilatonic braneworld, the AdS/CFT correspondence seems to play an important role in the single-brane case.

8.10 Conclusion

In this lecture, I have reviewed the gradient expansion method in the context of braneworlds. Using the formalism, I have tried to explain how the AdS/CFT correspondence is related to the braneworld models.

In the case of the RS single-brane model, we clarified when the conventional Einstein equations hold at low energy. Moreover, we revealed the relation between the geometrical and the AdS/CFT correspondence approach using the gradient expansion method. We have shown that the high energy and the Weyl term corrections found in the geometrical approach correspond to the CFT matter corrections found in the AdS/CFT approach.

In the case of the RS two-brane system, we showed that the AdS/CFT correspondence plays an important role in the sense that the low energy effective field theory can be described by the conformally coupled scalar-tensor theory where the radion plays the role of the scalar field. We also presented the symmetry method to derive KK corrections in the two-brane system.

These effective theories for RS braneworlds can be used to make cosmological predictions. More importantly, it turned out that the gradient expansion method provides a unified view of RS braneworlds.

We have also considered the bulk scalar field with a nontrivial potential and derived the non-linear low energy effective action for the dilatonic two-brane model using the gradient expansion method. As a result, we have shown that the effective theory reduces to the scalar-tensor theory with the non-trivial coupling between the radion and the bulk scalar field. Since the radion has a conformal coupling, the conformal symmetry is relevant even for the dilatonic braneworlds. In this sense, the AdS/CFT correspondence is related to dilatonic braneworlds. However, the radion couples to the scalar field which is non-conformal. Hence, the conformal invariance is violated.

Our phenomenological motivation to consider dilatonic braneworlds was a possibility of the bulk inflaton. Concerning to this issue, taking into account the fact that $\chi_{\mu\nu}$ and C becomes zero when two branes get separated infinitely, one can conclude that the bulk inflaton can drive the inflation on the brane as far as the slow roll conditions are satisfied. We also obtained KK corrections in the single brane limit which contain CFT corrections.

These results tell us that there exist profound relations between braneworlds and the AdS/CFT correspondence, although the correspondence is slightly deformed in the dilatonic cases.

In this lecture, we have considered only codimension-one braneworlds. It is important to extend the analysis to higher codimension models [75–84]. It is intriguing to study a role of the AdS/CFT correspondence in these higher codimension braneworlds.

Acknowledgments I would like to thank Sugumi Kanno for collaborations on which this lecture is based and useful comments on the manuscript. I am grateful to the organizers of the 5th Aegean Summer school, especially Eleftherios Papantonopoulos, for inviting me to give a lecture and their kind hospitality during the summer school. The present work is supported by the Japan-U.K. Research Cooperative Program, Grant-in-Aid for Scientific Research Fund of the Ministry of Education, Science and Culture of Japan No. 18540262, Grant-in-Aid for Scientific Research on Innovative Area No. 21111006 and the Grant-in-Aid for the Global COE Program “The Next Generation of Physics, Spun from Universality and Emergence”.

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Part III: Condensed Matter and the AdS/CFT Correspondence

Chapter 9

Condensed Matter and AdS/CFT

Subir Sachdev

Abstract I review two classes of strong coupling problems in condensed matter physics, and describe insights gained by application of the AdS/CFT correspondence. The first class concerns non-zero temperature dynamics and transport in the vicinity of quantum critical points described by relativistic field theories. I describe how relativistic structures arise in models of physical interest, present results for their quantum critical crossover functions and magneto-thermoelectric hydrodynamics. The second class concerns symmetry breaking transitions of two-dimensional systems in the presence of gapless electronic excitations at isolated points or along lines (i.e. Fermi surfaces) in the Brillouin zone. I describe the scaling structure of a recent theory of the Ising-nematic transition in metals, and discuss its possible connection to theories of Fermi surfaces obtained from simple AdS duals.

9.1 Introduction

The past couple of decades have seen vigorous theoretical activity on the quantum phases and phase transitions of correlated electron systems in two spatial dimensions. Much of this work has been motivated by the cuprate superconductors, but the list of interesting materials continues to increase unabated [1].

Methods from field theory have had a strong impact on much of this work. Indeed, they have become part of the standard toolkit of condensed matter physicists. In these lectures, I focus on two classes of strong-coupling problems which have not yielded accurate solutions via the usual arsenal of field-theoretic methods. I will also discuss how the AdS/CFT correspondence, discovered by string

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theorists, has already allowed substantial progress on some of these problems, and offers encouraging prospects for future progress.

The first class of strong-coupling problems are associated with the real-time, finite temperature behavior of strongly interacting quantum systems, especially those near quantum critical points. Field-theoretic or numerical methods often allow accurate determination of the *zero* temperature or of *imaginary time* correlations at non-zero temperatures. However, these methods usual fail in the real-time domain at non-zero temperatures, particularly at times greater than $\hbar/k_B T$, where T is the absolute temperature. In systems near quantum critical points the natural scale for correlations is $\hbar/k_B T$ itself, and so lowering the temperature in a numerical study does not improve the situation.

The second class of strong-coupling problems arise near two-dimensional quantum critical points with fermionic excitations. When the fermions have a massless Dirac spectrum, with zero excitation energy at a finite number of points in the Brillouin zone, conventional field-theoretic methods do allow significant progress. However, in metallic systems, the fermionic excitations have zeros along a line in the Brillouin zone (the Fermi surface), allowing a plethora of different low energy modes. Metallic quantum critical points play a central role in many experimental systems, but the interplay between the critical modes and the Fermi surface has not been fully understood (even at zero temperature). Readers interested only in this second class of problems can jump ahead to [Sect. 9.7](#).

These lectures will start with a focus on the first class of strong-coupling problems. We will begin in [Sect. 9.2](#) by introducing a variety of model systems and their quantum critical points; these are motivated by recent experimental and theoretical developments. We will use these systems to introduce basic ideas on the finite temperature crossovers near quantum critical points in [Sect. 9.3](#). In [Sect. 9.4](#), we will focus on the important *quantum critical region* and present a general discussion of its transport properties. An important recent development has been the complete exact solution, via the AdS/CFT correspondence, of the dynamic and transport properties in the quantum critical region of a variety of (supersymmetric) model systems in two and higher dimensions: this will be described in [Sect. 9.5](#). The exact solutions are found to agree with the earlier general ideas discussed here in [Sect. 9.4](#). As has often been the case in the history of physics, the existence of a new class of solvable models leads to new and general insights which apply to a much wider class of systems, almost all of which are not exactly solvable. This has also been the case here, as we will review in [Sect. 9.6](#): a hydrodynamic theory of the low frequency transport properties has been developed, and has led to new relations between a variety of thermo-electric transport co-efficients.

The latter part of these lectures will turn to the second class of strong coupling problems, by describing the role of fermions near quantum critical points. In [Sect. 9.7](#) we will consider some simple symmetry breaking transitions in d -wave superconductors. Such superconductors have fermionic excitations with a massless Dirac spectrum, and we will show how they become critical near the quantum phase transition. We will review how the field-theoretic $1/N$ expansion does allow solution of a large class of such problems. Finally, in [Sect. 9.8](#) we will consider phase

transitions of metallic systems with Fermi surfaces. We will discuss how the $1/N$ expansion fails here, and review the results of recent work involving the AdS/CFT correspondence.

Some portions of the discussions below have been adapted from other review articles by the author [2, 3].

9.2 Model Systems and Their Critical Theories

9.2.1 Coupled Dimer Antiferromagnets

Some of the best studied examples of quantum phase transitions arise in insulators with unpaired $S = 1/2$ electronic spins residing on the sites, i , of a regular lattice. Using S_i^a ($a = x, y, z$) to represent the spin $S = 1/2$ operator on site i , the low energy spin excitations are described by the Heisenberg exchange Hamiltonian

$$H_J = \sum_{i < j} J_{ij} S_i^a \cdot S_j^a + \dots \quad (9.1)$$

where $J_{ij} > 0$ is the antiferromagnetic exchange interaction. We will begin with a simple realization of this model is illustrated in Fig. 9.1. The $S = 1/2$ spins reside on the sites of a square lattice, and have nearest neighbor exchange equal to either J or J/λ . Here $\lambda \geq 1$ is a tuning parameter which induces a quantum phase transition in the ground state of this model.

At $\lambda = 1$, the model has full square lattice symmetry, and this case is known to have a Néel ground state which breaks spin rotation symmetry. This state has a checkerboard polarization of the spins, just as found in the classical ground state, and as illustrated on the left side of Fig. 9.1. It can be characterized by a vector order parameter φ^a which measures the staggered spin polarization

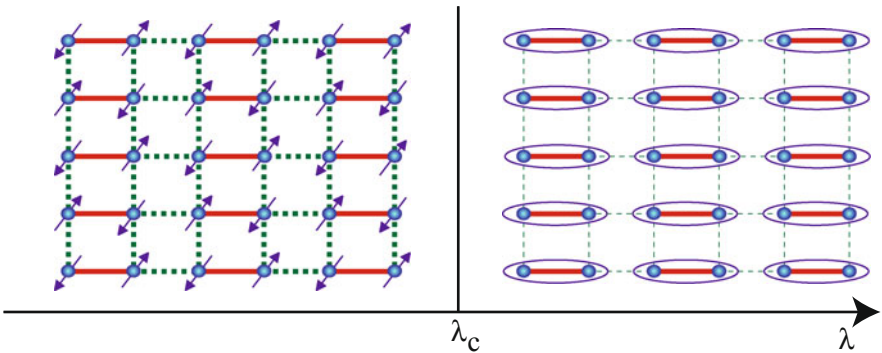


Fig. 9.1 The coupled dimer antiferromagnet. The full red lines represent an exchange interaction J , while the dashed green lines have exchange J/λ . The ellipses represent a singlet valence bond of spins $(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle)/\sqrt{2}$

$$\varphi^a = \eta_i S_i^a \quad (9.2)$$

where $\eta_i = \pm 1$ on the two sublattices of the square lattice. In the Néel state we have $\langle \varphi^a \rangle \neq 0$, and we expect that the low energy excitations can be described by long wavelength fluctuations of a field $\varphi^a(x, \tau)$ over space, x , and imaginary time τ .

On the other hand, for $\lambda \gg 1$ it is evident from Fig. 9.1 that the ground state preserves all symmetries of the Hamiltonian: it has total spin $S = 0$ and can be considered to be a product of nearest neighbor singlet valence bonds on the J links. It is clear that this state cannot be smoothly connected to the Néel state, and so there must at least one quantum phase transition as a function λ .

Extensive quantum Monte Carlo simulations [4–6] on this model have shown there is a direct phase transition between these states at a critical λ_c , as in Fig. 9.1. The value of λ_c is known accurately, as are the critical exponents characterizing a second-order quantum phase transition. These critical exponents are in excellent agreement with the simplest proposal for the critical field theory, [6] which can be obtained via conventional Landau-Ginzburg arguments. Given the vector order parameter φ^a , we write down the action in d spatial and one time dimension,

$$S_{LG} = \int d^d r d\tau \left[\frac{1}{2} [(\partial_\tau \varphi^a)^2 + v^2 (\nabla \varphi^a)^2 + s(\varphi^a)^2] + \frac{u}{4} [(\varphi^a)^2]^2 \right], \quad (9.3)$$

as the simplest action expanded in gradients and powers of φ^a which is consistent with all the symmetries of the lattice antiferromagnet. The transition is now tuned by varying $s \sim (\lambda - \lambda_c)$. Notice that this model is identical to the Landau-Ginzburg theory for the thermal phase transition in a $d + 1$ dimensional ferromagnet, because time appears as just another dimension. As an example of the agreement: the critical exponent of the correlation length, ν , has the same value, $\nu = 0.711\dots$, to three significant digits in a quantum Monte Carlo study of the coupled dimer antiferromagnet, [6] and in a 5-loop analysis [7] of the renormalization group fixed point of S_{LG} in $d = 2$. Similar excellent agreement is obtained for the double-layer antiferromagnet [8, 9] and the coupled-plaquette antiferromagnet [10].

In experiments, the best studied realization of the coupled-dimer antiferromagnet is TiCuCl_3 . In this crystal, the dimers are coupled in all three spatial dimensions, and the transition from the dimerized state to the Néel state can be induced by application of pressure. Neutron scattering experiments by Ruegg and collaborators [11] have clearly observed the transformation in the excitation spectrum across the transition, and these observations are in good quantitative agreement with theory [1].

9.2.2 Deconfined Criticality

We now consider an analog of transition discussed in Sect. 9.2.1, but for a Hamiltonian $H = H_0 + \lambda H_1$ which has full square lattice symmetry at all λ . For H_0 , we choose a form of H_J , with $J_{ij} = J$ for all nearest neighbor links. Thus at

$\lambda = 0$ the ground state has Néel order, as in the left panel of Fig. 9.1. We now want to choose H_1 so that increasing λ leads to a spin singlet state with spin rotation symmetry restored. A large number of choices have been made in the literature, and the resulting ground state invariably [12] has valence bond solid (VBS) order; a VBS state has been observed in the organic antiferromagnet $\text{EtMe}_3\text{P}[\text{Pd}(\text{dmit})_2]_2$ [13, 14]. The VBS state is superficially similar to the dimer singlet state in the right panel of Fig. 9.1: the spins primarily form valence bonds with near-neighbor sites. However, because of the square lattice symmetry of the Hamiltonian, a columnar arrangement of the valence bonds as in Fig. 9.1, breaks the square lattice rotation symmetry; there are 4 equivalent columnar states, with the valence bond columns running along different directions. More generally, a VBS state is a spin singlet state, with a non-zero degeneracy due to a spontaneously broken lattice symmetry. Thus a direct transition between the Néel and VBS states involves two distinct broken symmetries: spin rotation symmetry, which is broken only in the Néel state, and a lattice rotation symmetry, which is broken only in the VBS state. The rules of Landau-Ginzburg-Wilson theory imply that there can be no generic second-order transition between such states.

It has been argued that a second-order Néel-VBS transition can indeed occur [15], but the critical theory is not expressed directly in terms of either order parameter. It involves a fractionalized bosonic spinor z_α ($\alpha = \uparrow, \downarrow$), and an emergent gauge field A_μ . The key step is to express the vector field φ^a in terms of z_α by

$$\varphi^a = z_\alpha^* \sigma_{\alpha\beta}^a z_\beta \quad (9.4)$$

where σ^a are the 2×2 Pauli matrices. Note that this mapping from φ^a to z_α is redundant. We can make a spacetime-dependent change in the phase of the z_α by the field $\theta(x, \tau)$

$$z_\alpha \longrightarrow e^{i\theta} z_\alpha \quad (9.5)$$

and leave φ^a unchanged. All physical properties must therefore also be invariant under Eq. 9.5, and so the quantum field theory for z_α has a $U(1)$ gauge invariance, much like that found in quantum electrodynamics. The effective action for the z_α therefore requires introduction of an ‘emergent’ $U(1)$ gauge field A_μ (where $\mu = x, \tau$ is a three-component spacetime index). The field A_μ is unrelated the electromagnetic field, but is an internal field which conveniently describes the couplings between the spin excitations of the antiferromagnet. As we did for $\mathcal{S}_{LG}$, we can write down the quantum field theory for z_α and A_μ by the constraints of symmetry and gauge invariance, which now yields

$$\mathcal{S}_z = \int d^2 r d\tau \left[|(\partial_\mu - iA_\mu)z_\alpha|^2 + s|z_\alpha|^2 + u(|z_\alpha|^2)^2 + \frac{1}{2g^2} (\epsilon_{\mu\nu\lambda} \partial_\nu A_\lambda)^2 \right] \quad (9.6)$$

For brevity, we have now used a “relativistically” invariant notation, and scaled away the spin-wave velocity v ; the values of the couplings s, u are different from, but related to, those in $\mathcal{S}_{LG}$. The Maxwell action for A_μ is generated from short

distance z_α fluctuations, and it makes A_μ a dynamical field; its coupling g is unrelated to the electron charge. The action S_z is a valid description of the Néel state for $s < 0$ (the critical upper value of s will have fluctuation corrections away from 0), where the gauge theory enters a Higgs phase with $\langle z_\alpha \rangle \neq 0$. This description of the Néel state as a Higgs phase has an analogy with the Weinberg-Salam theory of the weak interactions—in the latter case it is hypothesized that the condensation of a Higgs boson gives a mass to the W and Z gauge bosons, whereas here the condensation of z_α quenches the A_μ gauge boson. As written, the $s > 0$ phase of S_z is a ‘spin liquid’ state with a $S = 0$ collective gapless excitation associated with the A_μ photon. Non-perturbative effects [12] associated with the monopoles in A_μ (not discussed here), show that this spin liquid is ultimately unstable to the appearance of VBS order.

Numerical studies of the Néel-VBS transition have focussed on a specific lattice antiferromagnet proposed by Sandvik [16–19]. There is strong evidence for VBS order proximate to the Néel state, along with persuasive evidence of a second-order transition. However, some studies [20, 21] support a very weak first order transition.

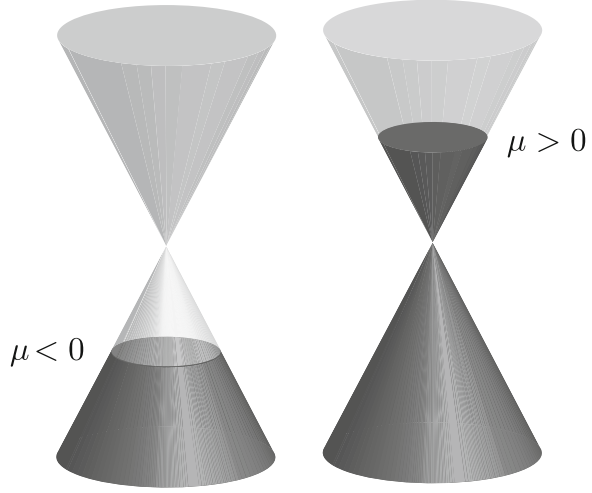
9.2.3 Graphene

The last few years have seen an explosion in experimental and theoretical studies [22] of graphene: a single hexagonal layer of carbon atoms. At the currently observed temperatures, there is no evident broken symmetry in the electronic excitations, and so it is not conventional to think of graphene as being in the vicinity of a quantum critical point. However, graphene does indeed undergo a bona fide quantum phase transition, but one without any order parameters or broken symmetry. This transition may be viewed as being ‘topological’ in character, and is associated with a change in nature of the Fermi surface as a function of carrier density.

Pure, undoped graphene has a conical electronic dispersion spectrum at two points in the Brillouin zone, with the Fermi energy at the particle-hole symmetric point at the apex of the cone. So there is no Fermi surface, just a Fermi point, where the electronic energy vanishes, and pure graphene is a ‘semi-metal’. By applying a gate voltage, the Fermi energy can move away from this symmetric point, and a circular Fermi surface develops, as illustrated in Fig. 9.2. The Fermi surface is electron-like for one sign of the bias, and hole-like for the other sign. This change from electron to hole character as a function of gate voltage constitutes the quantum phase transition in graphene. As we will see below, with regard to its dynamic properties near zero bias, graphene behaves in almost all respects like a canonical quantum critical system.

The field theory for graphene involves fermionic degrees of freedom. Representing the electronic orbitals near one of the Dirac points by the two-component

Fig. 9.2 Dirac dispersion spectrum for graphene showing a ‘topological’ quantum phase transition from a hole Fermi surface for $\mu < 0$ to an electron Fermi surface for $\mu > 0$



fermionic spinor Ψ_s , where s is a sublattice index (we suppress spin and ‘valley’ indices), we have the effective electronic action

$$\begin{aligned} \mathcal{S}_\Psi = & \int d^2r \int d\tau \Psi_s^\dagger [(\partial_\tau + iA_\tau - \mu)\delta_{ss'} + iv_F \tau_{ss'}^x \partial_x + iv_F \tau_{ss'}^y \partial_y] \Psi_{s'} \\ & + \frac{1}{2g^2} \int \frac{d^2q}{4\pi^2} \int d\tau \frac{q}{2\pi} |A_\tau(\mathbf{q}, \tau)|^2, \end{aligned} \quad (9.7)$$

where $\tau_{ss'}^i$ are Pauli matrices in the sublattice space, μ is the chemical potential, v_F is the Fermi velocity, and A_τ is the scalar potential mediating the Coulomb interaction with coupling $g^2 = e^2/\epsilon$ (ϵ is a dielectric constant). This theory undergoes a quantum phase transition as a function of μ , at $\mu = 0$, similar in many ways to that of $\mathcal{S}_{LG}$ as a function of s . The interaction between the fermionic excitations here has coupling g^2 , which is the analog of the non-linearity u in $\mathcal{S}_{LG}$. The strength of the interactions is determined by the dimensionless ‘fine structure constant’ $\alpha = g^2/(\hbar v_F)$ which is of order unity in graphene. While u flows to a non-zero fixed point value under the renormalization group, α flows logarithmically slowly to zero. For many purposes, it is safe to ignore this flow, and to set α equal to a fixed value.

9.3 Finite Temperature Crossovers

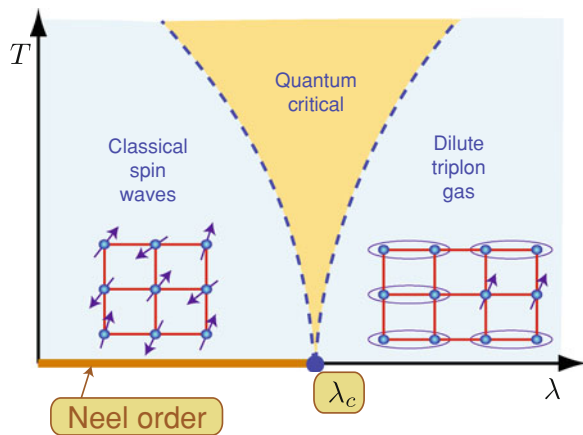
The previous section has described four model systems at $T = 0$: we examined the change in the nature of the ground state as a function of some tuning parameter, and motivated a quantum field theory which describes the low energy excitations on both sides of the quantum critical point.

We now turn to the important question of the physics at non-zero temperatures. All of the models share some common features, which we will first explore for the coupled dimer antiferromagnet. For $\lambda > \lambda_c$ (or $s > 0$ in $\mathcal{S}_{LG}$), the excitations consist of a triplet of $S = 1$ particles (the ‘triplons’), which can be understood perturbatively in the large λ expansion as an excited $S = 1$ state on a dimer, hopping between dimers (see Fig. 9.3). The mean field theory tells us that the excitation energy of this dimer vanishes as $\sqrt{s}$ upon approaching the quantum critical point. Fluctuations beyond mean field, described by $\mathcal{S}_{LG}$, show that the exponent is modified to s^{zv} , where $z = 1$ is the dynamic critical exponent, and v is the correlation length exponent. Now imagine turning on a non-zero temperature. As long as T is smaller than the triplon gap, i.e. $T < s^{zv}$, we expect a description in terms of a dilute gas of thermally excited triplon particles. This leads to the behavior shown on the right-hand-side of Fig. 9.3, delimited by the crossover indicated by the dashed line. Note that the crossover line approaches $T = 0$ only at the quantum critical point.

Now let us look at the complementary behavior at $T > 0$ on the Néel-ordered side of the transition, with $s < 0$. In two spatial dimensions, thermal fluctuations prohibit the breaking of a non-Abelian symmetry at all $T > 0$, and so spin rotation symmetry is immediately restored. Nevertheless, there is an exponentially large spin correlation length, ξ , and at distances shorter than ξ we can use the ordered ground state to understand the nature of the excitations. Along with the spin-waves, we also found the longitudinal ‘Higgs’ mode with energy $\sqrt{-2s}$ in mean field theory. Thus, just as was this case for $s > 0$, we expect this spin-wave+Higgs picture to apply at all temperatures lower than the natural energy scale; i.e. for $T < (-s)^{zv}$. This leads to the crossover boundary shown on the left-hand-side of Fig. 9.3.

Having delineated the physics on the two sides of the transition, we are left with the crucial *quantum critical* region in the center of Fig. 9.3. This is present for

Fig. 9.3 Finite temperature crossovers of the coupled dimer antiferromagnet in Fig. 9.1



$T > |s|^{\text{zv}}$, i.e. at *higher* temperatures in the vicinity of the quantum critical point. To the left of the quantum critical region, we have a description of the dynamics and transport in terms of an effectively classical model of spin waves: this is the ‘renormalized classical’ regime of [23]. To the right of the quantum critical region, we again have a regime of classical dynamics, but now in terms of a Boltzmann equation for the triplon particles. A key property of quantum critical region is that there is no description in terms of either classical particles or classical waves at the times of order the typical relaxation time, τ_r , of thermal excitations. Instead, quantum and thermal effects are equally important, and involve the non-trivial dynamics of the fixed-point theory describing the quantum critical point. Note that while the fixed-point theory applies only at a single point ($\lambda = \lambda_c$) at $T = 0$, its influence broadens into the quantum critical region at $T > 0$. Because there is no characterizing energy scale associated with the fixed-point theory, $k_B T$ is the only energy scale available to determine τ_r at non-zero temperatures. Thus, in the quantum critical region [24, 25]

$$\tau_r = \mathcal{C} \frac{\hbar}{k_B T} \quad (9.8)$$

where $\mathcal{C}$ is a universal constant dependent only upon the universality class of the fixed point theory *i.e.* it is universal number just like the critical exponents. This value of τ_r determines the ‘friction coefficients’ associated with the dissipative relaxation of spin fluctuations in the quantum critical region. It is also important for the transport co-efficients associated with conserved quantities, and this will be discussed in Sect. 9.4

Let us now consider the similar $T > 0$ crossovers for the other models of Sect. 9.2.

The Néel-VBS transition of Sect. 9.2.2 has crossovers very similar to those in Fig. 9.3, with one important difference. The VBS state breaks a discrete lattice symmetry, and this symmetry remains broken for a finite range of non-zero temperatures. Thus, within the right-hand ‘triplon gas’ regime of Fig. 9.3, there is a phase transition line at a critical temperature T_{VBS} . The value of T_{VBS} vanishes very rapidly as $s \searrow 0$, and is controlled by the non-perturbative monopole effects which were briefly noted in Sect. 9.2.2.

For graphene, the discussion above applied to Fig. 9.2 leads to the crossover diagram shown in Fig. 9.4, as noted by Sheehy and Schmalian [26]. We have the Fermi liquid regimes of the electron- and hole-like Fermi surfaces on either side of the critical point, along with an intermediate quantum critical Dirac liquid. A new feature here is related to the logarithmic flow of the dimensionless ‘fine structure constant’ α controlling the Coulomb interactions, which was noted in Sect. 9.2.3. In the quantum critical region, this constant takes the typical value $\alpha \sim 1/\ln(1/T)$. Consequently for the relaxation time in Eq. 9.8 we have $\mathcal{C} \sim \ln^2(1/T)$. This time determines both the width of the electron spectral functions, and also the transport co-efficients, as we will see in Sect. 9.4.

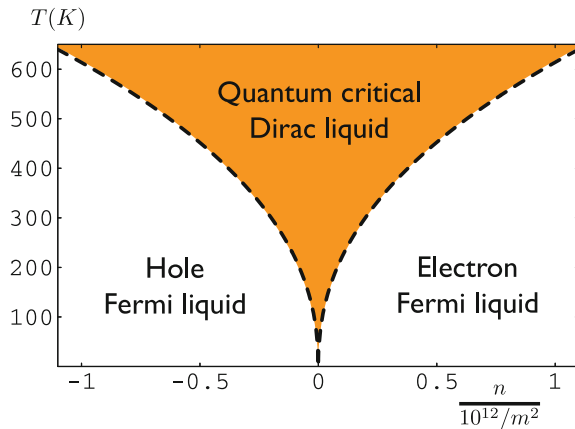
9.4 Quantum Critical Transport

We now turn to the ‘transport’ properties in the quantum critical region: we consider the response functions associated with any globally conserved quantity. For the antiferromagnetic systems in Sect. 9.2.1 and 9.2.2, this requires consideration of the transport of total spin, and the associated spin conductivities and diffusivities. For graphene, we can consider charge and momentum transport. Our discussion below will also apply to the superfluid-insulator transition: for bosons in a periodic potential, this transition is described [27] by a field theory closely related to that in Eq. 9.3. However, we will primarily use a language appropriate to charge transport in graphene below. We will describe the properties of a generic strongly-coupled quantum critical point and mention, where appropriate, the changes due to the logarithmic flow of the coupling in graphene.

In traditional condensed matter physics, transport is described by identifying the low-lying excitations of the quantum ground state, and writing down ‘transport equations’ for the conserved charges carried by them. Often, these excitations have a particle-like nature, such as the ‘triplon’ particles of Fig. 9.3 or the electron or hole quasiparticles of the Fermi liquids in Fig. 9.4. In other cases, the low-lying excitations are waves, such as the spin-waves in Fig. 9.3, and their transport is described by a non-linear wave equation (such as the Gross-Pitaevski equation). However, as we have discussed in Sect. 9.3 neither description is possible in the quantum critical region, because the excitations do not have a particle-like or wave-like character.

Despite the absence of an intuitive description of the quantum critical dynamics, we can expect that the transport properties should have a universal character determined by the quantum field theory of the quantum critical point. In addition to describing single excitations, this field theory also determines the S -matrix of these excitations by the renormalization group fixed-point value of the couplings, and these should be sufficient to determine transport properties [28].

Fig. 9.4 Finite temperature crossovers of graphene as a function of electron density n (which is tuned by μ in Eq. 9.7) and temperature, T . Adapted from [26]



The transport co-efficients, and the relaxation time to local equilibrium, are not proportional to a mean free scattering time between the excitations, as is the case in the Boltzmann theory of quasiparticles. Such a time would typically depend upon the interaction strength between the particles. Rather, the system behaves like a “perfect fluid” in which the relaxation time is as short as possible, and is determined universally by the absolute temperature, as indicated in Eq. 9.8. Indeed, it was conjectured in [29] that the relaxation time in Eq. 9.8 is a generic lower bound for interacting quantum systems. Thus the non-quantum-critical regimes of all the phase diagrams in Sect. 9.3 have relaxation times which are all longer than Eq. 9.8.

The transport co-efficients of this quantum-critical perfect fluid also do not depend upon the interaction strength, and can be connected to the fundamental constants of nature. In particular, the electrical conductivity, σ , is given by (in two spatial dimensions) [28]

$$\sigma_Q = \frac{e^{*2}}{h} \Phi_\sigma, \quad (9.9)$$

where Φ_σ is a universal dimensionless constant of order unity, and we have added the subscript Q to emphasize that this is the conductivity for the case of graphene with the Fermi level at the Dirac point (for the superfluid-insulator transition, this would correspond to bosons at integer filling) with no impurity scattering, and at zero magnetic field. Here e^* is the charge of the carriers: for a superfluid-insulator transition of Cooper pairs, we have $e^* = 2e$, while for graphene we have $e^* = e$. The renormalization group flow of the ‘fine structure constant’ α of graphene to zero at asymptotically low T , allows an exact computation in this case [30–32]: $\Phi_\sigma \approx 0.05 \ln^2(1/T)$. For the superfluid-insulator transition, Φ_σ is T -independent (this is the generic situation with non-zero fixed point values of the interaction [33]) but it has only been computed [28, 29] to leading order in expansions in $1/N$ (where N is the number of order parameter components) and in $3 - d$ (where d is the spatial dimensionality). However, both expansions are neither straightforward nor rigorous, and require a physically motivated resummation of the bare perturbative expansion to all orders. It would therefore be valuable to have exact solutions of quantum critical transport where the above results can be tested, and we turn to such solutions in the next section.

In addition to charge transport, we can also consider momentum transport. This was considered in the context of applications to the quark-gluon plasma [34]; application of the analysis of [28] shows that the viscosity, η , is given by

$$\frac{\eta}{s} = \frac{\hbar}{k_B} \Phi_\eta, \quad (9.10)$$

where s is the entropy density, and again Φ_η is a universal constant of order unity. The value of Φ_η has recently been computed [35] for graphene, and again has a logarithmic T dependence because of the marginally irrelevant interaction: $\Phi_\eta \approx 0.008 \ln^2(1/T)$.

We conclude this section by discussing some subtle aspects of the physics behind the seemingly simple result quantum-critical in Eq. 9.9. For simplicity, we will consider the case of a “relativistically” invariant quantum critical point in $2 + 1$ dimensions (such as the field theories of Sects. 9.2.1 and 9.2.2, but marginally violated by graphene, a subtlety we ignore below). Consider the retarded correlation function of the charge density, $\chi(k, \omega)$, where $k = |\mathbf{k}|$ is the wave-vector, and ω is frequency; the dynamic conductivity, $\sigma(\omega)$, is related to χ by the Kubo formula,

$$\sigma(\omega) = \lim_{k \rightarrow 0} \frac{-i\omega}{k^2} \chi(k, \omega). \quad (9.11)$$

It was argued in [28] that despite the absence of particle-like excitations of the critical ground state, the central characteristic of the transport is a crossover from collisionless to collision-dominated transport. At high frequencies or low temperatures, the limiting form for χ reduces to that at $T = 0$, which is completely determined by relativistic and scale invariance and current conservation upto an overall constant

$$\chi(k, \omega) = \frac{e^{*2}}{h} K \frac{k^2}{\sqrt{v^2 k^2 - (\omega + i\eta)^2}}, \quad \sigma(\omega) = \frac{e^{*2}}{h} K; \quad \hbar\omega \gg k_B T, \quad (9.12)$$

where K is a universal number [36, 37]. However, phase-randomizing collisions are intrinsically present in any strongly interacting critical point (above one spatial dimension) and these lead to relaxation of perturbations to local equilibrium and the consequent emergence of hydrodynamic behavior. So at low frequencies, we have instead an Einstein relation which determines the conductivity with

$$\chi(k, \omega) = e^{*2} \chi_c \frac{Dk^2}{Dk^2 - i\omega}, \quad \sigma(\omega) = e^{*2} \chi_c D = \frac{e^{*2}}{h} \Theta_1 \Theta_2; \quad \hbar\omega \ll k_B T, \quad (9.13)$$

where χ_c is the compressibility and D is the charge diffusion constant. Quantum critical scaling arguments show that the latter quantities obey

$$\chi_c = \Theta_1 \frac{k_B T}{\hbar^2 v^2}, \quad D = \Theta_2 \frac{\hbar v^2}{k_B T}, \quad (9.14)$$

where $\Theta_{1,2}$ are universal numbers. A large number of papers in the literature, particularly those on critical points in quantum Hall systems, have used the collisionless method of Eq. 9.12 to compute the conductivity. However, the correct d.c. limit is given by Eq. 9.13, and the universal constant in Eq. 9.9 is given by $\Phi_\sigma = \Theta_1 \Theta_2$. Given the distinct physical interpretation of the collisionless and collision-dominated regimes, we expect that $K \neq \Theta_1 \Theta_2$. This has been shown in a resummed perturbation expansion for a number of quantum critical points [29].

9.5 Exact Results for Quantum Critical Transport

The results of Sect. 9.4 were obtained by using physical arguments to motivate resummations of perturbative expansions. Here we shall support the ad hoc assumptions behind these results by examining an exactly solvable model of quantum critical transport.

The solvable model may be viewed as a generalization of the gauge theory in Eq. 9.6 to the maximal possible supersymmetry. In $2 + 1$ dimensions, this is known as $\mathcal{N} = 8$ supersymmetry. Such a theory with the $U(1)$ gauge group is free, and so we consider the non-Abelian Yang-Mills theory with a $SU(N)$ gauge group. The resulting supersymmetric Yang-Mills (SYM) theory has only one coupling constant, which is the analog of the electric charge g in Eq. 9.6. The matter content is naturally more complicated than the complex scalar z_α in Eq. 9.6, and also involves relativistic Dirac fermions as in Eq. 9.7. However all the terms in the action for the matter fields are also uniquely fixed by the single coupling constant g . Under the renormalization group, it is believed that g flows to an attractive fixed point at a non-zero coupling $g = g^*$; the fixed point then defines a supersymmetric conformal field theory in $2 + 1$ dimensions (a SCFT3), and we are interested here in the transport properties of this SCFT3.

A remarkable recent advance has been the exact solution of this SCFT3 in the $N \rightarrow \infty$ limit using the AdS/CFT correspondence [38]. The solution proceeds by a dual formulation as a four-dimensional supergravity theory on a spacetime with uniform negative curvature: anti-de Sitter space, or AdS_4 . Remarkably, the solution is also easily extended to non-zero temperatures, and allows direct computation of the correlators of conserved charges in real time. At $T > 0$ a black hole appears in the gravity, resulting in an AdS-Schwarzschild spacetime, and T is also the Hawking temperature of the black hole; the real time solutions also extend to $T > 0$.

The results of a full computation [39] of the density correlation function, $\chi(k, \omega)$ are shown in Figs. 9.5 and 9.6. The most important feature of these results is that the expected limiting forms in the collisionless (Eq. 9.12) and collision-dominated (Eq. 9.13) are obeyed. Thus the results do display the collisionless to collision-dominated crossover at a frequency of order $k_B T / \hbar$, as was postulated in Sect. 9.4.

An additional important feature of the solution is apparent upon describing the full structure of both the density and current correlations. Using spacetime indices ($\mu, \nu = t, x, y$) we can represent these as the tensor $\chi_{\mu\nu}(\mathbf{k}, \omega)$, where the previously considered $\chi \equiv \chi_{tt}$. At $T > 0$, we do not expect $\chi_{\mu\nu}$ to be relativistically covariant, and so can only constrain it by spatial isotropy and density conservation. Introducing a spacetime momentum $p_\mu = (\omega, \mathbf{k})$, and setting the velocity $v = 1$, these two constraints lead to the most general form

$$\chi_{\mu\nu}(\mathbf{k}, \omega) = \frac{e^{*2}}{h} \sqrt{p^2} \left(P_{\mu\nu}^T K^T(k, \omega) + P_{\mu\nu}^L K^L(k, \omega) \right) \quad (9.15)$$

Fig. 9.5 Spectral weight of the density correlation function of the SCFT3 with $\mathcal{N} = 8$ supersymmetry in the collisionless regime

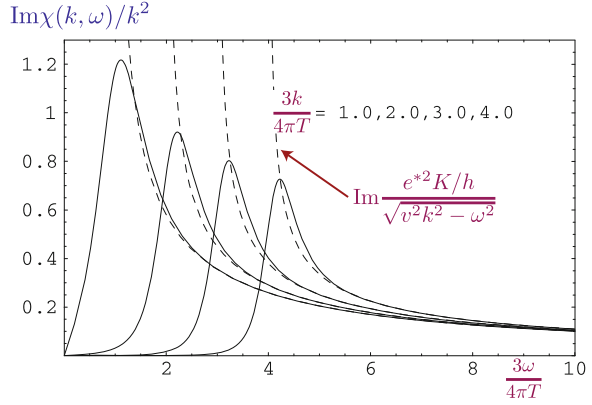
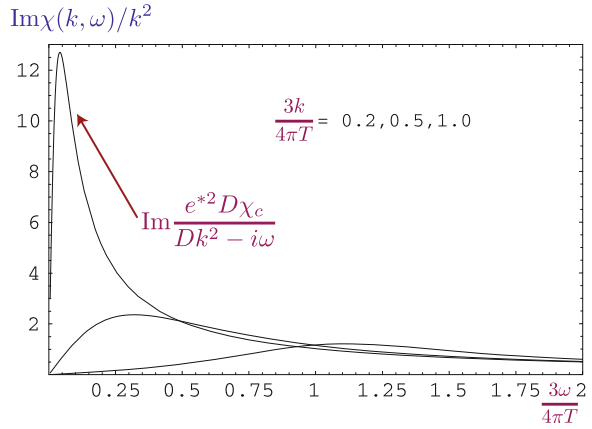


Fig. 9.6 As in Fig. 9.5, but for the collision-dominated regime



where $p^2 = \eta^{\mu\nu} p_\mu p_\nu$ with $\eta_{\mu\nu} = \text{diag}(-1, 1, 1)$, and $P_{\mu\nu}^T$ and $P_{\mu\nu}^L$ are orthogonal projectors defined by

$$P_{00}^T = P_{0i}^T = P_{i0}^T = 0, \quad P_{ij}^T = \delta_{ij} - \frac{k_i k_j}{k^2}, \quad P_{\mu\nu}^L = \left(\eta_{\mu\nu} - \frac{p_\mu p_\nu}{p^2} \right) - P_{\mu\nu}^T, \quad (9.16)$$

with the indices i, j running over the two spatial components. The two functions $K^{T,L}(k, \omega)$ define all the correlators of the density and the current, and the results in Eqs. 9.13 and 9.12 are obtained by taking suitable limits of these functions. We will also need below the general identity

$$K^T(0, \omega) = K^L(0, \omega), \quad (9.17)$$

which follows from the analyticity of the $T > 0$ current correlations at $\mathbf{k} = 0$.

The relations of the previous paragraph are completely general and apply to any theory. Specializing to the AdS-Schwarzschild solution of SYM3, the results were found to obey a simple and remarkable identity [39]:

$$K^L(k, \omega)K^T(k, \omega) = \mathcal{K}^2 \quad (9.18)$$

where $\mathcal{K}$ is a known pure number, independent of ω and k . It was also shown that such a relation applies to any theory which is equated to classical gravity on AdS_4 , and is a consequence of the electromagnetic self-duality of its four-dimensional Maxwell sector. The combination of Eqs. 9.17 and 9.18 fully determines the $\chi_{\mu\nu}$ correlators at $\mathbf{k} = 0$: we find $K^L(0, \omega) = K^T(0, \omega) = \mathcal{K}$, from which it follows that the $\mathbf{k} = 0$ conductivity is frequency independent and that $\Phi_\sigma = \Theta_1\Theta_2 = K = \mathcal{K}$. These last features are believed to be special to theories which are equivalent to classical gravity, and not hold more generally.

We can obtain further insight into the interpretation of Eq. 9.18 by considering the field theory of the superfluid-insulator transition of lattice bosons at integer filling. As we noted earlier, this is given by the field theory in Eq. 9.3 with the field φ^a having two components. It is known that this two-component theory of relativistic bosons is equivalent to a dual relativistic theory, $\tilde{\mathcal{S}}$ of vortices, under the well-known ‘particle-vortex’ duality [39, 40] considered the action of this particle-vortex duality on the correlation functions in Eq. 9.15, and found the following interesting relations:

$$K^L(k, \omega)\tilde{K}^T(k, \omega) = 1, \quad K^T(k, \omega)\tilde{K}^L(k, \omega) = 1 \quad (9.19)$$

where $\tilde{K}^{L,T}$ determine the vortex current correlations in $\tilde{\mathcal{S}}$ as in Eq. 9.15. Unlike Eq. 9.18, Eq. 9.19 does *not* fully determine the correlation functions at $\mathbf{k} = 0$: it only serves to reduce the four unknown functions $K^{L,T}$, $\tilde{K}^{L,T}$ to two unknown functions. The key property here is that while the theories $\mathcal{S}_{LG}$ and $\tilde{\mathcal{S}}$ are dual to each other, they are not equivalent, and the theory $\mathcal{S}_{LG}$ is not self-dual.

We now see that Eq. 9.18 implies that the classical gravity theory of SYM3 is self-dual under an analog of particle-vortex duality [39]. It is not expected that this self-duality will hold when quantum gravity corrections are included; equivalently, the SYM3 at finite N is expected to have a frequency dependence in its conductivity at $\mathbf{k} = 0$. If we apply the AdS/CFT correspondence to the superfluid-insulator transition, and approximate the latter theory by classical gravity on AdS_4 , we immediately obtain the self-dual prediction for the conductivity, $\Phi_\sigma = 1$. This value is not far from that observed in numerous experiments, and we propose here that the AdS/CFT correspondence offers a rationale for understanding such observations.

9.6 Hydrodynamic Theory

The successful comparison between the general considerations of Sect. 9.4, and the exact solution using the AdS/CFT correspondence in Sect. 9.5, emboldens us to

seek a more general theory of low frequency ($\hbar\omega \ll k_B T$) transport in the quantum critical regime. We will again present our results for the special case of a relativistic quantum critical point in $2 + 1$ dimensions (a CFT3), but it is clear that similar considerations apply to a wider class of systems. Thus we can envisage applications to the superfluid-insulator transition, and have presented scenarios under which such a framework can be used to interpret measurements of the Nernst effect in the cuprates [41]. We have also described a separate set of applications to graphene [30–32]: while graphene is strictly not a CFT3, the Dirac spectrum of electrons leads to many similar results, especially in the inelastic collision-dominated regime associated with the quantum critical region. These results on graphene are reviewed in a separate paper [42], where explicit microscopic computations are also discussed.

Our idea is to relax the restricted set of conditions under which the results of Sect. 9.4 were obtained. We will work within the quantum critical regimes of the phase diagrams of Sect. 9.3 but now allow a variety of additional perturbations. First, we will move away from the particle-hole symmetric case, allow a finite density of carriers. For graphene, this means that μ is no longer pinned at zero; for the antiferromagnets, we can apply an external magnetic field; for the superfluid-insulator transition, the number density need not be commensurate with the underlying lattice. For charged systems, such as the superfluid-insulator transition or graphene, we allow application of an external magnetic field. Finally, we also allow a small density of impurities which can act as a sink of the conserved total momentum of the CFT3. In all cases, the energy scale associated with these perturbations is assumed to be smaller than the dominant energy scale of the quantum critical region, which is $k_B T$. The results presented below were obtained in two separate computations, associated with the methods described in Sects. 9.4 and 9.5, and are described in the two subsections below.

9.6.1 Relativistic Magnetohydrodynamics

With the picture of relaxation to local equilibrium at frequencies $\hbar\omega \ll k_B T$ developed in [28], we postulate that the equations of relativistic magnetohydrodynamics should describe the low frequency transport. The basic principles involved in such a hydrodynamic computation go back to the nineteenth century: conservation of energy, momentum, and charge, and the constraint of the positivity of entropy production. Nevertheless, the required results were not obtained until our recent work [41]: the general case of a CFT3 in the presence of a chemical potential, magnetic field, and small density of impurities is very intricate, and the guidance provided by the dual gravity formulation was very helpful to us. In this approach, we do not have quantitative knowledge of a few transport co-efficients, and this is complementary to our ignorance of the effective couplings in the dual gravity theory to be discussed in Sect. 9.6.2.

The complete hydrodynamic analysis can be found in [41]. The analysis is intricate, but is mainly a straightforward adaption of the classic procedure outlined by Kadanoff and Martin [43] to the relativistic field theories which describe quantum critical points. We list the steps:

1. Identify the conserved quantities, which are the energy-momentum tensor, $T^{\mu\nu}$, and the particle number current, J^μ .
2. Obtain the real time equations of motion, which express the conservation laws:

$$\partial_\nu T^{\mu\nu} = F^{\mu\nu} J_\nu, \quad \partial_\mu J^\mu = 0; \quad (9.20)$$

here $F^{\mu\nu}$ represents the externally applied electric and magnetic fields which can change the net momentum or energy of the system, and we have not written a term describing momentum relaxation by impurities.

3. Identify the state variables which characterize the local thermodynamic state—we choose these to be the density, ρ , the temperature T , and an average velocity u^μ .
4. Express $T^{\mu\nu}$ and J^μ in terms of the state variables and their spatial and temporal gradients; here we use the properties of the observables under a boost by the velocity u^μ , and thermodynamic quantities like the energy density, ε , and the pressure, P , which are determined from T and ρ by the equation of state of the CFT. We also introduce transport co-efficients associated with the gradient terms.
5. Express the equations of motion in terms of the state variables, and ensure that the entropy production rate is positive [44]. This is a key step which ensures relaxation to local equilibrium, and leads to important constraints on the transport co-efficients. In $d = 2$, it was found that situations with the velocity u^μ spacetime independent are characterized by only a *single* independent transport co-efficient [41]. This we choose to be the longitudinal conductivity at $B = 0$.
6. Solve the initial value problem for the state variables using the linearized equations of motion.
7. Finally, translate this solution to the linear response functions, as described in [43].

9.6.2 Dyonic Black Hole

Given the success of the AdS/CFT correspondence for the specific supersymmetric model in Sect. 9.5, we boldly assume a similar correspondence for a generic CFT3. We assume that each CFT3 is dual to a strongly-coupled theory of gravity on AdS_4 . Furthermore, given the operators associated with the perturbations away from the pure CFT3 we want to study, we can also deduce the corresponding perturbations away from the dual gravity theory. So far, this correspondence is purely formal and not of much practical use to us. However, we now restrict our

attention to the hydrodynamic, collision dominated regime, $\hbar\omega \ll k_B T$ of the CFT3. We would like to know the corresponding low energy effective theory describing the quantum gravity theory on AdS_4 . Here, we make the simplest possible assumption: the effective theory is just the Einstein-Maxwell theory of general relativity and electromagnetism on AdS_4 . As in Sect. 9.5, the temperature T of CFT3 corresponds to introducing a black hole on AdS_4 whose Hawking temperature is T . The chemical potential, μ , of the CFT3 corresponds to an electric charge on the black hole, and the applied magnetic field maps to a magnetic charge on the black hole. Such a dyonic black hole solution of the Einstein-Maxwell equations is, in fact, known: it is the Reissner-Nordstrom black hole.

We solved the classical Einstein-Maxwell equations for linearized fluctuations about the metric of a dyonic black hole in a space which is asymptotically AdS_4 . The results were used to obtain correlators of a CFT3 using the prescriptions of the AdS/CFT mapping. As we have noted, we have no detailed knowledge of the strongly-coupled quantum gravity theory which is dual to the CFT3 describing the superfluid-insulator transition in condensed matter systems, or of graphene. Nevertheless, given our postulate that its low energy effective field theory essentially captured by the Einstein-Maxwell theory, we can then obtain a powerful set of results for CFT3s.

9.6.3 Results

In the end, we obtained complete agreement between the two independent computations in Sects. 9.6.1 and 9.6.2, after allowing for their distinct equations of state. This agreement demonstrates that the assumption of a low energy Einstein-Maxwell effective field theory for a strongly coupled theory of quantum gravity is equivalent to the assumption of hydrodynamic transport for $\hbar\omega \ll k_B T$ in a strongly coupled CFT3.

Finally, we turn to our explicit results for quantum critical transport with $\hbar\omega \ll k_B T$.

First, consider adding a chemical potential, μ , to the CFT3. This will induce a non-zero number density of carriers ρ . The value of ρ is defined so that the total charge density associated with ρ is $e^* \rho$. Then the electrical conductivity at a frequency ω is

$$\sigma(\omega) = \frac{e^{*2}}{h} \Phi_\sigma + \frac{e^{*2} \rho^2 v^2}{(\varepsilon + P)} \frac{1}{(-i\omega + 1/\tau_{\text{imp}})}. \quad (9.21)$$

In this section, we are again using the symbol v to denote the characteristic velocity of the CFT3 because we will need c for the physical velocity of light below. Here ε is the energy density and P is the pressure of the CFT3. We have assumed a small density of impurities which lead to a momentum relaxation time τ_{imp} [41, 45]. In general, Φ_σ , ρ , ε , P , and $1/\tau_{\text{imp}}$ will be functions of $\mu/k_B T$ which

cannot be computed by hydrodynamic considerations alone. However, apart from Φ_σ , these quantities are usually amenable to direct perturbative computations in the CFT3, or by quantum Monte Carlo studies. The physical interpretation of Eq. 9.21 should be evident: adding a charge density ρ leads to an additional Drude-like contribution to the conductivity. This extra current cannot be relaxed by collisions between the unequal density of particle and hole excitations, and so requires an impurity relaxation mechanism to yield a finite conductivity in the d.c. limit.

Now consider thermal transport in a CFT3 with a non-zero μ . The d.c. thermal conductivity, κ , is given by

$$\kappa = \Phi_\sigma \left(\frac{k_B^2 T}{h} \right) \left(\frac{\varepsilon + P}{k_B T \rho} \right)^2, \quad (9.22)$$

in the absence of impurity scattering, $1/\tau_{\text{imp}} \rightarrow 0$. This is a Wiedemann-Franz-like relation, connecting the thermal conductivity to the electrical conductivity in the $\mu = 0$ CFT. Note that κ diverges as $\rho \rightarrow 0$, and so the thermal conductivity of the $\mu = 0$ CFT is infinite.

Next, turn on a small magnetic field B ; we assume that B is small enough that the spacing between the Landau levels is not as large as $k_B T$. The case of large Landau level spacing is also experimentally important, but cannot be addressed by the present analysis. Initially, consider the case $\mu = 0$. In this case, the result Eq. 9.22 for the thermal conductivity is replaced by

$$\kappa = \frac{1}{\Phi_\sigma} \left(\frac{k_B^2 T}{h} \right) \left(\frac{\varepsilon + P}{k_B T B / (hc/e^*)} \right)^2 \quad (9.23)$$

also in the absence of impurity scattering, $1/\tau_{\text{imp}} \rightarrow 0$. This result relates κ to the electrical *resistance* at criticality, and so can be viewed as Wiedemann-Franz-like relation for the vortices. A similar $1/B^2$ dependence of κ appeared in the Boltzmann equation analysis of [46, 47], but our more general analysis applies in a wider and distinct regime [30–32], and relates the co-efficient to other observables.

We have obtained a full set of results for the frequency-dependent thermo-electric response functions at non-zero B and μ . The results are lengthy and we refer the reader to [41] for explicit expressions. Here we only note that the characteristic feature [41, 48] of these results is a new *hydrodynamic cyclotron resonance*. The usual cyclotron resonance occurs at the classical cyclotron frequency, which is independent of the particle density and temperature; further, in a Galilean-invariant system this resonance is not broadened by electron-electron interactions alone, and requires impurities for non-zero damping. The situation for our hydrodynamic resonance is very different. It occurs in a collision-dominated regime, and its frequency depends on the density and temperature: the explicit expression for the resonance frequency is

$$\omega_c = \frac{e^* B \rho v^2}{c(\varepsilon + P)}. \quad (9.24)$$

Further, the cyclotron resonance involves particle and hole excitations moving in opposite directions, and collisions between them can damp the resonance even in the absence of impurities. Our expression for this intrinsic damping frequency is [41, 48]

$$\gamma = \frac{e^{*2}}{h} \Phi_\sigma \frac{B^2 v^2}{c^2(\varepsilon + P)}, \quad (9.25)$$

relating it to the quantum-critical conductivity as a measure of collisions between counter-propagating particles and holes. We refer the reader to a separate discussion [30–32] of the experimental conditions under which this hydrodynamic cyclotron resonance may be observed.

9.7 *d*-Wave Superconductors

We now turn to the second class of strong-coupling problems outlined in Sect. 9.1: those involving quantum critical points with fermionic excitations. This section will consider the simpler class of problems in which the fermions have a Dirac spectrum, and the field-theoretic $1/N$ expansion does allow for substantial progress.

We will begin in Sect. 9.7.1 by an elementary discussion of the origin of these Dirac fermions. Then we will consider two quantum phase transitions, both involving a simple Ising order parameter. The first in Sect. 9.7.2, with time-reversal symmetry breaking, leads to a relativistic quantum field theory closely related to the Gross-Neveu model. The second model of Sect. 9.7.3 involves breaking of a lattice rotation symmetry, leading to “Ising-nematic” order. The theory for this model is not relativistically invariant: it is strongly coupled, but can be controlled by a traditional $1/N$ expansion.

We note that symmetry breaking transitions in graphene are also described by field theories similar to those discussed in this section [49, 50].

9.7.1 *Dirac Fermions*

We begin with a review of the standard BCS mean-field theory for a *d*-wave superconductor on the square lattice, with an eye towards identifying the fermionic Bogoliubov quasiparticle excitations. For now, we assume we are far from any QPT associated with SDW, Ising-nematic, or other broken symmetries. We consider the Hamiltonian

$$H_{IJ} = \sum_k \varepsilon_k c_{k\alpha}^\dagger c_{k\alpha} + J_1 \sum_{\langle ij \rangle} \mathbf{S}_i \cdot \mathbf{S}_j \quad (9.26)$$

where $c_{j\alpha}$ is the annihilation operator for an electron on site j with spin $\alpha = \uparrow, \downarrow$, $c_{k\alpha}$ is its Fourier transform to momentum space, ε_k is the dispersion of the electrons (it is conventional to choose $\varepsilon_k = -2t_1(\cos(k_x) + \cos(k_y)) - 2t_2(\cos(k_x + k_y) + \cos(k_x - k_y)) - \mu$, with $t_{1,2}$ the first/second neighbor hopping and μ the chemical potential), and the J_1 term is similar to that in Eq. 9.1 with

$$S_{ja} = \frac{1}{2} c_{j\alpha}^\dagger \sigma_{\alpha\beta}^a c_{j\beta} \quad (9.27)$$

and σ^a the Pauli matrices. We will consider the consequences of the further neighbor exchange interactions for the superconductor in Sect. 9.7.2 below. Applying the BCS mean-field decoupling to H_{IJ} we obtain the Bogoliubov Hamiltonian

$$H_{BCS} = \sum_k \varepsilon_k c_{k\alpha}^\dagger c_{k\alpha} - \frac{J_1}{2} \sum_{j\mu} \Delta_\mu \left(c_{j\uparrow}^\dagger c_{j+\hat{\mu},\downarrow}^\dagger - c_{j\downarrow}^\dagger c_{j+\hat{\mu},\uparrow}^\dagger \right) + \text{h.c.} \quad (9.28)$$

For a wide range of parameters, the ground state energy optimized by a $d_{x^2-y^2}$ wavefunction for the Cooper pairs: this corresponds to the choice $\Delta_x = -\Delta_y = \Delta_{x^2-y^2}$. The value of $\Delta_{x^2-y^2}$ is determined by minimizing the energy of the BCS state

$$E_{BCS} = J_1 |\Delta_{x^2-y^2}|^2 - \int \frac{d^2k}{4\pi^2} [E_k - \varepsilon_k] \quad (9.29)$$

where the fermionic quasiparticle dispersion is

$$E_k = \left[\varepsilon_k^2 + \left| J_1 \Delta_{x^2-y^2} (\cos k_x - \cos k_y) \right|^2 \right]^{1/2}. \quad (9.30)$$

The energy of the quasiparticles, E_k , vanishes at the four points $(\pm Q, \pm Q)$ at which $\varepsilon_k = 0$. We are especially interested in the low energy quasiparticles in the vicinity of these points, and so we perform a gradient expansion of H_{BCS} near each of them. We label the points $\mathcal{Q}_1 = (Q, Q)$, $\mathcal{Q}_2 = (-Q, Q)$, $\mathcal{Q}_3 = (-Q, -Q)$, $\mathcal{Q}_4 = (Q, -Q)$ and write

$$c_{j\alpha} = f_{1\alpha}(\mathbf{r}_j) e^{i\mathcal{Q}_1 \cdot \mathbf{r}_j} + f_{2\alpha}(\mathbf{r}_j) e^{i\mathcal{Q}_2 \cdot \mathbf{r}_j} + f_{3\alpha}(\mathbf{r}_j) e^{i\mathcal{Q}_3 \cdot \mathbf{r}_j} + f_{4\alpha}(\mathbf{r}_j) e^{i\mathcal{Q}_4 \cdot \mathbf{r}_j}, \quad (9.31)$$

while assuming the $f_{1-4,\alpha}(\mathbf{r})$ are slowly varying functions of x . We also introduce the bispinors $\Psi_1 = (f_{1\uparrow}, f_{3\downarrow}, f_{1\downarrow}, -f_{3\uparrow})$, and $\Psi_2 = (f_{2\uparrow}, f_{4\downarrow}, f_{2\downarrow}, -f_{4\uparrow})$, and then express H_{BCS} in terms of $\Psi_{1,2}$ while performing a spatial gradient expansion. This yields the following effective action for the fermionic quasiparticles:

$$\begin{aligned} \mathcal{S}_\Psi = \int d\tau d^2r \Big[& \Psi_1^\dagger \left(\partial_\tau - i \frac{v_F}{\sqrt{2}} (\partial_x + \partial_y) \tau^z - i \frac{v_\Delta}{\sqrt{2}} (-\partial_x + \partial_y) \tau^x \right) \Psi_1 \\ & + \Psi_2^\dagger \left(\partial_\tau - i \frac{v_F}{\sqrt{2}} (-\partial_x + \partial_y) \tau^z - i \frac{v_\Delta}{\sqrt{2}} (\partial_x + \partial_y) \tau^x \right) \Psi_2 \Big]. \end{aligned} \quad (9.32)$$

where the $\tau^{x,z}$ are 4×4 matrices which are block diagonal, the blocks consisting of 2×2 Pauli matrices. The velocities $v_{F,\Delta}$ are given by the conical structure of E_k near the Q_{1-4} : we have $v_F = |\nabla_k \varepsilon_k|_{k=Q_a}|$ and $v_\Delta = |J_1 \Delta_{x^2-y^2} \sqrt{2} \sin(Q)|$. In this limit, the energy of the Ψ_1 fermionic excitations is $E_k = (v_F^2(k_x + k_y)^2/2 + v_\Delta^2(k_x - k_y)^2/2)^{1/2}$ (and similarly for Ψ_2), which is the spectrum of massless Dirac fermions.

9.7.2 Time-Reversal Symmetry Breaking

We will consider a simple model in which the pairing symmetry of the superconductor changes from $d_{x^2-y^2}$ to $d_{x^2-y^2} \pm id_{xy}$. The choice of the phase between the two pairing components leads to a breaking of time-reversal symmetry. Studies of this transition were originally motivated by the cuprate phenomenology, but we will not explore this experimental connection here because the evidence has remained sparse.

The mean field theory of this transition can be explored entirely within the context of BCS theory, as we will review below. However, fluctuations about the BCS theory are strong, and lead to non-trivial critical behavior involving both the collective order parameter and the Bogoliubov fermions: this is probably the earliest known example [23, 51, 52] of the failure of BCS theory in two (or higher) dimensions in a superconducting ground state. At $T > 0$, this failure broadens into the “quantum critical” region.

We extend H_{IJ} in Eq. 9.26 so that BCS mean-field theory permits a region with d_{xy} superconductivity. With a J_2 second neighbor interaction, Eq. 9.26 is modified to:

$$\tilde{H}_{IJ} = \sum_k \varepsilon_k c_{k\sigma}^\dagger c_{k\sigma} + J_1 \sum_{\langle ij \rangle} \mathbf{S}_i \cdot \mathbf{S}_j + J_2 \sum_{\text{nnn } ij} \mathbf{S}_i \cdot \mathbf{S}_j. \quad (9.33)$$

We will follow the evolution of the ground state of $\tilde{H}_{IJ}$ as a function of J_2/J_1 .

The mean-field Hamiltonian is now modified from Eq. 9.28 to

$$\begin{aligned} \tilde{H}_{BCS} = & \sum_k \varepsilon_k c_{k\sigma}^\dagger c_{k\sigma} - \frac{J_1}{2} \sum_{j,\mu} \Delta_\mu (c_{j\uparrow}^\dagger c_{j+\hat{\mu},\downarrow}^\dagger - c_{j\downarrow}^\dagger c_{j+\hat{\mu},\uparrow}^\dagger) + \text{h.c.} \\ & - \frac{J_2}{2} \sum'_{j,\nu} \Delta_\nu (c_{j\uparrow}^\dagger c_{j+\hat{\nu},\downarrow}^\dagger - c_{j\downarrow}^\dagger c_{j+\hat{\nu},\uparrow}^\dagger) + \text{h.c.}, \end{aligned} \quad (9.34)$$

where the second summation over v is along the diagonal neighbors $\hat{x} + \hat{y}$ and $-\hat{x} + \hat{y}$. To obtain d_{xy} pairing along the diagonals, we choose $\Delta_{x+y} = -\Delta_{-x+y} = \Delta_{xy}$. We summarize our choices for the spatial structure of the pairing amplitudes (which determine the Cooper pair wavefunction) in Fig. 9.7. The values of $\Delta_{x^2-y^2}$ and Δ_{xy} are to be determined by minimizing the ground state energy (generalizing Eq. 9.29)

$$E_{BCS} = J_1 |\Delta_{x^2-y^2}|^2 + J_2 |\Delta_{xy}|^2 - \int \frac{d^2k}{4\pi^2} [E_k - \varepsilon_k] \quad (9.35)$$

where the quasiparticle dispersion is now (generalizing Eq. 9.30)

$$E_k = \left[\varepsilon_k^2 + \left| J_1 \Delta_{x^2-y^2} (\cos k_x - \cos k_y) + 2J_2 \Delta_{xy} \sin k_x \sin k_y \right|^2 \right]^{1/2}. \quad (9.36)$$

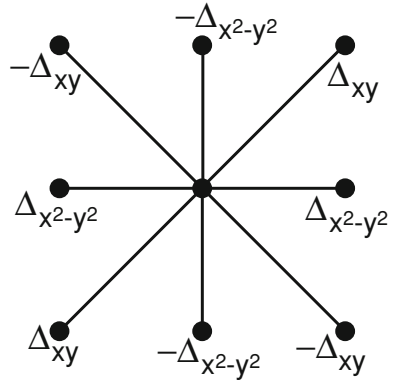
Notice that the energy depends upon the relative phase of $\Delta_{x^2-y^2}$ and Δ_{xy} : this phase is therefore an observable property of the ground state.

It is a simple matter to numerically carry out the minimization of Eq. 9.36, and the results for a typical choice of parameters are shown in Fig. 9.8 as a function J_2/J_1 . One of the two amplitudes $\Delta_{x^2-y^2}$ or Δ_{xy} is always non-zero and so the ground state is always superconducting. The transition from pure $d_{x^2-y^2}$ superconductivity to pure d_{xy} superconductivity occurs via an intermediate phase in which *both* order parameters are non-zero. Furthermore, in this regime, their relative phase is found to be pinned to $\pm\pi/2$ i.e.

$$\arg(\Delta_{xy}) = \arg(\Delta_{x^2-y^2}) \pm \pi/2. \quad (9.37)$$

The reason for this pinning can be intuitively seen from Eq. 9.36: only for these values of the relative phase does the equation $E_k = 0$ never have a solution. In other words, the gapless nodal quasiparticles of the $d_{x^2-y^2}$ superconductor acquire a finite energy gap when a secondary pairing with relative phase $\pm\pi/2$ develops. By

Fig. 9.7 Values of the pairing amplitudes, $-\langle c_{i\uparrow} c_{j\downarrow} - c_{i\downarrow} c_{j\uparrow} \rangle$ with i the central site, and j is one of its eight near neighbors



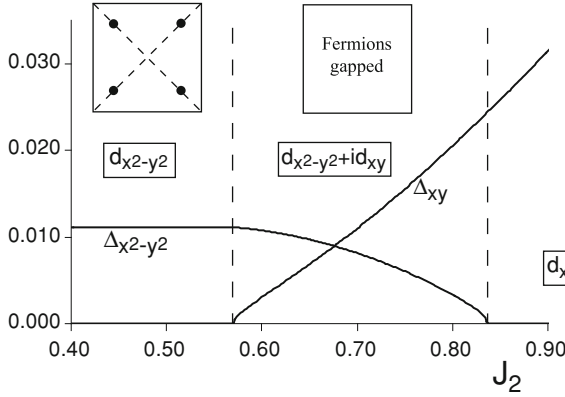


Fig. 9.8 BCS solution of the phenomenological Hamiltonian $\tilde{H}_{IJ}$ in Eq. 9.33. Shown are the optimum values of the pairing amplitudes $|\Delta_{x^2-y^2}|$ and $|\Delta_{xy}|$ as a function of J_2 for $t_1 = 1$, $t_2 = -0.25$, $\mu = -1.25$, and J_1 fixed at $J_1 = 0.4$. The relative phase of the pairing amplitudes was always found to obey Eq. 9.37. The dashed lines denote locations of phase transitions between $d_{x^2-y^2}$, $d_{x^2-y^2} + id_{xy}$, and d_{xy} superconductors. The pairing amplitudes vanishes linearly at the first transition corresponding to the exponent $\beta_{BCS} = 1$ in Eq. 9.40. The Brillouin zone location of the gapless Dirac points in the $d_{x^2-y^2}$ superconductor is indicated by filled circles. For the dispersion ε_k appropriate to the cuprates, the d_{xy} superconductor is fully gapped, and so the second transition is ordinary Ising

a level repulsion picture, we can expect that gapping out the low energy excitations should help lower the energy of the ground state. The intermediate phase obeying Eq. 9.37 is called a $d_{x^2-y^2} + id_{xy}$ superconductor.

The choice of the sign in Eq. 9.37 leads to an overall two-fold degeneracy in the choice of the wavefunction for the $d_{x^2-y^2} + id_{xy}$ superconductor. This choice is related to the breaking of time-reversal symmetry, and implies that the $d_{x^2-y^2} + id_{xy}$ phase is characterized by the non-zero expectation value of a Z_2 Ising order parameter; the expectation value of this order vanishes in the two phases (the $d_{x^2-y^2}$ and d_{xy} superconductors) on either side of the $d_{x^2-y^2} + id_{xy}$ superconductor. As is conventional, we will represent the Ising order by a real scalar field ϕ . Fluctuations of ϕ become critical near both of the phase boundaries in Fig. 9.8. As we will explain below, the critical theory of the $d_{x^2-y^2}$ to $d_{x^2-y^2} + id_{xy}$ transition is *not* the usual ϕ^4 field theory which describes the ordinary Ising transition in three spacetime dimensions. (For the dispersion ε_k appropriate to the cuprates, the d_{xy} superconductor is fully gapped, and so the $d_{x^2-y^2} + id_{xy}$ to d_{xy} transition in Fig. 9.8 will be ordinary Ising.)

Near the phase boundary from $d_{x^2-y^2}$ to $d_{x^2-y^2} + id_{xy}$ superconductivity it is clear that we can identify

$$\phi = i\Delta_{xy}, \quad (9.38)$$

(in the gauge where $\Delta_{x^2-y^2}$ is real). We can now expand E_{BCS} in Eq. 9.35 for small ϕ (with $\Delta_{x^2-y^2}$ finite) and find a series with the structure [53, 54]

$$E_{BCS} = E_0 + s\phi^2 + v|\phi|^3 + \dots, \quad (9.39)$$

where s, v are coefficients and the ellipses represent regular higher order terms in even powers of ϕ ; s can have either sign, whereas v is always positive. Notice the non-analytic $|\phi|^3$ term that appears in the BCS theory—this arises from an infrared singularity in the integral in Eq. 9.35 over E_k at the four nodal points of the $d_{x^2-y^2}$ superconductor, and is a preliminary indication that the transition differs from that in the ordinary Ising model, and that the Dirac fermions play a central role. We can optimize ϕ by minimizing E_{BCS} in Eq. 9.39—this shows that $\langle\phi\rangle = 0$ for $s > 0$, and $\langle\phi\rangle \neq 0$ for $s < 0$. So $s \sim (J_2/J_1)_c - J_2/J_1$ where $(J_2/J_1)_c$ is the first critical value in Fig. 9.8. Near this critical point, we find

$$\langle\phi\rangle \sim (s_c - s)^\beta, \quad (9.40)$$

where we have allowed for the fact that fluctuation corrections will shift the critical point from $s = 0$ to $s = s_c$. The present BCS theory yields the exponent $\beta_{BCS} = 1$; this differs from the usual mean-field exponent $\beta_{MF} = 1/2$, and this is of course due to the non-analytic $|\phi|^3$ term in Eq. 9.39.

We can now write down the required field theory of the onset of d_{xy} order. In addition to the order parameter ϕ , the field theory should also involve the low energy nodal fermions of the $d_{x^2-y^2}$ superconductor, as described by $\mathcal{S}_\Psi$ in Eq. 9.32. For the ϕ fluctuations, we write down the usual terms permitted near a phase transition with Ising symmetry:

$$\mathcal{S}_\phi = \int d^2r d\tau \left[\frac{1}{2} \left((\partial_\tau \phi)^2 + c^2 (\partial_x \phi)^2 + c^2 (\partial_y \phi)^2 + s\phi^2 \right) + \frac{u}{24} \phi^4 \right]. \quad (9.41)$$

Note that, unlike Eq. 9.39, we do not have any non-analytic $|\phi|^3$ terms in the action: this is because we have not integrated out the low energy Dirac fermions, and the terms in Eq. 9.41 are viewed as arising from high energy fermions away from the nodal points. Finally, we need to couple the ϕ and $\Psi_{1,2}$ excitations. Their coupling is already contained in the last term in Eq. 9.34: expressing this in terms of the $\Psi_{1,2}$ fermions using Eq. 9.31 we obtain

$$\mathcal{S}_{\Psi\phi} = \vartheta_{xy} \int d^2r d\tau \left[\phi \left(\Psi_1^\dagger \tau^y \Psi_1 - \Psi_2^\dagger \tau^y \Psi_2 \right) \right], \quad (9.42)$$

where ϑ_{xy} is a coupling constant. The partition function of the full theory is now

$$\mathcal{Z}_{did} = \int \mathcal{D}\phi \mathcal{D}\Psi_1 \mathcal{D}\Psi_2 \exp(-\mathcal{S}_\Psi - \mathcal{S}_\phi - \mathcal{S}_{\Psi\phi}), \quad (9.43)$$

where $\mathcal{S}_\Psi$ was in Eq. 9.32. It can now be checked that if we integrate out the $\Psi_{1,2}$ fermions for a spacetime independent ϕ , we do indeed obtain a $|\phi|^3$ term in the effective potential for ϕ .

We begin our analysis of Z_{did} by assuming that the transition is described by a fixed point with $\vartheta_{xy} = 0$: then the theory for the transition would be the ordinary ϕ^4 field theory $\mathcal{S}_\phi$, and the nodal fermions would be innocent spectators. The scaling dimension of ϕ at such a fixed point is $(1 + \eta_I)/2$ (where η_I is the anomalous order parameter exponent at the critical point of the ordinary three dimensional Ising model), while that of $\Psi_{1,2}$ is 1. Consequently, the scaling dimension of ϑ_{xy} is $(1 - \eta_I)/2 > 0$. This positive scaling dimension implies that ϑ_{xy} is relevant and the $\vartheta_{xy} = 0$ fixed point is unstable: the Dirac fermions are fully involved in the critical theory.

Determining the correct critical behavior now requires a full renormalization group analysis of Z_{did} . This has been described in some detail in [55], and we will not reproduce the details here. The main result we need for our purposes is that couplings ϑ_{xy} , u , v_F/c and v_Δ/c all reach *non-zero* fixed point values which define a critical point in a new universality class. These fixed point values, and the corresponding critical exponents, can be determined in expansions in either $(3 - d)$ [51, 52, 55] (where d is the spatial dimensionality) or $1/N$ [56] (where N is the number of fermion species). An important simplifying feature here is that the fixed point is actually relativistically invariant. Indeed the fixed point has the structure of the so-called Higgs-Yukawa (or Gross-Neveu) model which has been studied extensively in the particle physics literature [57] in a different physical context: quantum Monte Carlo simulation of this model also exist [58], and provide probably the most accurate estimate of the exponents.

The non-trivial fixed point has strong implications for the correlations of the Bogoliubov fermions. The fermion correlation function $G_1 = \langle \Psi_1 \Psi_1^\dagger \rangle$ obeys

$$G_1(k, \omega) = \frac{\omega + v_F k_x \tau^z + v_\Delta \tau^x}{(v_F^2 k_x^2 + v_\Delta^2 k_y^2 - \omega^2)^{(1-\eta_f)/2}} \quad (9.44)$$

at low frequencies for $s \geq s_c$. Away from the critical point in the $d_{x^2-y^2}$ superconductor with $s > s_c$, Eq. 9.44 holds with $\eta_f = 0$, and this is the BCS result, with sharp quasi-particle poles in the Green's function. At the critical point $s = s_c$ Eq. 9.44 holds with the fixed point values for the velocities (which satisfy $v_F = v_\Delta = c$) and with the anomalous dimension $\eta_f \neq 0$ —the $(3 - d)$ expansion [51, 52] estimate is $\eta_f \approx (3 - d)/14$, and the $1/N$ expansion estimate [56] is $\eta_f \approx 1/(3\pi^2 N)$, with $N = 2$. This is clearly non-BCS behavior, and the fermionic quasiparticle pole in the spectral function has been replaced by a branch-cut representing the continuum of critical excitations. The corrections to BCS extend also to correlations of the Ising order ϕ : its expectation value vanishes as Eq. 9.40 with the Monte Carlo estimate $\beta \approx 0.877$ [58]. The critical point correlators of ϕ have the anomalous dimension $\eta \approx 0.754$ [58], which is clearly different from the

very small value of the exponent η_l at the unstable $\vartheta_{xy} = 0$ fixed point. The value of β is related to η by the usual scaling law $\beta = (1 + \eta)v/2$, with $v \approx 1.00$ the correlation length exponent (which also differs from the exponent v_l of the Ising model).

9.7.3 Nematic Ordering

We now consider an Ising transition associated with “Ising-nematic” ordering in the d -wave superconductor. This is associated with a spontaneous reduction of the lattice symmetry of the Hamiltonian from “square” to “rectangular”. Our study is motivated by experimental observations of such a symmetry breaking in the cuprate superconductors [59–61].

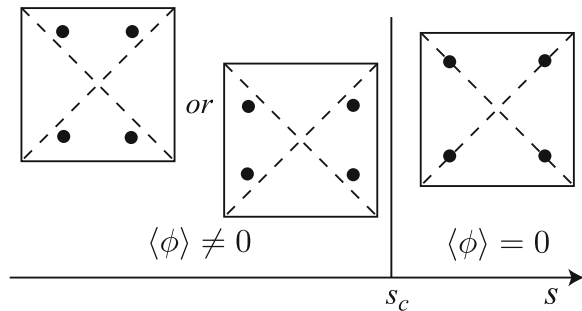
The ingredients of such an ordering are actually already present in our simple review of BCS theory in Sect. 9.7.1. In Eq. 9.28, we introduce 2 variational pairing amplitudes Δ_x and Δ_y . Subsequently, we assumed that the minimization of the energy led to a solution with $d_{x^2-y^2}$ pairing symmetry with $\Delta_x = -\Delta_y = \Delta_{x^2-y^2}$. However, it is possible that upon including the full details of the microscopic interactions we are led to a minimum where the optimal solution also has a small amount of s -wave pairing. Then $|\Delta_x| \neq |\Delta_y|$, and we can expect all physical properties to have distinct dependencies on the x and y co-ordinates. Thus, one measure of the the Ising nematic order parameter is $|\Delta_x|^2 - |\Delta_y|^2$.

The derivation of the field theory for this transition follows closely our presentation in Sect. 9.7.2. We allow for small Ising-nematic ordering by introducing a scalar field ϕ and writing

$$\Delta_x = \Delta_{x^2-y^2} + \phi; \Delta_y = -\Delta_{x^2-y^2} + \phi. \quad (9.45)$$

The evolutions of the Dirac fermion spectrum under such a change is indicated in Fig. 9.9. We now develop an effective action for ϕ and the Dirac fermions $\Psi_{1,2}$. The result is essentially identical to that in Sect. 9.7.2, apart from a change in the structure of the Yukawa coupling. Thus we obtain a theory $\mathcal{S}_\Psi + \mathcal{S}_\phi + \bar{\mathcal{S}}_{\Psi\phi}$, defined by Eqs. 9.32 and 9.41, and where Eq. 9.42 is now replaced by

Fig. 9.9 Phase diagram of Ising nematic ordering in a d -wave superconductor as a function of the coupling s in $\mathcal{S}_\phi$. The filled circles indicate the location of the gapless fermionic excitations in the Brillouin zone. The two choices for $s < s_c$ are selected by the sign of $\langle \phi \rangle$.



$$\overline{\mathcal{S}}_{\Psi\phi} = \vartheta_I \int d^2r d\tau \left[\phi \left(\Psi_1^\dagger \tau^x \Psi_1 + \Psi_2^\dagger \tau^x \Psi_2 \right) \right]. \quad (9.46)$$

The seemingly innocuous change between Eqs. 9.42 and 9.46 however has strong consequences. This is partly linked to the fact with $\overline{\mathcal{S}}_{\Psi\phi}$ cannot be relativistically invariant even after all velocities are adjusted to equal. A weak-coupling renormalization group analysis in powers of the coupling ϑ_I was performed in $(3-d)$ dimensions in [51, 52, 55], and led to flows to strong coupling with no accessible fixed point: thus no firm conclusions on the nature of the critical theory were drawn.

This problem remained unsolved until the recent works of [62, 63]. It is essential that there not be any expansion in powers of the coupling ϑ_I . This is because it leads to strongly non-analytic changes in the structure of the ϕ propagator, which have to be included at all stages. In a model with N fermion flavors, the $1/N$ expansion does avoid any expansion in ϑ_I . The renormalization group analysis has to be carried out within the context of the $1/N$ expansion, and this involves some rather technical analysis which is explained in [63]. In the end, an asymptotically exact description of the vicinity of the critical point was obtained. It was found that the velocity ratio v_F/v_Δ diverged logarithmically with energy scale, leading to strongly anisotropic ‘arc-like’ spectra for the Dirac fermions. Associated singularities in the thermal conductivity have also been computed [64].

9.8 Metals

This section considers symmetry breaking transitions in two-dimensional metals. Away from the quantum critical point, the phases will be ordinary Fermi liquids. We will be interested in the manner in which the Fermi liquid behavior breaks down at the quantum critical point. Our focus will be exclusively on two spatial dimensions: quantum phase transitions of metals in three dimensions are usually simpler, and the traditional perturbative theory appears under control.

In Sect. 9.7 the fermionic excitations had vanishing energy only at isolated nodal points in the Brillouin zone: see Figs. 9.8 and 9.9. Metals have fermionic excitations with vanishing energy along a *line* in the Brillouin zone. Thus we can expect them to have an even stronger effect on the critical theory. This will indeed be the case, and we will be led to problems with a far more complex structure. Unlike the situation in insulators and d -wave superconductor, many basic issues associated with ordering transition in two dimensional metals have not been fully resolved. The problem remains one of active research and is being addressed by many different approaches. In recent papers [65, 66], Metlitski and the author have argued that the problem is strongly coupled, and proposed field theories and scaling structures for the vicinity of the critical point. We will review the main ingredients for the transition involving Ising-nematic ordering in a metal. Thus the

symmetry breaking will be just as in Sect. 9.7.3, but the fermionic spectrum will be quite different. Our study here is also motivated by experimental observations in the cuprate superconductors [59–61].

As in Sect. 9.7, let us begin by a description of the non-critical fermionic sector, before its coupling to the order parameter fluctuations. We use the band structure describing the cuprates in the over-doped region, well away from the Mott insulator. Here the electrons $c_{k\alpha}$ are described by the kinetic energy in Eq. 9.26, which we write in the following action

$$\mathcal{S}_c = \int d\tau \sum_k c_{k\alpha}^\dagger \left(\frac{\partial}{\partial \tau} + \varepsilon_k \right) c_{k\alpha}, \quad (9.47)$$

As in Sect. 9.7.3, we will have an Ising order parameter represented by the real scalar field ϕ , which is described as before by $\mathcal{S}_\phi$ in Eq. 9.41. Its coupling to the electrons can be deduced by symmetry considerations, and the most natural coupling (the analog of Eqs. 9.46) is

$$\mathcal{S}_{c\phi} = \frac{1}{V} \int d\tau \sum_{k,q} (\cos k_x - \cos k_y) \phi(q) c_{k+q/2,\alpha}^\dagger c_{k-q/2,\alpha}. \quad (9.48)$$

where V is the volume. The momentum dependent form factor is the simplest choice which changes sign under $x \leftrightarrow y$, as is required by the symmetry properties of ϕ . The sum over q is over small momenta, while that over k extends over the entire Brillouin zone. The theory for the nematic ordering transition is now described by $\mathcal{S}_c + \mathcal{S}_\phi + \mathcal{S}_{c\phi}$. The phase diagram as a function of the coupling s in $\mathcal{S}_\phi$ and temperature T is shown in Fig. 9.10. Note that there is a line of Ising phase transitions at $T = T_c$: this transition is in the same universality class as the classical two-dimensional Ising model. However, quantum effects and fermionic excitations are crucial at $T = 0$ critical point at $s = s_c$ and its associated quantum critical region.

A key property of Eq. 9.48 is that small momentum critical ϕ fluctuations can efficiently scatter fermions at every point on the Fermi surface. Thus the non-Fermi singularities in the fermion Green's function will extend to all points on the Fermi surface. This behavior is dramatically different from all the field theories we have met so far, all of which had singularities only at isolated points in momentum space. We evidently have to write down a long-wavelength theory which has singularities along a line in momentum space.

We describe the construction of [65] of a field theory with this unusual property. Pick a fluctuation of the order parameter ϕ at a momentum q . As shown in Fig. 9.11 this fluctuation will couple most efficiently to fermions near two points on the Fermi surface, where the tangent to the Fermi surface is parallel to q . A fermion absorbing momentum q at these points, changes its energy only by $\sim q^2$; at all other points on the Fermi surface the change is $\sim q$. Thus we are led to focus on different points on the Fermi surface for each direction of q . In the continuum limit, we will therefore need a separate field theory for each pair of points $\pm k_0$ on

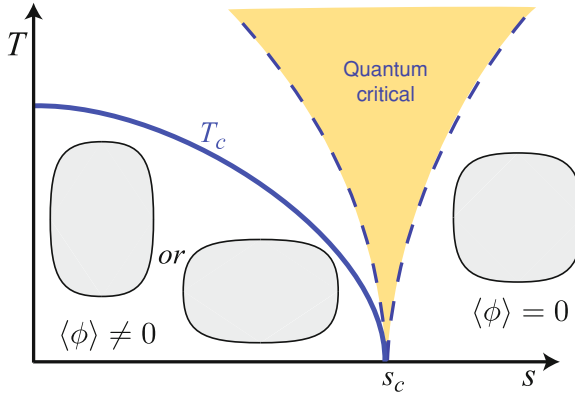
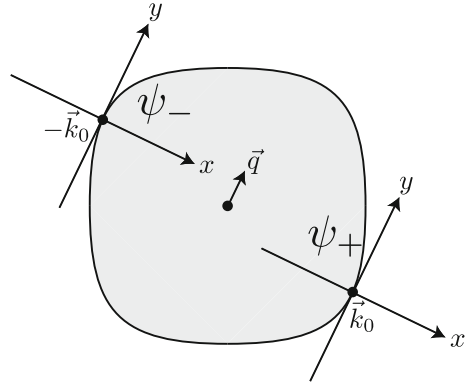


Fig. 9.10 Phase diagram of Ising nematic ordering in a metal as a function of the coupling s in S_ϕ and temperature T . The Fermi surface for $s > 0$ is as in the overdoped region of the cuprates, with the shaded region indicating the occupied hole (or empty electron) states. The choice between the two quadrupolar distortions of the Fermi surface is determined by the sign of $\langle \phi \rangle$. The line of $T > 0$ phase transitions at T_c is described by Onsager's solution of the classical two-dimensional Ising model. We are interested here in the quantum critical point at $s = s_c$, which controls the quantum-critical region

Fig. 9.11 A ϕ fluctuation at wavevector $\mathbf{q}$ couples most efficiently to the fermions $\psi_\pm$ near the Fermi surface points $\pm \mathbf{k}_0$



the Fermi surface. We conclude that the quantum critical point is described by an infinite number of field theories.

There have been earlier descriptions of Fermi surfaces by an infinite number of field theories [67–73]. However many of these earlier works differ in a crucial respect from the theory to be presented here. They focus on the motion of fermions transverse to Fermi surface, and so represent each Fermi surface point by a $1 + 1$ dimensional chiral fermion. Thus they have an infinite number of $1 + 1$ dimensional field theories, labelled by points on the one dimensional Fermi surface. The original problem was $2 + 1$ dimensional, and so this conserves the total dimensionality and the number of degrees of freedom. However, we have already

argued above that the dominant fluctuations of the fermions are in a direction transverse to the Fermi surface, and so we believe this earlier approach is not suited for the vicinity of the nematic quantum critical point. As we will see in [Sect. 9.8.1](#), we will need an infinite number of $2 + 1$ dimensional field theories labelled by points on the Fermi surface. (Some of the earlier works included fluctuations transverse to the Fermi surface [72, 73], but did not account for the curvature of the Fermi surface; it is important to take the scaling limit at fixed curvature, as we will see.) Thus we have an emergent dimension and a redundant description of the degrees of freedom. We will see in [Sect. 9.8.2](#) how compatibility conditions ensure that the redundancy does not lead to any inconsistencies. The emergent dimensionality suggests a connection to the AdS/CFT correspondence, as will be discussed in [Sect. 9.8.5](#).

9.8.1 Field Theories

Let us now focus on the vicinity of the points $\pm \mathbf{k}_0$, by introducing fermionic field $\psi_{\pm}$ by

$$\psi_{+}(\mathbf{k}) = c_{\mathbf{k}_0+\mathbf{k}}, \quad \psi_{-}(\mathbf{k}) = c_{-\mathbf{k}_0+\mathbf{k}}. \quad (9.49)$$

Then we expand all terms in $\mathcal{S}_c + \mathcal{S}_{\phi} + \mathcal{S}_{c\phi}$ in spatial and temporal gradients. Using the co-ordinate system illustrated in [Fig. 9.11](#), performing appropriate rescaling of co-ordinates, and dropping terms which can later be easily shown to be irrelevant, we obtain the $2 + 1$ dimensional Lagrangian

$$\begin{aligned} \mathcal{L} = & \psi_{+\alpha}^{\dagger} \left(\zeta \partial_{\tau} - i \partial_x - \partial_y^2 \right) \psi_{+\alpha} + \psi_{-\alpha}^{\dagger} \left(\zeta \partial_{\tau} + i \partial_x - \partial_y^2 \right) \psi_{-\alpha} \\ & - \lambda \phi \left(\psi_{+\alpha}^{\dagger} \psi_{+\alpha} + \psi_{-\alpha}^{\dagger} \psi_{-\alpha} \right) + \frac{N}{2} (\partial_y \phi)^2 + \frac{Ns}{2} \phi^2. \end{aligned} \quad (9.50)$$

Here ζ , λ and s are coupling constants, with s the tuning parameter across the transition; we will see that all couplings apart from s can be scaled away or set equal to unity. We now allow the spin index $\alpha = 1, \dots, N$, as we will be interested in the structure of the large N expansion. Note that [Eq. 9.50](#) has the same basic structure as the models considered in [Sect. 9.7](#), apart from differences in the spatial gradients and the matrix structure. We will see that these seemingly minor differences will completely change the physical properties and the nature of the large N expansion.

9.8.2 Symmetries

A first crucial property of $\mathcal{L}$ is that the fermion Green's functions do indeed have singularities along a line in momentum space, as was required by our discussion

above. This singularity is a consequence of the invariance of $\mathcal{L}$ under the following transformation

$$\phi(x, y) \rightarrow \phi(x, y + \theta x), \quad \psi_s(x, y) \rightarrow e^{-is(\frac{\theta}{2}y + \frac{\theta^2}{4}x)} \psi_s(x, y + \theta x), \quad (9.51)$$

where θ is a constant. Here we have dropped time and α indices because they play no role, and $s = \pm$. We can view this transformation as one which performs a ‘rotation’ of spatial co-ordinates, moving the point $\mathbf{k}_0$ to neighboring points on the Fermi surface. We are not assuming the Fermi surface is circular, and so the underlying model is not rotationally invariant. However, we are considering a limiting case of a rotation, precisely analogous to the manner in which Galilean transformations emerge as a limiting case of a relativistic transformation (x behaves like time, and y as space, in this analogy). This ‘Galilean’ symmetry is an emergent symmetry of $\mathcal{L}$ for arbitrary shapes of the Fermi surface. It is not difficult to now show from (9.51) that the ϕ Green’s function D , and the fermion Green’s functions G_s obey the exact identities

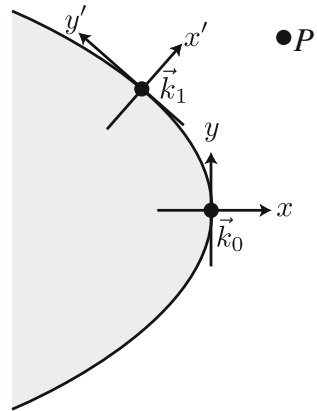
$$D(q_x, q_y) = D(q_y) \quad (9.52)$$

$$G_s(q_x, q_y) = G(sq_x + q_y^2). \quad (9.53)$$

So we see that the Ψ_+ Green’s function depends only on $q_x + q_y^2$. The singularities of this function appear when $q_x + q_y^2 = 0$, and this is nothing but the equation of the Fermi surface passing through the point $\mathbf{k}_0$ in Figs. 9.11 and 9.12. Thus we have established the existence of a line of singularities in momentum space.

The identities in Eq. 9.51 also help establish the consistency of our description in terms of an infinite number of $2 + 1$ dimensional field theories. Consider the fermion Green’s function at the point P in Fig. 9.12. This can be computed in terms of the $2 + 1$ dimensional field theory defined at the point $\mathbf{k}_0$, or from that at a

Fig. 9.12 The fermion correlator at the point P can be described either in terms of the $2 + 1$ dimensional field theory at $\mathbf{k}_0$, or that at $\mathbf{k}_1$



neighboring point $\mathbf{k}_1$. Equation 9.53 ensures that both methods yield the same result. A little geometry [65] shows that $q_x + q_y^2$ is an invariant that measures the distance between P and the closest point on the Fermi surface: thus it takes the same value in the co-ordinates systems at $\mathbf{k}_0$ and $\mathbf{k}_1$, with $q_x + q_y^2 = q'_x + q_y'^2$, establishing the identity of the two computations.

9.8.3 Scaling Theory

Let us now discuss the behavior of $\mathcal{L}$ under renormalization group scaling transformations. The structure of the spatial gradient terms in the Lagrangian indicates that the rescaling of spatial co-ordinates should be defined by

$$x' = x/b^2, \quad y' = y/b. \quad (9.54)$$

The invariance in Eq. 9.53 implies that these scalings are exact, and the spatial anisotropy acquires no fluctuation corrections. Or, in other words

$$\dim[y] = -1, \quad \dim[x] = -2. \quad (9.55)$$

For now, let us keep the rescaling of the temporal co-ordinate general:

$$\dim[\tau] = -z. \quad (9.56)$$

Note that the dynamic critical exponent z is defined relative to the spatial co-ordinate y tangent to the Fermi surface (other investigators sometimes define it relative to the co-ordinate x normal to the Fermi surface, leading to a difference in a factor of 2). We define the engineering dimensions of the fields so that co-efficients of the y derivatives remain constant. Allowing for anomalous dimensions η_ϕ and η_ψ from loop effects we have

$$\dim[\phi] = (1 + z + \eta_\phi)/2, \quad \dim[\psi] = (1 + z + \eta_\psi)/2. \quad (9.57)$$

Using these transformations, we can examine the scaling dimensions of the couplings in $\mathcal{L}$ at tree level

$$\dim[\zeta] = 2 - z - \eta_\psi, \quad \dim[\lambda] = (3 - z - \eta_\phi - 2\eta_\psi)/2. \quad (9.58)$$

We will see in Sect. 9.8.4 that low order loop computations suggest that the anomalous dimensions η_ϕ and η_ψ are small, and that $z \approx 3$. Assuming these estimates are approximately correct, we see that the coupling ζ is strongly irrelevant. Thus we can send $\zeta \rightarrow 0$ in all our computations. However, we do not set $\zeta = 0$ at the outset, because the temporal derivative term is needed to define the proper analytic structure of the frequency loop integrals [74]. As we will discuss later, the limit $\zeta \rightarrow 0$, also dramatically changes the counting of powers of $1/N$ in the loop expansion [74].

Also note that these estimates of the scaling dimensions imply $\dim[\lambda] \approx 0$. Thus the fermion and order parameter fluctuations remain strongly coupled at all scales. Conversely, we can also say that the requirement of working in a theory with fixed λ implies that $z \approx 3$; this circumvents the appeal to loop computations for taking the $\zeta \rightarrow 0$ limit. With a near zero scaling dimension for λ , we cannot expand perturbatively in powers of λ . This feature was also found in Sect. 9.7.3, but there we were able to use the $1/N$ expansion to circumvent this problem.

Moving beyond tree level considerations, we note that another Ward identity obeyed by the theory $\mathcal{L}$ allows us to fix the scaling dimension of ϕ exactly. This Ward identity is linked to the fact that ϕ appears in the Yukawa coupling like the x component of a gauge field coupled to the fermions [65]. The usual arguments associated with gauge invariance then imply that $\dim[\phi] = 2$ (the same as the scaling dimension of ∂_x), and that we can work in theory in which the “gauge coupling” λ set equal to unity at all scales. Note that with this scaling dimension, we have the exact relation

$$\eta_\phi = 3 - z. \quad (9.59)$$

Note also that Eq. 9.57 now implies that $\dim[\lambda] = \eta_\psi$ at tree level, which is the same as the tree level transformation of the spatial derivative terms. The latter terms have been set equal to unity by rescaling the fermion field, and so it is also consistent to set $\lambda = 1$ from now on.

We reach the remarkable conclusion that at the critical point $s = s_c$, $\mathcal{L}$ is independent of all coupling constants. The only parameter left is N , and we have no choice but to expand correlators in powers of $1/N$. The characterization of the critical behavior only requires computations of the exponents z and η_ψ , and associated scaling functions.

We can combine all the above results into scaling forms for the ϕ and Ψ Green’s functions at the quantum critical point at $T = 0$. These are, respectively

$$D^{-1}(q_x, q_y, \omega) = q_y^{z-1} \mathcal{F}_D \left(\frac{\omega}{q_y^z} \right) \quad (9.60)$$

$$G^{-1}(q_x, q_y, \omega) = (q_x + q_y^2)^{1-\eta_\psi/2} \mathcal{F}_G \left(\frac{\omega}{(q_x + q_y^2)^{z/2}} \right), \quad (9.61)$$

where $\mathcal{F}_D$ and $\mathcal{F}_G$ are scaling functions.

9.8.4 Large N Expansion

We have come as far as possible by symmetry and scaling analyses alone on $\mathcal{L}$. Further results require specific computations of loop corrections, and these can

only be carried out within the context of the $1/N$ expansion. At $N = \infty$, the ϕ propagator at criticality is

$$\frac{D^{-1}}{N} = q_y^2 + \frac{|\omega|}{4\pi|q_y|} \quad (9.62)$$

for imaginary frequencies ω . This is clearly compatible with Eq. 9.60 with $z = 3$. The leading correction to the fermion propagator comes from the self energy associated with one ϕ exchange, and this leads to

$$G_+^{-1} = -i\zeta\omega + q_x + q_y^2 - i\text{sgn}(\omega) \frac{2}{\sqrt{3}(4\pi)^{2/3}N} |\omega|^{2/3}, \quad (9.63)$$

which is also compatible with Eq. 9.61 with $z = 3$ and $\eta_\psi = 0$. Notice also that as $\omega \rightarrow 0$, the $\zeta\omega$ term in Eq. 9.63 is smaller than the $|\omega|^{2/3}$ term arising from the self energy at order $1/N$; this relationship is equivalent to our earlier claim that ζ is irrelevant at long scales, and so we should take the limit $\zeta \rightarrow 0$ to obtain our leading critical scaling functions.

The structure of Eq. 9.63 also illustrates a key difficulty associated with the $\zeta \rightarrow 0$ limit. At $\zeta = 0$, the leading ω dependence of G^{-1} is $\sim |\omega|^{2/3}/N$. Feynman graphs which are sensitive to this ω dependence will therefore acquire additional factors of N , leading to a breakdown of the conventional counting of powers of $1/N$ in the higher loop graphs.

This breakdown of the $1/N$ expansion was investigated by Lee [74] for a ‘single patch’ theory with fermions only at $\mathbf{k}_0$ (and not at $-\mathbf{k}_0$). We see from Eq. 9.63 that the $1/N$ term in G^{-1} becomes important when $q_x = q_y = 0$ i.e. the fermion is precisely on the Fermi surface. Thus the power of N is maximized when fermions in all internal lines are on the Fermi surface. Such a Fermi surface restriction is satisfied only in a subspace of reduced dimension in the momentum space integral of any Feynman graph. Lee presented an algorithm for computing the dimensionality of this restricted subspace: he demonstrated that the power of $1/N$ was determined by the genus of the surface obtained after drawing the graph in a double-line representation. So determining the leading $1/N$ terms in Eqs. 9.62 and 9.63 requires summation of the infinite set of planar graphs. This problem remains unsolved, but the unexpected appearance of planar graphs does suggest an important role for the AdS/CFT correspondence.

The structure of the loop expansion for the ‘two patch’ theory with fermions at $\pm\mathbf{k}_0$, as written in Eq. 9.50, was studied in [65]. It was found that $z = 3$ was preserved up to three loops, but a small non-zero value for η_ψ did appear at three loop order. Also, the genus counting of powers of $1/N$ was found to break down, with larger powers of N appearing in some three loop graphs.

9.8.5 AdS/CFT Correspondence

There has been a great deal of recent work [75–93] investigating the structure of Fermi surfaces using the AdS/CFT correspondence. The results obtained so far do have features that resemble our results above for the Ising-nematic transition in two dimensional metals. However, the precise connection remains obscure, and is an important topic for future research. In particular, a microscopic understanding of the field content of the CFT dual of the AdS theory is lacking, although there has been interesting progress very recently [88, 92].

One of the main results of the analysis of [77, 78] is the general structure of the fermion Green's function obtained in a theory dual to a Reissner-Nordstrom black hole in AdS₄. This had the form

$$G^{-1}(\mathbf{k}, \omega) = -i\omega + v_F(|\mathbf{k}| - k_F) - c_1 \omega^\theta, \quad (9.64)$$

where the momentum $\mathbf{k}$ is now measured from the origin of momentum space (and not from a Fermi surface), and the complex number c_1 and exponent θ are computable functions of the ultraviolet scaling dimension of the fermion field. The AdS theory only considers a circular Fermi surface, and for this geometry (after appropriate rescaling)

$$v_F(|\mathbf{k}| - k_F) = v_F(|\mathbf{q} + \mathbf{k}_0| - k_F) \approx q_x + q_y^2; \quad (9.65)$$

now Eq. 9.64 is seen to be strikingly similar to Eq. 9.63. Liu et al. [77], and Faulkner et al. [78] also argued that Eq. 9.64 was a generic property of the near horizon geometry of the Reissner-Nordstrom black hole: the geometry changes from AdS₄ near the boundary to AdS₂ $\times$ R^2 near the black hole horizon.

It is interesting to compare the structure of the critical theory in the AdS/CFT framework to that found in the subsections above for the Ising-nematic transition in a metal. The latter was described by an infinite set of 2 + 1 dimensional field theories labelled by pairs of momenta on a one-dimensional Fermi surface *i.e.* a S^1/Z_2 set of 2 + 1 dimensional field theories. In the low-energy limit, the AdS/CFT approach yields [77, 78] a AdS₂ $\times$ R^2 geometry: this can be interpreted as an infinite set of chiral 1 + 1 dimensional theories labelled by a R^2 set of two-dimensional momenta $\mathbf{k}$. It is notable, and perhaps significant, that both approaches have an emergent dimension not found in the underlying degrees of freedom. The Ising nematic theory began with a 2 + 1 dimensional Hamiltonian $\mathcal{S}_c + \mathcal{S}_\phi + \mathcal{S}_{c\phi}$ in Eqs. (9.41, 9.47, 9.48), and ended up with a S^1/Z_2 set of 2+1 dimensional field theories. In AdS/CFT, there is the emergent radial direction representing energy scale. These emergent dimensions imply redundant descriptions, and require associated consistency conditions: we explored such consistency conditions in Sect. 9.8.2, while in AdS/CFT the consistency conditions are Einstein's equations representing the renormalization group flow under changes of energy scale. It would be interesting to see if fluctuations about the classical gravity theory

yield corrections to the $\text{AdS}_2 \times R^2$ geometry which clarify the connection to our Ising-nematic theory.

Acknowledgements Many of the ideas reviewed in Sect. 9.8 are due to Max Metliski [65]; I also thank him for valuable comments on the manuscript. I thank the participants of the school for their interest, and for stimulating discussions. This research was supported by the National Science Foundation under grant DMR-0757145, by the FQXi foundation, and by a MURI grant from AFOSR.

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Chapter 10

Introduction to Holographic Superconductors

Gary T. Horowitz

Abstract These lectures give an introduction to the theory of holographic superconductors. These are superconductors that have a dual gravitational description using gauge/gravity duality. After introducing a suitable gravitational theory, we discuss its properties in various regimes: the probe limit, the effects of backreaction, the zero temperature limit, and the addition of magnetic fields. Using the gauge/gravity dictionary, these properties reproduce many of the standard features of superconductors. Some familiarity with gauge/gravity duality is assumed. A list of open problems is included at the end.

10.1 Introduction

The name “holographic superconductor” suggests that one can look at a two (spatial) dimensional superconductor and see a three dimensional image. We will see that there is a class of superconductors for which this is true, but the image one “sees” is quite striking. It involves a charged black hole with nontrivial “hair”. This remarkable connection between condensed matter and gravitational physics was discovered just a few years ago. It grew out of the gauge/gravity duality which has emerged from string theory [54, 26]. Although this duality was first formulated as an equivalence between a certain gauge theory and a theory of quantum gravity (and provided new insights into each of these theories), over the past decade it has been applied with notable success to other areas of physics as well. Many of these new applications are discussed in other lectures in this school. I will focus

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on the application to superconductivity. These lectures will be heavily based on [36, 37, 44, 38]. For a more general discussion of applying gauge/gravity duality to condensed matter, see the excellent reviews by Hartnoll [33], Herzog [39] and McGreevy [55].

I will start with a brief introduction to superconductivity. In Sect. 10.2, I will introduce a simple model for a holographic superconductor. Most of our discussion will be devoted to exploring the consequences of this model, beginning in Sect. 10.3 with the probe limit—a simplification of the model which preserves most of the physics. In Sect. 10.4 we will discuss the full solution with backreaction. We next examine the ground state of the superconductor Sect. 10.5, and study its behavior when magnetic fields are added (Sect. 10.6). We conclude with a brief discussion of recent developments (Sect. 10.7) and a list of open problems (Sect. 10.8).

10.1.1 Superconductivity

It was noticed in the early part of the 20th century that the electrical resistivity of most metals drops suddenly to zero as the temperature is lowered below a critical temperature T_c . These materials were called superconductors.¹ A second independent property of these materials was the Meissner effect: A magnetic field is expelled when $T < T_c$. This is perfect diamagnetism and does not follow from the perfect conductivity (which alone would imply that a pre-existing magnetic field is trapped inside the sample). A phenomenological description of both of these properties was first given by the London brothers in 1935 with the simple equation $J_i \propto A_i$ [51]. Taking a time derivative yields $E_i \propto \partial J_i / \partial t$, showing that electric fields accelerate superconducting electrons rather than keeping their velocity constant as in Ohm's law with finite conductivity. Taking the curl of both sides and combining with Maxwell's equations yields $\nabla^2 B_i \propto B_i$ showing the decay of magnetic fields inside a superconductor.

In 1950, Landau and Ginzburg described superconductivity in terms of a second order phase transition whose order parameter is a complex scalar field φ [22]. The density of superconducting electrons is given by $n_s = |\varphi(x)|^2$. The contribution of φ to the free energy is assumed to take the form

$$F = \alpha(T - T_c)|\varphi|^2 + \frac{\beta}{2}|\varphi|^4 + \dots \quad (10.1)$$

where α and β are positive constants and the dots denote gradient terms and higher powers of φ . Clearly for $T > T_c$ the minimum of the free energy is at $\varphi = 0$ while for $T < T_c$ the minimum is at a nonzero value of φ . This is just like the Higgs

¹ A good general reference is [63].

mechanism in particle physics, and is associated with breaking a $U(1)$ symmetry. The London equation follows from this spontaneous symmetry breaking [66].

A more complete theory of superconductivity was given by Bardeen, Cooper and Schrieffer in 1957 and is known as BCS theory [7]. They showed that interactions with phonons can cause pairs of electrons with opposite spin to bind and form a charged boson called a Cooper pair. Below a critical temperature T_c , there is a second order phase transition and these bosons condense. The DC conductivity becomes infinite producing a superconductor. The pairs are not bound very tightly and typically have a size which is much larger than the lattice spacing. In the superconducting ground state, there is an energy gap Δ for charged excitations. This gap is typically related to the critical temperature by $\Delta \approx 1.7T_c$. The charged excitations are “dressed electrons” called quasiparticles. The gap in the spectrum results in a gap in the (frequency dependent) optical conductivity. If a photon of frequency ω hits the superconductor, it must produce two quasiparticles. The binding energy of the Cooper pair is very small, but the energy of each quasiparticle is Δ , so the gap in the optical conductivity is $\omega_g = 2\Delta \approx 3.5T_c$.

It was once thought that the highest T_c for a BCS superconductor was around 30 K. But in 2001, MgB_2 was found to be superconducting at 40 K and is believed to be described by BCS. Some people now speculate that BCS could describe a superconductor with $T_c = 200$ K².

A new class of high T_c superconductors were discovered in 1986 [9]. They are cuprates and the superconductivity is along the CuO_2 planes. The highest T_c known today (at atmospheric pressure) is $T_c = 134$ K for a mercury, barium, copper oxide compound. If you apply pressure, T_c climbs to about 160 K. Another class of superconductors were discovered in 2008 based on iron and not copper [47]. The highest T_c so far is 56 K. These materials are also layered and the superconductivity is again associated with the two dimensional planes. They are called iron pnictides since they involve other elements like arsenic in the nitrogen group of the periodic table.

There is evidence that electron pairs still form in these high T_c materials, but the pairing mechanism is not well understood. Unlike BCS theory, it involves strong coupling. Gauge/gravity duality is a new tool to study strongly coupled field theories. In particular, it allows one to compute dynamical transport properties of strongly coupled systems at nonzero temperature. Condensed matter theorists have very few tools to do this. We will describe below the first steps toward applying this new tool to better understand high T_c superconductivity. I must stress at the beginning that we are still at the early stages of this endeavor. We will construct simple gravity models and show that they reproduce basic properties of superconductors. But our models are too crude to make detailed comparisons with any real-world material.

² Discussion at the KITP program on Quantum Criticality and the AdS/CFT Correspondence, July 2009.

10.2 A Gravitational Dual

How do we go about constructing a holographic dual for a superconductor? The minimal ingredients are the following. In the superconductor we need a notion of temperature. On the gravity side, that role is played by a black hole. Recall that in the 1970s Hawking (following work by Bekenstein and others) showed that stationary black holes are thermodynamic objects with a temperature T related to the surface gravity κ via $T = \kappa/2\pi$. In gauge/gravity duality, the Hawking temperature of the black hole is identified with the temperature of the dual field theory.³ Since gauge/gravity duality traditionally requires that spacetime asymptotically approach anti de Sitter (AdS) space at infinity, we will be studying black holes in AdS. Unlike asymptotically flat black holes, these black holes have the property that at large radius, their temperature increases with their mass, i.e., they have positive specific heat, just like familiar nongravitational systems. There are also planar AdS black holes, which will be of most interest. These black holes always have positive specific heat.

In the superconductor, we also need a condensate. In the bulk, this is described by some field coupled to gravity. A nonzero condensate corresponds to a static nonzero field outside a black hole. This is usually called black hole “hair”. So to describe a superconductor, we need to find a black hole that has hair at low temperatures, but no hair at high temperatures. More precisely, we need the usual Schwarzschild or Reissner–Nordstrom AdS black hole (which exists for all temperatures) to be unstable to forming hair at low temperature. At first sight, this is not an easy task. There are “no-hair theorems” which say that certain matter fields must be trivial outside a black hole (see, e.g., [10, 42]). The idea behind these theorems is simply that matter outside a black hole wants to fall into the horizon (or radiate out to infinity in the asymptotically flat case). However, there is no general “no-hair theorem”. Each matter field must be considered separately. The result is a set of black hole uniqueness theorems showing that when gravity is coupled to certain matter fields, stationary black holes are uniquely characterized by their conserved charges: mass, angular momentum and electromagnetic charge. These theorems usually require linear matter fields or scalars with certain potentials $V(\phi)$. Counterexamples to a general no-hair theorem have been known since the early 1990s. (So our task is not impossible.) For example, it was shown that static Yang–Mills fields can exist outside the horizon [65].

String theory has many dilatonic black holes with scalar hair, and one might be tempted to try to use one of these to model a superconductor. But if the action includes a term like $e^{2a\phi}F_{\mu\nu}F^{\mu\nu}$, this is doomed to failure. In this case, F^2 is a source for ϕ , so all charged black holes have nonzero ϕ . This “secondary hair” is not what we want, since we want the hair to go away at high temperatures. (Theories with more general coupling $f(\phi)F_{\mu\nu}F^{\mu\nu}$ are possible candidates provided

³ I will assume that the reader is familiar with the basics of gauge/gravity duality. If not, see other contributions in this book.

f does not have a linear term in ϕ . However, the example we will study uses a standard Maxwell action.) A general argument against AdS black holes developing scalar hair was given by Hertog [38]. He considered a real scalar field with arbitrary potential $V(\phi)$ (with negative extremum so AdS is a solution), and showed that neutral AdS black holes can have scalar hair if and only if AdS itself is unstable. This is clearly unacceptable.

A surprisingly simple solution to this problem was found by Gubser [25]. He argued that a charged scalar field around a *charged* black hole in AdS would have the desired property. Consider

$$S = \int d^4x \sqrt{-g} \left(R + \frac{6}{L^2} - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} - |\nabla\Psi - iqA\Psi|^2 - m^2|\Psi|^2 \right). \quad (10.2)$$

This is just general relativity with a negative cosmological constant $\Lambda = -3/L^2$, coupled to a Maxwell field and charged scalar with mass m and charge q . It is easy to see why black holes in this theory might be unstable to forming scalar hair: For an electrically charged black hole, the effective mass of Ψ is $m_{\text{eff}}^2 = m^2 + q^2 g^{tt} A_t^2$. But the last term is negative, so there is a chance that m_{eff}^2 becomes sufficiently negative near the horizon to destabilize the scalar field. Furthermore, as one lowers the temperature of a charged black hole, it becomes closer to extremality which means that g_{tt} is closer to developing a double zero at the horizon. This means that $|g^{tt}|$ becomes larger and the potential instability becomes stronger at low temperature.

We will see that black holes in this theory indeed develop scalar hair at low temperature. One might wonder why such a simple type of hair was not noticed earlier. One reason is that this does not work for asymptotically flat black holes. The AdS boundary conditions are crucial. One way to understand the difference is by the following quantum argument. Let Q_i be the initial charge on the black hole. If qQ_i is large enough, even maximally charged black holes with zero Hawking temperature create pairs of charged particles. This is simply due to the fact that the electric field near the horizon is strong enough to pull pairs of oppositely charged particles out of the vacuum via the Schwinger mechanism of ordinary field theory. The particle with opposite charge to the black hole falls into the horizon, reducing Q_i while the particle with the same sign charge as the black hole is repelled away. In asymptotically flat spacetime, these particles escape to infinity, so the final result is a standard Reissner–Nordstrom black hole with final charge $Q_f < Q_i$. In AdS, the charged particles cannot escape since the negative cosmological constant acts like a confining box, and they settle outside the horizon. This gas of charged particles is the quantum description of the hair. This quantum process has an entirely classical analog in terms of superradiance of the charged scalar field.

The four dimensional bulk theory (10.2) is dual to a $2 + 1$ dimensional boundary theory. This is the right context to try to understand the superconductivity associated with two dimensional planes in the high T_c cuprates or iron pnictides. One can also study this theory in five dimensions to describe $3 + 1$ dimensional superconductors. I should emphasize that at the moment we are not trying to derive the gravitational theory from string theory. The idea is to find a simple gravity

theory with the properties we want, and analyze it using standard entries in the gauge/gravity duality dictionary. However, we will see later that this simple model can, in fact, be realized as a consistent truncation of string theory.

Before we proceed, I should comment on the following confusing point. In gauge/gravity duality, gauge symmetries in the bulk correspond to global symmetries in the dual field theory. So although the scalar hair breaks a local $U(1)$ symmetry in the bulk, the dual description consists of a condensate breaking a global $U(1)$ symmetry. Thus strictly speaking, the dual theory is a superfluid rather than a superconductor. The superfluid properties of this model have indeed been investigated [4, 8, 41]. However, one can still view the dual theory as a superconductor in the limit that the $U(1)$ symmetry is “weakly gauged”.⁴ In fact, most of condensed matter physics does not include dynamical photons, since their effects are usually small. In particular, BCS theory only includes the electrons and phonons. Electromagnetic fields are usually introduced as external fields, as we will do here. (Unfortunately, our electromagnetic fields will not be dynamical on the boundary.)

10.3 Probe Limit

If one rescales $A_\mu = \tilde{A}_\mu/q$ and $\Psi = \tilde{\Psi}/q$, then the matter action in (10.2) has a $1/q^2$ in front, so the backreaction of the matter fields on the metric is suppressed when q is large. The limit $q \rightarrow \infty$ with $\tilde{A}_\mu$ and $\tilde{\Psi}$ fixed is called the probe limit. It simplifies the problem but retains most of the interesting physics since the non-linear interactions between the scalar and Maxwell field are retained. In this section, we will explore this probe limit. We first discuss the formation of the condensate and then compute the conductivity. To simplify the presentation, we will drop the tildes.

10.3.1 Condensate

We start with the planar Schwarzschild anti-de Sitter black hole in four dimensions

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2(dx^2 + dy^2), \quad (10.3)$$

where

⁴ This means that one imagines that the dual action includes terms like $|\nabla_i - ieA_i)\varphi|^2$ with very small charge e .

$$f = \frac{r^2}{L^2} \left(1 - \frac{r_0^3}{r^3} \right). \quad (10.4)$$

L is the AdS radius, and the Schwarzschild radius r_0 determines the Hawking temperature of the black hole:

$$T = \frac{3r_0}{4\pi L^2}. \quad (10.5)$$

In the probe approximation this metric is a fixed background in which we solve the Maxwell-scalar equations.

We assume a plane symmetric ansatz,

$$\Psi = \psi(r), \quad A_t = \phi(r). \quad (10.6)$$

With $A_r = A_x = A_y = 0$, the Maxwell equations imply that the phase of ψ must be constant. Without loss of generality we therefore take ψ to be real. The Maxwell-scalar field equations reduce to the following coupled, nonlinear, ordinary differential equations:

$$\psi'' + \left(\frac{f'}{f} + \frac{2}{r} \right) \psi' + \frac{\phi^2}{f^2} \psi - \frac{m^2}{f} \psi = 0, \quad (10.7)$$

$$\phi'' + \frac{2}{r} \phi' - \frac{2\psi^2}{f} \phi = 0. \quad (10.8)$$

The key term in the first equation is $(\phi^2/f^2)\psi$. This comes in with the opposite sign of the mass term and will cause the scalar hair to form at low temperature.

We first consider the case $m^2 = -2/L^2$. It might seem strange to make the scalar field tachyonic, but this mass is perfectly allowed in gauge/gravity duality. First note that a tachyonic mass in field theory does not describe particles moving faster than the speed of light. Instead, it usually describes an instability. The value $\psi = 0$ is unstable and the field rolls off the potential. However, Breitenlohner and Freedman [9] showed that AdS_{d+1} spacetime is stable even with scalar fields with $m^2 < 0$ provided $m^2 \geq m_{BF}^2$ with

$$m_{BF}^2 = -\frac{d^2}{4L^2}. \quad (10.9)$$

This is because there is so much volume at large radius that the positive gradient energy can compensate for a negative m^2 . Our choice of mass satisfies this bound and in fact corresponds to a conformally coupled scalar in AdS_4 . The standard compactification of 11D supergravity on S^7 produces many fields with this mass.

We now consider the boundary conditions. At the horizon, one often argues that $\phi = A_t$ must vanish in order for $g^{\mu\nu}A_\mu A_\nu$ to remain finite. This is correct, but A_μ is gauge dependent, so it is not obvious that a diverging vector potential is a problem

if the Maxwell field remains finite. A better argument for setting $A_t = 0$ at the horizon is the following.⁵ The source for Maxwell's equations in the bulk is, of course, gauge invariant. But in a gauge in which ψ is real, the current is just $\psi^2 A_\mu$. Since the current must remain finite at the horizon, we need A_μ to remain finite, and hence $\phi = A_t = 0$. Even in Einstein–Maxwell theory (without charged sources) A_t must vanish at a static black hole horizon by the following argument: For describing thermal properties of the black hole, one should use the Euclidean solution. The Wilson loop of A_μ around the Euclidean time circle is finite and gauge invariant. If A_t is nonzero at the horizon, the Wilson loop is nonzero around a vanishing circle which implies that the Maxwell field is singular.

There is another constraint that must be satisfied in order for the solution to be smooth at the horizon. Multiplying (10.7) by f and evaluating at $r = r_0$ one finds $f'\psi' = m^2\psi$, so $\psi(r_0)$ and $\psi'(r_0)$ are not independent. As a result, even though we start with two second order equations which have a four parameter family of solutions, there is only a two parameter subfamily which is regular at the horizon. They can be labelled by $\psi(r_0)$ and $\phi'(r_0)$. (Note that $\phi'(r_0)$ is essentially the electric field at the horizon: $(\phi')^2 = F_{\mu\nu}F^{\mu\nu}$.)

We now turn to the boundary condition at infinity. Asymptotically:

$$\psi = \frac{\psi^{(1)}}{r} + \frac{\psi^{(2)}}{r^2} + \dots \quad (10.10)$$

and

$$\phi = \mu - \frac{\rho}{r} + \dots \quad (10.11)$$

Usually, normalizability requires that the leading coefficient in ψ must vanish. However, since we have chosen a mass close to the BF bound, even the leading term in ψ is normalizable. In this case, one has a choice: One can consider solutions with $\psi^{(1)} = 0$ or $\psi^{(2)} = 0$. For definiteness, we will mostly consider standard boundary conditions, $\psi^{(1)} = 0$.

After imposing this asymptotic boundary condition, we have a one parameter family of solutions, which can be found numerically [36]. (See [24] for an accurate analytic approximation and [50] for an analytic solution of a related model.) We will not present any plots of $\psi(r)$ and $\phi(r)$ since they look rather boring. It is easy to see from (10.8) that ϕ is a monotonic function. It starts at zero at the horizon and at any local extremum $\phi'' \propto \phi$. So it cannot have a positive maximum or a negative minimum. If it starts increasing away from the horizon, it continues to increase and asymptotically approaches the constant μ . $\psi(r)$ does not have to be monotonic. There is a discrete infinite family of solutions for $\psi(r)$ that satisfy our asymptotic boundary conditions. They can be labelled by the number of times ψ vanishes. It is believed that only the lowest solution which monotonically decreases from $\psi(r_0)$ to

⁵ I thank Karl Landsteiner for suggesting this.

zero is stable, although I do not think a stability analysis has been performed yet. Despite the fact that the solutions of interest are simple monotonic functions, they have important consequences for the dual field theory as we now discuss.

Since we have not started with a consistent truncation of string theory, we do not have a detailed microscopic description of the dual field theory. Nevertheless, basic elements of the gauge/gravity duality dictionary allow us to say the following. The dual theory is a $2 + 1$ dimensional conformal field theory (CFT) at temperature T given by (10.5). The local gauge symmetry in the bulk corresponds to a global $U(1)$ symmetry in the CFT. The asymptotic behavior of the bulk solution determines certain properties of the dual field theory. For example, from (10.11), μ is the chemical potential and ρ is the charge density. It may seem strange that our superconductor has a nonzero charge density. After all, ordinary superconductors are electrically neutral. Perhaps the best analogy is to say that our holographic superconductor is modeling the electrons, but does not include the (positively charged) atomic lattice. Indeed, our model has translational symmetry, and there is no sign of any lattice. From a practical standpoint, we need the charge since neutral black holes in the bulk are not unstable to forming scalar hair. In addition, without the charge (or chemical potential), the dual theory is scale invariant and cannot have a phase transition.

The dual theory also has an operator charged under the $U(1)$ which is dual to ψ . Since we have chosen m^2 close to the BF bound, there are two possible operators depending on how one quantizes ψ in the bulk.⁶ [49] If the modes are defined with the standard boundary condition (faster falloff) for ψ in the bulk, the dual operator has dimension two. In this case, a nonzero $\psi^{(1)}$ corresponds to adding a source for this operator in the CFT, and a nonzero $\psi^{(2)}$ corresponds to a nonzero expectation value⁷

$$O_2 = \psi^{(2)}. \quad (10.12)$$

Since we want the condensate to turn on without being sourced, we have set $\psi^{(1)} = 0$. There is an alternative quantization of ψ in the bulk in which the roles of $\psi^{(1)}$ and $\psi^{(2)}$ are reversed. ψ is now dual to a dimension one operator. If one wishes to study this case, one should impose the boundary condition $\psi^{(2)} = 0$.

We want to know how the condensate O_2 behaves as a function of temperature. Before presenting the results we have to discuss an important scaling symmetry of our problem. In any conformal field theory on R^n , one can change the temperature by a simple rescaling. In the bulk, this is reflected in the statement that the rescaling

$$r \rightarrow ar, \quad (t, x, y) \rightarrow (t, x, y)/a, \quad r_0 \rightarrow ar_0 \quad (10.13)$$

⁶ We will work in the classical limit, but the correspondence applies to the full quantum theory.

⁷ This normalization of O differs from that of [36] by a factor of $\sqrt{2}$.

leaves the form of the black hole (10.3) invariant with $f \rightarrow a^2 f$. It is easy to check that the Maxwell-scalar field equations (10.7, 10.8) are invariant under this rescaling if $\phi \rightarrow a\phi$ (so $A = \phi dt$ is invariant) and ψ is unchanged $\psi \rightarrow \psi$. Rather than discuss this trivial dependence on temperature which simply reflects the scaling dimension, we are interested in the dependence of a dimensionless measure of the condensate as a function of a dimensionless measure of the temperature. It is convenient to use the chemical potential to fix a scale and consider $\sqrt{O_2}/\mu$ as a function of T/μ . When one does this, one finds that the condensate is nonzero only when T/μ is small enough. Setting T_c equal to the critical temperature when the condensate first turns on, we get Fig. 10.1.

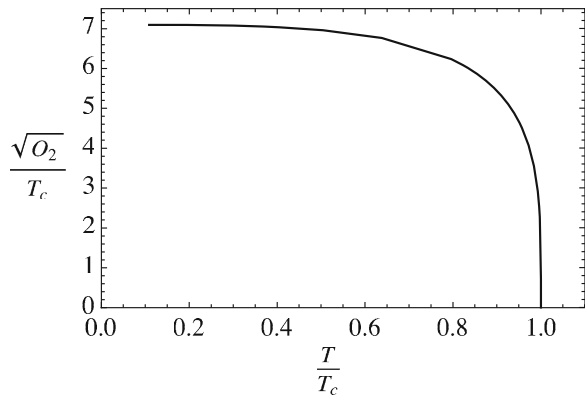
This curve is qualitatively similar to that obtained in BCS theory, and observed in many materials, where the condensate rises quickly as the system is cooled below the critical temperature and goes to a constant as $T \rightarrow 0$. Near T_c , there is a square root behavior $O_2 = 100T_c^2(1 - T/T_c)^{1/2}$. This is the standard behavior predicted by Landau–Ginzburg theory.

A nonzero condensate means that the black hole in the bulk has developed scalar hair. One can compute the free energy (Euclidean action) of these hairy configurations and compare with the solution $\psi = 0, \phi = \rho(1/r_0 - 1/r)$ which describes a black hole with the same charge or chemical potential, but no scalar hair. It turns out that the free energy is always lower for the hairy configurations and becomes equal as $T \rightarrow T_c$ [36]. The difference of free energies scales like $(T_c - T)^2$ near the transition, showing that this is a second order phase transition. Actually, you can compare the free energy at fixed charge or fixed chemical potential. These correspond to two different ensembles. In both cases, the free energy is lower for the hairy configuration.

We now generalize to other masses. Recall that the asymptotic behavior of a scalar field of mass m in AdS_4 is

$$\psi = \frac{\psi_-}{r^{\lambda_-}} + \frac{\psi_+}{r^{\lambda_+}} + \dots \quad (10.14)$$

Fig. 10.1 The condensate as a function of temperature. The critical temperature is proportional to the chemical potential



where

$$\lambda_{\pm} = \frac{1}{2} \left(3 \pm \sqrt{9 + 4(mL)^2} \right). \quad (10.15)$$

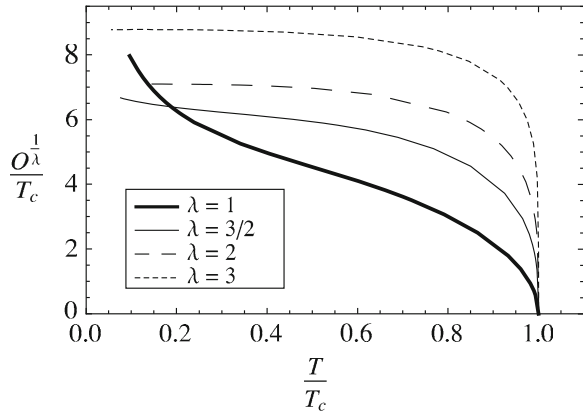
For $m^2 \geq m_{BF}^2 + L^{-2}$, only the mode with λ_+ falloff is normalizable, and so we interpret $\psi_+ = O$, where O is the expectation value of the operator dual to the scalar field, and λ_+ is the dimension of the operator. ψ_- is dual to a source for this operator, so we only consider solutions with the standard boundary condition $\psi_- = 0$. For $m_{BF}^2 < m^2 < m_{BF}^2 + L^{-2}$ one can also consider the alternative boundary condition $\psi_+ = 0$, and the dual operator has dimension λ_- .

Figure 10.2 shows the results for the condensate for various masses. It is convenient to label the curves in terms of the dimension of the condensate λ , rather than the mass to distinguish the two possible cases when $m^2 = -2/L^2$. The masses range from the BF bound to $m^2 = 0$. Similar behavior is found for $m^2 > 0$ [48].

The qualitative behavior is the same as before. In all cases, there is a critical temperature T_c (proportional to the chemical potential) above which the condensate is zero. Near the critical temperature, $O \propto (T_c - T)^{1/2}$. In all but one case, the condensate saturates as $T \rightarrow 0$. The exceptional case is the dimension one curve which starts to grow at low temperature. When the condensate becomes very large, the backreaction on the bulk metric can no longer be neglected. We will see later that in the full solution with backreaction, the condensate approaches a finite limit at zero temperature.

It turns out that the ratio T_c/μ increases as the dimension of the operator decreases. For $\lambda > 3/2$, one can understand this since a smaller λ corresponds to a smaller m^2 making it easier for the scalar hair to form. However, this continues for $\lambda < 3/2$ when one must increase m^2 (and use the alternative boundary conditions) to decrease the dimension. In fact, as observed in [14], T_c/μ appears to diverge as λ approaches the unitarity bound, $1/2$. The reason for this is not clear. But the

Fig. 10.2 Condensates with different dimension, λ , as a function of temperature. The condensate tends to increase with λ . Figure is taken from [44]



lesson is that, at least in this simple model, to have a higher temperature superconductor, one should lower the dimension of the condensate.

10.3.2 Conductivity

We want to compute the optical conductivity, i.e. the conductivity as a function of frequency. By symmetry, it suffices to consider just the conductivity in the x direction. According to the gauge/gravity dictionary, this is obtained by solving for fluctuations in the Maxwell field in the bulk. Maxwell's equation with zero spatial momentum and time dependence $e^{-i\omega t}$ gives the following equation for $A_x(r)$:

$$A_x'' + \frac{f'}{f} A_x' + \left(\frac{\omega^2}{f^2} - \frac{2\psi^2}{f} \right) A_x = 0. \quad (10.16)$$

We want to solve this with ingoing wave boundary conditions at the horizon, since this corresponds to causal propagation on the boundary, i.e., yields the retarded Green's function [61]. Asymptotically,

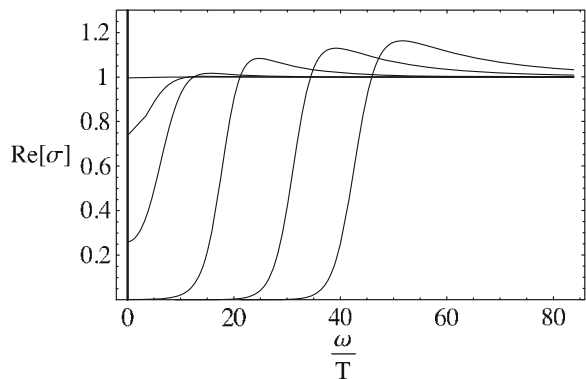
$$A_x = A_x^{(0)} + \frac{A_x^{(1)}}{r} + \dots \quad (10.17)$$

The gauge/gravity dictionary says the limit of the electric field in the bulk is the electric field on the boundary: $E_x = -\dot{A}_x^{(0)}$, and the expectation value of the induced current is the first subleading term: $J_x = A_x^{(1)}$. From Ohm's law we get:

$$\sigma(\omega) = \frac{J_x}{E_x} = -\frac{J_x}{\dot{A}_x^{(0)}} = -\frac{iJ_x}{\omega A_x^{(0)}} = -\frac{iA_x^{(1)}}{\omega A_x^{(0)}}. \quad (10.18)$$

The real part of the conductivity is given in Fig. 10.3 for the case $\lambda = 2$. Above the critical temperature, the conductivity is constant [40]. As you start to lower the

Fig. 10.3 The formation of a gap in the real part of the conductivity as the temperature is lowered below the critical temperature. The curves describe successively lower temperatures. There is also a delta function at $\omega = 0$. Figure is for the dimension two condensate and is taken from [36]



temperature below T_c a gap opens up at low frequency. When these curves were first obtained, it was thought that $\text{Re}[\sigma]$ was given by $e^{-\Delta/T}$ at small ω , which is what one expects from a BCS type description with an energy gap Δ . This would imply that at $T = 0$, $\text{Re}[\sigma]$ should vanish inside the gap. We will see that this is not the case.

There is also a delta function at $\omega = 0$ for all $T < T_c$. This cannot be seen from a numerical solution of the real part, but it can be seen by looking for a pole in $\text{Im}[\sigma]$. A simple argument for this comes from the Drude model of a conductor. Suppose we have charge carriers with mass m , charge e , and number density n in a normal conductor. They satisfy

$$m \frac{dv}{dt} = eE - m \frac{v}{\tau} \quad (10.19)$$

where the last term is a damping term and τ is the relaxation time due to scattering. The current is $J = env$, so if $E(t) = Ee^{-i\omega t}$, the conductivity is

$$\sigma(\omega) = \frac{k\tau}{1 - i\omega\tau} \quad (10.20)$$

where $k = ne^2/m$. So

$$\text{Re}[\sigma] = \frac{k\tau}{1 + \omega^2\tau^2}, \quad \text{Im}[\sigma] = \frac{k\omega\tau^2}{1 + \omega^2\tau^2}. \quad (10.21)$$

For superconductors, $\tau \rightarrow \infty$, so $\text{Re}[\sigma] \propto \delta(\omega)$ and $\text{Im}[\sigma] \propto 1/\omega$. A more general derivation comes from the Kramers–Kronig relations. These relate the real and imaginary parts of any causal quantity, such as the conductivity, when expressed in frequency space. One of the relations is

$$\text{Im}[\sigma(\omega)] = -\frac{1}{\pi} \mathcal{P} \int_{-\infty}^{\infty} \frac{\text{Re}[\sigma(\omega')]}{\omega' - \omega} d\omega'. \quad (10.22)$$

From this formula we can see that the real part of the conductivity contains a delta function, if and only if the imaginary part has a pole. One finds that there is indeed a pole in $\text{Im}[\sigma]$ at $\omega = 0$ for all $T < T_c$.

Figure 10.4 shows the low temperature limit of the optical conductivity. The solid line denotes the real part and the dashed line denotes the imaginary part. The pole at $\omega = 0$ is clearly visible.

A finite conductivity implies dissipation. In the bulk, this is reflected by the ingoing wave boundary conditions at the horizon. A standard normalizable perturbation of the Maxwell field would decay and get swallowed by the black hole. We maintain a constant amplitude and purely harmonic time dependence by driving the mode with an applied electric field at the boundary. Dissipation normally causes a system to heat up, and indeed if we had included the backreaction of the Maxwell perturbation on the metric, the flow of energy into the horizon would cause the black hole to grow and increase its temperature.

Fig. 10.4 The low temperature limit of the optical conductivity for the dimension two condensate. The solid line denotes the real part and the dashed line denotes the imaginary part. Figure is taken from [44]

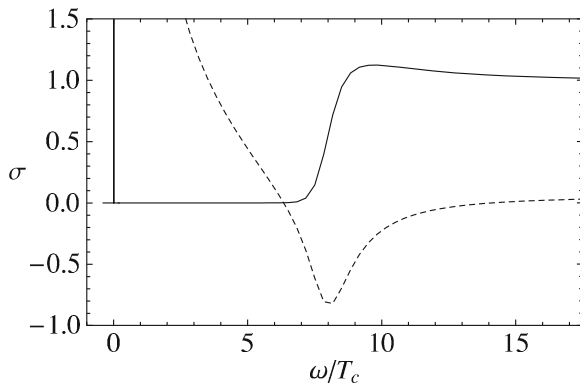
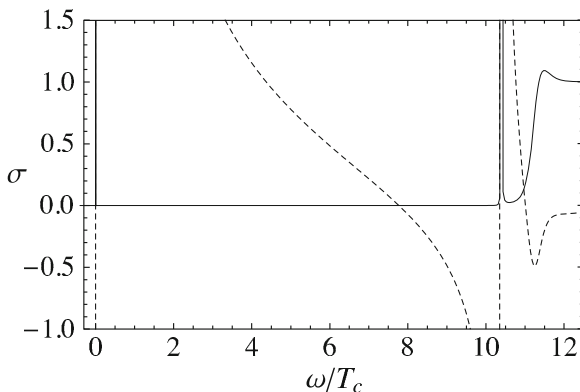


Fig. 10.5 The low temperature limit of the conductivity for the dimension 3/2 condensate. Note the extra spike that appears inside the gap. Figure is taken from [44]



When you approach the BF bound something interesting happens. As shown in Fig. 10.5, a new spike appears inside the gap. This looks like a bound state of quasiparticles with the binding energy given by distance between the pole and the edge of the gap. However, one must keep in mind that quasiparticles are a weak coupling concept and may not be well defined at strong coupling. We will have more to say about this spike in the next section.

Although we have focussed on the case of $2 + 1$ dimensional superconductors with a $3 + 1$ dimensional bulk dual, everything we have done is easily generalized to higher dimensions [44]. The results are qualitatively the same.

A robust feature that holds in both $2 + 1$ and $3 + 1$ superconductors and all $\lambda > \lambda_{BF}$ is that

$$\frac{\omega_g}{T_c} \approx 8 \quad (10.23)$$

with deviations of less than 10%.⁸ This is more than twice the weakly coupled BCS value of 3.5. Remarkably, measurements of this ratio in the high T_c superconductors give roughly this value [23]!

10.4 Full Solution with Backreaction

We now discuss the full solution to action (10.2) including the backreaction on the spacetime geometry. We start with the following ansatz for the metric

$$ds^2 = -g(r)e^{-\chi(r)}dt^2 + \frac{dr^2}{g(r)} + r^2(dx^2 + dy^2) \quad (10.24)$$

and use the same ansatz for the matter fields as before:

$$A = \phi(r)dt, \quad \Psi = \psi(r). \quad (10.25)$$

The equations of motion are⁹:

$$\psi'' + \left(\frac{g'}{g} - \frac{\chi'}{2} + \frac{2}{r}\right)\psi' + \frac{q^2\phi^2e^\chi}{g^2}\psi - \frac{m^2}{g}\psi = 0, \quad (10.26)$$

$$\phi'' + \left(\frac{\chi'}{2} + \frac{2}{r}\right)\phi' - \frac{2q^2\psi^2}{g}\phi = 0, \quad (10.27)$$

$$\chi' + r\psi'^2 + \frac{rq^2\phi^2\psi^2e^\chi}{g^2} = 0, \quad (10.28)$$

$$g' + \left(\frac{1}{r} - \frac{\chi'}{2}\right)g + \frac{r\phi'^2e^\chi}{4} - 3r + \frac{rm^2\psi^2}{2} = 0. \quad (10.29)$$

The first two are the scalar and Maxwell equations as before. The last two are the two independent components of Einstein's equation. (There are three nonzero components of Einstein's equation but only two are independent.) Note that the equations for g and χ are first order, and χ is monotonically decreasing. These equations are invariant under a scaling symmetry analogous to (10.13):

$$r \rightarrow ar, \quad (t, x, y) \rightarrow (t, x, y)/a, \quad g \rightarrow a^2g, \quad \phi \rightarrow a\phi. \quad (10.30)$$

⁸ A few caveats should be noted: If one adds scalar interactions in the bulk by introducing a more general potential $V(\Psi)$, the gap in the low temperature optical conductivity can become much less pronounced, so that this ratio becomes ill defined [28, 29]. We will see in the next section that it also becomes ill defined at small q . Even when it is well defined, it is modified by higher order curvature corrections in bulk [24].

⁹ We will set $L = 1$ for the rest of our discussion.

When the horizon is at nonzero r , this can be used to set $r_0 = 1$. These equations are also invariant under

$$e^\chi \rightarrow a^2 e^\chi, \quad t \rightarrow at, \quad \phi \rightarrow \phi/a. \quad (10.31)$$

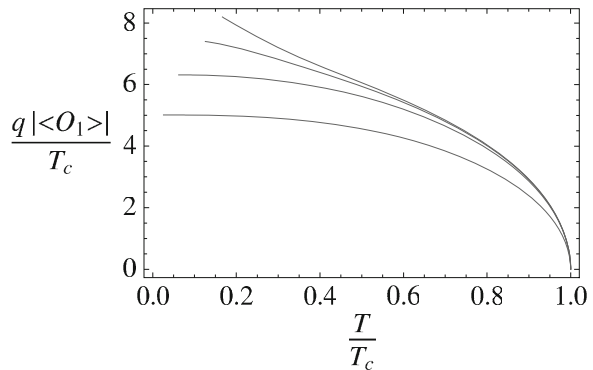
This symmetry can be used to set $\chi = 0$ at the boundary at infinity, so the metric takes the standard AdS form asymptotically.

The qualitative behavior of the solutions is similar to the probe limit, i.e., the probe limit indeed captures most of the physics. However, there are a few important differences [37]. First, the apparent divergence in the dimension one condensate at low T is gone. Figure 10.6 shows a plot of qO_1 as a function of temperature. We have multiplied the condensate by the charge q since this is the quantity which is represented by the probe limit at large q . For q of order one, the curves are similar to the other condensates in the probe limit. But for large q , they grow at low temperature. One can show that for all q , the condensate still approaches a finite limit at $T = 0$. It is not clear why the dimension one condensate behaves differently in the probe limit from other masses.

The second difference is more surprising. As explained earlier, the origin of the instability responsible for the scalar hair is the coupling of the charged scalar to the charge of the black hole. It was therefore expected that as $q \rightarrow 0$ the instability would turn off. This is not what happens. A nearly extremal Reissner–Nordstrom AdS black hole remains unstable to forming neutral scalar hair, provided that m^2 is close to the Breitenlohner–Freedman (BF) bound. This means that there is a new source of instability which can be understood as follows: An extremal Reissner–Nordstrom AdS black hole has a near horizon geometry $AdS_2 \times R^2$. The BF bound for AdS_{d+1} is $m_{BF}^2 = -d^2/4$. So scalars which are slightly above the BF bound for AdS_4 , can be below the bound for AdS_2 . This instability to forming neutral scalar hair is not associated with superconductivity (or superfluidity) since it does not break a $U(1)$ symmetry. At most it breaks a Z_2 symmetry corresponding to $\psi \rightarrow -\psi$. Its interpretation in the dual field theory is not clear.

One can make the following general argument for when an extremal Reissner–Nordstrom AdS black hole will be unstable to forming scalar hair [14]. Consider a

Fig. 10.6 From bottom to top, the various curves correspond to $q = 1, 3, 6$, and 12. Figure is taken from [37]



scalar field with mass m and charge q in the near horizon geometry of an extremal Reissner–Nordstrom AdS black hole. Its field equation reduces to a wave equation in AdS_2 with effective mass $m_{eff}^2 = (m^2 - 2q^2)/6$. The $-2q^2$ is the usual coupling to the electric charge and the factor of six comes from the difference between the radius of curvature in AdS_4 and AdS_2 . The instability to form scalar hair at low temperature is then just the instability of scalar fields below the BF bound for AdS_2 : $m_{BF}^2 = -1/4$. Thus the condition for instability is

$$m^2 - 2q^2 < -3/2. \quad (10.32)$$

Of course, the mass must be above the four-dimensional BF bound, $m^2 > -9/4$, so that the asymptotic AdS_4 geometry remains stable. Equation 10.32 is a sufficient condition to guarantee instability, but it is not necessary.

To compute the conductivity, we again perturb the Maxwell field in the bulk. Assuming zero momentum and harmonic time dependence, the perturbed Maxwell field A_x now couples to the perturbed metric component g_{tx} . Physically, this is what one expects from the standpoint of the dual field theory. We are applying an electric field and inducing a current. The current carries momentum, so T_{tx} should be nonzero. This requires that the metric perturbation g_{tx} must also be nonzero.¹⁰ However, this also means that one has to solve for the thermal conductivity at the same time as the electrical conductivity. Fortunately, for homogeneous perturbations with harmonic time dependence, one can solve for g_{tx} in terms of A_x and get

$$A_x'' + \left[\frac{g'}{g} - \frac{\chi'}{2} \right] A_x' + \left[\left(\frac{\omega^2}{g^2} - \frac{\phi'2}{g} \right) e^\chi - \frac{2q^2\psi^2}{g} \right] A_x = 0. \quad (10.33)$$

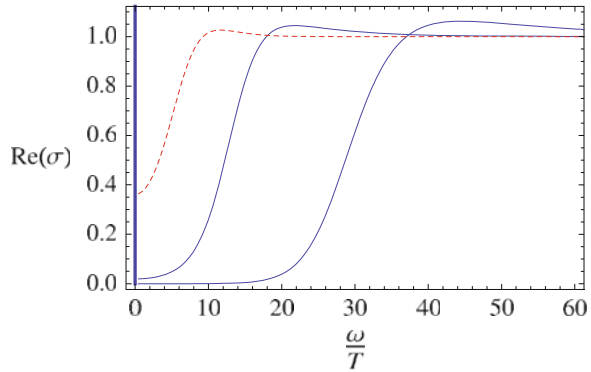
This is similar to (10.16) except for the form of the metric and the extra term $\phi'2/g$ coming from the metric perturbation g_{tx} . The conductivity is still

$$\sigma(\omega) = -\frac{i}{\omega} \frac{A_x^{(1)}}{A_x^{(0)}}. \quad (10.34)$$

The results for the optical conductivity are qualitatively similar to the probe limit, and shown in Fig. 10.7. There are a few differences. One is that the conductivity in the normal phase is no longer constant. Another difference is that in the low temperature limit, the gap in $\text{Re}\sigma(\omega)$ for small ω becomes less pronounced at small q . The robust feature $\omega_g/T_c \approx 8$ seen in the probe limit continues to hold for $q > 3$, but is less robust for $q < 3$ mainly because the gap becomes less well defined. Perhaps the most important difference is that $\text{Re}\sigma$ now has a delta function contribution at $\omega = 0$ even in the normal phase when $T > T_c$. This infinite DC conductivity is not superconductivity, but just a consequence of translational invariance. A translationally invariant, charged system does not have a finite DC conductivity because application of an electric field will cause uniform acceleration. One does not see this

¹⁰ I thank Hong Liu for this comment.

Fig. 10.7 Conductivity for the dimension two condensate with $q = 3$. The dashed line is the real part of the conductivity at $T = T_c$ and the solid lines are the same conductivities at successively lower temperature $T/T_c = 0.651, 0.304$. There is a delta function at the origin in all cases. Figure is taken from [37]



in the probe limit, since we fixed the gravitational background which implicitly breaks translation invariance. One indication of this is that we get an electric current without momentum flow as we mentioned earlier. Although there is a delta function at $\omega = 0$ for all temperature, its coefficient grows as the temperature is lowered below T_c . This indicates the presence of a new contribution coming from the onset of superconductivity.

10.4.1 Reformulation of the Conductivity

Equation 10.33 can be simplified by introducing a new radial variable

$$dz = \frac{e^{\chi/2}}{g} dr. \quad (10.35)$$

At large r , $dz = dr/r^2$, and we can choose the additive constant so that $z = -1/r + O(1/r^2)$. Since g vanishes at least linearly at a horizon and χ is monotonically decreasing, the horizon corresponds to $z = -\infty$. In terms of z , (10.33) takes the form of a standard Schrödinger equation with energy ω^2 :

$$-A_{x,zz} + V(z)A_x = \omega^2 A_x \quad (10.36)$$

where

$$V(z) = g[\phi_{,r}^2 + 2q^2\psi^2 e^{-\chi}]. \quad (10.37)$$

From the known asymptotic behavior of the solution near infinity, one can show that $V(0) = 0$ if the dimension of the condensate, λ , is greater than one, $V(0)$ is a nonzero constant if $\lambda = 1$, and $V(z)$ diverges as $z \rightarrow 0$ if $1/2 < \lambda < 1$. One can also show that the potential always vanishes at the horizon [45].

We want to solve (10.36) with boundary conditions at $z = -\infty$ corresponding to waves propagating to the left. This corresponds to ingoing boundary conditions

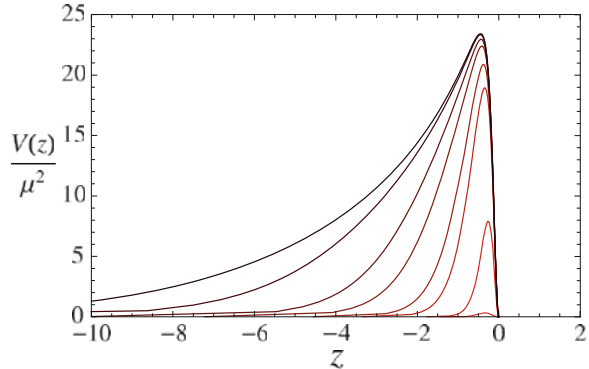
at the horizon which is needed to extract causal results. The easiest way to do this is to first extend the definition of the potential to all z by setting $V = 0$ for $z > 0$. An incoming wave from the right will be partly transmitted and partly reflected by the potential barrier. Since the transmitted wave is purely ingoing at the horizon, this satisfies our desired boundary conditions. Writing the solution for $z > 0$ as $A_x = e^{-i\omega z} + \mathcal{R}e^{i\omega z}$, we clearly have $A_x(0) = 1 + \mathcal{R}$ and $A_{x,z}(0) = -i\omega(1 - \mathcal{R})$. In terms of z , $A_x^{(1)} = -A_{x,z}(0)$, so from (10.34)

$$\sigma(\omega) = \frac{1 - \mathcal{R}}{1 + \mathcal{R}}. \quad (10.38)$$

The conductivity is directly related to the reflection coefficient, with the frequency simply giving the incident energy [45]! The qualitative behavior of $\sigma(\omega)$ is now clear. Let us first assume that the dimension of the condensate is $\lambda \geq 1$ so that V is bounded. At frequencies below the height of the barrier, the probability of transmission will be small, $\mathcal{R}$ will be close to one, and $\sigma(\omega)$ will be small. At frequencies above the height of the barrier, $\mathcal{R}$ will be very small and $\sigma(\omega) \sim 1$ (the normal state value). Clearly the size of the gap in $\sigma(\omega)$ is set by the height of the barrier: $\omega_g \sim \sqrt{V_{\max}}$. The case $1/2 < \lambda < 1$ is qualitatively similar. Even though the potential is not bounded, $\sqrt{V}$ is integrable, so there is still tunneling through the barrier.

As one lowers the temperature, the condensate ψ increases and the potential becomes both higher and wider (see Fig. 10.8). This causes an increasing exponential suppression which was erroneously interpreted as evidence for $\text{Re } \sigma \sim e^{-\Delta/T}$ in the original papers on holographic superconductors. However, as $T \rightarrow 0$, the potential approaches a finite, limiting form which still vanishes at $z = -\infty$. This means that there will always be nonzero tunnelling probability and hence a nonzero conductivity even at small frequency. In other words, there is no hard gap in the optical conductivity at $T = 0$. This conclusion continues to hold if the $m^2|\Psi|^2$ term in our bulk lagrangian is replaced by a general potential $V(\Psi)$. The fact that there is not a hard gap is consistent with calculations of the specific heat which show that it vanishes as a power law at low temperature and not exponentially.

Fig. 10.8 Schrödinger potential for $\lambda = 2, q = 10$. The *top* curve as $T = 0$ and the *bottom* curve has $T = T_c$. Figure is taken from [45]



As discussed in Sect. 3.2, in order for $\text{Re}\sigma$ to have a delta function at $\omega = 0$ representing the infinite DC conductivity, one needs $\text{Im}\sigma$ to have a pole at $\omega = 0$. It is easy to see that this is indeed the case, for any positive Schrödinger potential $V(z)$ that vanishes at $z = -\infty$. Imagine solving (10.36) with $\omega = 0$, and $A_x = 1$ at $z = -\infty$. (This represents the normalizable solution and is the zero frequency limit of the ingoing wave boundary condition.) Since $A_{x,zz} > 0$, the solution will be monotonically increasing. At $z = 0$, $A_x^{(0)} = A_x(0)$ and $A_x^{(1)} = -A_{x,z}(0)$. These are both real and nonzero. From (10.34) it then follows that $\text{Im}\sigma$ has a pole at $\omega = 0$. Even in the normal phase when $\psi = 0$, the potential is nonzero due to the contribution from the electric field. This is precisely the term which arises due to the backreaction on the metric and is absent in the probe approximation. This is another way to see why the probe limit does not have a delta function at $\omega = 0$ in the normal phase while the full backreacted solution does. Note that the Schrödinger potential remains finite in the probe limit $q \rightarrow \infty$, since $q\psi$ is held fixed.

This approach also explains the spike in the conductivity that was seen in Fig. 10.5. At low frequency, the incoming wave from the right is almost entirely reflected. If the potential is high enough, one can raise the frequency so that about one wavelength fits between the potential and $z = 0$. In this case, the reflected wave can interfere destructively with the incident wave and cause its amplitude at $z = 0$ to be exponentially small. This produces a spike in the conductivity. If one can raise the frequency so that two wavelengths fit between the potential and $z = 0$ one gets a second spike, etc. More precisely, using standard WKB matching formula, spikes will occur when there exists ω satisfying

$$\int_{-z_0}^0 \sqrt{\omega^2 - V(z)} dz + \frac{\pi}{4} = n\pi \quad (10.39)$$

for some integer n , where $V(-z_0) = \omega^2$. It is now clear that the spike which we saw in the probe approximation with m^2 saturating the BF bound, also appears in the full backreacted solution, and for some m^2 slightly above the BF bound. When the spikes were first seen, it was speculated that they corresponded to vector normal modes of the hairy black hole. It is now clear that they are not true normal modes even at $T = 0$, since A_x does not actually vanish at infinity. The actual modes all have complex frequency and correspond to familiar quasinormal modes. In other words, there are no bound states in this potential (with boundary condition $A_x = 0$ at $z = 0$) since the potential vanishes at $z = -\infty$.

10.5 Zero Temperature Limit

The extremal Reissner–Nordstrom AdS black hole has large entropy at $T = 0$. If this was dual to a condensed matter system, it would mean the ground state was highly degenerate. The extremal limit of the hairy black holes dual to the

superconductor is not like Reissner–Nordstrom. It has zero horizon area, consistent with a nondegenerate ground state [45]. One might wonder how this could be the limit of the $T > 0$ black holes since we have seen that for all $T \neq 0$, one can scale r so that $r_0 = 1$. The point is simply that the dimensionless ratio $r_0/\mu \rightarrow 0$ as $T \rightarrow 0$.

The extremal limit also has zero charge inside the horizon (except in the case $q = 0$ when the scalar hair cannot carry the charge). This is an expected consequence of the horizon having zero area. If the black hole tried to keep a nonzero charge as the temperature was lowered and its horizon shrunk to zero, the electric field on the surface would grow arbitrarily large. Eventually this would pair create charged particles via the Schwinger mechanism and neutralize the black hole. This is a quantum argument involving pair creation, but it has a classical analog in terms of a superradiant instability for charged scalar fields.

The near horizon behavior of the zero temperature solution depends on the mass and charge of the bulk scalar field. We will discuss two classes of solutions below. In both cases, one can solve for the solutions analytically near $r = 0$. These leading order solutions depend on a free parameter which can be adjusted so that the solution asymptotically satisfies the desired boundary condition. In the cases we study, the solution is typically not smooth at $r = 0$, but since they are the limit of smooth black holes, they are physically allowed. If one modifies the potential for Ψ so that it has more than one extremum, then smooth, zero temperature solutions exist in which Ψ rolls from one extremum to another [28, 29].

10.5.1 $m^2 = 0$

This corresponds to a marginal, dimension three operator developing a nonzero expectation value in the dual superconductor. To determine the leading order behavior near $r = 0$, we try an ansatz

$$\phi = r^{2+\alpha}, \quad \psi = \psi_0 - \psi_1 r^{2(1+\alpha)}, \quad \chi = \chi_0 - \chi_1 r^{2(1+\alpha)}, \quad g = r^2(1 - g_1 r^{2(1+\alpha)}) \quad (10.40)$$

We have used the scaling symmetries (10.30) and (10.31) to set the coefficients in ϕ and g to one. Substituting this into the field equations and equating the dominant terms for small r (assuming $\alpha > -1$), one finds:

$$q\psi_0 = \left(\frac{\alpha^2 + 5\alpha + 6}{2}\right)^{1/2}, \quad \chi_1 = \frac{\alpha^2 + 5\alpha + 6}{4(\alpha + 1)} e^{\chi_0} \quad (10.41)$$

$$g_1 = \frac{\alpha + 2}{4} e^{\chi_0}, \quad \psi_1 = \frac{qe^{\chi_0}}{2(2\alpha^2 + 7\alpha + 5)} \left(\frac{\alpha^2 + 5\alpha + 6}{2}\right)^{1/2}. \quad (10.42)$$

Fig. 10.9 Values of α for various charges. Note that it approaches a constant as $q \rightarrow \infty$. Figure taken from [45]

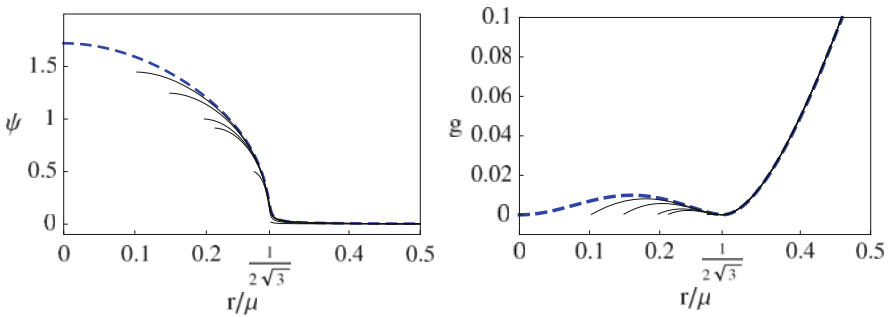
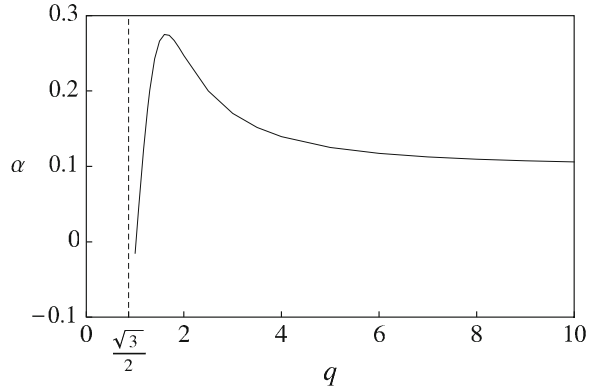


Fig. 10.10 Zero temperature, $\lambda = 3$ and $q = 1$ solution (*dashed line*), compared to successively lower temperature hairy black holes (*solid lines*). Note that g almost has a double zero at $r/\mu = 1/2\sqrt{3}$ which is the extremal horizon for Reissner–Nordstrom AdS. In the limit $q \rightarrow \sqrt{3}/2$ the solution becomes extremal Reissner–Nordstrom AdS with all hair behind the horizon. Figure is taken from [45]

One can now numerically integrate this solution to large radius and adjust α so that the solution for ψ is normalizable. One finds that this is possible provided $q^2 > 3/4$. This is consistent with the condition for instability (10.32). The value of α depends weakly on q (see Fig. 10.9). In all cases, $|\alpha| < .3$. Figure 10.10 shows the results for ψ and g for the zero temperature solution, and shows how the $T > 0$ solutions approach it as $T \rightarrow 0$.

Near $r = 0$, χ approaches a constant and $g = r^2$. Thus the metric approaches AdS_4 with the same value of the cosmological constant as the asymptotic region. The extremal horizon is just the Poincare horizon of AdS_4 . The scalar field approaches a constant and the Maxwell field vanishes. In terms of the dual field theory, this means that the full conformal symmetry is restored in the infrared.

These solutions are not singular at $r = 0$ since all curvature invariants remain finite. They can be viewed as static, charged domain walls. Even though they are

not singular, when $\alpha \neq 0$ the solutions are not C^∞ across $r = 0$. Some derivatives of the curvature will blow up. However, there is a special value of the charge, $q = 1.018$, where $\alpha = 0$. This solution is completely smooth across the horizon. It is not clear what the significance of this value is for the dual field theory.

10.5.2 $q^2 > |m^2|/6$ ($m^2 < 0$)

We try the following ansatz near $r = 0$:

$$\psi = A(-\log r)^{1/2}, \quad g = g_0 r^2(-\log r), \quad \phi = \phi_0 r^\beta(-\log r)^{1/2} \quad (10.43)$$

where we have used the radial scaling symmetry (10.30) to set an arbitrary length scale in the logarithm to one. The behavior of χ is determined by (10.28) and whether we expect $r\psi'/2$ or $r q^2 \phi^2 \psi^2 e^\chi / g^2$ to dominate. We assume $r\psi'/2$ dominates, so

$$e^\chi = K(-\log r)^{A^2/4} \quad (10.44)$$

with K a constant of integration. The second term in (10.28) is indeed negligible provided $\beta > 1$. Equating the dominant terms in the equations of motion leads to

$$A = 2, \quad g_0 = -\frac{2}{3}m^2, \quad \beta = -\frac{1}{2} \pm \frac{1}{2} \left(1 - \frac{48q^2}{m^2} \right)^{1/2}. \quad (10.45)$$

It is clear that for appropriate metric signature we need positive g_0 which tells us this ansatz is only appropriate for negative m^2 . Since we require $\beta > 1$, only the plus sign in (10.45) is allowed and we require

$$q^2 > -m^2/6. \quad (10.46)$$

With these restrictions, our near horizon solution is

$$\psi = 2(-\log r)^{1/2}, \quad g = (2m^2/3)r^2 \log r, \quad e^\chi = -K \log r \quad (10.47)$$

$$\phi = \phi_0 r^\beta (-\log r)^{1/2}, \quad (10.48)$$

The one remaining free parameter is ϕ_0 . This parameter can be tuned so that the asymptotic boundary condition on ψ is satisfied.

The horizon at $r = 0$ has a mild singularity. The scalar field diverges logarithmically and the metric takes the form (after rescaling t)

$$ds^2 = r^2(-dt^2 + dx_i dx^i) + \frac{dr^2}{g_0 r^2 (-\log r)}. \quad (10.49)$$

Notice that Poincare invariance is restored near the horizon, but not the full conformal invariance. (Some early indications of emergent Poincare symmetry were found in [28, 29].) Introducing a new radial coordinate, $\tilde{r} = -2(-\log r)^{1/2}/g_0^{1/2}$ the metric becomes

$$ds^2 = e^{-g_0\tilde{r}^2/2}(-dt^2 + dx_i dx^i) + d\tilde{r}^2 \quad (10.50)$$

near the horizon which is now at $\tilde{r} = -\infty$.

10.6 Adding Magnetic Fields

As we mentioned in the introduction, one of the characteristic properties of superconductors is that they expel magnetic fields. However, a superconductor can expel a magnetic field only up to a point. A sufficiently strong field will destroy the superconductivity. One defines the thermodynamic critical field B_c by setting the energy it takes to expel the magnetic field equal to the difference in free energy between the normal and superconducting states:

$$\frac{B_c^2(T) V}{8\pi} = F_n(T) - F_s(T) \quad (10.51)$$

where V is the volume. Superconductors are divided into two classes depending on how they make the transition from a superconducting to a normal state as the magnetic field is increased. In type I superconductors, there is a first order phase transition at $B = B_c$, above which magnetic field lines penetrate uniformly and the material no longer superconducts. In type II superconductors, there is a more gradual second order phase transition. The magnetic field starts to penetrate the superconductor in the form of vortices with quantized flux when $B = B_{c1} < B_c$. The vortices become more dense as the magnetic field is increased, and at an upper critical field strength, $B = B_{c2} > B_c$, the material ceases to superconduct.

We now show that our $2 + 1$ dimensional holographic superconductors are type II. (Interestingly enough, the high T_c superconductors are also type II.) We will view this superconductor as a thin film superconductor in $3 + 1$ dimensions with a perpendicular magnetic field. Suppose the $2 + 1$ dimensional sample is a disk of radius R . In order for the disk to expel the magnetic field, it must produce a current circulating around the perimeter. Solving Maxwell's equations, this current will expel a field not only in the area πR^2 of the disk but in a larger volume of size $V_3 \sim R^3$. Since the difference in free energies between the normal and superconducting state scales like R^2 , in the large R (thermodynamic) limit, the superconductor does not have enough free energy available to expel a magnetic field. Thus magnetic fields of any non-vanishing strength will penetrate a thin superconducting film and $B_c = 0$.

To show that the holographic superconductor is type II, we start in the normal phase with a large applied magnetic field, and slowly lower B . We will see that parts of the system start to become superconducting at a value $B_{c2} > 0$ [1, 44]. To model the normal phase with an applied magnetic field, we use a black hole in AdS with both electric and magnetic charges (but no scalar hair). The magnetic field in the bulk corresponds to applying a background magnetic field on the boundary. The metric takes the form (10.24) with $\chi = 0$ and

$$g(r) = r^2 - \frac{1}{4rr_0} (4r_0^4 + \rho^2 + B^2) + \frac{1}{4r^2} (\rho^2 + B^2). \quad (10.52)$$

The black hole temperature is

$$T = \frac{12r_0^4 - \rho^2 - B^2}{16\pi r_0^3}. \quad (10.53)$$

The vector potential is

$$A = \rho \left(\frac{1}{r_0} - \frac{1}{r} \right) dt + Bx dy. \quad (10.54)$$

Since we are interested in the onset of superconductivity which is a second order phase transition, the condensate will be small and we can ignore the backreaction of ψ on the metric. (Note that this is different from the probe approximation since the backreaction of the Maxwell field is included here.) Since the gauge potential now depends on x , we separate variables as follows

$$\psi(r, x, y) = R(r)X(x)e^{iky}. \quad (10.55)$$

The equation for $X(x)$ turns out to be a simple harmonic oscillator centered at k/qB with width $(qB)^{-1/2}$. The eigenvalues are $2qB(n + \frac{1}{2})$ but only the lowest mode is expected to be stable. Note that the momentum k shifts the origin of the harmonic oscillator, but does not change its energy. This is just the degeneracy of Landau levels. The radial equation becomes:

$$(r^2 g R')' + \left[\frac{q^2 \rho^2}{g} \left(\frac{r}{r_0} - 1 \right) + qB - m^2 r^2 \right] R = 0. \quad (10.56)$$

One can now choose B so that there is a normalizable solution. This defines the upper critical field B_{c2} . One finds that B_{c2} is a decreasing function of temperature and vanishes at $T = T_c$. This is what one expects: At lower temperatures, a superconductor can support a larger magnetic field. Due to the degeneracy in k , the actual condensate involves a superposition of these modes for all k and forms a lattice of vortices. Maeda et al. [53] have recently studied the free energy as a function of the two parameters which govern the shape of the vortex array and shown that the minimum of the free energy at long wavelength corresponds to a triangular array which is what is predicted by the Landau–Ginzburg model.

10.6.1 London Equation

As mentioned in Sect. 10.2, the boundary theory does not have dynamical Maxwell fields, so one cannot actually see the Meisner effect. However, one can show that in the superconducting state, static magnetic fields induce currents whose backreaction would cancel the applied magnetic field [52, 37]. In fact, one can show that these currents are proportional to A_i and thus reproduce London's equation.

To make life easier, in this section we shall work in the probe limit ($q \rightarrow \infty$) in which the metric is kept fixed to be simply the Schwarzschild AdS black hole (10.3). Assume that we have solved for A_t and ψ in this background as described in Sect. 3.1. We then wish to add a static perturbation A_x that has nonzero momentum. We will take the momentum to lie in a direction orthogonal to A_μ , so we can consistently perturb the gauge field without sourcing any other fields. This perturbation corresponds to adding a static magnetic field in the bulk, and the asymptotic value of this magnetic field corresponds to the magnetic field added to the boundary theory. As before, the first subleading term in $A_x(r)$ corresponds to the induced current.

Rather than doing this calculation explicitly, we can obtain the answer by relating it to the calculation in Sect. 3.2. Let A_x have both momentum and frequency dependence of the form $A_x \sim e^{-i\omega t + iky}$. The radial equation for $A_x(r)$ reduces to

$$(fA'_x)' + \left(\frac{\omega^2}{f} - \frac{k^2}{r^2}\right)A_x = 2\psi^2 A_x, \quad (10.57)$$

where $'$ denotes differentiation with respect to r . We studied this equation with $k = 0$ in Sect. 3.2, and found a pole in $\text{Im}\sigma[\omega]$ at $\omega = 0$. Let the residue of this pole be n_s , the superfluid density. From (10.18) this means that $A_x^{(1)} = -n_s A_x^{(0)}$. But $J_x = A_x^{(1)}$ and $A_x^{(0)}$ is just the vector potential in the boundary theory, so

$$J_x(\omega, k = 0) = -n_s A_x(\omega, k = 0). \quad (10.58)$$

It is clear from the form of Eq. 10.57 that the limits $\omega \rightarrow 0$ and $k \rightarrow 0$ must commute. To compute n_s , we simply set both ω and k to zero and solve (10.57). Thus for small k , we directly obtain the London equation

$$J_i(\omega, k) = -n_s A_i(\omega, k) \quad (10.59)$$

where we have used the isotropy of space. This equation is clearly not gauge invariant. It is expected to hold in a gauge where $\nabla_i A^i = 0$.

10.6.2 Correlation Length

The gauge/gravity dictionary says that the retarded Green's function (for J_x) in the dual field theory is

$$G^R(\omega, k) = \frac{A_x^{(1)}(\omega, k)}{A_x^{(0)}(\omega, k)}. \quad (10.60)$$

We can define a correlation length ξ by expanding

$$\text{Re } G^R(0, k) = -n_s(1 + \xi^2 k^2 + \dots) \quad (10.61)$$

By solving (10.57) numerically, including the k dependence, one can compute this correlation length. One finds that it diverges near the critical temperature like $\xi T_c \approx 0.1(1 - T/T_c)^{-1/2}$ [44]. A similar divergence is found in a correlation length obtained from fluctuations in the condensate [52].

10.6.3 Vortices

As we discussed above, at the onset of superconductivity when $B = B_{c2}$, there is a lattice of vortices. However, at this point, the condensate is small everywhere and the equations could be linearized. It is of interest to find the bulk solution describing a vortex lattice away from $B = B_{c2}$. The first step is to find the single vortex solution. This corresponds to a something like a cosmic string stretching from the black hole horizon to infinity. However, there is a crucial difference between a cosmic string and the dual of a vortex. A standard cosmic string has a fixed proper radius. This means that its size in the x, y coordinates used in (10.24) goes to zero at large r . In terms of the boundary theory, this is a point-like object. We want a solution in which the cosmic string is a fixed size in the x, y coordinates, so it has a finite radius in the superconductor. This looks more like a funnel in the bulk with a proper radius that grows as one increases r .

Such a solution has recently been found in the probe limit [56].¹¹ Let us write the background metric using polar coordinates for the flat transverse space:

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2(d\tilde{r}^2 + \tilde{r}^2 d\phi^2). \quad (10.62)$$

Assume an ansatz

$$\Psi = \psi(r, \tilde{r})e^{in\phi}, \quad A_t = A_t(r, \tilde{r}), \quad A_\phi = A_\phi(r, \tilde{r}). \quad (10.63)$$

¹¹ See [2, 3] for another approach.

Substituting this into the field equations, one obtains a set of nonlinear PDE's. The condensate is now a function of radius $\tilde{r}$. It vanishes at $\tilde{r} = 0$ which represents the center of the vortex, and approaches a constant at large $\tilde{r}$. This shows that there is no superconductivity inside the vortex. Since the Maxwell field is not dynamical on the boundary, the magnetic field is not localized inside the vortex as one would expect. Instead, the authors of [56] work in finite volume and assume a uniform magnetic field on the boundary with total flux equal to $2\pi n$ as expected for the vortex.

10.7 Recent Developments

So far, we have discussed planar black holes in the bulk. Spherical black holes have the same instability and describe spherical superconductors. One can add rotation to a spherical black hole, and the effect of this rotation on the dual superconductor has recently been studied [62]. It was found that for $T < T_c$ (the transition temperature at zero rotation), there is a critical value of the rotation which destroys the superconductivity in analogy to the critical magnetic field.

The bulk gravitational theory that we have studied was not derived from string theory. It was just the simplest model that could describe a dual superconductor, and we have seen that it works rather well. It predicts that a charged condensate forms at low temperature (in a second order phase transition), that the DC conductivity is infinite, and that there is a gap in the optical conductivity at low frequency—all basic properties of superconductors. However, to go beyond this and have a more detailed microscopic understanding of the dual superconductor, one needs to embed this model in string theory. A linearized instability of the type needed to describe a holographic superconductor was found in a string compactification in [14]. This has now been extended to a full description by two different groups. Gauntlett et al. [20, 21] realized the $m^2 = -2, q = 2$ model in M theory. In other words, they found a consistent truncation of eleven dimensional supergravity in which the four dimensional fields were just a metric, Maxwell field and charged scalar with this mass and charge.¹² Gubser et al. [32] realized the same model in one higher dimension (a five dimensional bulk which is dual to a $3 + 1$ dimensional superconductor) with $m^2 = -3, q = 2$ in type IIB string theory. Both groups used Sasaki–Einstein compactifications with $U(1)$ symmetry, where the charged scalar is related to the size of the $U(1)$ fibration.

There are two main differences between the models obtained from string theory and our simple model. The first is that the kinetic term for the charged scalar involves nonminimal coupling. (This was investigated earlier by Franco et al. [18, 19].) The second difference is that the scalar potentials $V(\psi)$ coming from

¹² In general, there is an additional neutral scalar, but for purely electrically charged black holes, this field can be set to zero.

supergravity are more complicated than the simple mass terms we have considered so far. This does not affect the onset of superconductivity: since the scalar field is small near T_c , the critical temperature is just determined by the mass term in the potential. However, the low temperature limit is different. The potentials coming from supergravity have more than one extremum. In this case, the zero temperature limit of the hairy black hole is a smooth domain wall in which the scalar rolls from one extremum in the UV to another extremum in the IR [49]. The IR limit is another copy of AdS with, in general, a different cosmological constant. These examples all have emergent conformal symmetry in the IR.

The optical conductivity in these theories never show a pronounced gap at low temperature. Typically, $\text{Re}\sigma(\omega) = k\omega^\delta$ for small ω with a coefficient k of order one. From our Schrödinger equation reformulation this is a bit surprising. Why is not there an exponential suppression due to tunnelling through the barrier? The answer is simply that the effective potential in the Schrödinger equation never gets very large in these solutions since all fields are order one.

Even though embedding in string theory makes it possible to study the dual field theory in more detail, this is still difficult in $2 + 1$ dimensions. It is somewhat easier in $3 + 1$ dimensions since the dual theories are related to ordinary gauge theory. One particularly interesting aspect of the $3 + 1$ superconductor is that the condensate includes a term bilinear in the fermions, like a Cooper pair [26].

It was shown in [27] that one can replace the Maxwell field and charged scalar in the bulk with an $SU(2)$ gauge field and still have a dual description of a superconductor. The main difference is the following: The symmetry of a superconductor refers to the energy gap above the Fermi surface. The holographic superconductors we have discussed here all have S-wave symmetry since they do not prefer a direction. With an $SU(2)$ gauge field in the bulk, one obtains a P-wave superconductor.¹³ This leads to another way to realize holographic superconductors (in the probe limit) in string theory: one can use the $SU(2)$ gauge field realized on a pair of branes [5, 6, 58].

This is a rapidly evolving field and there have been several developments since the Milos summer school in September 2009, including:

- A key property of condensed matter systems is, of course, the atomic lattice. This gives rise to the phonons in the BCS theory which cause the electrons to pair. Our simple model has translational symmetry and no sign of a lattice. However, some interesting ideas on how to include a lattice have been discussed by Kachru et al. [46].
- Some superconductors exhibit striped phases associated with charged density waves. It was shown in [57] that the Chern–Simons term in five dimensional

¹³ One cannot compute the energy gap directly. Indeed, as we have seen, there probably is not a strict gap. The interpretation as a P-wave superconductor comes from the fact that there is a vector order parameter and the conductivity is strongly anisotropic in a manner consistent with P-wave nodes on the Fermi surface.

supergravity can lead to an instability at nonzero momentum in which a spatially modulated condensate forms.

- The response of fermions to the superconducting phase has been studied [13, 17, 31]. It was found that with a suitable coupling between the bulk fermions and scalar, there are stable quasiparticles with a gap [17].

10.8 Conclusions and Open Problems

For 90 years after its discovery in 1915, general relativity was viewed as a theory of gravity. It has proved very successful in describing a wide range of gravitational phenomena from the bending of light to gravitational waves and black holes. But in the last few years we have seen that this same theory can describe other areas of physics as well, including superconductivity. This is all due to the magic of gauge/gravity duality. Although the full power of gauge/gravity duality relates a quantum theory of gravity (indeed string theory) in the bulk to a nongravitational theory on the boundary, we have worked in a large N limit in which the bulk theory is just classical general relativity. This large N limit also explains how we can have spontaneous symmetry breaking in a $2 + 1$ dimensional field theory, in apparent contradiction to the Coleman–Mermin–Wagner theorem. The large N limit suppresses fluctuations in the fields.¹⁴

It is natural to ask how surprised one should be that general relativity can reproduce the basic properties of superconductors. After all, Weinberg [66] has shown that much of the phenomenology of superconductivity follows just from the spontaneous breaking of the $U(1)$ symmetry. Once we have found the instability that leads to charged scalar hair, does not everything else follow? There are indications that something deeper is going on. For example, order one dimensionless ratios can be computed and compared with experiment. In particular, the ratio $\omega_g/T_c \approx 8$ discussed in Sect. 3.2, is close to the observed value. This does not follow from symmetry arguments alone.

Since our bulk dual of a superconductor just involves gravity interacting with a Maxwell field and a charged scalar, there is a superficial similarity to a Landau–Ginzburg description. However, it is important to keep in mind two key differences. First, the low temperature instability must be put in by hand in the Landau–Ginzburg model, whereas it arises naturally in our gravitational description. Indeed, we have seen that there are two physically distinct instabilities which can trigger the phase transition. Second, the Landau–Ginzburg model is only valid near the transition temperature, since it involves a power series in the order parameter φ . To go beyond $T \approx T_c$, one would need to specify an entire potential $V(\varphi)$. Initially, our bulk theory also had the freedom to add an arbitrary potential $V(\Psi)$. We chose just a mass term for simplicity. However,

¹⁴ This is not an issue for the $3 + 1$ dimensional superconductor, which can be holographically described by the bulk gravitational theory (10.2) in one higher dimension.

once one embeds the bulk theory into string theory, the potential is fixed and is no longer arbitrary.

10.8.1 Open Problems

We close with a list of open problems.¹⁵ They are roughly ordered in difficulty with the easier problems listed first. (Of course, this is my subjective impression. With the right approach, an apparently difficult problem may become easy!)

1. In the probe limit below the critical temperature, there is an infinite discrete set of solutions for ψ which are all regular on the horizon and satisfy the required boundary condition at infinity. (When backreaction is included, there is only a finite number of such solutions.) They can be labelled by the number of nodes (zeros) they contain. It is widely believed that only the lowest solution with no nodes is stable. Is this true in the theory we have discussed where the scalar potential $V(\psi)$ just contains a mass term? Is it true if the theory contains scalar self interactions, like the potentials derived from string theory?
2. We have seen that an extreme Reissner–Nordstrom AdS black hole is unstable to forming neutral scalar hair if the scalar has mass close to the BF bound (9). Under evolution, as the black hole is developing scalar hair, the total charge and mass at infinity are conserved. Since the scalar field can carry energy, but not charge, the final black hole (which need not be extremal) has the same charge, but less mass than the initial extremal limit. How is this consistent with the original black hole being extremal? What is the extremal limit in this Einstein–Maxwell–scalar theory? A related question is: what is the interpretation of this neutral scalar hair in the dual field theory?
3. We studied magnetic fields in $2 + 1$ dimensional superconductors in Sect. 10.6. What happens if you add a magnetic field to $3 + 1$ superconductors? One immediate difference is that B_c is no longer zero. Are the holographic superconductors still type II? The new magnetic brane solutions in [15, 16] may be useful.
4. As discussed in Sect. 6.2, a vortex solution has recently been found in the probe approximation. Find the vortex solution with backreaction. More generally, find the solution describing a lattice of vortices in the bulk. This would correspond to a magnetically charged black hole in which the magnetic flux is confined to “cosmic strings” stretching from the horizon to infinity. To solve this problem, one must solve nonlinear partial differential equations.
5. Recall that the symmetry of a superconductor refers to the energy gap on the Fermi surface. The holographic superconductors we have discussed here all have S-wave symmetry since they do not prefer a direction. P-wave holographic superconductors have been found [27, 60]. The high T_c cuprates are

¹⁵ I thank Sean Hartnoll for suggesting some of these problems.

known to have D-wave symmetry. Can one find a holographic superconductor with D-wave symmetry? It is natural to try to condense a charged spin two field. Interestingly enough, the newly discovered iron pnictides appear to have S-wave symmetry, but it is complicated by multiple Fermi surfaces [12].

6. It was discovered recently that there are materials in which the (DC) conductivity goes to zero for $T < T_c$ [64]. This is the opposite of a superconductor and is called a “superinsulator”. Can one find a holographic description of a superinsulator? It is tempting to use the fact that under electromagnetic duality in AdS_4 , the conductivity transforms as $\sigma \rightarrow -1/\sigma$ [35]. However, one must start with a bulk action which is invariant under electromagnetic duality. The action (10.2) is not, since it explicitly involves A_μ .
7. As we discussed in Sect. 10.7, it has recently been shown that the holographic superconductors can be derived from string theory, so in some cases the dual microscopic theory is known. Can one understand the pairing mechanism in these cases (if indeed “pairing mechanism” is the right concept at strong coupling)? This remains one of the main open questions in understanding the high T_c materials.
8. As mentioned above, the large N limit is responsible for allowing spontaneous symmetry breaking in our $2 + 1$ dimensional field theory. Can one show that away from this limit, massless fluctuations lead to infrared divergences which destroy the long range order?
9. In gauge/gravity duality, one takes a large N limit to justify using classical general relativity in the bulk. What is the analog of this large N limit in condensed matter systems? In other words, what types of materials are likely to have a (tractable) dual gravitational description? (See [59] for a discussion of some of the issues.)
10. The high temperature cuprate superconductors satisfy a simple scaling law relating the superfluid density, the normal state (DC) conductivity and the critical temperature [43]. Can this be given a dual gravitational interpretation?

Acknowledgement It is a pleasure to thank my collaborators, Sean Hartnoll, Chris Herzog, and Matt Roberts for teaching me many of the results described here. I also thank Hartnoll and Roberts for comments on these lecture notes. Finally, I thank the organizers and participants of the 5th Aegean Summer School, “From Gravity to Thermal Gauge Theories: the AdS/CFT Correspondence” for stimulating discussions. This work is supported in part by NSF grant number PHY-0855415.

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Chapter 11

Flavor Superconductivity and Superfluidity

Matthias Kaminski

Abstract In these lecture notes we derive a generic holographic string theory realization of a p-wave superconductor/superfluid. For this purpose we also review basic D-brane physics, gauge/gravity methods at finite temperature, key concepts of superconductivity and recent progress in distinct realizations of holographic superconductors and superfluids. The gravity dual is a D3/D7-brane construction yielding a superconducting or superfluid vector-condensate. The corresponding gauge theory is 3 + 1-dimensional $\mathcal{N} = 2$ supersymmetric Yang-Mills theory with $SU(N_c)$ color and $SU(2)$ flavor symmetry. It shows a second order phase transition to a phase in which a $U(1)$ subgroup of the $SU(2)$ symmetry is spontaneously broken and typical superconductivity signatures emerge. For example an infinite dc conductivity, a conductivity (pseudo-)gap and the Meissner-Ochsenfeld effect are found. Condensates of this nature are comparable to those recently found experimentally in p-wave superconductors such as a *ruthenate compound*. A string picture of the pairing mechanism and condensation is given using the exact knowledge of the corresponding field theory degrees of freedom.

11.1 Introduction

In these lecture notes¹ we generically construct a holographic p-wave superconductor. This introductory section serves to explain why this particular setup is of

¹ These lecture notes are based on joint research work with Martin Ammon, Johanna Erdmenger, Patrick Kerner and Felix Rust.

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interest both from a string-theoretical as well as from a condensed matter physics point of view. In order to guide the unexperienced reader in [Sect. 11.2](#) we will describe the basic concepts in words and motivate the project. Detailing this overview we build the full holographic setup from scratch in a self-contained manner in [Sect. 11.3](#). [Section 11.4](#) briefly introduces common holographic methods and then summarizes the major results known for the thermodynamics and the spectrum of D-brane systems. With these outcomes in the back of our minds we will understand how our system develops a new ground state and why it can be interpreted as a ρ meson superfluid or superconductor. We compute and discuss thermodynamic observables, conductivities and the spectrum of our holographic superconductor in [Sect. 11.5](#). Finally, we put together all of our findings in order to draw a string theory picture for the *pairing mechanism* in our p-wave superconductor. These notes are self-contained and widely complementary to the notes focusing on holographic s-wave superconductors [[1](#), [2](#)].

11.1.1 String Motivation

The setup of intersecting D-branes and especially the particular D3-D7 construction presented here is interesting because of its diverse applications. It has been successfully used to model strongly coupled particle physics phenomena such as a *deconfinement*, or rather *meson melting* phase transition for fundamental matter and transport coefficients in the quark gluon plasma experimentally created at the RHIC collider in Brookhaven (see [[3](#)] for an introductory review and further references). On the other hand it has recently been used to model strongly correlated electron systems as they appear in superfluids and superconductors in the realm of condensed matter physics [[4](#), [5](#)]. All these applications are also crucial checks of the basic principles and methods coming from the conjectured gauge/gravity correspondence. Receiving physically meaningful outcomes when applying these holographic methods to various systems accessible by experiment, strengthens our confidence in the gauge/gravity conjecture.

From the *string-theoretic point of view* this particular setup is interesting because of its simplicity, uniqueness (explained below) and the naturalness with which the symmetry is broken spontaneously here. The flavor² superfluid/superconductor³ examined here was the first generic *top-down* string theory realization of a superfluid/superconducting phase. In contrast to this the pioneering papers on

² The term “flavor” superconductor stems from earlier applications of this setup to model strongly correlated high energy systems such as the quark gluon plasma. In the present case the name is not important and possibly misleading, since it is really only essential that the system has a non-Abelian $SU(2)$ symmetry.

³ We are using the terms superfluid and superconductor interchangeably here. For the considered phenomena this distinction does not make any difference. Some details to this distinction are given in [Sect. 11.2.1](#).

holographic superconductors [6, 7] had exclusively treated gravity toy models which were not directly obtained from string theory, i.e. *bottom-up models*. Therefore the gauge/gravity correspondence could not be used to identify the exact gauge theory dual. This fact obstructed the interpretation of the outcomes. Furthermore these toy models had few restrictions on their parameters such that in principle a large parameter space needed to be scanned, see e.g. [8]. Our string-derived flavor superfluid/superconductor overcomes those two problems: First the field theory degrees of freedom are exactly known since it is simply $\mathcal{N} = 2$ supersymmetric Yang-Mills theory with an $SU(2)$ flavor-symmetry. Second, the values of parameters, such as for example the dimension of the condensing operator, in our setup are severely restricted by their string theoretic derivation. In this sense this setup is “unique” compared to the big parameter space to be scanned in bottom-up toy models. Later, other top-down string realizations have been suggested and for example involve a consistent truncation of type IIB supergravity with a chemical potential for the R-charge [9], and domain-wall solutions interpolating between AdS solutions with distinct radii which may be lifted to IIB supergravity or eleven-dimensional supergravity [10].

11.1.2 Condensed Matter Motivation

From a *condensed-matter physics point of view* our flavor superconductor is highly interesting because it reproduces features which have been measured in experiments with unconventional superconductors, such as p-wave superconductivity, a system of strongly-correlated particles, a pseudo-gap in the frequency-dependent conductivity (found in high temperature d-wave superconductors). Most of these phenomena lack a widely-accepted microscopic explanation by conventional approaches, so there is the hope that gauge/gravity can shed some light on the nature of these systems. These systems usually contain strongly correlated electrons, so the dual weak gravity description is in principle accessible. Our system has a vector operator which condenses upon breaking a *residual Abelian* flavor symmetry spontaneously. This gives the vector order parameter as described below. So there is a preferred spatial direction in the superfluid/superconducting condensate. This is exactly the situation recently found experimentally in the *p-wave* (explained below in Sect. 11.2.1) superconductor Sr_2RuO_4 [11]. These materials are investigated with great excitement in the condensed-matter community because they are hoped to be usable for *quantum computing* [12]. The reason is that the p-wave structure implies the presence of non-Abelian quasi-particles in the *ruthenate compound*. These non-Abelian quasi-particles can be used as the states to be manipulated in a *topological quantum computer*. The biggest practical obstacle for quantum computation are the errors which may occur during a calculation due to materials being not ideal. Topological quantum computers minimize this source of error because they carry out operations by *braiding* the non-Abelian quasi-particles in a Hilbert-subspace containing degenerate

ground states. Due to an energy gap between this subspace and the rest of the Hilbert space it virtually decouples from all local perturbations [13]. Another confirmed and well-studied condensed-matter example for an emerging p-wave structure is superfluid $^3\text{He-A}$ [14].

11.2 Superconductivity and Holography

This section is a primer on the subject of spontaneous symmetry breaking, superconductivity and superfluidity in the holographic context of the gauge/gravity correspondence. Little formalism is used, while we introduce all the necessary concepts.

11.2.1 Basics of Superconductivity and Our Field Theory Idea

Let us review the essential concepts of superconductivity and understand how to build a p-wave superconductor. We are going to need this knowledge in order to appreciate the fact that our holographic setup reproduces this behavior in great detail.

11.2.1.1 Superconductor Basics and the P-Wave

Superconductivity is the phenomenon associated with infinite dc conductivity in materials at low temperatures. It is caused by the formation of a *charged condensate* in which directed currents do not experience resistivity. A defining criterion for superconductivity is the *Meissner-Ochsenfeld effect* described below. In *conventional superconductors* the superconducting condensate consists of electron pairs called *Cooper pairs*. So there are two simultaneous steps: the fermionic electrons have to pair up to form bosonic Cooper pairs, and these pairs do condense. This condensation happens in a second order phase transition (at vanishing magnetic field) when the temperature is lowered through its critical value T_c . Due to the requirement for the fermionic state to be antisymmetric there are only certain symmetry combinations allowed for the two electron state describing a Cooper pair. As seen from Table 11.1 the name “*p-wave*” superconductor refers to those pairs in which the relative orbital angular momentum between the two electrons is $L = 1$, the spatial part of the parity is odd and the spin state is a triplet.

The mechanism pairing electrons in conventional superconductors is well described by a mean field theory approach and the microscopic *BCS-theory*. Recall that condensed matter systems are conveniently described in terms of lattices with many (about 10^{23}) sites. Then conventional BCS-theory tells us that the lattice is slightly deformed by the presence of an electron. This deformation can be

Table 11.1 Nomenclature of superconducting states

Orbital angular momentum	Name	Parity of spatial part	Spin state
0	s-wave	even	singlet
1	p-wave	odd	triplet
2	d-wave	even	singlet

described by a quasi-particle excitation, a *phonon*. We can imagine the phonon to create a small potential well near the electron in which another electron can be caught. So lattice vibrations (phonons) mediate a weakly attractive interaction between the electrons which then form pairs. Conventional BCS-theory with phonons is only valid at low temperatures because around 30 K the phononic lattice vibrations caused by the temperature already destroy the weakly attractive interaction between the electrons. Note that BCS-theory itself does not depend on the origin of the attractive interaction.

A superconductor is simply a charged superfluid

The crucial defining property for any kind of superconductivity or superfluidity is that a symmetry is spontaneously broken. In superfluids this symmetry is global while in superconductors it is local, i.e. a gauge symmetry. Therefore in superconductors there are a few additional effects related to the gauge symmetry and the corresponding gauge field. But besides that those two phenomena are very similar. Especially in both cases there is a *Goldstone boson* created for each spontaneously broken symmetry of the describing field theory. For a broken global symmetry this Goldstone boson survives and is visible in the spectrum as a hydrodynamic mode [15]. For a broken local symmetry however the Goldstone boson is eaten by the gauge field which couples to the charge belonging to the broken symmetry. In superconductors this causes the gauge boson, i.e. the photon for the broken electromagnetic $U(1)$ to become massive. Since these heavy photons can travel only an exponentially small distance, the electromagnetic interaction becomes short-ranged. Therefore magnetic fields, which can be thought of as consisting of photons, can only penetrate the system up to a certain distance, the *penetration depth*. This is called *Meissner-Ochsenfeld effect* and it is a defining criterion for superconductivity. In the Anderson-Higgs mechanism particles acquire a mass by the same mechanism. Thus it is sometimes described as the superfluidity of the vacuum. See [16] for a more precise review.

There is a class of experimentally well-studied but theoretically less understood *unconventional superconductors*, such as copper or ruthenate compounds. Some of these materials show superconducting phases at high temperatures⁴ up to 138 K.

⁴ Sr_2RuO_4 is superconducting at low temperatures around 2 K but the pairing mechanism is not microscopically understood. The simplified argument is that the phonon-interaction of conventional Cooper pairing is isotropic, thus not providing an anisotropic p-wave structure.

The conventional BCS-theory does not apply to such high temperatures as mentioned above. So the biggest mystery remains to understand the *pairing mechanism* of electrons in these high temperature superconductors. Pairing and condensation need not occur at the same temperature here. Most important for the application of gauge/gravity duality: In these *unconventional superconductors* the coupling strength of the electrons to each other is strong. Thus these are experimentally accessible systems governed by strongly coupled field theory.

Another clear signature for superconductivity is of course an infinite dc conductivity. Together with that there is a conductivity gap in the frequency-dependent conductivity. This is caused by the fact that the conductivity at small frequencies or energies vanishes until there is enough energy to break up one Cooper pair. From that energy on the material is a normal conductor with individual electrons being the charge carriers. In unconventional superconductors there is a surprising phenomenon called *pseudo-gap*. This means that the experiments carried out in unconventional superconductors show a gap in the conductivity even at and above the transition temperature where the superconducting condensate forms. Inside this pseudo-gap the conductivity does not drop to zero but to a small finite value.

11.2.1.2 How to Build a Feld Theory with P-wave Superconductivity

As stressed above the crucial thing to do in order to get a superconductor or superfluid is to spontaneously break a symmetry. We are going to accomplish this by allowing our system to develop a charged condensate.

Let us assume a particle physics point of view for a while. In these notes we will focus on a $3 + 1$ -dimensional $\mathcal{N} = 2$ supersymmetric $SU(N_c)$ Yang-Mills theory at temperature T , consisting of a $\mathcal{N} = 4$ gauge multiplet as well as N_f massive $\mathcal{N} = 2$ supersymmetric hypermultiplets (ψ_i, ϕ_i) . The hypermultiplets give rise to the flavor degrees transforming in the fundamental representation of the gauge group. The action is written down explicitly for instance in [17]. In particular, we work in the large N_c limit with $N_f \ll N_c$ at strong coupling, i.e. with $\lambda \gg 1$, where $\lambda = g_{\text{YM}}^2 N_c$ is the 't Hooft coupling constant. In the following we will consider only two flavors, i.e. $N_f = 2$. The flavor degrees of freedom are called u and d . If the masses of the two flavor degrees are degenerate, the theory has a global $U(2)$ flavor symmetry, whose overall $U(1)_B$ subgroup can be identified with the baryon number.

In the following we will consider the theory at finite isospin chemical potential μ , which is introduced as the source of the operator

$$J_0^3 \propto \bar{\psi} \sigma^3 \gamma_0 \psi + \phi \sigma^3 \partial_0 \phi = n_u - n_d, \quad (11.1)$$

where $n_{u/d}$ is the charge density of the isospin fields, $(\phi_u, \phi_d) = \phi$ and $(\psi_u, \psi_d) = \psi$. σ^i are the Pauli matrices. A non-zero vev $\langle J_0^3 \rangle$ introduces an isospin density as discussed in [18]. The isospin chemical potential μ explicitly breaks the $U(2) \simeq U(1)_B \times SU(2)_I$ flavor symmetry down to $U(1)_B \times U(1)_3$, where $U(1)_3$ is

generated by the unbroken generator σ^3 of the $SU(2)_I$. Under the $U(1)_3$ symmetry the fields with index u and d have positive and negative charge, respectively.

However, the theory is unstable at large isospin chemical potential [18]. The new phase is sketched in Fig. 11.1. We show in this lecture (see also [4]), that the new phase is stabilized by a non-vanishing vacuum expectation value of the current component

$$J_3^1 \propto \bar{\psi} \sigma^1 \gamma_3 \psi + \phi \sigma^1 \partial_3 \phi = \bar{\psi}_u \gamma_3 \psi_d + \bar{\psi}_d \gamma_3 \psi_u + \text{bosons} . \quad (11.2)$$

This current component breaks both the $SO(3)$ rotational symmetry as well as the remaining Abelian $U(1)_3$ flavor symmetry spontaneously. The rotational $SO(3)$ is broken down to $SO(2)_3$, which is generated by rotations around the x^3 axis. Due to the non-vanishing vev for J_3^1 , flavor charged vector mesons condense and form a superfluid. Let us emphasize that we do not describe a color superconductor on the field theory side, since the condensate is a gauge singlet. Figure 11.1 shows a “not accessible” parameter region in which we get divergent quantities. This is due to the fact that at such large charge densities we would need to take in account the backreaction of the D7-branes on the AdS geometry.

In a condensed matter context our model can be considered as a holographic p-wave superconductor in the following way. The global $U(1)_3$ in our model is the analog of the local $U(1)_{\text{em}}$ symmetry of electromagnetic interactions. So far in all holographic models of superconductors the breaking of a global symmetry on the field theory side is considered. In our model, the current J^3 corresponds to the electric current J_{em} . The condensate $\langle J_3^1 \rangle$ breaks the $U(1)_3$ spontaneously. Therefore it can be viewed as the superconducting condensate, which is analogous

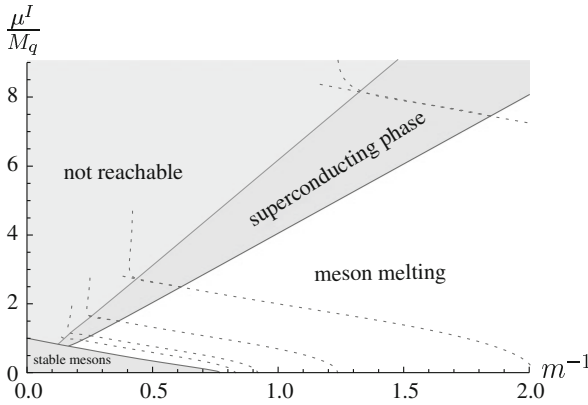


Fig. 11.1 Phase diagram of our field theory depending on the temperature parametrized by $m^{-1} \propto T$ and on the isospin chemical potential μ^I scaled by the quark mass M_q . At low temperatures and chemical potentials a phase of stable mesons forms which melts at higher μ^I and m^{-1} . Above a critical density we find a superconducting or superfluid phase with a vector order parameter

to the Cooper pairs. Since the condensate $\langle J_3^1 \rangle$ transforms as a vector under spatial rotations, it is a p-wave superconductor. Strictly speaking, for a superconductor interpretation it would be necessary to charge the superfluid, i.e. gauge the global $U(1)_3$ symmetry which is broken spontaneously in our model. A spontaneously broken global symmetry corresponds to a superfluid. However, as we mentioned before many features of superconductivity do not depend on whether the $U(1)_3$ is gauged. One exception to this is the Meissner–Ochsenfeld effect. To generate the currents expelling the magnetic field, the $U(1)_3$ symmetry has to be gauged. This matter is discussed further below in [Sect. 11.5.4](#) where we will be able to see the onset of the Meissner–Ochsenfeld effect within our holographic setup.

11.2.2 Holographic Realization

Holographic superconductors have first been studied in [6, 7, 19, 20]. The initial idea presented in [19] was that the Abelian Higgs model coupled to gravity with a negative cosmological constant provides a charged scalar condensate near but outside a charged black hole horizon. The charged condensate spontaneously breaks the Abelian gauge symmetry of the theory. Later studies revealed that this breaking also occurs in setups with neutral black holes but with a negative mass for the charged scalar [6]. The basic idea for a holographic p-wave superconductor has been outlined in [7]. Let us review the basic ideas.

11.2.2.1 Holographic Superconductor Basics

In order to spontaneously break a symmetry we need a gravity theory with a gauge symmetry. Note that this is not the usual gauge symmetry $SU(N)$ of the correspondence where $N \rightarrow \infty$, but an additional one with a finite rank, for example an Abelian $U(1)$. Furthermore we need a black hole background in order to introduce finite temperature in the dual field theory, see [3] for details. Most importantly we need a charged condensate hovering in the bulk over the horizon. To be more precise, we need a bulk field ϕ charged under the gauge symmetry which is to be broken. ϕ has to have the typical expansion $\phi = \phi_{\text{normalizable}} + \phi_{\text{non-normalizable}} + \dots$ near the AdS boundary, but with $\phi_{\text{non-normalizable}} \equiv 0$. Why do we require this particular structure? Recall that in the gauge/gravity correspondence the normalizable mode is identified with the field theory *expectation value* $\langle \mathcal{O}_\phi \rangle$ of the operator $\mathcal{O}_\phi$ dual to the gravity field ϕ . This is our condensate in the field theory, which we want to be non-zero. The normalizable mode on the other hand is dual to a *source* in the field theory. Therefore we want it to vanish since it would break the symmetry *explicitly*. The role of the field ϕ providing the condensate could be played for example by a charged scalar or by one component of a gauge field. These concepts are illustrated in the following example.

Example Consider the bottom-up toy model p-wave superconductor introduced in [7]. There we have an Einstein Yang-Mills theory with a negative cosmological constant

$$S \sim \int \mathcal{D}^4x \left[R - \frac{1}{4} (F_{\mu\nu}^a)^2 + \frac{6}{L^2} \right], \quad (11.3)$$

with F being the field strength of an $SU(2)$ gauge field A . This $SU(2)$ in this case is the gauge group that we want to break. Actually we break only a $U(1)$ subgroup of it. But let us not worry about the details now. Only note that this theory is placed in an AdS_4 charged black hole background. Our gauge field A now plays the role of the field ϕ providing the condensate. The operator dual to the gauge field A is the electromagnetic current $\mathcal{O}_\phi = J^\mu$. The boundary behavior of the gravity field's components according to the equations of motion derived from Eq. 11.3 is given by

$$A_t^3 = \mu + \frac{d}{r}, \quad A_x^1 = 0 + \frac{\langle J_x^1 \rangle}{r}, \quad (11.4)$$

with the radial AdS coordinate r . In principle we could allow more non-vanishing components but this combination turns out to be both sufficient and consistent. As usual in thermal AdS/CFT the temporal component A_t introduces a chemical potential μ in the dual field theory. This chemical potential sources the corresponding operator $J^t = d$ which is simply the charge density of the $SU(2)$ charge in the field theory at the boundary. This explicitly breaks $SU(2) \rightarrow U(1)_3$. However the spatial component A_x^1 spontaneously breaks the remaining $U(1)_3$ with the condensate $\langle J_x^1 \rangle$.

11.2.2.2 How to Build a Gravity Dual to P-Wave Superconductivity

As mentioned several times before, we need a vector condensate for our p-wave superconductor. Conveniently the previous example already showed us what the structure for the dual gravity theory has to be in order to realize a vector condensate. Unfortunately that example has not been derived from string theory. Probably the most difficult task in building a holographic superconductor is finding a gravity setup with all the necessary features, which is actually stable and thermodynamically favored. We are going to see that the system of intersecting D3 and D7 branes provides exactly such a stable configuration. Since we are going to review Dp/Dq-brane systems below, let us here spot only those features which are of importance to superconductivity. The $SU(2)$ “flavor” gauge group which we are going to break (partly spontaneously) is created by using two coincident D7 branes. By introducing a gauge field A_μ^a living on these D7-branes and giving its temporal component a non-trivial profile in one of the flavor directions, we introduce a chemical potential in the dual field theory. This is completely analogous to the introducing the non-trivial A_t^3 in the previous example. All this is going

to take place in the background of a stack of N_c black D3-branes. They introduce the temperature into the dual field theory. Finally, the gravity field which is going to break the residual flavor symmetry is going to be a spatial component of the gauge field just as A_x^1 in the previous example. Our analysis of the thermodynamic potentials is going to show that this phase is thermodynamically preferred over the phase without the symmetry-breaking condensate. Furthermore it is stable against all obvious gauge field fluctuations.

11.3 Holographic Setup

In this section we carry out exactly the D3-D7 brane construction outlined in [Sect. 11.2.2](#). The result will be a gravity setup being holographically dual to a p-wave superfluid/superconductor. But let us first review how to add flavor to the gauge/gravity correspondence.

11.3.1 Flavor from Intersecting Branes

Let us imagine for this subsection that we want to use gauge/gravity in order to model QCD or the quark gluon plasma state of matter produced at the RHIC Brookhaven heavy-ion collider. The original AdS/CFT conjecture does not include matter in the fundamental representation of the gauge group but only adjoint matter. In order to come closer to a QCD-like behavior one can therefore investigate how to incorporate quarks and their bound states in this section. We focus on the main results of [\[21\]](#) and [\[22\]](#), however for a concise review the reader is referred to [\[17\]](#).

Since AdS/CFT has been discovered a lot of modifications of the original conjecture have been proposed and analyzed. This is always achieved by modifying the gravity theory in an appropriate way. For example the metric on which the gravity theory is defined may be changed to produce chiral symmetry breaking in the dual gauge theory [\[23, 24\]](#). Other modifications put the gauge theory at finite temperature and produce confinement [\[25\]](#). Besides the introduction of finite temperature the inclusion of fundamental matter, i.e. quarks, is the most relevant extension for us since we are aiming at a qualitative description of strongly coupled QCD effects at finite temperature. These effects are similar to the ones observed at the RHIC heavy ion collider.

11.3.1.1 Adding Flavor to AdS/CFT

The change we have to make on the gravity side in order to produce fundamental matter on the gauge theory side is the introduction of a small number N_f of

D7-branes. These are also called *probe branes* since their backreaction on the geometry originally produced by the stack of N D3-branes is neglected. Strings within this D3/D7-setup now have the choice of starting (ending) on the D3- or alternatively on the D7-brane. Note that the two types of branes share the four Minkowski directions 0, 1, 2, 3 in which also the dual gauge theory will extend on the boundary of AdS as visualized in Fig. 11.2.

The configuration of one string ending on N coincident D3-branes produces an $SU(N)$ gauge symmetry of rotations in color space. Similarly the N_f D7-branes generate a $U(N_f)$ flavor gauge symmetry. We will call the strings starting on the stack of Dp -branes and ending on the stack of Dq -branes p - q strings. The original 3-3 strings are unchanged while the 3-7- or equivalently 7-3 strings are interpreted as quarks on the gauge theory side of the correspondence. This can be understood by looking at the 3-3 strings again. They come in the adjoint representation of the gauge group which can be interpreted as the decomposition of a bifundamental representation $(N^2 - 1) \oplus 1 = N \otimes \bar{N}$. So the two string ends on the D3-brane are interpreted as one giving the fundamental, the other giving the anti-fundamental representation in the gauge theory. In contrast to this the 3-7 string has only one end on the D3-brane stack corresponding to a single fundamental representation which we interpret as a single quark in the gauge theory.

We can also give mass to these quarks by separating the stack of D3-branes from the D7-branes in a direction orthogonal to both branes. Now 3-7 strings are forced to have a finite length L which is the minimum distance between the two brane stacks. On the other hand a string is an object with tension and if it assumes a minimum length, it needs to have a minimum energy being the product of its length and tension. The dual gauge theory object is the quark and it now also has a minimum energy which we interpret as its mass $M_q = L/(2\pi\alpha')$.

The 7-7 strings decouple from the rest of the theory since their effective coupling is suppressed by N_f/N . In the dual gauge theory this limit corresponds to neglecting quark loops which is often called *the quenched approximation*. Nevertheless, they are important for the description of mesons as we will see below.

Let us be a bit more precise about the fundamental matter introduced by 3-7 strings. The gauge theory introduced by these strings (in addition to the original setup) gives a $\mathcal{N} = 2$ supersymmetric $U(N)$ gauge theory containing N_f fundamental hypermultiplets.

	0	1	2	3	4	5	6	7	8	9
D3	x	x	x	x						
D7	x	x	x	x	x	x	x	x		

Fig. 11.2 Coordinate directions in which the Dp -branes extend are marked by ‘x’. D3- and D7-branes always share the four Minkowski directions and may be separated in the 8, 9-directions which are orthogonal to both brane types

Fig. 11.3 This figure summarizes our coordinates and indices

coord. names	AdS_5					S^3
	x^0	x^1	x^2	x^3	ϱ	—
indices	$\mu, \nu, \dots$					
	$i, j, \dots$				ϱ	
	0	1	2	3	4	

11.3.1.2 D7 Embeddings and Meson Excitations

Mesons correspond to fluctuations of the D7-branes⁵ embedded in the $AdS_5 \times S^5$ -background generated by the D3-branes. From the string-point of view these fluctuations are fluctuations of the hypersurface on which the $7-7$ strings can end, hence these are small oscillations of the $7-7$ string ends. The $7-7$ strings again lie in the adjoint representation of the flavor gauge group for the same reason which we employed above to argue that $3-3$ strings are in the adjoint of the (color) gauge group. Mesons are the natural objects in the adjoint flavor representation. Vector mesons correspond to fluctuations of the gauge field on the D7-branes.

Before we can examine mesons as D7-fluctuations we need to find out how the D7-branes are embedded into the 10-dimensional geometry without any fluctuations. Such a stable configuration needs to minimize the effective action. The effective action to consider is the world volume action of the D7-branes which is composed of a Dirac-Born-Infeld and a topological Chern-Simons part

$$S_{D7} = -T_{D7} \int \mathcal{D}^8 \sigma e^{-\Phi} \sqrt{-\det \left\{ P[g + B]_{\alpha\beta} + (2\pi\alpha') F_{\alpha\beta} \right\}} + \frac{(2\pi\alpha')^2}{2} T_{D7} \int P[C_4] \wedge F \wedge F. \quad (11.5)$$

The preferred coordinates to examine the fluctuations of the D7 are obtained from the standard AdS coordinates

$$\mathcal{D}s^2 = \frac{R^2}{\varrho^2} \mathcal{D}\varrho^2 + \frac{\varrho^2}{R^2} (-\mathcal{D}t^2 + \mathcal{D}\mathbf{x}^2), \quad (11.6)$$

with the AdS radius R and the dimensionful radial AdS coordinate ϱ .

Exercise Show that the standard AdS metric (11.6) transforms to

$$\mathcal{D}s^2 = \frac{r^2}{R^2} \mathcal{D}\mathbf{x}^2 + \frac{R^2}{r^2} (\mathcal{D}\varrho^2 + \varrho^2 \mathcal{D}\Omega_3^2 + \mathcal{D}w_5^2 + \mathcal{D}w_6^2), \quad (11.7)$$

⁵ To be precise the fluctuations correspond to the mesons with spins 0, 1/2 and 1 [22, 26].

under the transformation $q^2 = w_1^2 + \dots + w_4^2$, $r^2 = q^2 + w_5^2 + w_6^2$, where $\mathbf{x}$ is a four vector in Minkowski directions 0, 1, 2, 3 and R is the AdS radius. The coordinate r is the radial AdS coordinate while q is the radial coordinate on the coincident D7-branes.

Let us follow [22]: For a static D7 embedding with vanishing field strength F on the D7 world volume the equations of motion are

$$0 = \frac{D}{Dq} \left(\frac{q^3}{\sqrt{1 + w_5'^2 + w_6'^2}} \frac{Dw_{5,6}}{Dq} \right), \quad (11.8)$$

where $w_{5,6}$ denotes that these are two equations for the two possible directions of fluctuation. Since (11.8) is the equation of motion of a supergravity field in the bulk, the solution near the AdS boundary takes the standard form with a non-normalizable and a normalizable mode, or source and expectation value respectively

$$w_{5,6} = m + \frac{c}{q^2} + \dots, \quad (11.9)$$

with m being the quark mass acting as a source and c being the expectation value of the operator which is dual to the field $w_{5,6}$. While c can be related to the scaled quark condensate $c \propto \langle \bar{q}q \rangle (2\pi\alpha')^3$.

If we now separate the D7-branes from the stack of D3-branes the quarks become massive and the radius of the S^3 on which the D7 is wrapped becomes a function of the radial AdS coordinate r . The separation of stacks by a distance m modifies the metric induced on the D7 $P[g]$ such that it contains the term $R^2 q^2 / (q^2 + m^2) \mathcal{D}\Omega_3^2$. This expression vanishes at a radius $q^2 = r^2 - m^2 = 0$ such that the S^3 shrinks to zero size at a finite AdS radius.

Fluctuations about these w_5 and w_6 embeddings give scalar and pseudoscalar mesons. We take

$$w_5 = 0 + 2\pi\alpha'\chi, \quad w_6 = m + 2\pi\alpha'\varphi \quad (11.10)$$

After plugging these into the effective action (11.5) and expanding to quadratic order in fluctuations we can derive the equations of motion for φ and χ . As an example we consider scalar fluctuations using an Ansatz

$$\varphi = \phi(q) e^{i\mathbf{k}\cdot\mathbf{x}} \mathcal{Y}_l(S^3), \quad (11.11)$$

where $\mathcal{Y}_l(S^3)$ are the scalar spherical harmonics on the S^3 , ϕ solves the radial part of the equation and the exponential represents propagating waves with real momentum $\mathbf{k}$. We additionally have to assume that the mass-shell condition

$$M^2 = -\mathbf{k}^2 \quad (11.12)$$

is valid. Solving the radial part of the equation we get the hypergeometric function $\phi \propto F(-\alpha, -\alpha + l + 1; (l + 2); \frac{-q^2}{m^2})$ and the parameter

$$\alpha = -\frac{1 - \sqrt{1 - \mathbf{k}^2 R^4 / m^2}}{2} \quad (11.13)$$

summarizes a factor appearing in the equation of motion. In general this hypergeometric function may diverge if we take $q \rightarrow \infty$. But since this is not compatible with our linearization of the equation of motion in small fluctuations, we further demand normalizability of the solution. This restricts the sum of parameters appearing in the hypergeometric function to take the integer values

$$n = \alpha - l - 1, \quad n = 0, 1, 2, \dots \quad (11.14)$$

With this quantization condition we determine the scalar meson mass spectrum to be

$$M_{(\text{pseudo})\text{scalar}} = \frac{2m}{R^2} \sqrt{(n + l + 1)(n + l + 2)}, \quad (11.15)$$

where n is the radial excitation number found for the hypergeometric function. Similarly we can determine pseudoscalar masses and find the same formula (11.15). For vector meson masses we need to consider fluctuations of the gauge field A appearing in the field strength F in Eq. 11.5. The formula for vector mesons (corresponding to e.g. the ϱ -meson of QCD) is

$$M_v = \frac{2m}{R^2} \sqrt{(n + l + 1)(n + l + 2)}. \quad (11.16)$$

Note that the scalar, pseudoscalar and vector mesons computed within this framework show identical mass spectra. Further fluctuations corresponding to other mesonic excitations can be found in [22, 26].

11.3.1.3 Brane Embeddings at Finite Temperature

In order to get a finite temperature in the dual field theory, the gravity theory needs to be put into a black hole or black brane background geometry. It was found in [24, 27, 28] that at finite temperature our N_f probe flavor branes can be embedded in two distinct ways. There are high temperature configurations called *black hole embeddings* in which part of the brane falls into the black hole horizon. On the other hand there are low-temperature configurations called in which the brane stays outside the black hole horizon. These two configurations are separated by a geometric transition, i.e. the configuration in which the brane just barely touches the black hole horizon. This geometric transition corresponds to the meson melting transition for the fundamental matter of the dual field theory. Note that the adjoint matter of this field theory is always deconfined in this setup.

At finite charge density however there is only one kind of embedding and that is the black hole embedding. The heuristic argument is that introducing a finite charge on the brane there have to be field lines for the associated field strength. These lines have to end somewhere. Our setup has rotational symmetry in the spatial directions. Imagine the radial AdS coordinate and field lines running along it. If they are supposed to end somewhere, there has to be a horizon. Otherwise they would all meet in the origin at $\rho = 0$. Note that field lines ending at the horizon means, that we can interpret the horizon as being charged. In these lecture notes we will exclusively deal with non-vanishing density and thus only encounter black hole embeddings.

11.3.2 Background and Brane Configuration

We aim to have a field theory dual at finite temperature. This is holographically accomplished by placing the gravity theory in a black hole or black brane background. Here the black hole's Hawking temperature can be identified with the field theory temperature. We consider asymptotically $AdS_5 \times S^5$ space-time. The $AdS_5 \times S^5$ geometry is holographically dual to the $\mathcal{N} = 4$ Super Yang-Mills theory with gauge group $SU(N_c)$. The dual description of a finite temperature field theory is an AdS black hole. We use the coordinates of [29] to write the AdS black hole background in Minkowski signature as

$$\mathcal{D}s^2 = \frac{q^2}{2R^2} \left(-\frac{f^2}{f} dt^2 + \mathfrak{f} d\mathbf{x}^2 \right) + \left(\frac{R}{q} \right)^2 (dq^2 + q^2 d\Omega_5^2), \quad (11.17)$$

with $d\Omega_5^2$ the metric of the unit 5-sphere and

$$f(q) = 1 - \frac{q_H^4}{q^4}, \quad \tilde{f}(q) = 1 + \frac{q_H^4}{q^4}, \quad (11.18)$$

where R is the AdS radius, with $R^4 = 4\pi g_s N_c \alpha'^2 = 2\lambda \alpha'^2$.

Exercise Show that the metric (11.17) can be obtained from the standard finite temperature AdS metric

$$\mathcal{D}s^2 = \frac{(\pi TR)^2}{u} [-f(u) \mathcal{D}t^2 + \mathcal{D}\mathbf{x}^2] + \frac{R^2}{4u^2 f(u)} \mathcal{D}u^2 + R^2 \mathcal{D}\Omega_5^2; \quad f(u) = 1 - u^2, \quad (11.19)$$

by the transformation $(r_0 \rho)^2 = 2r^2 + \sqrt{r^4 - r_0^4}$, with r being the original radial AdS coordinate and r_0 the location of the black hole horizon.

Exercise The temperature of the black hole given by (11.17) may be determined by demanding regularity of the Euclidean section. Show that it is given by

$$T = \frac{Q_H}{\pi R^2}. \quad (11.20)$$

In the following we may use the dimensionless coordinate $\rho = Q/Q_H$, which covers the range from the event horizon at $\rho = 1$ to the boundary of the AdS space at $\rho \rightarrow \infty$. To include fundamental matter, we embed N_f coinciding D7-branes into the ten-dimensional space-time as illustrated in Fig. 11.2. These D7-branes host flavor gauge fields A_μ with gauge group $U(N_f)$. To write down the DBI action for the D7-branes, we introduce spherical coordinates $\{r, \Omega_3\}$ in the 4567-directions and polar coordinates $\{L, \phi\}$ in the 89-directions [29]. The angle between these two spaces is denoted by θ ($0 < \theta < \pi/2$). The six-dimensional space in the 456789-directions is given by

$$\begin{aligned} dQ^2 + Q^2 d\Omega_5^2 &= dr^2 + r^2 d\Omega_3^2 + dL^2 + L^2 d\phi^2 \\ &= dQ^2 + Q^2 (d\theta^2 + \cos^2 \theta d\phi^2 + \sin^2 \theta d\Omega_3^2), \end{aligned} \quad (11.21)$$

where $r = Q \sin \theta$, $Q^2 = r^2 + L^2$ and $L = Q \cos \theta$. D3- and D7-branes always share the four Minkowski directions and may be separated in the 8, 9-directions which are orthogonal to both brane types. That separation is dual to the mass of the fundamental flavor fields in the dual gauge theory, i.e. the *quarks*.

Due to the $SO(4)$ rotational symmetry in the 4567 directions, the embedding of the D7-branes only depends on the radial coordinate ρ . Defining $\chi = \cos \theta$, we parametrize the embedding by $\chi = \chi(\rho)$ and choose $\phi = 0$ using the $SO(2)$ symmetry in the 89-direction. The induced metric G on the D7-brane probes is then

$$ds^2(G) = \frac{Q^2}{2R^2} \left(-\frac{f^2}{f} dt^2 + f dx^2 \right) + \frac{R^2}{Q^2} \frac{1 - \chi^2 + Q^2 (\partial_\rho \chi)^2}{1 - \chi^2} dQ^2 + R^2 (1 - \chi^2) d\Omega_3^2. \quad (11.22)$$

The square root of the determinant of G is given by

$$\sqrt{-G} = \frac{\sqrt{h_3}}{4} Q^3 f \tilde{f} (1 - \chi^2) \sqrt{1 - \chi^2 + Q^2 (\partial_\rho \chi)^2}, \quad (11.23)$$

where h_3 is the determinant of the 3-sphere metric.

As in [18] we introduce a $SU(2)$ isospin chemical potential μ by a non-vanishing time component of the non-Abelian background field on the D7-brane. The generators of the $SU(2)$ gauge group are given by the Pauli matrices σ^i . Due to the gauge symmetry, we may rotate the flavor coordinates until the chemical potential lies in the third flavor direction,

$$\mu = \lim_{\rho \rightarrow \infty} A_0^3(\rho). \quad (11.24)$$

This non-zero gauge field breaks the $SU(2)$ gauge symmetry down to $U(1)_3$ generated by the third Pauli matrix σ^3 . The spacetime symmetry on the boundary is still $SO(3)$. Notice that the Lorentz group $SO(3,1)$ is already broken down to $SO(3)$ by the finite temperature. In addition, we consider a further non-vanishing background gauge field which stabilizes the system for large chemical potentials. Due to the symmetry of our setup we may choose $A_3 dx^3 \sigma^1$ to be non-zero. To obtain an isotropic configuration in the field theory, this new gauge field A_3^1 only depends on ρ . Due to this two non-vanishing gauge fields, the field strength tensor on the branes has the following non-zero components,

$$F_{\varrho 3}^1 = -F_{3\varrho}^1 = \partial_\varrho A_3^1 \quad (7-7\text{strings}), \quad (11.25)$$

$$F_{\varrho 0}^3 = -F_{0\varrho}^3 = \partial_\varrho A_0^3 \quad (3-7\text{strings}), \quad (11.26)$$

$$F_{03}^2 = -F_{30}^2 = \frac{\gamma}{\sqrt{\lambda}} A_0^3 A_3^1 \quad (\text{interaction}). \quad (11.27)$$

The labels behind those equations refer to the sort of strings which generate the corresponding gauge fields. The field strength F_{03}^2 can be understood as an interaction term between 7-7 and 3-7 strings. We derive this interpretation in [Sect. 11.6.1](#).

11.3.3 DBI Action and Equations of Motion

In this section we calculate the equations of motion which determine the profile of the D7-brane probes and of the gauge fields on these branes. A discussion of the gauge field profiles, which we use to give a geometrical interpretation of the stabilization of the system and the pairing mechanism, may be found in [Sect. 11.6.1](#).

The DBI action determines the shape of the brane embeddings, i.e. the scalar fields ϕ , as well as the configuration of the gauge fields A on these branes. We consider the case of $N_f = 2$ coincident D7-branes for which the non-Abelian DBI action reads [\[30\]](#)

$$S_{\text{DBI}} = -T_{D7} \text{Str} \int d^8 \xi \sqrt{\det Q} \left[\det \left(P_{ab} \left[E_{\mu\nu} + E_{\mu i} (Q^{-1} - \delta)^{ij} E_{j\nu} \right] + 2\pi\alpha' F_{ab} \right) \right]^{\frac{1}{2}} \quad (11.28)$$

with

$$Q_j^i = \delta_j^i + i2\pi\alpha' [\Phi^i, \Phi^k] E_{kj} \quad (11.29)$$

and P_{ab} the pullback to the Dp -brane, where for a Dp -brane in d dimensions we have $\mu, \nu = 0, \dots, (d-1), a, b = 0, \dots, p, i, j = (p+1), \dots, (d-1), E_{\mu\nu} = g_{\mu\nu} + B_{\mu\nu}$. In our case we set $p = 7, d = 10, B \equiv 0$. As in [\[18\]](#) we can simplify this action

significantly by using the spatial and gauge symmetries present in our setup. The action becomes

$$S_{\text{DBI}} = -T_{D7} \int d^8 \xi \text{Str} \sqrt{|\det(G + 2\pi\alpha' F)|} \quad (11.30)$$

$$= -T_{D7} \int d^8 \xi \sqrt{-G} \text{Str} \left[1 + G^{00} G^{44} (F_{\varrho 0}^3)^2 (\sigma^3)^2 + G^{33} G^{44} (F_{\varrho 3}^1)^2 (\sigma^1)^2 + G^{00} G^{33} (F_{03}^2)^2 (\sigma^2)^2 \right]^{\frac{1}{2}} \quad (11.31)$$

where in the second line the determinant is calculated. Due to the symmetric trace, all commutators between the matrices σ^i vanish. It is known that the symmetrized trace prescription in the DBI action is only valid up to fourth order in α' [31, 32]. However the corrections to the higher order terms are suppressed by N_f^{-1} [33] (see also [34]). Here we use two different approaches to evaluate the non-Abelian DBI action (11.30). First, we modify the symmetrized trace prescription by omitting the commutators of the generators σ^i and then setting $(\sigma^i)^2 = 1$ (see Sect. 11.3.3.1 below). This prescription makes the calculation of the full DBI action feasible. This prescription is not verified in general but we obtain physically reasonable results as discussed in Sects. 11.5.1 and 11.5.2. Second, we expand the non-Abelian DBI action to fourth order in the field strength F (see Sect. 11.3.3.2). Here it should be noted that in general the higher terms of this expansion need not be smaller than the leading ones. However, we again get physical results in our specific case which confirm this approach. We further motivate the validity of our two approaches below.

11.3.3.1 Adapted Symmetrized Trace Prescription

Using the adapted symmetrized trace prescription defined above, the action becomes

$$\begin{aligned} S_{\text{DBI}} &= -T_{D7} N_f \int d^8 \xi \sqrt{-G} \left[1 + G^{00} G^{44} (F_{\varrho 0}^3)^2 + G^{33} G^{44} (F_{\varrho 3}^1)^2 + G^{00} G^{33} (F_{03}^2)^2 \right]^{\frac{1}{2}} \\ &= -\frac{T_{D7} N_f}{4} \int d^8 \xi \varrho^3 f \tilde{f} (1 - \chi^2) \Upsilon(\rho, \chi, \tilde{A}), \end{aligned} \quad (11.32)$$

with

$$\Upsilon(\rho, \chi, \tilde{A}) = \left[1 - \chi^2 + \rho^2 (\partial_\rho \chi)^2 - \frac{2\tilde{f}}{f^2} (1 - \chi^2) (\partial_\rho \tilde{A}_0^3)^2 + \frac{2}{\tilde{f}} (1 - \chi^2) (\partial_\rho \tilde{A}_3^1)^2 - \frac{2\gamma^2}{\pi^2 \rho^4 f^2} (1 - \chi^2 + \rho^2 (\partial_\rho \chi)^2) (\tilde{A}_0^3 \tilde{A}_3^1)^2 \right]^{\frac{1}{2}}, \quad (11.33)$$

where the dimensionless quantities $\rho = q/q_H$ and $\tilde{A} = (2\pi\alpha')A/q_H$ are used. To obtain first order equations of motion for the gauge fields which are easier to solve numerically, we perform a Legendre transformation. Similarly to [18, 29] we calculate the electric displacement p_0^3 and the magnetizing field p_3^1 which are given by the conjugate momenta of the gauge fields A_0^3 and A_3^1 ,

$$p_0^3 = \frac{\delta S_{\text{DBI}}}{\delta (\partial_\rho A_0^3)}, \quad p_3^1 = \frac{\delta S_{\text{DBI}}}{\delta (\partial_\rho A_3^1)}, \quad (11.34)$$

In contrast to [18, 29, 35, 36], the conjugate momenta are not constant any more but depend on the radial coordinate q due to the non-Abelian term $A_0^3 A_3^1$ in the DBI action. For the dimensionless momenta $\tilde{p}_0^3$ and $\tilde{p}_3^1$ defined as

$$\tilde{p} = \frac{p}{2\pi\alpha' N_f T_{D7} q_H^3}, \quad (11.35)$$

we get

$$\tilde{p}_0^3 = \frac{\rho^3 \tilde{f}^2 (1 - \chi^2)^2 \partial_\rho \tilde{A}_0^3}{2f \Upsilon(\rho, \chi, \tilde{A})}, \quad \tilde{p}_3^1 = -\frac{\rho^3 f (1 - \chi^2)^2 \partial_\rho \tilde{A}_3^1}{2\Upsilon(\rho, \chi, \tilde{A})}. \quad (11.36)$$

Finally, the Legendre-transformed action is given by

$$\tilde{S}_{\text{DBI}} = S_{\text{DBI}} - \int d^8 \xi \left[(\partial_\rho A_0^3) \frac{\delta S_{\text{DBI}}}{\delta (\partial_\rho A_0^3)} + (\partial_\rho A_3^1) \frac{\delta S_{\text{DBI}}}{\delta (\partial_\rho A_3^1)} \right] \quad (11.37)$$

$$= -\frac{T_{D7} N_f}{4} \int d^8 \xi q^3 f \tilde{f} (1 - \chi^2) \sqrt{1 - \chi^2 + \rho^2 (\partial_\rho \chi)^2} V(\rho, \chi, \tilde{A}, \tilde{p}), \quad (11.38)$$

with

$$V(\rho, \chi, \tilde{A}, \tilde{p}) = \left[\left(1 - \frac{2\gamma^2}{\pi^2 \rho^4 f^2} (\tilde{A}_0^3 \tilde{A}_3^1)^2 \right) \times \left(1 + \frac{8(\tilde{p}_0^3)^2}{\rho^6 \tilde{f}^3 (1 - \chi^2)^3} - \frac{8(\tilde{p}_3^1)^2}{\rho^6 \tilde{f}^2 (1 - \chi^2)^3} \right) \right]^{\frac{1}{2}}. \quad (11.39)$$

Then the first order equations of motion for the gauge fields and their conjugate momenta are

$$\partial_\rho \tilde{A}_0^3 = \frac{2f\sqrt{1-\chi^2+\rho^2(\partial_\rho\chi)^2}}{\rho^3\tilde{f}^2(1-\chi^2)^2}\tilde{p}_0^3W(\rho,\chi,\tilde{A},\tilde{p}), \quad (11.40)$$

$$\partial_\rho \tilde{A}_3^1 = -\frac{2\sqrt{1-\chi^2+\rho^2(\partial_\rho\chi)^2}}{\rho^3f(1-\chi^2)^2}\tilde{p}_3^1W(\rho,\chi,\tilde{A},\tilde{p}), \quad (11.41)$$

$$\partial_\rho \tilde{p}_0^3 = \frac{\tilde{f}(1-\chi^2)\sqrt{1-\chi^2+\rho^2(\partial_\rho\chi)^2}c^2}{2\pi^2\rho fW(\rho,\chi,\tilde{A},\tilde{p})}(\tilde{A}_3^1)^2\tilde{A}_0^3, \quad (11.42)$$

$$\partial_\rho \tilde{p}_3^1 = \frac{\tilde{f}(1-\chi^2)\sqrt{1-\chi^2+\rho^2(\partial_\rho\chi)^2}c^2}{2\pi^2\rho fW(\rho,\chi,\tilde{A},\tilde{p})}(\tilde{A}_0^3)^2\tilde{A}_3^1, \quad (11.43)$$

with

$$W(\rho,\chi,\tilde{A},\tilde{p}) = \sqrt{\frac{1 - \frac{2\gamma^2}{\pi^2\rho^4\tilde{f}^2}(\tilde{A}_0^3\tilde{A}_3^1)^2}{1 + \frac{8(\tilde{p}_0^3)^2}{\rho^6\tilde{f}^3(1-\chi^2)^3} - \frac{8(\tilde{p}_3^1)^2}{\rho^6\tilde{f}f^2(1-\chi^2)^3}}}. \quad (11.44)$$

For the embedding function χ we get the second order equation of motion

$$\begin{aligned} \partial_\rho \left[\frac{\rho^5 f \tilde{f} (1-\chi^2) (\partial_\rho \chi)}{\sqrt{1-\chi^2+\rho^2(\partial_\rho\chi)^2}} V \right] = & -\frac{\rho^3 f \tilde{f} \chi}{\sqrt{1-\chi^2+\rho^2(\partial_\rho\chi)^2}} \left[\left[3(1-\chi^2) + 2\rho^2(\partial_\rho\chi)^2 \right] V \right. \\ & \left. - \frac{24(1-\chi^2+\rho^2(\partial_\rho\chi)^2)}{\tilde{f}^3\rho^6(1-\chi^2)^3} W \left((\tilde{p}_0^3)^2 - \frac{\tilde{f}^2}{f^2} (\tilde{p}_3^1)^2 \right) \right]. \end{aligned} \quad (11.45)$$

We solve the equations numerically and determine the solution by integrating the equations of motion from the horizon at $\rho = 1$ to the boundary at $\rho = \infty$. The initial conditions may be determined by the asymptotic expansion of the gravity fields near the horizon

$$\tilde{A}_0^3 = \frac{c_0}{\sqrt{(1-\chi_0^2)^3 + c_0^2}}(\rho-1)^2 + \mathcal{O}\left((\rho-1)^3\right), \quad (11.46)$$

$$\tilde{A}_3^1 = b_0 + \mathcal{O}\left((\rho-1)^3\right), \quad (11.47)$$

$$\tilde{p}_0^3 = c_0 + \frac{\gamma^2 b_0^2 c_0}{8\pi^2}(\rho-1)^2 + \mathcal{O}\left((\rho-1)^3\right), \quad (11.48)$$

$$\tilde{p}_3^1 = +\mathcal{O}\left((\rho - 1)^3\right), \quad (11.49)$$

$$\chi = \chi_0 - \frac{3\chi_0(1 - \chi_0^2)^3}{4[(1 - \chi_0^2)^3 + c_0^2]}(\rho - 1)^2 + \mathcal{O}\left((\rho - 1)^3\right), \quad (11.50)$$

where the terms in the expansions are arranged according to their order in $\rho - 1$. For the numerical calculation we consider the terms up to sixth order in $\rho - 1$. The three independent parameter b_0, c_0 and χ_0 may be determined by field theory quantities defined via the asymptotic expansion of the gravity fields near the boundary,

$$\tilde{A}_0^3 = \tilde{\mu} - \frac{\tilde{d}_0^3}{\rho^2} + \mathcal{O}(\rho^{-4}), \quad (11.51)$$

$$\tilde{A}_3^1 = -\frac{\tilde{d}_3^1}{\rho^2} + \mathcal{O}(\rho^{-4}), \quad (11.52)$$

$$\tilde{p}_0^3 = \tilde{d}_0^3 + \mathcal{O}(\rho^{-4}), \quad (11.53)$$

$$\tilde{p}_3^1 = -\tilde{d}_3^1 + \frac{\gamma^2 \tilde{\mu}^2 \tilde{d}_3^1}{4\pi^2 \rho^2} + \mathcal{O}(\rho^{-4}), \quad (11.54)$$

$$\chi = \frac{m}{\rho} + \frac{c}{\rho^3} + \mathcal{O}(\rho^{-4}). \quad (11.55)$$

According to the AdS/CFT dictionary, μ is the isospin chemical potential. The parameters $\tilde{d}$ are related to the vev of the flavor currents J by

$$\tilde{d}_0^3 = \frac{2^{\frac{5}{2}} \langle J_0^3 \rangle}{N_f N_c \sqrt{\lambda} T^3}, \quad \tilde{d}_3^1 = \frac{2^{\frac{5}{2}} \langle J_3^1 \rangle}{N_f N_c \sqrt{\lambda} T^3} \quad (11.56)$$

and m and c to the bare quark mass M_q and the quark condensate $\langle \bar{\psi} \psi \rangle$,

$$m = \frac{2M_q}{\sqrt{\lambda} T}, \quad c = -\frac{8 \langle \bar{\psi} \psi \rangle}{\sqrt{\lambda} N_f N_c T^3}, \quad (11.57)$$

respectively. There are two independent physical parameters, e. g. m and μ , in the grand canonical ensemble. From the boundary asymptotics (11.51), we also obtain that there is no source term for the current J_3^1 . Therefore as a non-trivial result we find that the $U(1)_3$ symmetry is always broken spontaneously. In contrast, in the related works on p-wave superconductors in 2 + 1 dimensions [7, 37], the spontaneous breaking of the $U(1)_3$ symmetry has to be put in by hand by setting the source term for the corresponding operator to zero. With the constraint $\tilde{A}_3^1|_{\rho \rightarrow \infty} = 0$ and the two independent physical parameters, we may fix the three independent

parameters of the near-horizon asymptotics and obtain a solution to the equations of motion.

11.3.3.2 Expansion of the DBI Action

We now outline the second approach which we use. Expanding the action (11.30) to fourth order in the field strength F yields

$$S_{\text{DBI}} = -T_{D7} N_f \int d^8 \xi \sqrt{-G} \left[1 + \frac{T_2}{2} - \frac{T_4}{8} + \dots \right], \quad (11.58)$$

where $\mathcal{T}_i$ consists of the terms with order i in F . To calculate the $\mathcal{T}_i$, we use the following results for the symmetrized traces

$$2\sigma : \quad \text{Str} \left[(\sigma^i)^2 \right] = N_f, \quad (11.59)$$

$$4\sigma : \quad \text{Str} \left[(\sigma^i)^4 \right] = N_f, \quad \text{Str} \left[(\sigma^i)^2 (\sigma^j)^2 \right] = \frac{N_f}{3}, \quad (11.60)$$

where the indices i, j are distinct. Notice that the symmetric trace of terms with unpaired σ matrices vanish, e.g. $\text{Str}[\sigma^i \sigma^j] = N_f \delta^{ij}$. The $\mathcal{T}_i$ are given in the appendix of [5].

To perform the Legendre transformation of the above action, we determine the conjugate momenta as in (11.34). However, we cannot easily solve these equations for the derivative of the gauge fields since we obtain two coupled equations of third degree. Thus we directly calculate the equations of motion for the gauge fields on the D7-branes. The equations are given in the appendix of [5].

To solve these equations, we use the same strategy as in the adapted symmetrized trace prescription discussed above. We integrate the equations of motion from the horizon at $\rho = 1$ to the boundary at $\rho = \infty$ numerically. The initial conditions may be determined by the asymptotic behavior of the gravity fields near the horizon

$$\tilde{A}_0^3 = a_2(\rho - 1)^2 + \mathcal{O}((\rho - 1)^3), \quad (11.61)$$

$$\tilde{A}_3^1 = b_0 + \mathcal{O}((\rho - 1)^3), \quad (11.62)$$

$$\chi = \chi_0 + \frac{3(a_2^4 + 4a_2^2 - 8)\chi_0}{4(3a_2^4 + 4a_2^2 + 8)}(\rho - 1)^2 + \mathcal{O}((\rho - 1)^3). \quad (11.63)$$

For the numerical calculation we use the asymptotic expansion up to sixth order. As in the adapted symmetrized trace prescription, there are again three independent parameters a_2, b_0, χ_0 . Since we have not performed a Legendre transformation, we trade the independent parameter c_0 in the asymptotics of the conjugate

momenta $\tilde{p}_0^3$ in the symmetrized trace prescription with the independent parameter a_2 (cf. asymptotics in Eq. 11.46). However, the three independent parameters may again be determined in field theory quantities which are defined by the asymptotics of the gravity fields near the boundary

$$\tilde{A}_0^3 = \mu - \frac{\tilde{d}_0^3}{\rho^2} + \mathcal{O}(\rho^4), \quad (11.64)$$

$$\tilde{A}_3^1 = -\frac{\tilde{d}_3^1}{\rho^2} + \mathcal{O}(\rho^4), \quad (11.65)$$

$$\chi = \frac{m}{\rho} + \frac{c}{\rho^3} + \mathcal{O}(\rho^4). \quad (11.66)$$

The independent parameters $\mu, \tilde{d}_0^3, \tilde{d}_3^1, m, c$ are given by field theory quantities as presented in (11.56) and (11.57). Again we find that there is no source term for the current J_3^1 , which implies spontaneous symmetry breaking. Therefore the independent parameters in both prescriptions are the same and we can use the same strategy to solve the equations of motion as described below (11.57).

Exercise In [38] only the leading order of the action (11.58) quadratic in field fluctuations was considered. For a specific chemical potential an analytic solution with non-zero A_0^3 and A_3^1 can be found. Show that the analytic solution found there (adapted to our AdS_5 case)

$$A_0^3 = \mu(1 - \rho^4), \quad A_3^1 = \epsilon \frac{\rho^4}{(1 + \rho^4)^2}, \quad (11.67)$$

indeed solves the equations of motion derived from the action expanded to quadratic order for the chemical potential $\mu = 4$. ϵ is a constant formed from the coupling constant and the vacuum expectation value of the dual current $\langle J_3^1 \rangle$. Note: This exercise requires some work.

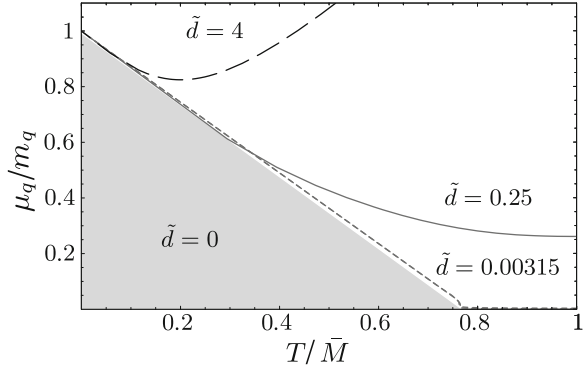
11.4 D-Brane Thermodynamics and Spectrum

In this section we briefly review the results obtained for the thermodynamics and the spectrum of our setup with a baryonic, and later an isospin chemical potential.

11.4.1 Baryon Chemical Potential

Figure 11.4 shows the phase diagram of the field theory dual to the gravity setup we have constructed in the previous Sect. 11.3. In this field theory we have introduced a

Fig. 11.4 The phase diagram in the canonical ensemble plotted against the variables of the grand canonical ensemble. On the axes the scaled chemical potential μ_q/M_q , with the quark mass M_q is shown versus the scaled temperature $T/\bar{M} = m^{-1}$ from [39]



baryonic chemical potential μ_q which is shown on the vertical axis scaled by the quark mass M_q defined by Eq. 11.57. It is introduced by a non trivial background gauge field solving the equation of motion derived from the DBI-action and asymptoting to the chemical potential near the boundary as $\lim_{\rho \rightarrow \rho_B} A_t(\rho) = \mu_q$. The horizontal axis shows the scaled temperature $T/\bar{M} = m^{-1}$. Recall that m is the asymptotic value of the D7-brane embedding near the AdS boundary. In other words it is the source term for a quark condensate, or the non-normalizable mode of the brane embedding function. Here however we use it merely as a temperature scale. The lines in the diagram in Fig. 11.4 are lines of equal baryon density $\tilde{d}$. At low temperature and chemical potential there is a triangle-shaped phase of zero density. Its diagonal borderline to the white region with finite baryon density is the location of the so-called *meson-melting transition*. This is the transition where the fundamental matter melts, i.e. quark bound states, the mesons, acquire an increasing decay width becoming quasi-particles. In the grey phase we therefore have stable mesons with zero decay width. The deeper we go into the white phase away from the transition line, the mesons melt. We will also see this in the spectral functions computed below. As explained at the end of Sect. 11.3.1 we are going to stay entirely in the white phase at finite temperature where the brane embeddings are of black hole type, which means that part of them falls into the black hole horizon (see Sect. 11.3.1).

11.4.1.1 Correlator Recipe

We need to examine fluctuations of the fields in our setup in order to determine its spectrum and stability. There might be a fluctuation which is tachyonic and therefore could destabilize the whole system. For this purpose we quickly review how to compute real-time correlation functions in gauge/gravity.

Let us work along the example of a gauge field fluctuation a_μ . This appears in the action, in our case the Dirac-Born-Infeld action, in the field strength $F = \mathcal{D}a + a \wedge a$. Note that a background gauge field A would appear in this field

strength too, as we will see in later sections. For simplicity we consider Abelian gauge field fluctuations without background A , i.e. $F_{\mu\nu} = \partial_\mu a_\nu - \partial_\nu a_\mu$. The action then reads something like

$$S \sim \int \mathcal{D}^8 \xi \sqrt{P[g_{\mu\nu}] + F_{\mu\nu}^2}, \quad (11.68)$$

with the P being the pullback of the metric g to the flavor brane. Expanding this action to quadratic order in fluctuations a , we get the linearized equation of motion for example for the spatial fluctuation a_y which in Fourier space looks like this

$$0 = \partial_\rho^2 a_y + \frac{\partial_\rho [\sqrt{-g} g^{yy} g^{\rho\rho}]}{\sqrt{-g} g^{yy} g^{\rho\rho}} \partial_\rho a_y + \frac{g^{tt}}{g^{\rho\rho}} w^2 a_y, \quad (11.69)$$

where w is the dimensionless frequency of the fluctuation. Also for other fields we end up with a second order differential equation that we need to solve. Usually these equations have singular (at the horizon ρ_H) coefficients which need to be regularized by an appropriate ansatz. In order to find the most singular behavior solving this equation, we plug the ansatz $(\rho - \rho_H)^\beta$ into Eq. 11.69 and expand. The leading order is a quadratic equation which can be solved for β giving β_{in} or β_{out} . Only one of those two solutions, β_{in} describes a fluctuation falling into the horizon, the other one is outgoing. We discard the outgoing one because nothing is supposed to leave a classical black hole. β is sometimes called the *indicial exponent*. Now we plug $a = (\rho - \rho_H)^{\beta_{\text{in}}} F(\rho)$, with $F(\rho) = \sum_{n=0}^{\infty} f_n (\rho - \rho_H)^n$ being a regular function of ρ , into the equation of motion (11.69). Note that there might be logarithmic terms present as explained in general for example in [40] and discussed in detail in [3]. Picking the fixes one of the two boundary conditions. The other boundary condition may be fixed by choosing the normalization of $F(\rho)$, i.e. the value f_0 . All the higher f_n depend recursively on f_0 and the indicial exponent β . Now we solve the equation for F analytically or numerically. The correlator is then obtained from the quadratic part of the on-shell action, which in our case has the structure $S_{\text{on-shell}} \sim \int \mathcal{D}\rho B(\rho) a \partial_\rho a$, where B is a function of ρ depending on metric coefficients. The recipe developed in [41, 42] tells us to strip the boundary values a^{bdy} off from the fields $a = a^{\text{bdy}} \mathcal{A}$ and identify what remains of the integrand with the Green function at the boundary $\rho = \rho_B$

$$G_{\mu\nu}^R = \lim_{\rho \rightarrow \rho_B} B(\rho) \mathcal{A}_\mu \partial_\rho \mathcal{A}_\nu. \quad (11.70)$$

Now we only need to plug in our solutions $a = a^{\text{bdy}} \mathcal{A}$. More details on the analytic and numerical procedures are explained in [3].

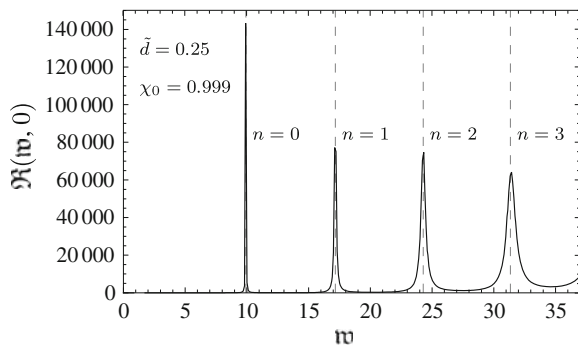
11.4.1.2 Spectral Function and Quasi-Normal Modes

The spectral function $\mathcal{R}$ is obtained from the imaginary part of the Green function $\mathcal{R} = -2\text{Im}G^R$. It encodes the spectrum of our thermal field theory. In particular in Fig. 11.5 we are able to identify pretty stable quasi-particle excitations at a low finite temperature parametrized by χ_0 and at finite baryon density $\tilde{d} = 0.25$. Lowering the temperature these quasiparticles approach the line spectrum (11.16) indicated here as dashed vertical lines. The reason is that at lower temperature our theory restores its original supersymmetry. This also tells us that the vector quasi-particles we see in the thermal spectral function are vector mesons. Our mesons melt in the finite temperature, finite density phase as mentioned above in the discussion of the phase diagram (11.4).

Quasi Normal Modes (QNM)

The difference between the zero temperature line spectrum and the finite temperature spectrum of quasi-particle excitations lies in the nature of the corresponding eigenmodes. In the zero temperature case there is no black hole, thus no dissipation on the gravity side. The system has well-defined normal modes at real frequencies $\omega \in \mathbf{R}$. At finite temperature however the (quasi) eigenfrequencies are complex $\omega \in \mathbf{C}$ owing to the dissipation into the field theory plasma or in the dual gravity picture: dissipation into the black hole. The modes travelling with these (quasi) eigenfrequencies are called *quasi-normal modes (QNM)*. We can roughly think of quasi-normal modes as being those solutions to the gravity fluctuation equations which vanish at the AdS boundary. These QNMs have been found to be identical to the poles in the field theory Green function. Therefore the location of the QNMs is closely related to the location of the peaks in the spectral function. At least some of the quasi-normal modes create quasi-particle peaks in the spectral function. In the zero temperature limit we can think of those QNMs as approaching the real frequency axis and reaching it in the limit, becoming real-valued.

Fig. 11.5 The spectral function $\mathcal{R}$ (in units of $N_f N_c T^2/4$) at finite baryon density $\tilde{d}$ versus dimensionless frequency ω . At large χ_0 here the peaks approach the dashed drawn line spectrum given by (11.16)



The corresponding quasiparticles become stable which means line-shaped in the spectrum. A more complete picture of QNMs is given in [43].

11.4.2 Isospin Chemical Potential

We can equally well introduce a chemical potential with $SU(2)$ —let’s call it isospin—structure, represented by the Pauli matrices σ^i . Choosing the chemical potential to point along the third flavor direction this background boils down to having two copies of the Abelian background gauge field $A_t(\rho)$ explored above in the following way

$$\mu^3 = A_t(\rho)\sigma^3 = \begin{pmatrix} A_t(\rho) & 0 \\ 0 & -A_t(\rho) \end{pmatrix}. \quad (11.71)$$

However we get some interesting new signatures through the $SU(2)$ structure. For example the spectrum shown in Fig. 11.6 shows a triplet splitting for our mesons. In particular we observe a splitting of the line expected at the lowest meson mass at $w = 4.5360$ ($n = 0$). The resonance is shifted to lower frequencies for $\mathcal{R}_{XY}$ and to higher ones for $\mathcal{R}_{YX}$, while it remains in place for $\mathcal{R}_{E^3E^3}$. The second meson resonance peak ($n = 1$) shows a similar behavior. So the different flavor combinations propagate differently and have distinct quasi-particle resonances. This behavior is analogous to that of QCD’s ρ meson which is vector meson and a triplet under the isospin $SU(2)$ of QCD. Thus we have modelled the melting process of vector mesons in a quark gluon plasma at finite isospin density.

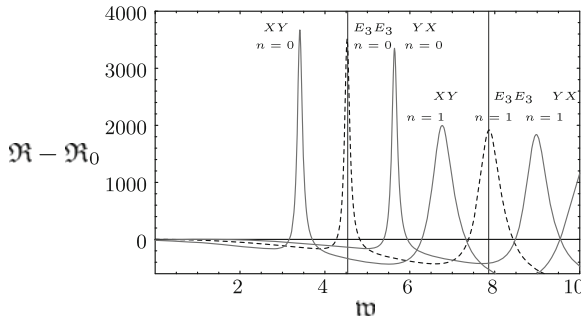


Fig. 11.6 A comparison between the finite temperature part of the spectral functions $\mathcal{R}_{XY}$ and $\mathcal{R}_{YX}$ (solid lines) in the two flavor directions X and Y transversal to the chemical potential is shown in units of $N_c T^2 T_r / 4$ for large quark mass to temperature ratio $\chi_0 = 0.99$ and $\tilde{d} = 0.25$. The spectral function $\mathcal{R}_{E^3E^3}$ along the $a = 3$ -flavor direction is shown as a dashed line

11.4.3 Instabilities and the New Phase

What does all this have to do with our condensed matter motivation? The crucial thing is that the isospin setup described above develops an instability at large enough isospin density. This means that the modes X and Y develop quasi-normal modes (QNM) which have a positive imaginary part, i.e. which are enhanced instead of being damped. This suggests that exactly the mesons corresponding to the X , Y fluctuations condense. This instability comes about naturally if we think about the fact, that we are trying to push more and more charge density into a confined volume. In particular the 3–7 strings charging the D7-brane are located at the black hole horizon as motivated earlier in [Sect. 11.3.1](#). Our current setup does not allow the strings to move into the bulk because we are forcing all the background fields they would create there to be zero. Up to now we have required $A_0^3 \neq 0$ and $A_\mu^a \equiv 0$ for all other field components. But we are going to relax that restriction by allowing a non-trivial A_3^1 . We will see below that this is sufficient to stabilize the theory in a new phase which we will prove to be superconducting/superfluid. This setup naturally produces a p-wave structure since we have seen above that the condensing vector mesons have a triplet structure. According to [Table 11.1](#) this implies a p-wave (at least to leading order).

11.5 Signatures of Super-Something

Finally we put together everything we have learned about D-branes, superconductors and holographic methods. We discuss the holographic results and interpret them in the condensed matter context. [Section 11.5.1](#) starts with the thermodynamics and [Sect. 11.5.2](#) continues with the details of the fluctuation computation. The spectrum and conductivity is Examined in [Sect. 11.5.3](#) and all the signatures will be pointing to the fact that *we have indeed created a holographic p-wave superconductor/superfluid*.

11.5.1 Thermodynamics of the Broken Phase

[Figure 11.7](#) shows the background field configuration. The different curves correspond to the temperatures $T = T_c$ and $T \approx 0.9T_c$. The plots are obtained at zero quark mass $m = 0$ and by using the adapted symmetrized trace prescription. Similar plots may also be obtained at finite mass $m \neq 0$ and by using the DBI action expanded to fourth order in F . These plots show the same features: (top left) The gauge field $\tilde{A}_0^3$ increases monotonically towards the boundary. At the boundary, its value is given by the dimensionless chemical potential $\tilde{\mu}$. The gauge field $\tilde{A}_3^1$ is zero for $T \geq T_c$. For $T < T_c$, its value is non-zero at the horizon and

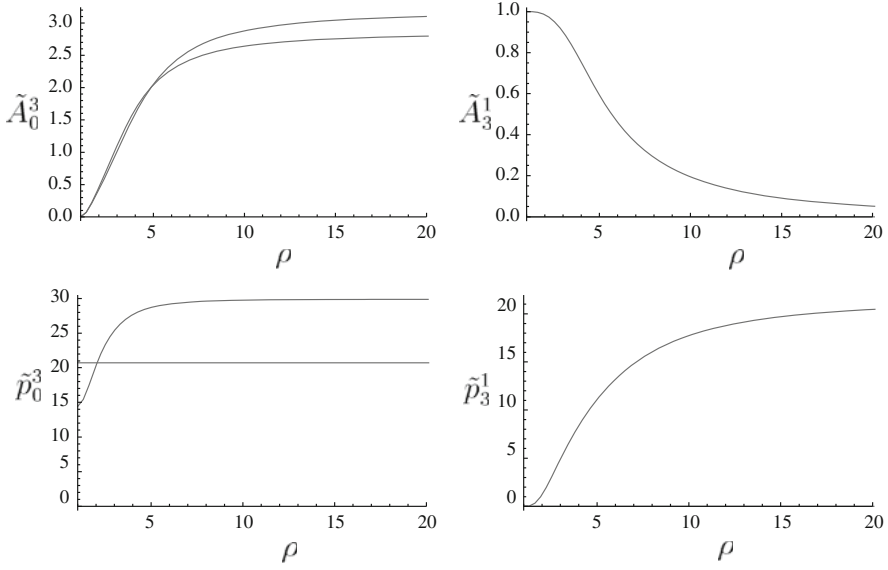


Fig. 11.7 Profiles of the relevant dimensionless gauge fields $\tilde{A}$ on the D7-branes and their dimensionless conjugate momenta $\tilde{p}$ versus the dimensionless AdS radial coordinate ρ near the horizon at $\rho = 1$. (top left) the lower curve is $T = 0.9T_c$, (bottom left) the constant is at $T = T_c$ since there are no charges in the bulk, the other curve shows that the charge proportional to $\tilde{p}_0^3$ increases towards the boundary

decreases monotonically towards the boundary where its value has to be zero. (top right) The conjugate momentum $\tilde{p}_0^3$ of the gauge field $\tilde{A}_0^3$ is constant for $T \geq T_c$. For $T < T_c$, its value increases monotonically towards the boundary. Its boundary value is given by the dimensionless density $\tilde{d}_0^3$. (bottom left) The conjugate momentum $\tilde{p}_3^1$ of the gauge field $\tilde{A}_3^1$ is zero for $T \geq T_c$. For $T < T_c$, its value increases monotonically towards the boundary. Its boundary value is given by the dimensionless density $-\tilde{d}_3^1$. (bottom right)

All thermodynamic quantities are determined in terms of the relevant thermodynamic potential. We use the grand potential W_7 in the grand canonical and the free energy F_7 in the canonical ensemble. Both stem from the D7-brane action. They are related through a Legendre transformation in the background gauge field A_0^3 on the gravity side. We obtain both potentials due to the gauge/gravity dictionary from the Euclideanized gravity on-shell action according to $Z = \mathcal{E}^{-I_{\text{on-shell}}}$, with the partition function Z of the boundary field theory. So the for example $W_7 = TS_{\text{D7, on-shell}}^{\text{euclideanized}}$.

Comparison of the grand potentials in Fig. 11.8 then shows that the new phase with a finite value for A_3^1 is thermodynamically preferred below a temperature $T = T_c$ and does not exist above. The transition seems numerically smooth, i.e. it is a *continuous phase transition*. In these respects both computation schemes, the

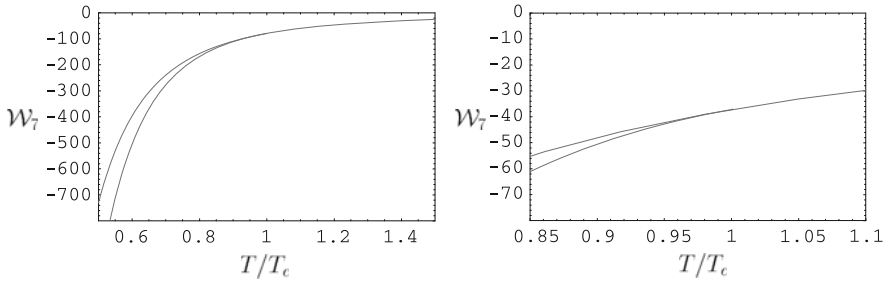


Fig. 11.8 Grand canonical potential computed **a** from the symmetrized trace prescription and **b** from the expanded DBI action, both at vanishing quark mass $M_q = 0$. The qualitative behavior agrees for both prescriptions: At $T = T_c$ the energy curve splits into a lower and a higher energy branch. The branch with lower energy in both cases is the one with a finite new condensate $\tilde{d}_3^1$ below T_c

symmetrized trace prescription as well as the DBI expansion to fourth order give the same qualitative behavior. Figure 11.9 confirms this result and at least numerically determines the transition to be of second order. The order parameter $\tilde{d}_3^1$ vanishes with a critical exponent of $1/2$ to numerical accuracy. It has been explicitly verified that this condensation of vector particles occurs before, i.e. at higher temperature than the condensation of the scalars in our theory (see remarks in [5]). If this was QCD the scalar pions would condense before the vector mesons did. However, there might be other instabilities and also other symmetry breaking configurations may be possible.

Using some intuition to define the superconducting density $\tilde{d}_s = (\tilde{d}_0^3 - c_0)/\tilde{d}_0^3$, we find that it vanishes linearly at the critical temperature as shown in Fig. 11.10

All these signatures are those of a superconducting/superfluid phase transition. The order parameter $\tilde{d}_3^1$ has vector structure by construction, which implies the p-wave.

We can also compute the specific heat which the flavor branes contribute to the theory as $C_7 = -T\partial^2 F_7/\partial T^2$. In Fig. 11.11 the higher curve corresponds to the normal phase with $A_3^1 = 0$, while the lower one corresponds to the superconducting phase with $A_3^1 \neq 0$. Note that the total specific heat is always positive although the flavor brane contribution is negative. The divergences near $T = 0$ in both phases can be attributed to the missing backreaction in our setup.⁶ We read off from the numerical result that near the critical temperature, the dimensionless specific heat $C_7 = 32C_7/(\lambda N_f N_c T^3)$ is constant in the superconducting phase.

This implies that the dimensionful specific heat C_7 is proportional to T^3 . This temperature dependence is characteristic for Bose liquids. There is also a finite jump in the specific heat at T_c .

⁶ The backreaction of the gauge field on the geometry, i.e. on the Einstein equations for metric components has been considered in [44].

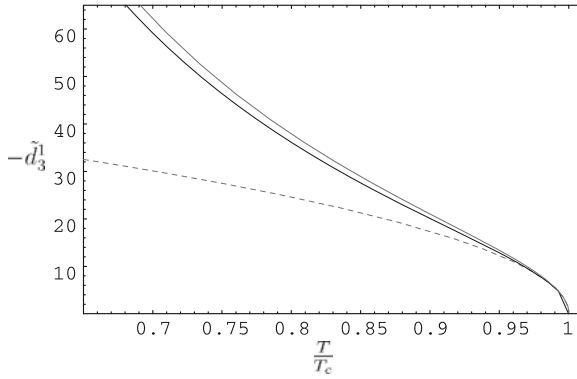


Fig. 11.9 The order parameter $\tilde{d}_3^1$ defined in (11.51) as obtained from the adapted symmetrized trace prescription versus temperature: The case of vanishing quark mass (*top solid curve*) shows the same behavior near T_c as that where $\mu/M_q = 3$ is fixed (*lower solid curve*). Both curves go to zero with a critical exponent of $1/2$ near T_c , as visualized by the fit $55(1 - T/T_c)^{1/2}$ (*dashed curve*)

Fig. 11.10 Superconducting density $\tilde{d}_s = (\tilde{d}_0^3 - c_0)/\tilde{d}_0^3$ versus temperature T : In both, the massless and the massive case at $\mu/M_q = 3$ (both curves coincide on the plot), the superconducting density $\tilde{d}_s$ vanishes linearly at the critical temperature. This is visualized by the fit $6.8(1 - T/T_c)$ (*dashed blue curve*)

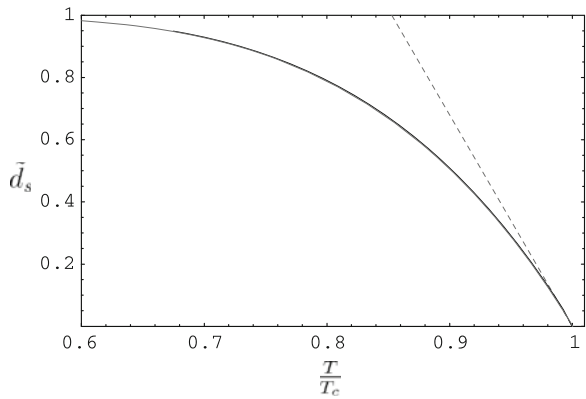
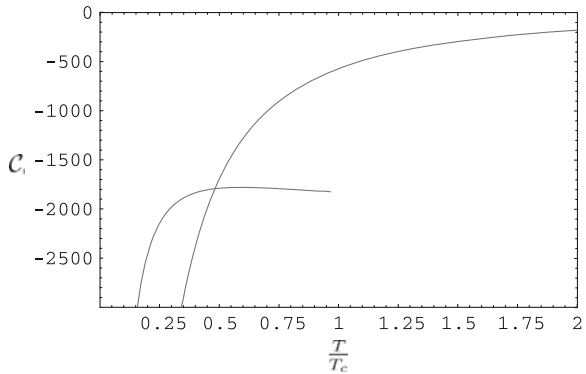


Fig. 11.11 The flavor brane contribution to the specific heat as computed from the adapted symmetrized trace prescription in the massless case. The lower curve starting at $T = T_c$ is the heat capacity in the superconducting phase. The other curve is computed in the normal phase



11.5.2 Fluctuations in the Broken Phase

Let us now investigate fluctuations of this system. We are going to see that the computation of the non-Abelian DBI-action is a very subtle issue. We argue for a novel prescription which actually gives reasonable results. The full gauge field $\hat{A}$ on the branes consists of the field A and fluctuations a ,

$$\hat{A} = A_0^3 \tau^3 \mathcal{D}t + A_3^1 \tau^1 \mathcal{D}x_3 + a_\mu^a \tau^a \mathcal{D}x^\mu, \quad (11.72)$$

where τ^a are the $SU(2)$ generators. The linearized equations of motion for the fluctuations a are obtained by expanding the DBI action in a to second order. We will analyze the fluctuations a_2^3 and $X = a_2^1 + ia_2^2, Y = a_2^1 - ia_2^2$.

Including these fluctuations, the DBI action reads

$$S = -T_7 \int \mathcal{D}^8 \xi \text{Str} \sqrt{\det [G + (2\pi\alpha') \hat{F}]}, \quad (11.73)$$

with the non-Abelian field strength tensor

$$\hat{F}_{\mu\nu}^a = F_{\mu\nu}^a + \check{F}_{\mu\nu}^a, \quad (11.74)$$

where the background is collected in

$$F_{\mu\nu}^a = 2\partial_{[\mu} A_{\nu]}^a + \frac{\gamma}{\sqrt{\lambda}} f^{abc} A_\mu^b A_\nu^c, \quad (11.75)$$

and all terms containing fluctuations in the gauge field are summed in

$$\check{F}_{\mu\nu}^a = 2\partial_{[\mu} a_{\nu]}^a + \frac{\gamma}{\sqrt{\lambda}} f^{abc} a_\mu^b a_\nu^c + \frac{\gamma}{\sqrt{\lambda}} f^{abc} (A_\mu^b a_\nu^c + a_\mu^b A_\nu^c). \quad (11.76)$$

Index anti-symmetrization is always defined with a factor of two in the following way $\partial_{[\mu} A_{\nu]} = (\partial_\mu A_\nu - \partial_\nu A_\mu)/2$.

11.5.2.1 Adapted Symmetrized Trace Prescription

In this section we use the adapted symmetrized trace prescription to determine the fluctuations about the background we discussed in [Sect. 11.3.3.1](#). To obtain the linearized equations of motion for the fluctuations a , we expand the action (11.73) to second order in fluctuations,

$$S^{(2)} = -T_7 \int \mathcal{D}^8 \xi \text{Str} \left[\sqrt{-\mathcal{G}} + \frac{(2\pi\alpha')}{2} \sqrt{-\mathcal{G}} \mathcal{G}^{\mu\nu} F_{\nu\mu} \right. \\ \left. - \frac{(2\pi\alpha')^2}{4} \sqrt{-\mathcal{G}} \mathcal{G}^{\mu\mu'} \check{F}_{\mu'\nu} \mathcal{G}^{\nu\nu'} \check{F}_{\nu'\mu} + \frac{(2\pi\alpha')^2}{8} \sqrt{-\mathcal{G}} (\mathcal{G}^{\mu\nu} \check{F}_{\nu\mu})^2 \right]. \quad (11.77)$$

As in [39], we collect the metric and gauge field background in the tensor $\mathcal{G} = G + (2\pi\alpha')F$. Using the Euler-Lagrange equation, we get the linearized equation of motion for fluctuations a_κ^d in the form

$$\begin{aligned} 0 = & \partial_\lambda \text{Str} \left[\sqrt{-\mathcal{G}} \tau^d \left\{ \mathcal{G}^{[\kappa\lambda]} + (2\pi\alpha') \left(\mathcal{G}^{\mu[\kappa} \mathcal{G}^{\lambda]\nu} + \frac{1}{2} \mathcal{G}^{\mu\nu} \mathcal{G}^{[\kappa\lambda]} \right) F_{\nu\mu} \right\} \right] \\ & - \text{Str} \left[\frac{c}{\sqrt{\lambda}} f^{\text{abd}} \tau^a \sqrt{-\mathcal{G}} \times \left\{ \mathcal{G}^{[\kappa\lambda]} (a + A)_\lambda^b + (2\pi\alpha') \left(\mathcal{G}^{\mu[\kappa} \mathcal{G}^{\lambda]\nu} \right. \right. \right. \\ & \left. \left. \left. + \frac{1}{2} \mathcal{G}^{\mu\nu} \mathcal{G}^{[\kappa\lambda]} \right) \tilde{F}_{\nu\mu} A_\lambda^b \right\} \right]. \end{aligned} \quad (11.78)$$

Note that the linearized version of the fluctuation field strength used in Eq. 11.78 is given by

$$\tilde{F}_{\mu\nu}^a = 2\partial_{[\mu} a_{\nu]}^a + \frac{\gamma}{\sqrt{\lambda}} f^{abc} (A_\mu^b a_\nu^c + a_\mu^b A_\nu^c) + \mathcal{O}(a^2). \quad (11.79)$$

In our specific case the background tensor in its covariant form is given by

$$\mathcal{G}_{\mu\nu} = G_{\mu\nu} \tau^0 + (2\pi\alpha') \left(2\partial_\rho A_0^3 \delta_{4[\mu} \delta_{\nu]0} \tau^3 + 2\partial_\rho A_3^1 \delta_{4[\mu} \delta_{\nu]3} \tau^1 + 2\frac{\gamma}{\sqrt{\lambda}} A_0^3 A_3^1 \delta_{0[\mu} \delta_{\nu]3} \tau^2 \right). \quad (11.80)$$

Inversion yields the contravariant form needed to compute the explicit equations of motion. The inverse of $\mathcal{G}$ is defined as $\mathcal{G}^{\mu\nu} \mathcal{G}_{\nu\mu'} = \delta_\mu^{\mu'} \tau^0$.⁷ The non-zero components of $\mathcal{G}^{\mu\nu}$ may be found in the appendix of [5].

Fluctuations in a_2^3 : For the fluctuation a_2^3 with zero spatial momentum, we obtain the equation of motion

$$\begin{aligned} 0 = & (a_2^3)'' + \frac{\partial_\rho H}{H} (a_2^3)' - \left[\frac{4Q_H^4}{R^4} \left(\frac{\mathcal{G}^{33}}{\mathcal{G}^{44}} (\mathcal{M}_3^1)^2 + \frac{\mathcal{G}^{00}}{\mathcal{G}^{44}} w^2 \right) \right. \\ & \left. - 16 \frac{\partial_\rho \left(\frac{H}{\rho^4 f^2} \tilde{A}_0^3 (\partial_\rho \tilde{A}_0^3) (\mathcal{M}_3^1)^2 \right)}{H \left(1 - \frac{2c^2}{\pi^2 \rho^4 f^2} (\tilde{A}_3^1 \tilde{A}_0^3)^2 \right)} \right] a_2^3, \end{aligned} \quad (11.81)$$

with $\mathcal{M}_3^1 = \gamma \tilde{A}_3^1 / (2\sqrt{2}\pi)$ and $H = \sqrt{\mathcal{G}} G^{22} \mathcal{G}^{44}$.

Fluctuations in $X = a_2^1 + ia_2^2, Y = a_2^1 - ia_2^2$: For the fluctuations X and Y with zero spatial momentum, we obtain the coupled equations of motion

⁷ We calculate the inverse of $\mathcal{G}$ by ignoring the commutation relation of the τ 's because of the symmetrized trace. It is important that $\tau^a \tau^b$ must not be simplified to $\epsilon^{abc} \tau^c$ since the symmetrization is not the same for these two expressions.

$$\begin{aligned}
0 = & X'' + \frac{\partial_\rho H}{H} X' - \frac{4\varrho_H^4}{R^4} \left[\frac{\mathcal{G}^{00}}{\mathcal{G}^{44}} (w - \mathcal{M}_0^3)^2 + \frac{\mathcal{G}^{\{03\}}}{\mathcal{G}^{44}} \mathcal{M}_3^1 w \right] X + \frac{4\varrho_H^4}{R^4} \left[\frac{\mathcal{G}^{\{03\}}}{\mathcal{G}^{44}} \mathcal{M}_3^1 \mathcal{M}_0^3 \right. \\
& + \frac{R^2}{4\varrho_H^2} \frac{\partial_\rho \left[\sqrt{-G} G^{22} \mathcal{G}^{\{34\}} \mathcal{M}_3^1 \right]}{H} - \frac{\mathcal{G}^{33}}{2\mathcal{G}^{44}} (\mathcal{M}_3^1)^2 \left. \right] (X - Y) + \frac{4\varrho_H^2}{R^2} \frac{\mathcal{G}^{\{04\}}}{\mathcal{G}^{44}} w Y' \\
& + \frac{2\varrho_H^2}{R^2} \frac{\partial_\rho \left[\sqrt{-G} G^{22} \mathcal{G}^{\{04\}} (w + \mathcal{M}_0^3) \right]}{H} Y,
\end{aligned} \tag{11.82}$$

where the component of the inverse background tensor may be found in the appendix of [5] (just like the corresponding formula for Y which is only different from (11.82) by a few signs), index symmetrization is defined $\mathcal{G}^{\{ij\}} = (\mathcal{G}^{ij} + \mathcal{G}^{ji})/2$ and $\mathcal{M}_0^3 = \gamma \tilde{A}_0^3 / (2\sqrt{2}\pi)$.

11.5.2.2 Expansion of the DBI Action

In this section we determine the equation of motion for the fluctuation a_2^3 in the background determined by the DBI action expanded to fourth order in F (see Sect. 11.3.3.2). To obtain the quadratic action in the field a_2^3 , we first have to expand the DBI action (11.73) to fourth order in the full gauge field strength $\hat{F}$, and expand the result to second order in a_2^3 . Due to the symmetries of our setup, the equation of motion for the fluctuation a_2^3 at zero spatial momentum decouples from the other equations of motion, such that we can write down an effective Lagrangian for the fluctuation a_2^3 . This effective Lagrangian is given in the appendix of [5]. The equation of motion for a_2^3 with zero spatial momentum determined by the Euler-Lagrange equation is given by

$$\begin{aligned}
0 = & (a_2^3)'' + \frac{\partial_\rho \mathcal{H}}{\mathcal{H}} (a_2^3)' - \frac{\varrho_H^4}{R^4} \left[4 \left(\frac{\mathcal{H}^{00}}{\mathcal{H}^{44}} w^2 + \frac{\mathcal{H}^{33}}{\mathcal{H}^{44}} (\mathcal{M}_3^1)^2 \right) \right. \\
& + \frac{8}{3} \frac{\partial_\rho [\sqrt{-G} G^{00} G^{22} G^{33} G^{44} \tilde{A}_0^3 (\partial_\rho \tilde{A}_0^3) (\mathcal{M}_3^1)^2]}{\mathcal{H}} \left. \right] a_2^3,
\end{aligned} \tag{11.83}$$

where $\mathcal{H} = \sqrt{-G} G^{22} \mathcal{H}^{44}$. We introduce the factors $\mathcal{H}^{ij}$ which may be found in the appendix of [5] to emphasize the similarity to the equation of motion obtained by the adapted symmetrized trace prescription (11.81).

11.5.3 Conductivity and Spectrum

We calculate the frequency-dependent conductivity $\sigma(\omega)$ using the Kubo formula,

$$\sigma(\omega) = \frac{i}{\omega} G^R(\omega, q = 0), \quad (11.84)$$

where G^R is the retarded Green function of the current J_2^3 dual to the fluctuation a_2^3 , which we calculate using the method obtained in [41]. The current J_2^3 is the analog to the electric current since it is charged under the $U(1)_3$ symmetry. In real space it is transverse to the condensate. Since this fluctuation is the only one which transforms as a vector under the $SO(2)$ rotational symmetry, it decouples from the other fluctuations of the system.

Exercise Prove Eq. 11.84 for the current J_2^3 assuming that the gauge gravity correspondence is correct. Recall that in regular electrodynamics $\sigma = J/E$ with the electric current J and the electric field E . Also recall that the two-point Green function for a current J dual to the gauge field A can holographically be written in the form $G^R \propto \partial_\rho A/A$.

11.5.3.1 (Pseudo) Gap

An analysis of the imaginary part of the conductivity using Kramers-Kronig relations shows a delta peak at vanishing frequency $w = 0$ in the real part. The frequency-dependent conductivity is shown for our two distinct computation schemes in Fig. 11.12. Independent from the scheme we see an energy gap develop and grow while the temperature is decreased. The temperature of the black hole horizon induced on the D7-branes is proportional to the inverse particle mass parameter $m^{-1} \propto T$. Therefore from the trivial flat brane embedding at $m = 0$ we get an infinite temperature in Fig. 11.12b. Both schemes show the development of peaks in the conductivities. The peaks coming from fluctuations around the

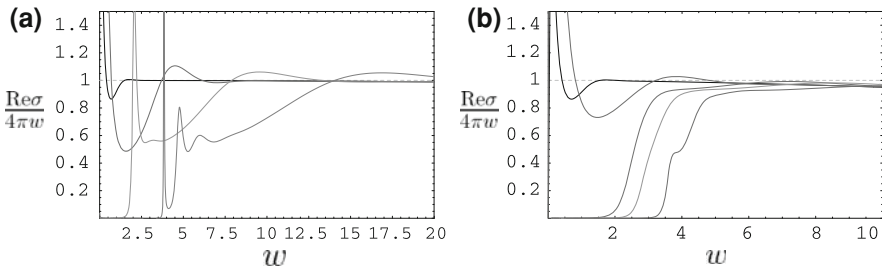


Fig. 11.12 Conductivity **a** from the symmetrized trace prescription at $T/T_c = 10, 1, 0.5, 0.28$ (from left to right), **b** from the DBI action expanded to fourth order at $T/T_c = \infty, 1, 0.6, 0.5, 0.39$

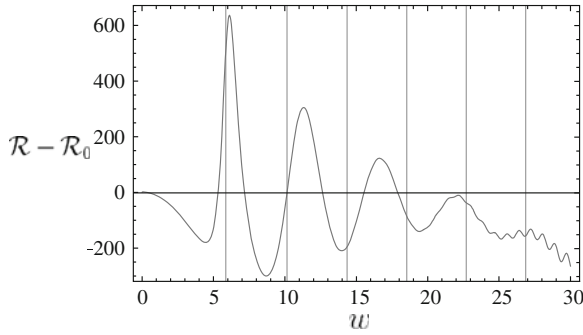


Fig. 11.13 Finite temperature part of the spectral function $\mathcal{R} - \mathcal{R}_0$ with $\mathcal{R}_0 = 4\pi w^2$ in units of $N_f N_c T^2/8$ versus the dimensionless frequency $w = \omega/(2\pi T)$ at finite quark mass $m = 2.842$ and chemical potential $\tilde{\mu} = 3.483$. The grey lines correspond to the supersymmetric mass spectrum calculated in [22]

symmetrized trace prescription background are a lot more pronounced. Taking into account only second order terms in the expanded DBI action as in [38] would hide the peak structures completely. Therefore we conclude that these peaks are higher order effects. Since the conductivity is closely related to the spectral functions, we interpret the peaks as quasiparticles just as described in Sects. 11.4.1.2 and 11.4.2. In particular these are again vector mesons. This identification is confirmed by the spectral function in Fig. 11.13. The peaks in that figure are identical with those in the conductivity and they approach the supersymmetric line spectrum for vector mesons, as in Sect. 11.4.2.

11.5.3.2 Dynamical Mass Generation

Even if we choose the two D7 branes to coincide with the stack of D3-branes, i.e. if we choose the quark mass to vanish, we observe the quasi-particle peaks mentioned before. This is due to a Higgs-like mechanism which dynamically generates masses for the bulk field fluctuations, which in turn give massive quasi-particles in the boundary theory. In the bulk our fields A_0^3 and A_3^1 break the $SU(2)$ symmetry spontaneously since the bulk action is still $SU(2)$ -invariant. Thus there are three Nambu-Goldstone bosons which are immediately eaten by the bulk gauge fields, giving them mass. This can be seen explicitly in the action where the following mass terms for the gauge field fluctuations appear: $(A_0^3)^2 (a^{1,2})^2$ and $(A_3^1)^2 (a^3)^2$.

11.5.4 Meissner-Ochsenfeld-Effect

The Meissner effect is a distinct signature of conventional and unconventional superconductors. It is the phenomenon of expulsion of external magnetic fields. An induced current in the superconductor generates a magnetic field counter-acting

the external magnetic field H . In AdS/CFT we are not able to observe the generation of counter-fields since the symmetries on the boundary are always global. Nevertheless, we can study their cause, i.e. the current induced in the superconductor. As usual [45–48] the philosophy here is to weakly gauge the boundary theory afterwards. In order to investigate how an external magnetic field influences our p-wave superconductor, we have two choices. Either we introduce the field along the spatial z -direction $H_3^3\tau^3$ or equivalently one along the y -direction i.e. $H_2^3\tau^3$. Both are “aligned” with the spontaneously broken $U(1)_3$ -flavor direction.

As an example here we choose a non-vanishing $H_3^3\tau^3$. This requires inclusion of some more non-vanishing field strength components in addition to those given in Eq. 11.25. In particular we choose $A_0^3(q, x)$, $A_3^1(q, x)$ and $A_2^3(x) = x^1 H_3^3$ yielding the additional components⁸

$$H_3^3 = F_{12}^3 = -F_{21}^3 = \partial_1 A_2^3, \quad (11.85)$$

$$F_{13}^1 = -F_{31}^1 = \partial_1 A_3^1, \quad (11.86)$$

$$F_{23}^2 = -F_{32}^2 = \frac{\gamma}{\sqrt{\lambda}} A_2^3 A_3^1, \quad (11.87)$$

$$F_{10}^3 = -F_{01}^3 = \partial_1 A_0^3. \quad (11.88)$$

Recall that the radial AdS-direction is designated by the indices q or 4 synonymously. Amending the DBI-action (11.30) with the additional components (11.85), we compute the determinant in analogy to Eq. 11.30. We then choose to expand the new action to second order in F , i.e. we only consider terms being at most quadratic in the fields. This procedure gives the truncated DBI action

$$\begin{aligned} S_{\text{DBI}} = & -T_{D7} N_f \int d^8 \xi \sqrt{-G} \left[1 + \frac{(2\pi\alpha')^2}{2} \left(G^{00} G^{33} (F_{03}^2)^2 + G^{33} G^{44} (F_{q3}^1)^2 \right. \right. \\ & + G^{00} G^{44} (F_{q0}^3)^2 + (G^{33})^2 (F_{13}^1)^2 + (G^{33})^2 (F_{23}^2)^2 \\ & \left. \left. + G^{00} G^{33} (F_{10}^3)^2 + (G^{33})^2 (F_{12}^3)^2 \right) + \dots \right]. \end{aligned} \quad (11.89)$$

Respecting the symmetries and variable dependencies in our specific system, this can be written as

⁸ Close to the phase transition, it is consistent to drop the dependence of the field H_3^3 on q . Away from the phase transition the q dependence must be included. From the boundary asymptotics it will be possible to extract the magnetic field and the magnetization of the superconductor.

$$\begin{aligned}
S_{\text{DBI}} = & -T_{D7} N_f \int d^8 \xi \sqrt{-G} \left[1 + \frac{1}{2} \left(G^{00} G^{33} (\partial_1 \mathbf{A}_0^3)^2 + G^{33} G^{44} (\partial_\varrho \mathbf{A}_3^1)^2 \right. \right. \\
& + G^{00} G^{44} (\partial_\varrho \tilde{A}_0^3)^2 + (G^{33})^2 (\partial_{\bar{1}} \tilde{A}_3^1)^2 + (G^{33})^2 (\bar{H}_3^3)^2 \\
& \left. \left. + G^{00} G^{33} \frac{\varrho_H^4 \gamma^2}{(2\pi\alpha')^2 \lambda} (\tilde{A}_0^3 \tilde{A}_3^1)^2 + (G^{33})^2 \frac{\varrho_H^4 \gamma^2}{(2\pi\alpha')^2 \lambda} (\tilde{A}_3^1 \bar{H}_3^3)^2 \tilde{x}^2 \right) + \dots \right],
\end{aligned} \tag{11.90}$$

with the convenient redefinitions

$$\tilde{A} = \frac{2\pi\alpha'}{\varrho_H} A, \quad x = \varrho_H \bar{x}, \quad \bar{H}_3^3 = 2\pi\alpha' H_3^3, \quad \varrho = \varrho_H \rho. \tag{11.91}$$

Rescaling the $\bar{x}$ -coordinate once more

$$\tilde{x} = \sqrt{\frac{\bar{H}_3^3 \varrho_H^2 \gamma}{2\pi\alpha' \sqrt{\lambda}}} \bar{x}, \tag{11.92}$$

the equations of motion derived from the action (11.90) take a simple form

$$\begin{aligned}
0 = & \partial_\rho^2 \tilde{A}_0^3 + \frac{\partial_\rho [\sqrt{-G} G^{00} G^{44}]}{\sqrt{-G} G^{00} G^{44}} \partial_\rho \tilde{A}_0^3 + \frac{\gamma \tilde{H}_3^3}{2\sqrt{2\pi}} \frac{G^{33}}{G^{44}} \partial_{\tilde{x}} \tilde{A}_0^3 - \frac{\gamma^2}{2\pi^2} \frac{G^{33}}{G^{44}} \tilde{A}_0^3 (\tilde{A}_3^1)^2, \\
0 = & \partial_\rho^2 \tilde{A}_3^1 + \frac{\partial_\rho [\sqrt{-G} G^{33} G^{44}]}{\sqrt{-G} G^{33} G^{44}} \partial_\rho \tilde{A}_3^1 + \frac{\gamma \tilde{H}_3^3}{2\sqrt{2\pi}} \frac{G^{33}}{G^{44}} [\partial_{\tilde{x}}^2 \tilde{A}_3^1 - \tilde{x}^2 \tilde{A}_3^1] \\
& - \frac{\gamma^2}{2\pi^2} \frac{G^{00}}{G^{44}} (\tilde{A}_0^3)^2 \tilde{A}_3^1.
\end{aligned} \tag{11.93}$$

Here all metric components are to be evaluated at $R = 1$ and $\varrho \rightarrow \rho$.

We aim at decoupling and solving the system of partial differential equations (11.93) by the product ansatz

$$\tilde{A}_3^1(\rho, \tilde{x}) = v(\rho) u(\tilde{x}). \tag{11.94}$$

For this ansatz to work, we need to make two assumptions: First we assume that $\tilde{A}_0^3$ is constant in $\tilde{x}$. Second we assume that $\tilde{A}_3^1$ is small, which clearly is the case near the transition $T \rightarrow T_c$. Our second assumption prevents $\tilde{A}_0^3$ from receiving a dependence on $\tilde{x}$ through its coupling to $\tilde{A}_3^1(\rho, \tilde{x})$. These assumptions allow to write the second equation in (11.93) as

$$\begin{aligned}
0 = & \partial_\rho^2 v(\rho) + \frac{\partial_\rho [\sqrt{-G} G^{33} G^{44}]}{\sqrt{-G} G^{33} G^{44}} \partial_\rho v(\rho) - \frac{\gamma^2}{2\pi^2} \frac{G^{00}}{G^{44}} (\tilde{A}_0^3)^2 v(\rho) \\
& + \frac{\gamma \tilde{H}_3^3}{2\sqrt{2\pi}} \frac{G^{33}}{G^{44}} v(\rho) \frac{\partial_{\tilde{x}}^2 u(\tilde{x}) - \tilde{x}^2 u(\tilde{x})}{u(\tilde{x})}.
\end{aligned} \tag{11.95}$$

All terms but the last one are independent of $\tilde{x}$, so the product ansatz (11.94) is consistent only if

$$\frac{\partial_{\tilde{x}}^2 u(\tilde{x}) - \tilde{x}^2 u(\tilde{x})}{u(\tilde{x})} = C, \quad (11.96)$$

where C is a constant. The differential Eq. 11.96 has a particular solution if $C = -(2n + 1)$, $n \in \mathbb{N}$. The solutions for $u(\tilde{x})$ are Hermite functions

$$u_n(\tilde{x}) = \frac{e^{-\frac{|\tilde{x}|^2}{2}}}{\sqrt{n!2^n\sqrt{\pi}}} H_n(\tilde{x}), \quad H_n(\tilde{x}) = (-1)^n e^{\frac{|\tilde{x}|^2}{2}} \frac{d^n}{d\tilde{x}^n} e^{-\frac{|\tilde{x}|^2}{2}}, \quad (11.97)$$

which have Gaussian decay at large $|\tilde{x}| \gg 1$. Choosing the lowest solution with $n = 0$ and $H_0 = 1$, which has no nodes, is most likely to give the configuration with lowest energy content. So the system we need to solve is finally given by

$$0 = \partial_\rho^2 \tilde{A}_0^3 + \frac{\partial_\rho [\sqrt{-G} G^{00} G^{44}]}{\sqrt{-G} G^{00} G^{44}} \partial_\rho \tilde{A}_0^3 \frac{\gamma^2}{2\pi^2} \frac{G^{33}}{G^{44}} \tilde{A}_0^3 (u_0(\tilde{x}) v(\rho))^2, \quad (11.98)$$

$$0 = \partial_\rho^2 v + \frac{\partial_\rho [\sqrt{-G} G^{33} G^{44}]}{\sqrt{-G} G^{33} G^{44}} \partial_\rho v - \frac{\gamma^2}{2\pi^2} \frac{G^{00}}{G^{44}} (\tilde{A}_0^3)^2 v - \frac{\gamma \tilde{H}_3^3}{2\sqrt{2}\pi} \frac{G^{33}}{G^{44}} v. \quad (11.99)$$

Asymptotically near the horizon the fields take the form

$$\tilde{A}_0^3 = +a_2(\rho - 1)^2 + \mathcal{O}((\rho - 1)^3), \quad (11.100)$$

$$v = b_0 + \frac{b_0 H_3^3}{4} (\rho - 1)^2 + \mathcal{O}((\rho - 1)^3), \quad (11.101)$$

while at the boundary we obtain

$$\tilde{A}_0^3 = \tilde{\mu} + \frac{\tilde{d}_0^3}{\rho^2} + \mathcal{O}(\rho^{-4}), \quad (11.102)$$

$$v = +\frac{\tilde{d}_3^1}{\rho^2} + \mathcal{O}(\rho^{-4}). \quad (11.103)$$

We succeed in finding numerical solutions $v(\rho)$ and $A_0^3(\rho)$ to the set of equations (11.98) obeying the asymptotics given by Eqs. 11.100 and 11.102. These numerical solutions are used to approach the phase transition from the superconducting phase by increasing the magnetic field. We map out the line of critical temperature-magnetic field pairs in Fig. 11.14. In this way we obtain a phase diagram displaying the Meissner effect. The critical line in Figs. 11.14 and 11.15 separates the phase with and without superconducting condensate $\tilde{d}_3^1$.

We emphasize that this is a background calculation involving no fluctuations. Complementary to the procedure described above we also confirmed the phase

Fig. 11.14 The line of critical magnetic field versus critical temperature. Below this line the external magnetic field coexists with the superconducting condensate. Above the line the superconducting condensate vanishes. We set the spatial position arbitrarily to $\tilde{x} = 0.1$ since the critical line does not depend on $\tilde{x}$

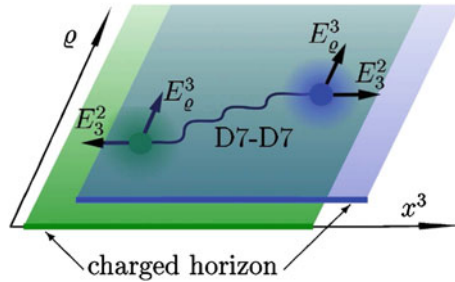
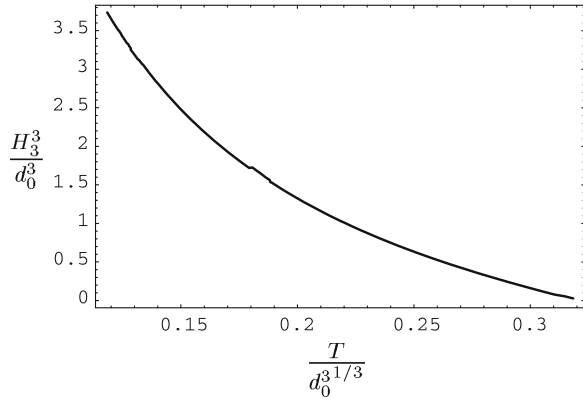


Fig. 11.15 Sketch of our string setup: The figure shows the two coincident D7 branes stretched from the black hole horizon to the boundary as a green and a blue plane, respectively. Strings spanned from the horizon of the AdS black hole to the D7-branes induce a charge at the horizon [18, 29, 35]. However, above a critical charge density, the strings charging the horizon recombine to D7–D7 strings. These D7–D7 strings are shown in the figure. Whereas the fundamental strings stretched between the horizon and the D7-brane are localized near the horizon, the D7–D7 strings propagate into the bulk balancing the flavorelectric and gravitational, i.e. tension forces (see text). Thus these D7–D7 strings distribute the isospin charges along the AdS radial coordinate, leading to a stable configuration of reduced energy. This configuration of D7–D7 strings corresponds to a superconducting condensate

diagram using the instability of the normal phase against fluctuations. Starting at large magnetic field and vanishing condensate $\tilde{a}_1^1$, we determine for a given magnetic field H_3^3 the temperature $T_c(H_3^3)$ at which the fluctuation a_1^1 becomes unstable. That instability signals the condensation process into the superconducting phase.

The presence of the coexistence phase below the critical line, where the system is still superconducting despite the presence of an external magnetic field, is the signal of the Meissner effect in the case of a global symmetry considered here. If we now weakly gauged the flavor symmetry at the boundary, the superconducting current J_3^1 would generate a magnetic field opposite to the external field. Thus the

phase observed is a necessary condition in the case of a global symmetry for finding the standard Meissner effect when gauging the symmetry.

11.6 Interpretation and Conclusion

Our findings merge to a string theory picture of the *pairing mechanism* and the subsequent condensation process in Sect. 11.6.1. Finally we summarize what we have learned about holographic p-wave super-somethings and propose some future territories to be conquered.

11.6.1 String Theory Picture

We now develop a string theory interpretation, i.e. a geometrical picture, of the formation of the superconducting/superfluid phase, for which the field theory is discussed in Sect. 11.2.1. We show that the system is stabilized by dynamically generating a non-zero vev of the current component J_3^1 dual to the gauge field A_3^1 on the brane. Moreover, we find a geometrical picture of the pairing mechanism which forms the condensate $\langle J_3^1 \rangle$, the Cooper pairs. Let us first describe the unstable configuration in absence of the field A_3^1 . As known from [18, 29, 35], the non-zero field A_0^3 is induced by fundamental strings which are stretched from the D7-brane to the horizon of the black hole. In the subsequent we call these strings ‘horizon strings’. Since the tension of these strings would increase as they move to the boundary, they are localized at the horizon, i.e. the horizon is effectively charged under the isospin charge given by (11.1). By increasing the horizon string density, the isospin charge on the D7-brane at the horizon and therefore the energy of the system grows. In [18], the critical density was found beyond which this setup becomes unstable. In this case, the strings would prefer to move towards the boundary due to the repulsive force on their charged endpoints generated by the flavorelectric field $E_q^3 = F_{0q}^3 = -\partial_q A_0^3$.

The setup is now stabilized by the new non-zero field A_3^1 . This field is induced by D7-D7 strings moving in the x^3 direction. This movement of the strings may be interpreted as a current in x^3 direction which induces the magnetic field $B_{3q}^1 = F_{3q}^1 = -\partial_q A_3^1$. Moreover, the non-Abelian interaction between the D7–D7 strings and the horizon strings induces a flavorelectric field $E_3^2 = F_{30}^2 = \gamma/\sqrt{\lambda} A_0^3 A_3^1$.

From the profile of the gauge fields and their conjugate momenta (see Fig. 11.7) we obtain the following: For $A_3^1 = 0$, i.e. in the normal phase ($T \geq T_c$), the isospin density $\tilde{d}_0^3$ is exclusively generated at the horizon by the horizon strings. This can also be understood by the profile of the conjugate momenta p_0^3 (see Fig. 11.7c). We interpret $p_0^3(\rho^*)$ as the isospin charge located between the

horizon at $\rho = 1$ and a fictitious boundary at $\rho = \rho^*$. In the normal phase, the momentum p_0^3 is constant along the radial direction ρ (see Fig. 11.7c, blue curve), and therefore the isospin density is exclusively generated at the horizon. In the superconducting phase where $A_3^1 \neq 0$, i.e. $T < T_c$, the momentum p_0^3 is not constant any more. Its value increases monotonically towards the boundary and asymptotes to $\tilde{d}_0^3$ (see Fig. 11.7c, red curve). Thus the isospin charge is also generated in the bulk and not only at the horizon. This decreases the isospin charge at the horizon and stabilizes the system.

Now we describe the string dynamics which distributes the isospin charge into the bulk. Since the field A_3^1 induced by the D7–D7 strings is non-zero in the superconducting phase, these strings must be responsible for stabilizing this phase. In the normal phase, there are only horizon strings. In the superconducting phase, some of these strings recombine to form D7–D7 strings which correspond to the non-zero gauge field A_3^1 and carry isospin charge.⁹ There are two forces acting on the D7–D7 strings, the flavorelectric force induced by the field E_ϱ^3 and the gravitational force between the strings and the black hole. The flavorelectric force points to the boundary while the gravitational force points to the horizon. The gravitational force is determined by the change in effective string tension, which contains the ϱ dependent warp factor. The position of the D7–D7 strings is determined by the equilibrium of these two forces. Therefore the D7–D7 strings propagate from the horizon into the bulk and distribute the isospin charge.

Since the D7–D7 strings induce the field A_3^1 , they also generate the density $\tilde{d}_3^1$ dual to the condensate $\langle J_3^1 \rangle$, the Cooper pairs. This density $\tilde{d}_3^1$ is proportional to the D7–D7 strings located in the bulk, in the same way as the density $\tilde{d}_0^3$ counts the strings which carry isospin charge [29]. This suggest that we can also interpret $p_3^1(\rho^*)$ as the number of D7–D7 strings which are located between the horizon at $\rho = 1$ and the fictitious boundary at $\rho = \rho^*$. The momentum p_3^1 is always zero at the horizon and increases monotonically in the bulk (see Fig. 11.7d). Thus there are no D7–D7 strings at the horizon.

The double importance of the D7–D7 strings is given by the fact that they are both responsible for stabilizing the superconducting phase by lowering the isospin charge density at the horizon, as well as being the dual of the Cooper pairs since they break the $U(1)_3$ symmetry. In QCD-language they correspond to quarks pairing up to form charged vector mesons which condense subsequently.

11.6.2 Summary

In conclusion we have derived from the top (string theory) down to the gravity theory a holographic p-wave superconductor. Thus we were able to directly

⁹ Note that the D7–D7 strings are of the same order as the horizon strings, namely N_f/N_c , since they originate from the DBI action [35].

identify the degrees of freedom in the boundary field theory, allowing us to translate geometric features directly into field theory features. In particular we have found a string theory picture of the pairing mechanism. The Cooper pairs are modeled by strings spanned between the two flavor D7-branes corresponding to quasi-particles in the vector bi-fundamental representation, i.e. vector mesons. The dual thermal field theory is $3 + 1$ -dimensional $\mathcal{N} = 2$ supersymmetric Yang-Mills theory with $SU(N_c)$ color and $SU(2)$ flavor symmetry coupled to an $\mathcal{N} = 4$ gauge multiplet. It shows a conductivity gap at low temperatures. A pseudo-gap forms even above T_c . The onset of the Meissner-Ochsenfeld effect is visible and in the conductivity spectrum we find massive quasi-particles even at vanishing quark mass. Their masses are generated through a Higgs-like mechanism in the bulk.

Note that our results can also be interpreted using this very setup as a model for the quark gluon plasma as introduced in [Sect. 11.3.1](#). In that case we have found a flavor superfluid phase. This should not be confused with the color-superconducting phase theoretically found in QCD at high *baryon density*.

11.6.3 Outlook

Some directions for promising future investigation are the study of critical exponents near the phase transition. Transport coefficients such as the speeds of second, fourth and other sounds can be determined. For some of these it will be necessary or convenient to use the backreacted solutions for this D3/D7 system given in [\[44\]](#). Similar to earlier semi-classical drag computations it might be instructive to compute the drag on the various strings near the superconducting phase transition. It would be interesting to see what happens to the Fermi surface formed by the background fermions when the superconducting/superfluid phase is entered. Adventurous spirits may also take this setup as a serious model for p-wave superconductors such as the ruthenate compounds mentioned in the introduction. The geometric insights we gain from the dual gravity description can in principle be directly translated to precise field theory statements. Again this pleasant fact is due to knowing the field theory degrees of freedom exactly by using the gauge/gravity correspondence. A different setup in which vector mesons condense is the Sakai-Sugimoto model as shown in [\[49\]](#). It could be illuminating to study all aforementioned effects in this model as well.

Acknowledgements I thank Johanna Erdmenger for discussions as well as kind support, and Martin Ammon, Patrick Kerner, Felix Rust for fruitful collaboration. I thank Ronny Thomale for instructive discussions, and especially Christopher Herzog and Andy O'Bannon for very helpful comments on these notes. This work is supported in part by the “*Deutsche Forschungsgemeinschaft*” (DFG). Last but certainly not least, I thank the organizers and participants of the *Fifth Aegean Summer School “From Gravity to Thermal Gauge Theory: The AdS/CFT Correspondence”*.

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Chapter 12

Holographic Torsion and the Prelude to Kalb–Ramond Superconductivity

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Abstract We discuss the holographic implications of torsional degrees of freedom in the context of $\text{AdS}_4/\text{CFT}_3$, emphasizing in particular the physical interpretation of the latter as carriers of the non-trivial gravitational magnetic field, i.e. the part of the magnetic field not determined by the frame field. As a concrete example we present a new exact four-dimensional gravitational background with torsion and argue that it corresponds to the holographic dual of a 3D system undergoing parity symmetry breaking. Finally, we compare our new gravitational background with known wormhole solutions—with and without cosmological constant—and argue that they can all be unified under an intriguing “Kalb–Ramond superconductivity” framework.

12.1 Introduction and Summary of the Results

$\text{AdS}_4/\text{CFT}_3$ is currently emerging as a novel paradigm of holography that has qualitatively different properties from the more familiar $\text{AdS}_5/\text{CFT}_4$ correspondence. Particularly intriguing is the recent accumulation of evidence that $\text{AdS}_4/\text{CFT}_3$ can be used to describe a plethora of phenomena in $2 + 1$ dimensional systems, such as quantum criticality [1, 2], Quantum Hall transitions [3–6], superconductivity [7–11], superfluidity [12, 13] and spontaneous symmetry breaking [14–16]. This has given rise to a whole new research area that goes under the name of $\text{AdS}/\text{C}(\text{ondensed})\text{M}(\text{atter})\text{T}(\text{heory})$. Furthermore, $\text{AdS}_4/\text{CFT}_3$ is the appropriate setup to study the holographic consequences of generalized electric–magnetic duality of gravity and higher-spin gauge fields [17–22].

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In the absence of an explicit $\text{AdS}_4/\text{CFT}_3$ correspondence example¹ various toy models have been used to study its general qualitative aspects. This work presents yet another model of $\text{AdS}_4/\text{CFT}_3$ which possesses a novel feature. Namely, it can describe the gravity dual of parity symmetry breaking in a 3D system. However, this is not our only aim. We also wish to shed light into torsion from a holographic point of view. The study of torsion is an interesting subject in itself that poses formal and phenomenological challenges.² In the context of string theory, torsion is omnipresent through antisymmetric tensor fields, therefore $\text{AdS}_4/\text{CFT}_3$ provides the basic setup where it can be holographically investigated.

This review presents in a slightly expanded form the results of [34]. We consider a simple toy model where torsion is introduced via the topological Nieh–Yan class. In particular, we consider the modification of the Einstein–Hilbert action with a negative cosmological constant by the Nieh–Yan class, the latter having a spacetime-dependent coefficient. In the context of the $3 + 1$ -split formalism for gravity [17] we emphasize that the torsional degrees of freedom carry the non-trivial ‘gravitational magnetic field.’ In pure gravity the gravitational magnetic field is fully determined by the frame field and hence torsion vanishes. In our model, the spacetime dependence of the Nieh–Yan coefficient makes some of the components of the magnetic field dynamical and as a consequence torsional degrees of freedom enter the theory. Our toy model is simple enough such that only one of the torsional degrees of freedom becomes dynamical. This degree of freedom can be either carried by a pseudoscalar, in which case our model is equivalent to a massless pseudoscalar coupled to gravity, or by a two-form gauge potential. In the latter case our model becomes equivalent to a Kalb–Ramond field coupled to gravity.

Next, we find an exact solution of the equations of motion in Euclidean signature. Our metric ansatz is that of a bulk domain wall (DW). The solution, the *torsion DW*, has two distinct asymptotically AdS_4 regimes along the “radial” coordinate. The pseudoscalar has a kink-like profile and it is finite at both of the asymptotic regimes. Our torsion DW can be viewed as a generalization of the axionic wormhole solution of [35] in the case of non-zero cosmological constant. See also [36] for recent work on AdS wormholes.

Having in mind the holographic interpretation of our model we focus mainly on the case where the torsional degree of freedom is carried by a pseudoscalar field. Following standard holographic recipes we find that the torsion DW is the gravity dual of a 3D system that possesses two distinct parity breaking vacua. The two vacua are distinguished by the relative sign of the pseudoscalar order parameter. Our bulk picture suggests that the transition from one vacuum to the other can be done by a marginal deformation of the boundary theory. In the Appendix we

¹ The recently suggested field theoretic models for M2 branes [23–28] are important steps towards the understanding of the boundary side of $\text{AdS}_4/\text{CFT}_3$.

² See [29–31] for recent reviews and [32, 33] for other recent works.

suggest that the above qualitative properties can be realized in the boundary by the 3D Gross–Neveu model coupled to $U(1)$ gauge fields.

Further, we point out that the bulk physics of our DW solution bears some intriguing resemblance to the standard Abrikosov vortex in superconducting systems. There is a natural mapping of the parameters of the torsion DW to those of the Abrikosov vortex. We show that the gravitational parameter that is interpreted as an order parameter satisfies a ϕ^4 -like equation and this motivates us to suggest that the cosmological constant is related to the “distance from the critical temperature” as $\Lambda \sim T_c - T$. However, there is an important difference in that the Abrikosov vortex is a one-dimensional defect while our DW is codimension one i.e. three-dimensional in AdS_4 . We also discuss multi-DW configurations and DW condensation and show that H -flux supports bubbles of flat spacetime.

Quite intriguing is our result that DW condensation occurs at a critical value of the magnetic field. This motivates us to reconsider the known Euclidean solutions of an Einstein-axion system with [36] and without [35] cosmological constant. These are wormhole solutions whose salient properties include a quantized electric and (possibly) magnetic flux. Moreover, the $\Lambda \neq 0$ solutions possess a lower bound on their electric flux. Using our intuition that Λ plays the role of “temperature”, we place the known wormhole and DW solutions on a (“Temperature”, “Magnetic Field”) graph and observe that it resembles a standard superconductivity graph. We call such a system a “Kalb–Ramond superconductor” and we will present more details on its properties in a forthcoming work [43].

12.2 Torsion as the Non-trivial Magnetic Field of Gravity

In this section, we discuss the physical interpretation of torsion which is that it carries the non-trivial magnetic degrees of freedom of gravity, namely those that are not determined by the frame field (or, equivalently, by the metric in a second order formulation). To motivate things we recall the first order formalism of electromagnetism in the presence of an x -dependent θ -angle, in a non-trivial background here taken to be AdS_4 . Then we present the 3 + 1-split formalism for gravity introduced in [17]. This formalism is a refined form of the standard ADM formalism, which however unveils the physical importance of the gravitational torsional d.o.f. As we will see, such a point of view is crucial in order to understand the holographic interpretation of torsion.

12.2.1 Electromagnetism with a $\theta(x)$ -Angle in AdS_4

The vierbeins and metric of AdS_4 are

$$e^0 = dt, \quad e^\alpha = e^{-t/L} dx^\alpha, \quad ds^2 = \sigma_\perp dt^2 + e^{-2t/L} \eta_{\alpha\beta} dx^\alpha dx^\beta, \quad (12.1)$$

with $\alpha, \beta = 1, 2, 3$. Throughout this work we are being flexible with the both the overall signature and also the nature (spacelike or timelike) of the t -direction i.e. we set $\eta_{\alpha\beta} = \text{diag}(1, 1, \sigma_3)$, $\sigma_\perp \sigma_3 = \sigma = \pm 1$. The gauge potential and field strength are one-forms

$$A = A_0 dt + \tilde{A} \quad F = -dt \wedge E + \tilde{F} = \frac{1}{2} F_{ab} e^a \wedge e^b, \quad (12.2)$$

where $a, b = 0, 1, 2, 3$ and $E = -F_{0\alpha} e^\alpha$ is the electric field. The tilde will always denote quantities along the three directions 1, 2, 3. With the above definitions we find

$$*_4 F = dt \wedge *_3 \tilde{F} - \sigma_\perp *_3 E, \quad dA = dt \wedge (\dot{\tilde{A}} - \tilde{d}A_0) + \tilde{d}\tilde{A}. \quad (12.3)$$

Recall e.g. that $*_3 e^i = \frac{1}{2} \epsilon_{jk}^i e^j \wedge e^k$ and $*_3 e^i \wedge e^k = \epsilon_k^{ij} e^j$. Note also that $*_4^2 = \sigma_\perp$, $*_3^2 = \sigma_3$ and $\epsilon_{0123} = 1$. The first order action is

$$I = \int -dA \wedge *_4 F + \frac{1}{2} F \wedge *_4 F + \frac{\theta}{2} dA \wedge dA. \quad (12.4)$$

Notice that due to the x -dependance of θ the last term in (12.4) is not a total derivative and will give contributions to the e.o.m. After some work the action above takes the more familiar form

$$I = \sigma_\perp \int dt \wedge \left[\dot{\tilde{A}} \wedge (*_3 E + \sigma_\perp \theta \tilde{d}\tilde{A}) + \frac{1}{2} (E \wedge *_3 E - \sigma_\perp \tilde{F} \wedge *_3 \tilde{F}) \right. \\ \left. + A_0 (\tilde{d} *_3 E + \sigma_\perp \tilde{d}\theta \wedge \tilde{d}\tilde{A}) \right]. \quad (12.5)$$

This gives the Hamiltonian e.o.m.

$$E = -\dot{\tilde{A}}, \quad \tilde{d}B = -\sigma_\perp (*_3 \dot{E}) - \dot{\theta} \sigma_3 *_3 B - \tilde{d}\theta \wedge E, \quad (12.6)$$

where we have defined the magnetic field (also a one-form) as

$$\tilde{F} = \tilde{d}\tilde{A} \equiv \sigma_3 (*_3 B). \quad (12.7)$$

We also have the Gauss law and Bianchi identity, respectively

$$\tilde{d} *_3 E + \sigma_\perp \tilde{d}\theta \wedge \tilde{d}\tilde{A} = 0, \quad \tilde{d} *_3 B = 0. \quad (12.8)$$

It is straightforward to show that the above give the Maxwell equations in the more familiar form

$$\nabla \times E = -\sigma_3 \frac{\partial B}{\partial t}, \quad \nabla \cdot E = -\sigma_\perp \sigma_3 \nabla \theta \cdot B, \quad (12.9)$$

$$\nabla \times B = -\sigma \frac{\partial E}{\partial t} - [\dot{\theta} B + \nabla \theta \times E], \quad \nabla \cdot B = 0. \quad (12.10)$$

We summarize the effects of an x -dependent θ -angle in electromagnetism:

- The modification of the canonical momentum as we see in (5)

$$*_3 E \mapsto *_3 E + \sigma \theta *_3 B. \quad (12.11)$$

- The presence of a source term for Gauss law.

In particular, there is no additional d.o.f. introduced by the θ -angle. We will compare this situation with gravity in the following.

12.3 Details on the 3 + 1-Split Formalism

In this section, we present a concise version of the 3 + 1-split formalism of [17] for gravity in the presence of non-zero cosmological constant. We consider a globally hyperbolic Lorentzian manifold $\mathcal{M}$ and take the Einstein–Hilbert action with cosmological constant in the first-order Palatini formalism as

$$S_{\text{EH}} = -\frac{1}{32\pi G} \int_{\mathcal{M}} \epsilon_{abcd} \left(R^{ab} + \frac{\Lambda}{2} e^a \wedge e^b \right) \wedge e^c \wedge e^d. \quad (12.12)$$

This is thus equivalent to the standard second-order gravitational action

$$S_{\text{second}} = -\frac{1}{16\pi G} \int d^4x \sqrt{-g} (R + 6\Lambda), \quad (12.13)$$

and hence the cosmological constant is related to the parameter Λ as $\Lambda_{\text{cosm.}} = -3\Lambda$. The curvature and torsion two-forms are defined in terms of the vielbein e^a and spin-connection ω^{ab} as

$$R^{ab} = d\omega^{ab} + \omega^a_c \wedge \omega^{cb}, \quad T^a = de^a + \omega^a_b \wedge e^b. \quad (12.14)$$

We define as before $\eta_{ab} = \text{diag}(\sigma_\perp, +, +, \sigma_3)$, where $\sigma_\perp \sigma_3 = \sigma = \pm 1$, $\sigma_\perp^2 = \sigma_3^2 = 1$ and set $\Lambda = \sigma_\perp / \ell^2$ such that $\Lambda < 0$ ($\Lambda > 0$) yields the de Sitter (Anti-de Sitter) vacuum. Next, we split the vielbein and the spin connection as

$$e^0 = N dt, \quad e^\alpha = N^\alpha dt + \tilde{e}^\alpha, \quad (12.15)$$

$$\omega^{0\alpha} = q^{0\alpha} dt + \sigma_\perp K^\alpha, \quad \omega^{\alpha\beta} = -\epsilon^{\alpha\beta\gamma} (Q_\gamma dt + B_\gamma). \quad (12.16)$$

The novelty of the formalism is the introduction of the gravitational electric K^α and magnetic fields B^α , which are both vector-valued one-forms on the slices. We then find for the torsion

$$T^\alpha = \tilde{T}^\alpha + dt \wedge \left\{ \dot{\tilde{e}}^\alpha - \tilde{d}N^\alpha + NK^\alpha - \sigma \epsilon^\alpha_{\beta\gamma} Q^\beta \tilde{e}^\gamma - \sigma \epsilon^\alpha_{\beta\gamma} N^\beta B^\gamma \right\}, \quad (12.17)$$

$$T^0 = \sigma_{\perp} K_{\alpha} \wedge \tilde{e}^{\alpha} + dt \wedge \{-\tilde{d}N - \sigma_{\perp} N_{\alpha} K^{\alpha} + q^0_{\beta} \tilde{e}^{\beta}\}, \quad (12.18)$$

and we write

$$R^a_b = \tilde{R}^a_b + dt \omega r^a_b, \quad (12.19)$$

$$\tilde{R}^0_{\alpha} = \sigma_{\perp} (\tilde{d}K_{\alpha} + K_{\beta} \wedge \tilde{\omega}^{\beta}_{\alpha}) \equiv \sigma_{\perp} (\tilde{D}K)_{\alpha}, \quad (12.20)$$

$$\tilde{R}^{\alpha}_{\beta} = {}^{(3)}R^{\alpha}_{\beta} - \sigma_{\perp} K^{\alpha} \wedge K_{\beta}, \quad (12.21)$$

with

$${}^{(3)}R^{\alpha\beta} = \sigma [\epsilon^{\alpha\beta\gamma} dB_{\gamma} - \sigma_{\perp} B^{\alpha} \omega B^{\beta}], \quad (12.22)$$

and

$$\begin{aligned} 2\epsilon_{\alpha\beta\gamma} r^{0\alpha} \wedge \tilde{e}^{\beta} \wedge \tilde{e}^{\gamma} &= 2\sigma_{\perp} \epsilon_{\alpha\beta\gamma} \dot{K}^{\alpha} \wedge \tilde{e}^{\beta} \wedge \tilde{e}^{\gamma} + 4Q_{\alpha} K_{\beta} \wedge \tilde{e}^{\beta} \wedge \tilde{e}^{\alpha} \\ &\quad + 4q^{0\alpha} [\epsilon_{\alpha\beta\gamma} \tilde{T}^{\beta} \wedge \tilde{e}^{\gamma}]. \end{aligned} \quad (12.23)$$

After some tedious but straightforward calculations we find

$$\begin{aligned} S_{\text{EH}} &= -\frac{\sigma_{\perp}}{8\pi G} \int dt \wedge \left\{ \dot{\tilde{e}}^{\alpha} \wedge (4\sigma_{\perp} \epsilon_{\alpha\beta\gamma} K^{\gamma} \wedge \tilde{e}^{\beta}) - 4\sigma_{\perp} N^{\alpha} \epsilon_{\alpha\beta\gamma} (\tilde{D}K)^{\beta} \wedge \tilde{e}^{\gamma} \right. \\ &\quad \left. + 2N \epsilon_{\alpha\beta\gamma} \left({}^{(3)}R^{\alpha\beta} - \sigma_{\perp} K^{\alpha} \wedge K^{\beta} - \frac{\Lambda}{3} \tilde{e}^{\alpha} \wedge \tilde{e}^{\beta} \right) \wedge \tilde{e}^{\gamma} \right. \\ &\quad \left. - 4q^{0\alpha} \epsilon_{\alpha\beta\gamma} \tilde{T}^{\beta} \omega \tilde{e}^{\gamma} + 4Q_{\alpha} \tilde{e}^{\alpha} \omega K_{\beta} \wedge \tilde{e}^{\beta} \right\} - S_{\text{GH}} \end{aligned} \quad (12.24)$$

where the last term is exactly the usual Gibbons–Hawking surface term

$$S_{\text{GH}} = -\frac{1}{16\pi G} \int_{\partial\mathcal{M}} (q^{0\alpha} dt + \sigma_{\perp} K^{\alpha}) \wedge \epsilon_{\alpha\beta\gamma} \tilde{e}^{\beta} \wedge \tilde{e}^{\gamma}. \quad (12.25)$$

Adding then the Gibbons–Hawking term in (12.24) we obtain

$$\begin{aligned} \hat{S}_{\text{EH}} &= -\frac{\sigma_{\perp}}{8\pi G} \int_{\mathcal{M}} dt \wedge \left\{ -K_{\alpha} \wedge \dot{\Sigma}^{\alpha} + N \tilde{W}_{\alpha} \wedge \tilde{e}^{\alpha} + \sigma_{\perp} \hat{Q} \wedge K_{\beta} \wedge \tilde{e}^{\beta} \right. \\ &\quad \left. + \sigma_{\perp} q^{0\alpha} \tilde{D}\Sigma_{\alpha} - N^{\alpha} \epsilon_{\alpha\beta\gamma} \tilde{D}K^{\beta} \wedge \tilde{e}^{\gamma} \right\}, \end{aligned} \quad (12.26)$$

where $\hat{Q} \equiv Q_{\alpha} \tilde{e}^{\alpha}$. We have introduced the two-form

$$\tilde{W}_{\alpha} \equiv \rho_{\alpha} - \frac{1}{2} \epsilon_{\alpha\beta\gamma} K^{\beta} \wedge K^{\gamma} + \frac{1}{\ell^2} \Sigma_{\alpha}. \quad (12.27)$$

and have defined the oriented surface element as

$$\Sigma^{\alpha} = *_3 \tilde{e}^{\alpha} = \frac{1}{2} \epsilon^{\alpha}_{\beta\gamma} \tilde{e}^{\beta} \wedge \tilde{e}^{\gamma}, \quad (12.28)$$

with $*_3$ the three-dimensional Hodge dual defined in terms of $\tilde{e}^\alpha$ only. The three-dimensional component of the curvature two-form

$$\rho_\alpha = \tilde{d}B_\alpha + \frac{1}{2}\epsilon_{\alpha\beta\gamma}B^\beta \wedge B^\gamma, \quad (12.29)$$

is made out of B^α only. Recall (i.e. 19) that $\tilde{D}$ is a covariant derivative with respect to the one-form field B^α as

$$\tilde{D}V^\alpha = \tilde{d}V^\alpha + \epsilon^\alpha_{\beta\gamma}B^\beta \wedge V^\gamma, \quad (12.30)$$

if V^α is a generic vector-valued one-form (with respect to either $\text{SO}(3)$ or $\text{SO}(2, 1)$ depending on whether $\sigma_\perp = \mp 1$, respectively) defined on Σ_t . Comparing the action (12.26) to the electromagnetic action (12.4) motivates calling the vector-valued one-forms K^α and B^α the “electric” and “magnetic” fields, respectively.

The action (12.26) is stationary on-shell when $\delta\tilde{e}^\alpha = 0$ in the boundary, i.e. it provides a good Dirichlet variational principle with respect to the vielbein. The form of the action (12.26) appears to indicate that the proper conjugate dynamical variables are Σ^α (or, equivalently, $\tilde{e}^\alpha$) and K^α . It has been shown in [19, 20] that the proper identification of the dynamical variables is slightly more involved than this. The remaining fields $\{N, N^\alpha, q^{0\alpha}, \hat{Q}, B^\alpha\}$ enter the action as Lagrange multipliers of the following constraints:

$$-8\pi G\sigma_\perp \frac{\delta S}{\delta N} = \tilde{W}_\alpha \wedge \tilde{e}^\alpha = 0, \quad (12.31)$$

$$-8\pi G\sigma_\perp \frac{\delta S}{\delta N^\alpha} = -\epsilon_{\alpha\beta\gamma}\tilde{D}K^\beta \wedge \tilde{e}^\gamma = 0, \quad (12.32)$$

$$-8\pi G\sigma_\perp \frac{\delta S}{\delta q^{0\alpha}} = \sigma_\perp \tilde{D}\Sigma_\alpha = \sigma_\perp \epsilon_{\alpha\beta\gamma}\tilde{T}^\beta \wedge \tilde{e}^\gamma = 0, \quad (12.33)$$

$$-8\pi G\sigma_\perp \frac{\delta S}{\delta \hat{Q}} = \sigma_\perp K_\alpha \wedge \tilde{e}^\alpha = 0, \quad (12.34)$$

$$-8\pi G\sigma_\perp \frac{\delta S}{\delta B^\alpha} = N\tilde{T}^\alpha + (\tilde{d}N + \sigma_\perp K_\beta N^\beta - \hat{q}) \wedge \tilde{e}^\alpha = 0, \quad (12.35)$$

where $\hat{q} \equiv q^{0\alpha}\tilde{e}^\alpha$. The exterior multiplication of (12.35) by $\epsilon_{\alpha\beta\gamma}\tilde{e}^\gamma$ gives, by virtue of (12.33),

$$\tilde{d}N + \sigma_\perp K_\beta N^\beta - \hat{q} = 0, \quad (12.36)$$

and hence we obtain the zero torsion condition

$$\tilde{T}^\alpha = \tilde{D}\tilde{e}^\alpha = 0, \quad (12.37)$$

The last equation unveils the physical meaning of the gravitational magnetic field B^α : it is a Lagrange multiplier which is algebraically related to the vielbein via the vanishing of torsion (12.37). This is exactly analogous to electromagnetism and gives an important hint regarding the relevance of torsion to holography and gravitational duality [34].

12.3.1 The Analog of θ -Angle in Gravity

There is a number of topological terms built from the gravitational dynamical variables that one may consider in four dimension. These are all of potential interest to holography because being total derivatives they may induce interesting boundary effects. We may parameterize these terms as follows (writing all possible $SO(3, 1)$ -invariant four-forms constructed from e^a, R^a_b, T^a):

$$I_{\text{top}} = n \int_M C_{\text{NY}} + 2\gamma^{-1} \int_M C_{\text{Im}} + p \int_M P_4 + q \int_M E_4, \quad (12.38)$$

where

$$C_{\text{NY}} = T^a \wedge T_a - R_{ab} \wedge e^a \wedge e^b = d(T^a \wedge e_a), \quad (12.39)$$

is the Nieh–Yan form. The parameter γ is often referred to as the Immirzi parameter with

$$C_{\text{Im}} = R^a_b \wedge e^b \wedge e_a. \quad (12.40)$$

The remaining objects are the Pontryagin form

$$P_4 = -\frac{1}{8\pi^2} R^a_b \omega R^b_a = -\frac{1}{8\pi^2} d\left(\omega^a_b \wedge R^b_a - \frac{1}{3} \omega^a_b \omega \omega^b_c \omega \omega^c_a\right), \quad (12.41)$$

and the Euler form

$$E_4 = -\frac{1}{32\pi^2} \epsilon_{abcd} R^{ab} \omega R^{cd}. \quad (12.42)$$

We note that $P_4 + \frac{\sigma_1 a^2}{4\pi^2} C_{\text{NY}}$ and $C_{\text{NY}} - C_{\text{Im}}$ are actually $SO(3, 2)$ invariants [30]. These terms become very interesting even in gravity if we allow the coefficients to become fields (for some interesting recent literature on the matter see [32, 33]).

12.3.2 Torsion and the Magnetic Field of Gravity

In pure Einstein–Hilbert gravity the torsional d.o.f. are not dynamical and are carried by the magnetic field B^α . This is seen for example if we recall the definition of the non-trivial ‘spatial’ torsion as

$$\tilde{T}^\alpha = \tilde{d}\tilde{e}^\alpha - \sigma\epsilon^{\alpha\beta\gamma}B_\beta \wedge \tilde{e}_\gamma. \quad (12.43)$$

Moreover, it is seen from (12.18) that the radial component of torsion T^0 is determined by $\tilde{e}^\alpha$ and K^α . Notice that (12.43) implies that the tensor $B_{\alpha\beta}$ is odd under ‘spatial’ parity, hence its trace B^α_α is a pseudoscalar. Hence, although a priori the torsional degrees of freedom are not connected with the pair of conjugate variables $\tilde{e}^\alpha$ and K^α , they are not dynamical as there is no kinetic term for B^α . Rather, they enter (12.26) algebraically and as such they give via (12.35) and (12.36) the algebraic zero torsion condition (12.37) by virtue of which the magnetic field is related to the frame field. This is the gravitational analogue of the electromagnetic case where the magnetic field is related to the gauge potential via the Bianchi identity.

Consider now adding to the Einstein–Hilbert action the Nieh–Yan class C_{NY} with a constant coefficient θ . Over a compact manifold, the NY class is a topological invariant and takes integer values³ [30]. Having in mind holography, we are interested here in manifolds with boundary. In particular, the $3 + 1$ split has been set up so that the boundary is a constant- t slice. The NY term reduces to a boundary contribution. The explicit calculation yields

$$\mathcal{I}_{\text{NY}} \equiv -2\sigma_\perp\theta \int C_{\text{NY}} = 2\sigma_\perp\theta \int dt \wedge [2\epsilon_{\alpha\beta\gamma}\dot{\tilde{e}}^\alpha \wedge \tilde{e}^\beta \wedge B^\gamma + \epsilon_{\alpha\beta\gamma}\dot{B}^\alpha \wedge \tilde{e}^\beta \wedge \tilde{e}^\gamma] \quad (12.44)$$

Adding (12.44) to (12.26) we obtain

$$\begin{aligned} \hat{S}_{\text{EH}} + \mathcal{I}_{\text{NY}} \propto \int dt \wedge (\dot{\tilde{e}}^\alpha \wedge (-4\sigma_\perp\epsilon_{\alpha\beta\gamma}\tilde{e}^\beta \wedge [K^\gamma - \theta B^\gamma]) \\ + 2\sigma_\perp\theta\epsilon_{\alpha\beta\gamma}\dot{B}^\alpha \wedge \tilde{e}^\beta \wedge \tilde{e}^\gamma + \text{constraint terms}). \end{aligned} \quad (12.45)$$

Notice that the $\mathcal{I}_{\text{NY}}$ term has two effects. One is to modify the canonical momentum variable $K^\alpha \mapsto K^\alpha - \theta B^\alpha$. This is analogous to the effect of the θ -angle in the canonical description of electromagnetism in Sect. 2.1. The other is to provide a kinetic term for the singlet component of the magnetic field [one easily verifies that only B^α_α contributes in the second term in the first line of (12.45)]. This second effect has no analogue in electromagnetism. Taking the variation of (12.45) with respect to B^α , one finds that the zero torsion condition still holds.

³ More precisely, $C_{\text{NY}}/(2\pi L)^2$ is integral, as it is equal to the difference of two Pontryagin forms, one $SO(3,2)$ and one $SO(3,1)$.

This is expected of course since the $\mathcal{I}_{\text{NY}}$ term is purely a boundary term. As a consequence, the true dynamical variables remain $\tilde{e}^\alpha$ and K^α . However, the holography is slightly modified. The variation of (12.45) gives on-shell

$$\delta(\hat{S}_{\text{EH}} + \mathcal{I}_{\text{NY}})_{\text{onshell}} \propto \int_{\partial\mathcal{M}} \delta\tilde{e}^\alpha \wedge (-4\sigma_\perp \epsilon_{\alpha\beta\gamma} \tilde{e}^\beta \wedge [K^\gamma - \theta B^\gamma])_{\text{onshell}}. \quad (12.46)$$

After the appropriate subtraction of divergences [19, 20], (12.46) yields a modified boundary energy momentum tensor. The modification is due to the term $4\sigma_\perp \theta \epsilon_{\alpha\beta\gamma} \tilde{e}^\beta \wedge B^\gamma$ which is parity odd and corresponds to the unique symmetric, conserved and traceless tensor of rank two and scaling dimension three that can be constructed from the three-dimensional metric [38]

$$T_{\alpha\beta}^{\text{bdry}} \mapsto T_{\alpha\beta}^{\text{bdry}} + \theta T_{\alpha\beta}^{\text{top}}, \quad T_{\alpha\beta}^{\text{top}} \propto \epsilon_{\ell m(x)} \partial_\ell \partial^2 g_{\beta}{}^m \quad (12.47)$$

where $g_{\alpha\beta}$ being the boundary metric. It is the exact analogue of the topological spin-1 current constructed from the 3D gauge potential [38, 42].

The form of the action (12.45) unveils an intriguing possibility. The above holographic interpretation was based on the zero torsion condition that connects B^α to the frame field. However, to get the zero torsion condition from (12.45) we needed to integrate by parts the last term in the first line. Hence, if θ were t -dependent, the torsion would no longer be zero and the trace B^α_α would become a proper dynamical degree of freedom independent of $\tilde{e}^\alpha$. In such a case the holographic interpretation of (12.45) would change. The new bulk degree of freedom would couple to a new pseudoscalar boundary operator. As a consequence, we have the possibility to probe additional aspects of the boundary physics and describe new $2 + 1$ dimensional phenomena. That we do in the next section.

12.4 The Nieh–Yan Models

12.4.1 General Aspects

In the previous section we sketched a mechanism by which torsional degrees of freedom become dynamical. In particular, we have argued that the addition of the Nieh–Yan class with a space–time-dependent coefficient in the Einstein–Hilbert action makes dynamical one pseudoscalar degree of freedom which is connected to the trace of the gravitational magnetic field. Adding boundary terms to the bulk action corresponds to a canonical transformation. Consequently, by adding boundary terms we can change the canonical interpretation and the variational principle. Consider first the action

$$I'_{\text{NY}} = \hat{S}_{\text{EH}}[e, \omega] + I_{\text{GH}}[e, \omega] + 2 \int_M F(x) C_{\text{NY}}, \quad (12.48)$$

where F is a pseudoscalar ‘axion’ field with no kinetic term. If $F \equiv -\sigma_\perp \theta$ were a constant, this theory would be equivalent to that studied in the last section. With $F = F(x)$, we have additional terms in the action involving gradients of F . If we perform the $3 + 1$ split on this action, we will find that $\tilde{e}^z$ and B^z are canonical coordinates, and their conjugate momenta will depend on F .

The action as given may be supplemented by additional boundary terms. Such boundary terms are analogous to the Gibbons–Hawking term in gravity, but here involve the torsional degrees of freedom. In particular, we can replace I'_{NY} by

$$I_{\text{NY}} = \hat{S}_{\text{EH}}[e, \omega] + I_{\text{GH}}[e, \omega] - 2 \int_M dF \omega T_a \omega e^a. \quad (12.49)$$

This action is such that $\tilde{e}^z$ and F are canonical coordinates with appropriate boundary conditions, while B^z appears in the momentum conjugate to F . To investigate this theory, we note that the variation of the action takes the form

$$\begin{aligned} \delta I_{\text{NY}} = & 2 \int_M \delta e^d \omega \left[\epsilon_{abcd} e^b \omega \left(R^{cd} - \frac{1}{3} \Lambda e^c \omega e^d \right) + 2dF \omega T_d \right] \\ & + 2 \int_M \delta \omega^{ab} \omega \left[\epsilon_{abcd} T^c \omega e^d + dF \omega e_b \omega e_a \right] + 2 \int_M \delta F C_{\text{NY}} \\ & + 2 \int_M d[\delta e^a \wedge (\epsilon_{abcd} e^b \omega \omega^{cd} - dF \omega e_a) - T_a \omega e^a \delta F]. \end{aligned} \quad (12.50)$$

A non-trivial configuration of F sources a particular component of the torsion. Indeed the classical equations of motion can be manipulated to yield in the bulk

$$T^a \omega e_a = 3 *_4 dF, \quad (12.51)$$

where $*_4$ denotes the Hodge- $*$ operation. However, as d’Auria and Regge [37] showed, this classical system is equivalent to a pseudoscalar coupled to torsionless gravity.

$$I_{\text{PS}} = \hat{S}_{\text{EH}}[e, \overset{\circ}{\omega}] + I_{\text{GH}}[e, \overset{\circ}{\omega}] - 3 \int_M dF \omega *_4 dF. \quad (12.52)$$

This comes about as follows. We write the connection as $\omega = \overset{\circ}{\omega} + \Omega$, where $\overset{\circ}{\omega}$ is torsionless, and insert the equation of motion (12.51). The latter becomes an equation⁴ for Ω , and we obtain (12.52).

A massless pseudoscalar field coupled to torsionless gravity is holographically dual to composite pseudoscalar operators of dimensions $\Delta = 3, 0$ in the boundary. The usual holographic dictionary then says that only the $\Delta = 3$ operator appears in

⁴ Explicitly this is $\Omega^a{}_b = \frac{g}{4} \epsilon^{acd}{}_b \partial_c F e_d$.

the boundary theory since only this is above the unitarity bound of the 3D conformal group $SO(3, 2)$. A scalar operator with dimension $\Delta = 0$ would simply correspond to a constant in the boundary. Hence, the sensible holographic interpretation of the massless bulk pseudoscalar is that its leading behaviour determines the marginal coupling of a $\Delta = 3$ operator; the expectation value of the operator itself is determined by the subleading behaviour of the bulk pseudoscalar.

Another equivalent formulation of this bulk theory is obtained by writing

$$*_4 dF = \frac{1}{3} H. \quad (12.53)$$

with H a three-form field. This is the parameterization that would be most familiar from string theory, as the system simply corresponds to an antisymmetric two-form field. In this formulation, we write

$$\begin{aligned} I_{\text{KR}} &= \hat{S}_{\text{EH}}[e, \overset{\circ}{\omega}] + I_{\text{GH}}[e, \overset{\circ}{\omega}] + \frac{1}{3} \int_M H \omega *_4 H + \sqrt{\frac{2}{3}} \int_M C \omega d *_4 H \\ &= \hat{S}_{\text{EH}}[e, \overset{\circ}{\omega}] + I_{\text{GH}}[e, \overset{\circ}{\omega}] - \frac{1}{2} \int_M dC \wedge *_4 dC + \int_M d(C \omega *_4 dC). \end{aligned} \quad (12.54)$$

In the first equation, C appears as a Lagrange multiplier for the ‘Gauss constraint’ and in the second expression, we have solved for the H equation of motion in the bulk, which is just $H = \sqrt{\frac{3}{2}} dC$.

12.4.2 The 3 + 1-Split of the Pseudoscalar Nieh–Yan Model

To investigate the holographic aspects of our model it is most useful to use the ‘radial quantization’ in which we think of the radial coordinate as ‘time’ t . The Nieh–Yan deformation gives

$$\begin{aligned} -2 \int dF \wedge T^a \wedge e_a &= 2 \int dt \omega \{ -\dot{F} \tilde{T}_\alpha \omega \tilde{e}^\alpha - \dot{\tilde{e}}_\alpha \omega \tilde{d}F \omega \tilde{e}^\alpha \\ &\quad + N[2\tilde{d}F \omega K_\alpha \omega \tilde{e}^\alpha] + N^\alpha[2\tilde{d}F \omega \tilde{T}_\alpha] + Q^\alpha[-\sigma_{\epsilon_{\alpha\beta\gamma}} \tilde{d}F \omega \tilde{e}^\beta \omega \tilde{e}^\gamma] \}. \end{aligned} \quad (12.55)$$

We see that the F field makes a contribution to the constraints, and has a conjugate momentum proportional to the scalar part of the torsion (the part transverse to the radial direction). The full bulk action becomes

$$\begin{aligned} I &= \int dt \wedge \left(\dot{\tilde{e}}^\alpha \wedge (4\sigma_\perp \epsilon_{\alpha\beta\gamma} K^\gamma \wedge \tilde{e}^\beta - 2\tilde{d}F \wedge \tilde{e}_\alpha) - 2\dot{F}(\tilde{e}^\alpha \wedge \tilde{T}_\alpha) \right. \\ &\quad \left. + N \left\{ 2\epsilon_{\alpha\beta\gamma} \left({}^{(3)}R^{\alpha\beta} - \sigma_\perp K^\alpha \wedge K^\beta - \frac{\Lambda}{3} \tilde{e}^\alpha \wedge \tilde{e}^\beta \right) \wedge \tilde{e}^\gamma + 4\tilde{d}F \wedge K_\alpha \wedge \tilde{e}^\alpha \right\} \right) \end{aligned}$$

$$\begin{aligned}
& + 4N^\alpha \left\{ -\sigma_\perp \epsilon_{\alpha\beta\gamma} (\tilde{D}K)^\beta \wedge \tilde{e}^\gamma + \tilde{d}F \wedge \tilde{T}_\alpha \right\} \\
& + 4Q^\alpha \left\{ (K_\beta \wedge \tilde{e}^\beta) \wedge \tilde{e}_\alpha - \frac{1}{2} \sigma \epsilon_{\alpha\beta\gamma} \tilde{d}F \wedge \tilde{e}^\beta \wedge \tilde{e}^\gamma \right\} \\
& + 4q^0_\alpha \left\{ \epsilon^\alpha_{\beta\gamma} \tilde{T}^\beta \wedge \tilde{e}^\gamma \right\} \Bigg). \tag{12.56}
\end{aligned}$$

We notice that the Q -constraint term can be written in the form

$$4Q_\alpha \tilde{e}^\alpha \omega (K_\beta \wedge \tilde{e}^\beta - \sigma *_3 \tilde{d}F). \tag{12.57}$$

Because of this constraint (which relates the antisymmetric part of the extrinsic curvature to the vorticity of F), the momentum conjugate to $\tilde{e}^\alpha$ is symmetric, i.e.

$$\begin{aligned}
\Pi_\alpha &= 4\sigma_\perp \epsilon_{\alpha\beta\gamma} K^\gamma \wedge \tilde{e}^\beta - 2\tilde{d}F \wedge \tilde{e}_\alpha \\
&= 4\sigma_\perp \left(\epsilon_{\alpha\beta\gamma} K^\gamma \wedge \tilde{e}^\beta - \frac{1}{2} \sigma_3 *_3 (K_\beta \omega \tilde{e}^\beta) \omega \tilde{e}_\alpha \right). \tag{12.58}
\end{aligned}$$

When written out in components, one finds that the antisymmetric part $K_{[\alpha\beta]}$ cancels

$$\Pi_\alpha = 4\sigma_\perp (K_{(\beta\alpha)} - \text{tr} K \eta_{\beta\alpha}) \tilde{e}^\beta. \tag{12.59}$$

This result is consistent with the fact noted above, that the system may be equivalently described as a pseudoscalar field coupled to torsionless gravity. Moreover, if we take the deDonder gauge $d^\dagger \tilde{e}^\alpha = 0$, the torsion constraint implies that B is symmetric.

The q^0_α constraint yields $\tilde{T}^\beta_{\alpha\beta} = 0$. Out of the nine components of $\tilde{T}$, which transform as $\mathbf{5} + \mathbf{3} + \mathbf{1}$ under $SO(3)$ (or $SO(2, 1)$), this sets the triplet to zero (the $\mathbf{5}$ also vanishes on an equation of motion). The momentum conjugate to F is given by

$$\Pi_F = -2\epsilon^{\alpha\beta\gamma} \tilde{T}_{\alpha\beta\gamma}. \tag{12.60}$$

This is the singlet part of the torsion, which has become dynamical in this description of the theory, in the sense that it is canonically conjugate to F .

12.5 The Torsion Domain Wall

We will now simplify the analysis by taking a coordinate basis and looking for solutions of the form

$$\tilde{e}^\alpha = e^{A(t)} dx^\alpha, \quad N = 1, \quad N^\alpha = 0, \tag{12.61}$$

and we will further suppose that $F = F(t)$. In this case K^α and B^α reduce to one degree of freedom each as a result of the constraints

$$K_\alpha = k\tilde{e}_\alpha, \quad B_\alpha = b\tilde{e}_\alpha, \quad (12.62)$$

and one finds $\Pi_A = -4\sigma_\perp k$ and $\Pi_F = 2\sigma b$. The action then takes the following relatively simple Hamiltonian form

$$I_{\text{NY}} \propto \int dt d^3x e^{3A(t)} \left[\dot{A}\Pi_A + \dot{F}\Pi_F - \left(\frac{1}{2}\sigma_3\Pi_F^2 + \frac{1}{8}\sigma_\perp\Pi_A^2 + \frac{2}{3}\Lambda \right) \right]. \quad (12.63)$$

and the equations of motion give

$$\begin{aligned} \dot{\Pi}_A &= 3\dot{F}\Pi_F, & \dot{\Pi}_F + 3\Pi_F\dot{A} &= 0, & \Pi_A &= 4\sigma_\perp\dot{A}, & \Pi_F &= \sigma_3\dot{F}, \\ \Pi_A^2 + 4\sigma\Pi_F^2 + \frac{16}{3}\sigma_\perp\Lambda &= 0. \end{aligned} \quad (12.64)$$

These equations of motion could of course alternatively be obtained by considering the theory in the form (12.52). It is convenient to rescale $F(t) = \frac{1}{3}\Theta(t)$. Then the equations of motion can be put in the form

$$\dot{A} + 3\dot{A}^2 - 3a^2 = 0, \quad \dot{A} = \frac{1}{12}\sigma\dot{\Theta}^2, \quad \dot{\Theta} + 3\dot{\Theta}\dot{A} = 0. \quad (12.65)$$

where we have set $\Lambda = -3\sigma_\perp a^2$ with $a = 1/L$. These are of the standard form of domain wall equations that have appeared numerous times in the AdS/CFT literature. However, there is a crucial difference. Notice that the first two of (12.65) imply

$$\dot{A}^2 + \frac{1}{36}\sigma\dot{\Theta}^2 - a^2 = 0. \quad (12.66)$$

For Euclidean signature ($\sigma = \sigma_3 = 1$) the second term in (12.66) has *positive* sign in contrast to most of the other holographic studies. This is due to the fact that in passing from Lorentzian to Euclidean signature the pseudoscalar kinetic term acquires the ‘wrong sign’ [39]. This property allows for a remarkable exact solution to the above system of non-linear equations in Euclidean signature, which we refer to as the *torsion DW*. To obtain it we define

$$h(t) = \dot{A}(t), \quad (12.67)$$

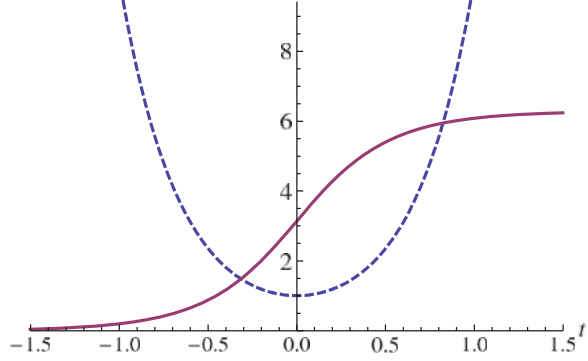
at which point we have

$$\dot{h} = \frac{1}{12}\dot{\Theta}^2, \quad \dot{h} + 3(h^2 - a^2) = 0. \quad (12.68)$$

The general solution is of the form (Fig. 12.1)

$$h(t) = a \tanh 3a(t - t_0) \quad (12.69)$$

Fig. 12.1 Horizon versus parameter r_0 for $g = 3$ (left panel) and $g = 0.0005$ (right panel)



and we then have

$$\Pi_F = \dot{F} = \pm 2\sqrt{a^2 - h^2(t)} = \pm 2a \operatorname{sech} 3a(t - t_0) \quad (12.70)$$

which gives

$$\Theta(t) = \Theta_0 \pm 4 \arctan\left(e^{3a(t-t_0)}\right). \quad (12.71)$$

The $\pm$ sign corresponds to kink/antikink and we will without loss of generality choose the $+$ sign. We may also solve for

$$e^{A(t)} = \alpha(2 \cosh 3a(t - t_0))^{1/3} \quad (12.72)$$

The parameter α is an arbitrary positive integration constant that sets the overall scale of the spatial part of the metric. t_0 may be interpreted as the position of the DW; when $t_0 = 0$ the torsion DW sits in the middle between the two asymptotically AdS_4 regimes. Below, we will discuss the interesting holographic interpretation of the torsion DW.

Note the curvature and torsion of this solution:

$$R^\alpha{}_\beta = -\dot{F}\dot{A}\epsilon^\alpha{}_{\beta\gamma}dt\omega e^\gamma - a^2 e^\alpha\omega e_\beta, \quad (12.73)$$

$$R^\alpha{}_0 = (\dot{h} + h^2)dt\omega e^\alpha - \frac{1}{2}\dot{F}\dot{A}\epsilon^\alpha{}_{\beta\gamma}e^\beta\omega e^\gamma, \quad (12.74)$$

$$T^\alpha = -\frac{1}{2}\dot{F}\epsilon^\alpha{}_{\beta\gamma}e^\beta\omega e^\gamma, \quad (12.75)$$

$$T^0 = 0. \quad (12.76)$$

These are non-singular for all $t \in (-\infty, \infty)$. The torsion DW solution has divergent action, but this divergence is cancelled by boundary counterterms, the same counterterms which render the action of AdS_4 finite. To see this, the energy

of the torsion DW can be computed by evaluating the Euclidean action on the solution. Introducing a cutoff at $t = \pm L$, we find

$$I_{IV, \text{on-shell}} = 4a^2 \int \epsilon_{\alpha\beta\gamma} dx^\alpha \omega dx^\beta \omega dx^\gamma \int dt e^{3A(t)} \quad (12.77)$$

$$= \left(6 \int \widehat{Vol}_3 \right) \cdot \left(\frac{4}{3} a \alpha^3 e^{3aL} + \dots \right), \quad (12.78)$$

where the ellipsis contains terms that vanish when the cutoff is removed. As in pure AdS_4 , an appropriate counterterm is of the form [40, 41]

$$I_{c.t.} = -\frac{4a}{3} \int_{\partial M} \epsilon_{\alpha\beta\gamma} \tilde{e}^\alpha \omega \tilde{e}^\beta \omega \tilde{e}^\gamma. \quad (12.79)$$

In the present case, we have such a counterterm on *each* asymptotic boundary, and thus we find

$$I_{c.t.} = -2 \frac{2a}{3} \alpha^3 e^{3aL} \cdot \left(6 \int \widehat{Vol}_3 \right), \quad (12.80)$$

which exactly cancels the divergent energy of the torsion DW.

Furthermore, we note that in the Kalb–Ramond representation, the solution has

$$H = \dot{\Theta} Vol_3 = \pm 6a \alpha^3 \widehat{Vol}_3 \equiv \hat{H} \widehat{Vol}_3, \quad (12.81)$$

where $\widehat{Vol}_3 = \frac{1}{6} \epsilon_{\alpha\beta\gamma} dx^\alpha \omega dx^\beta \omega dx^\gamma$. This corresponds to a ‘topological quantum number’ of the kink

$$\int *_4 H = \pm \Delta \Theta = \pm 2\pi. \quad (12.82)$$

12.6 The Gravity Dual of Parity Symmetry Breaking

The holographic interpretation of the torsion DW is rather interesting. To study this, we set to zero without loss of generality the integration constant $\Theta_0 = 0$ and pick the plus sign in (12.70), (12.71). Next we need the asymptotic expansion of the vierbein which reads

$$\tilde{e}^\alpha = 2^{-1/3} \alpha e^{\pm a(t-t_0)} \left(1 + \frac{1}{3} e^{\mp 6a(t-t_0)} + \dots \right) dx^\alpha \quad \text{for } t \rightarrow \pm\infty. \quad (12.83)$$

This shows that our solution is asymptotically anti-de Sitter for both $t \rightarrow \pm\infty$. The two asymptotic AdS spaces have the same cosmological constant. From this

expansion we could read the expectation value of the renormalized boundary energy momentum tensor which would be given by the coefficient of the $e^{\pm 3at}$ term (see e.g. [19, 20]). Such a term is missing in (12.83), hence the expectation value of the boundary energy momentum tensor is zero.

It is not immediately apparent how to interpret these two asymptotic regimes. Are they truly distinct, or should they be identified in some way? We note that the pseudoscalar behaves in these asymptotic regimes as

$$\Theta(t) \rightarrow 4e^{-3a(t-t_0)} - \frac{4}{3}e^{-9a(t-t_0)} + \dots \quad \text{for } t \rightarrow -\infty, \quad (12.84)$$

$$\Theta(t) \rightarrow 2\pi - 4e^{3a(t-t_0)} + \frac{4}{3}e^{9a(t-t_0)} + \dots \quad \text{for } t \rightarrow +\infty. \quad (12.85)$$

From the above we confirm that $\Theta(t)$ is dual to a dimension $\Delta = 3$ boundary pseudoscalar that we denote $\mathcal{O}_3$. In each one of the asymptotically AdS regimes, the leading constant behavior of $\Theta(t)$ corresponds to the source (i.e., coupling constant) for $\mathcal{O}_3$ and the subleading term proportional to $e^{\mp 3a(t-t_0)}$ to the expectation value $\langle \mathcal{O}_3 \rangle$. The two asymptotic regimes are distinguished by the behavior of Θ . In fact, the essential difference is *parity*.

We can now describe the holography of our torsion DW. In the $t \rightarrow -\infty$ boundary sits a three-dimensional CFT at a parity breaking vacuum state. The order parameter is the expectation value of the pseudoscalar which is $\langle \mathcal{O}_3 \rangle = 4$ in units of the AdS radius. The expectation value breaks of course the conformal invariance of the boundary theory. Then, the theory is deformed by the same pseudoscalar operator $g\mathcal{O}_3$ where g is a marginal coupling. The torsion DW provides the holographic description of that deformation. Nevertheless, our solution should not be interpreted in terms of the usual holographic renormalization group flow. In our case, at $t \rightarrow +\infty$ the space becomes AdS with the *same* radius as at $t \rightarrow -\infty$. Hence, the two boundary theories have the same ‘central charges’.⁵

We suggest that instead of interpreting the solution in terms of an RG flow, we should think of it as a domain wall transition between two inequivalent vacua of a single theory. This statement is supported by the behavior of $\Theta(t)$ in the two asymptotic regimes. For $t \rightarrow \infty$ the pseudoscalar asymptotes to the configuration (12.85). The interpretation is now that when the marginal coupling takes the fixed value $g_* = 2\pi$ we are back to the *same* CFT (i.e. having the same central charge) however in a distinct parity breaking vacuum such that $\langle \mathcal{O}_3 \rangle = -4$. In others words, the two asymptotic AdS regimes seem to describe two distinct parity breaking vacua of the same theory. The two vacua are distinguished by the expectation value of the parity breaking order parameter being $\langle \mathcal{O}_3 \rangle = \pm 4$. Quite remarkably, we also seem to find that starting in one of the two vacua, we can

⁵ We use “central charge” in $d = 3$ for a quantity that counts the massless degrees of freedom at the fixed point. Such a quantity may be taken to be the coefficient in the two-point function of the energy momentum tensor or the coefficient of the free energy density. There is no conformal anomaly in $d = 3$.

reach the other by a marginal deformation with a *fixed* value of the deformation parameter.

Since the marginal operator is of dimension $\Delta = 3$ and parity odd, we tentatively identify it with a Chern–Simons operator of a boundary gauge field. In this case the torsion DW induces the T-transformation in the boundary CFT [38, 42]. In [Appendix](#) we will argue that the 3D Gross–Neveu model coupled to abelian gauge fields exhibits a large- N vacuum structure that matches our holographic findings. Although our bulk model is extremely simple to provide details for its possible holographic dual, we regard this remarkable similarity as strong qualitative evidence that our torsion DW is the gravity dual of the ‘tunnelling’ between different parity breaking vacua in three dimensions. However, in a three-dimensional quantum field theory, we do not expect that tunnelling can occur because of large volume effects, and distinct vacua remain orthogonal. Thus, referring to the torsion DW as a tunnelling event should be taken figuratively. We leave to future work a more careful study of the boundary interpretation of the torsion DW solution. An interpretation will depend on the precise topology of the boundary [43]. Moreover, embedding our model into M-theory could provide additional clues regarding its holography.

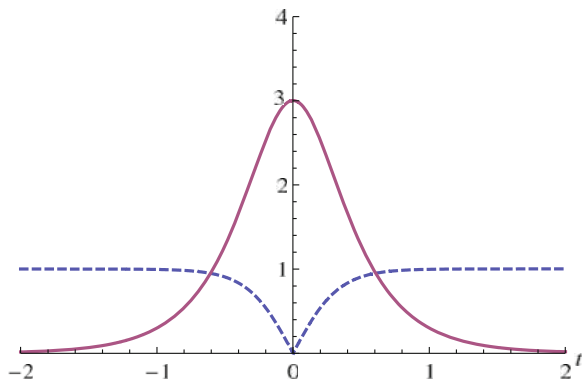
12.7 Physics in the Bulk: The Superconductor Analogy

The bulk interpretation of the exact solution is also interesting. Because the pseudoscalar field undergoes $\Theta(t) \rightarrow \Theta(t) + 2\pi$ under t goes from $-\infty$ to $+\infty$, the exact solution corresponds to a topological kink. It satisfies

$$\int dt \dot{\Theta} = 2\pi$$

In [Fig. 12.2](#), we plot the solution.

Fig. 12.2 The *blue dashed line* is $|h(t)|$, resembling the order parameter of a superconductor, while the *solid red line* is Π_F , analogous to the magnetic induction of an Abrikosov DW



12.7.1 Torsion Domain Wall Versus Abrikosov Vortex

The gravity DW solution (12.68–12.71) bears some resemblance to the Abrikosov vortex of superconducting systems. To avoid confusion we emphasize here that this is *not* a holographic interpretation i.e. the Abrikosov vortex (or more precisely, domain wall) is in the bulk. In this section, we will explore this and point out some possibly interesting features. The first thing to notice is that the plot in Fig. 12.2 is identical to the profile of an Abrikosov vortex (see for example Fig. 5.1 in Ref. [44].) The codimension differs and this is expected; the torsion DW supports a three-form field strength in contrast to a two-form field strength supported by the Abrikosov vortex. Nevertheless, there is a correspondence between our radial t -direction and the radial direction in the Abrikosov vortex, and $|h|$ and Π_F correspond to the condensate and magnetic induction of the superconductor, respectively. Table 12.1 summarizes the correspondence. In this correspondence, since the order parameter is $h = \dot{A}$, the superconducting phase (constant order parameter) corresponds to AdS_4 , while the normal phase corresponds to flat space ($h = 0$). Far away from the core of the torsion DW, the geometry is asymptotically AdS , but at the core the spatial slice (at $t \rightarrow t_0$) becomes flat. To see this, note that if we think of the system as a pseudoscalar coupled to torsionless gravity, the torsion DW has $\overset{\circ}{\omega}{}^\alpha{}_\beta = 0$ and $\overset{\circ}{\omega}{}^\alpha{}_0 = \dot{A}\tilde{e}^\alpha$, and so

$$\overset{\circ}{R}{}^\alpha{}_\beta = -h^2 \tilde{e}^\alpha \omega \tilde{e}_\beta, \quad (12.86)$$

$$\overset{\circ}{R}{}^\alpha{}_0 = (\dot{h} + h^2) dt \omega \tilde{e}^\alpha, \quad (12.87)$$

$$\overset{\circ}{T}{}^\alpha = 0. \quad (12.88)$$

Thus, at the core, we find that the Riemann tensor has components

$$R^\alpha{}_{0\alpha 0} \rightarrow -3a^2 \alpha, \quad (12.89)$$

$$R^\alpha{}_{\beta\alpha\beta} \rightarrow 0. \quad (12.90)$$

This behavior is in line with an Abrikosov vortex in which there is normal phase at the core and superconducting phase away from the core.

Table 12.1 Abrikosov vortex versus torsion DW

Abrikosov vortex	Torsion DW
Order parameter Φ	Order parameter $ h = \dot{A} $
$T - T_c$	Λ
Magnetic induction B	Π_F
Magnetic field H	$\hat{H}$
$\mathbb{Z}$ -quantized magnetic flux	$\mathbb{Z}_2$ -quantized electric flux

The analogue of the magnetic field is what we have called $\hat{H}$, proportional to the constant α^3 . In the DW, the magnetic induction, analogous to Π_F , has a penetration length $\lambda \sim 1/3a$, and the coherence length of the order parameter is $\xi \sim 1/6a$. The penetration and coherence lengths are obtained by looking at the exponential fall-off of these quantities in the core of the DW, away from their values in the superconducting phase.

The torsion DW also has a quantized flux $\int *_4 H = \Delta\Theta = 2\pi$. This flux is independent of any parameters of the solution and of any rescaling of fields in the theory. Thus, this is an analogue of the quantized magnetic flux in superconductivity.

Finally, note the following interesting feature. If we take a derivative of the second equation in (12.68), we arrive at

$$\ddot{h} - 6\Lambda h - 18h^3 = 0. \quad (12.91)$$

This looks like a Landau–Ginzburg equation of motion of an effective ϕ^4 theory. This leads us to interpret

$$\Lambda = -3\sigma_\perp a^2 = -3\sigma_\perp \frac{1}{L^2} \sim T_c - T \quad (12.92)$$

Of course, there is no real temperature in the case of the torsion DW, but we note that this implies that the penetration and coherence lengths diverge as

$$(T - T_c)^{1/2} \quad (12.93)$$

i.e. with a mean field theory critical exponent $1/2$, as in superconductivity.

12.7.2 Domain Wall Condensation

In the last section, we noted that there is a strong analogue between the torsion DW solution and superconductivity. It is intriguing to carry the analogy further and consider multi-DW configurations. We have noted that at the core of the torsion DW, the spatial sections are flat. Thus, one might imagine that if it was favourable for torsion DWs to condense, as DWs do in Type I superconductors, then finite regions of normal phase (corresponding to $\Lambda = 0$) would be obtained. We will argue below that this can in fact occur, although the system appears not to be unstable.

To understand the physics involved, the first step is to consider a configuration of two DWs. In the superconductivity literature, this is a standard computation. One takes two DWs separated by a distance ℓ and computes the Euclidean action. More precisely, we will treat this here as follows. Put the system in a box by restricting the $t \in [-L, L]$. In such a case, a DW located at t_0 has on-shell action

$$e^{A(t-t_0)} = \alpha(2 \cosh 3a(t-t_0))^{\frac{1}{3}} \Rightarrow I_{\text{on-shell}}^L = (6Vol_3) \frac{4}{3} a \alpha^3 e^{3a(L-t_0)}. \quad (12.94)$$

We then consider a piecewise solution of two DWs located at $t = \pm \frac{\ell}{2}$. This is not a solution of the e.o.m. because solutions of non-linear equations cannot be simply superimposed i.e. it fails at the midpoint between the DWs. However, if we simply evaluate the Euclidean action, we find

$$I_{\text{on-shell}}^{\pm(L/2)} = \left(\frac{4}{3} a^2 \hat{V} ol_3 \right) 4a\alpha^3 \sinh \frac{3aL}{2}. \quad (12.95)$$

Note that this is positive, so one might naively conclude that the DWs repel each other. However, recall that the DW profile exists not in flat space–time, but in the metric given by (12.72), which rises asymptotically. As a result, as we move the DWs further apart, there is a corresponding rise in the metric between the DWs. So, we should directly evaluate the force at a point $t = \ell$ as

$$F = - \frac{dI^{\pm(L/2)}}{d\ell} \propto \cosh(3a\ell/2) < 0. \quad (12.96)$$

Thus we conclude that the DWs in fact *attract* each other. In the superconducting analogue we would conclude that we have a *Type I superconductor* where the vortices clump together forming (potentially) finite regions of normal phase. In such a superconductor, the number of vortices is determined by the total magnetic flux.

We now describe the analogous situation in our gravitational system. We have noted that the constant $\hat{H}$ plays the role of the external magnetic induction, while H is the magnetic field, varying within the DW, with $\Delta\Theta = \int *_4 H$. Following the superconducting analogue, if we put the system in a box of size $2L$ (that is we impose a cutoff on each AdS asymptotic) the flux conservation equation is of the form

$$\Delta\Theta = 2L\hat{H} \quad (12.97)$$

The DWs carry the flux in the superconductor, and so it is natural to ask what is the lowest energy configuration satisfying (12.97)? To analyze this, consider an array of n DWs in a region of size L_0 . We take the DWs to be equally spaced, as one can show that deviating from such a configuration causes a rise in energy. For such a configuration, the flux quantization condition (12.97) gives a relation between n, L_0 and $\hat{H}$. Such a representative curve is shown in Fig. 12.3.

If we solve this equation for L_0 as a function of n and $\hat{H}$, we can then compute the energy as a function of n . One obtains a curve as in Fig. 12.4.

One notes that the energy is minimized for large n , and in that case, the size L_0 asymptotes to a fixed value, which is found to be

$$L_0 = \frac{\hat{H}}{6a} \cdot 2L = \alpha^3 \cdot 2L \quad (12.98)$$

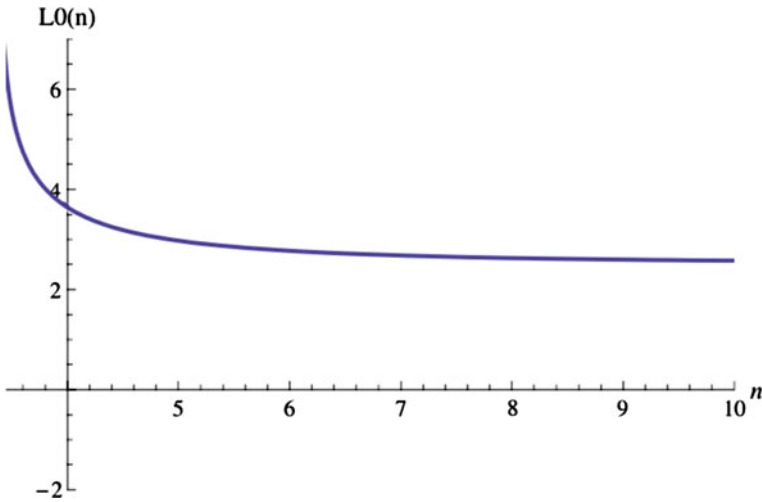
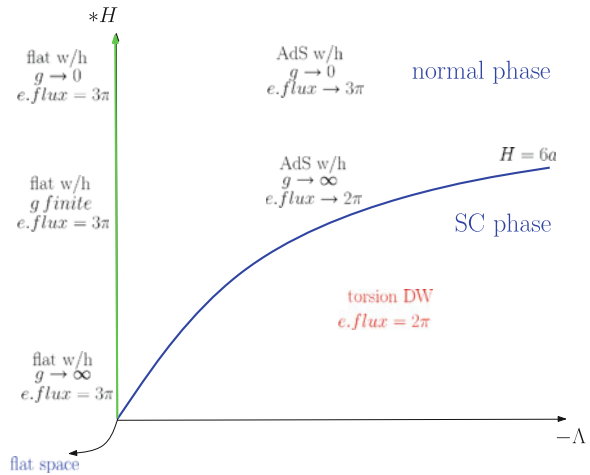


Fig. 12.3 Size of normal state droplet versus n for multi-domain walls

Fig. 12.4 A sketch of the phase diagram for a Kalb–Ramond superconductor



We conclude that the preferred configuration, given a fixed external flux, is a continuum of DWs arrayed over a finite size region. Within this droplet, the system is in the normal phase. We have noted that the DW core is spatially flat, and so we surmise that within the droplet, the space–time is flat. The asymptotic value of energy in Fig. 12.3 is precisely minus that contributed by the cosmological constant. Again, the size of the droplet is set by the value of the external H -flux, and the boundary conditions are AdS. Note that for a fixed cutoff, there is a critical field (given by $\hat{H} = 6a$) for which the entire spacetime is flat.

The result above motivate us to take a further step and suggest rather appealing physical picture that we may call *Kalb–Ramond superconductivity*. Here we present

a qualitative description of Kalb–Ramond superconductivity. A detailed description will appear shortly in [43]. As discussed above, *minus* the cosmological constant may be interpreted as $T - T_c$. This, together with the interpretation of H as a magnetic induction and $\hat{H} = 6a$ as a critical magnetic field above which the spacetime is flat due to DW condensation, motivates to draw a $-\Lambda, *H$ graph analogously with the $T - T_c$, *magnetic field* graph in superconductivity. To do so, we need to consider all known solutions of an Einstein-axion system in $d = 4$ with and without cosmological constant. For $\Lambda = 0$ we recall the axionic DW solutions of [35]. For $\Lambda < 0$ DW wormhole-like solutions were found in [36].

These solutions have the following properties (to be discussed in more detail in [43]).

The $\Lambda = 0$ wormhole solution [35] is asymptotically flat. Its magnetic flux is proportional to an arbitrary constant g which was speculated to be quantized via a string theory embedding of the Einstein-axion system in [35]. Its electric flux is $\mathbb{Z}_2$ -quantized to $\pm 3\pi$, but the electric field varies along the wormhole.

The $\Lambda < 0$ wormhole solution [36] is asymptotically AdS. Its magnetic flux is also proportional to the arbitrary constant g , and hence possibly quantized in a string theory embedding. Quite intriguingly, the electric flux on the wormhole solution interpolates between the value $\pm 3\pi$ for $g \rightarrow 0$ and $\pm 2\pi$ as $g \rightarrow \infty$. Moreover, there is a lower bound for magnetic field of the wormhole solution. Remarkably, this lower bound is the value $6a$ and coincides with the maximal magnetic field of the torsion DW.

In Fig. 5 we sketch the possible phase diagram of the Kalb–Ramond superconductor. Our torsion DW solution seems to play the role of the superconducting phase, while the wormhole solutions of [35] and [36] appear to correspond to the normal phase.

12.8 Conclusions

In this work we have presented in detail a simple toy model, the Nieh–Yan model, where torsion enters through the spacetime dependence of the coupling constant of the Nieh–Yan topological invariant. Although we have discussed the model directly in terms of torsion, it can classically be put into equivalent forms as either a massless pseudoscalar or a Kalb–Ramond field coupled to gravity. The model has an interesting and non-trivial holographic interpretation. In particular, we have shown that it possesses an exact bulk solution in Euclidean signature, termed the torsion DW, having two asymptotically AdS_4 regimes, while the pseudoscalar acquires a kink profile. We have argued then that the holographic interpretation of this torsion DW is a three-dimensional CFT with two distinct parity breaking vacua. Moreover, our bulk solution may imply that the deformation by a classically marginal pseudoscalar with a fixed coupling constant induces a transition between the two parity breaking vacua separated by a domain wall, which would be at infinity in the boundary components [43]. Remarkably, this qualitative behaviour is

seen already in the three-dimensional Gross–Neveu model coupled to $U(1)$ gauge fields. The economy of our bulk model does not allow a detailed identification of the bulk and boundary theories, nevertheless we believe that our results provide a strong base where an exact bulk/boundary dictionary for $\text{AdS}_4/\text{CFT}_3$ can be based. A further rather intriguing property of the torsion DW is that it can be mapped into the standard Abrikosov vortex of superconductivity. Such a map identifies flat spacetime with a superconductor’s normal phase, while AdS is identified with a superconducting phase. The cosmological constant would then measure the deviation from the ‘critical temperature’. A phenomenon of DW condensation is found, similar to the analogous case in type I superconductors. Finally, we have briefly discussed a picture of “Kalb–Ramond superconductivity” that emerges if we view in a unified way all known four-dimensional Euclidean solutions of the Einstein–axion system. This picture will be further analysed in [43].

Our results indicate that the torsional degrees of freedom of four dimensional gravity can provide holographic descriptions for a number of interesting properties of 3D critical systems. It would be interesting to extend our analysis to more elaborate models where more torsional degrees of freedom become dynamical. It is also of interest to discuss whether our simple model can be embedded into M-theory.

Acknowledgments I thank the organizers and in particular Eleftherios Papantonopoulos for the organization of a top quality workshop and the invitation to present this talk. This work is partially supported by the FP7-REGPOT-2008-1 “CreteHEPCosmo” No 228644 and also by the University of Crete ELKE grant with KA 2745. I wish to thank R. G. Leigh and N. N. Hoang for the very fruitful and pleasant collaboration that has led to the results presented in this report.

Appendix

A.1 Parity Breaking in Three Dimensions

Consider the 3D Gross–Neveu model coupled to abelian gauge fields. The Euclidean action is⁶

$$I = - \int d^3x \left[\bar{\psi}^a (\not{\partial} - ie\not{A}) \psi^a + \frac{G}{2N} (\bar{\psi}^a \psi^a)^2 + \frac{1}{4M} F_{\mu\nu} F_{\mu\nu} \right]. \quad (\text{A.1})$$

M is an UV mass scale. Introducing the usual Lagrange multiplier field σ , whose equation of motion is $\sigma = \frac{-2G}{N} \bar{\psi}^a \psi^a$ we can make the action quadratic in the fermions

⁶ We use $\bar{\psi}^i, \psi^i$ ($i = 1, 2, \dots, N$) two-component Dirac fermions. The γ -matrices are defined in terms of the usual Pauli matrices as $\gamma^i = \sigma^i$ $i = 1, 2, 3$.

$$I = - \int d^3x \left[\bar{\psi}^a (\not{\partial} + \sigma - ie\not{A}) \psi^a - \frac{N}{2G} \sigma^2 - \frac{1}{4M} F^{\mu\nu} F_{\mu\nu} \right]. \quad (\text{A.2})$$

The model possesses two parity breaking vacua distinguished by the value of the pseudoscalar order parameter $\langle \sigma \rangle$. This is seen as follows: switching off the gauge fields momentarily one integrates over the fermions to produce a large- N effective action as

$$\mathcal{Z} = \int (\mathcal{D}\sigma) e^N \left[\text{Tr} \log(\not{\partial} + \sigma) - \frac{1}{2G} \int d^3x \sigma^2 \right]. \quad (\text{A.3})$$

The path integral has a non-zero large- N extremum σ_* found by setting $\sigma = \sigma_* + \frac{1}{\sqrt{N}} \lambda$

$$\mathcal{Z} = \int (\mathcal{D}\lambda) e^N \left[\text{Tr} \log(\not{\partial} + \sigma_*) - \frac{1}{2G} \int d^3x \sigma_*^2 + \frac{1}{\sqrt{N}} \left\{ \text{Tr} \frac{\not{\lambda}}{\not{\partial} + \sigma_*} - \frac{\sigma_*}{G} \int d^3x \lambda \right\} + O(1/N) \right] \quad (\text{A.4})$$

The term in the curly brackets is the gap equation. To study it one considers a uniform momentum cutoff Λ to obtain

$$\frac{1}{G} = \int^\Lambda \frac{d^3p}{(2\pi)^3} \frac{2}{p^2 + \sigma_*^2} = (\text{Tr}1) \left[\frac{\Lambda}{\pi^2} - \frac{|\sigma_*|}{\pi^2} \arctan \frac{\Lambda}{|\sigma_*|} \right]. \quad (\text{A.5})$$

Defining the critical coupling as

$$\frac{1}{G_*} = \frac{\Lambda}{\pi^2}, \quad (\text{A.6})$$

(A.5) possesses a non-zero solution for σ_* when $G > G_*$ given by

$$|\sigma_*| = \frac{2\pi}{G} \left(\frac{G}{G_*} - 1 \right) \equiv m. \quad (\text{A.7})$$

The two distinct parity breaking vacua then have

$$\sigma_* = -\frac{2G}{N} \langle \bar{\psi}^a \psi^a \rangle = \pm m. \quad (\text{A.8})$$

Going back to (A.2) one can tune $G > G_*$ and start in any of the two parity breaking vacua. Suppose we start from $\sigma_* = +m$. To leading order in N we have

$$\begin{aligned} \mathcal{Z} &= \int (\mathcal{D}A_\mu) (\mathcal{D}\bar{\psi}^a) (\mathcal{D}\psi^a) \exp[\mathcal{S}] \\ \mathcal{S} &= \int d^3x \left[\bar{\psi}^a (\not{\partial} + m - ie\not{A}) \psi^a - \frac{N}{2G} m^2 + O(1/\sqrt{N}) - \frac{1}{4M} F^{\mu\nu} F_{\mu\nu} \right] \end{aligned} \quad (\text{A.9})$$

As is well known [45, 46] for N fermions the path integral (A.9) yields an effective action for the gauge fields which for low momenta is dominated by the Chern–Simons term i.e.

$$\mathcal{Z} \approx \int e^{S_{\text{CS}}}, \quad (\text{A.10})$$

with

$$S_{\text{CS}} = i \frac{ke^2}{4\pi} \int d^3x \epsilon^{\mu\nu\rho} A_\mu \partial_\nu A_\rho, \quad k = \frac{N}{2}. \quad (\text{A.11})$$

Had we started from the $\sigma_* = -m$ vacuum, we would have found again (A.10–A.11), however with $k = -\frac{N}{2}$, i.e. the vacuum with $\sigma_* = -m$ yields an effective Chern–Simons action with $k = -\frac{N}{2}$.

Consider now *deforming* the action (A.9) by the Chern–Simons term with a fixed coefficient as

$$\begin{aligned} \mathcal{Z}_{\text{deform}} &= \int (\mathcal{D}A_\mu) (\mathcal{D}\bar{\psi}^i) (\mathcal{D}\psi^i) \exp[S_{\text{def}}] \\ S_{\text{def}} &= \mathcal{S} - iq \int d^3x \epsilon^{\mu\nu\rho} A_\mu \partial_\nu A_\rho \end{aligned} \quad (\text{A.12})$$

If q is fixed to

$$q = \frac{Ne^2}{4\pi}, \quad (\text{A.13})$$

the effective action for the gauge fields resulting from the fermionic path integrals in (A.12) is going to be *equal* to the one obtained had we started at the $\sigma_* = -m$ vacuum. In other words, deforming the $\sigma_* = +m$ vacuum with a Chern–Simons term with a fixed coefficient is equivalent to being in the $\sigma_* = -m$ vacuum. This is exactly analogous to the holographic interpretation of our torsion DW.

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